View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

PERGAMON
Applied Mathematics Letters 15 (2002) 623–626
Applied Mathematics Letters
www.elsevier.com/locate/aml

Adjacent Strong Edge Coloring of Graphs

ZHONGFU ZHANG
Institute of Applied Mathematics, Lanzhou Railway Institute
Lanzhou 730070, P.R. China

LINZHONG LIU
Department of Transport. Management, Lanzhou Railway Institute,
Lanzhou 730070, P.R. China
LLZH@public.Lz.gs.cn

JIANFANG WANG
Institute of Applied Mathematics, Chinese Academy of Sciences
Beijing 100080, P.R. China

(Received April 1999; revised and accepted April 2001)

Abstract—For a graph $G(V,E)$, if a proper k-edge coloring f is satisfied with $C(u) \neq C(v)$ for $uv \in E(G)$, where $C(u) = \{f(uv) \mid uv \in E\}$, then f is called *k-adjacent strong edge coloring of G*, is abbreviated *k-ASEC*, and

$$\chi'_{as}(G) = \min\{k \mid k\text{-ASEC of } G\}$$

is called the *adjacent strong edge chromatic number of G*. In this paper, we discuss some properties of $\chi'_{as}(G)$, and obtain the $\chi'_{as}(G)$ of some special graphs and present a conjecture: if G are graphs whose order of each component is at least six, then $\chi'_{as}(G) \leq \Delta(G)+2$, where $\Delta(G)$ is the maximum degree of G. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Graph, Adjacent strong edge coloring, Adjacent strong edge coloring chromatic number.

1. INTRODUCTION

The coloring problem of graphs is widely applied in practice. In [1], some conditional coloring problems are introduced. Some network problem can be converted to the strong edge coloring and adjacent strong edge coloring.

DEFINITION 1. *For a graph $G(V,E)$, if a proper k-edge coloring f is satisfied with $C(u) \neq C(v)$ for $uv \in E(G)$, where $C(u) = \{f(uv) \mid uv \in E\}$, then f is called k-adjacent strong edge coloring of G, is abbreviated k-ASEC, and*

$$\chi'_{as}(G) = \min\{k \mid k\text{-ASEC of } G\}$$

is called the adjacent strong edge chromatic number of G.

*Author to whom all correspondence should be addressed.
This research is supported by NSFC (No. 19871036).

0893-9659/02/$ - see front matter © 2002 Elsevier Science Ltd. All rights reserved. Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX
PII: S0893-9659(01)00181-1

DEFINITION 2. *If T is a tree, $u, v \in V(T)$, if $d(u) = d(v) = \Delta(T)$ and $p = |V(T)| = 2\Delta(T)$, then T is called a double star.*

In this paper, we study the adjacent strong edge coloring of graphs.

The existence of $\chi'_{as}(G)$ for simple connected graphs whose order is at least three is obvious, and $\chi'_{as}(G) \geq \Delta(G)$, where $\Delta(G)$ is the naximum degree of graph G.

The other terminology can be found in [1–3].

2. MAIN RESULTS

THEOREM 1. *If G is composed of n connected components $G_1, G_2, \ldots, G_n$, and $|V(G_i)| \geq 3$ $(i = 1, 2, \ldots, n)$, then*

$$\chi'_{as}(G) = \max\{\chi'_{as}(G_i) \mid i = 1, 2, \ldots, n\}.$$

By this theorem, we can assume that all graphs are connected in following process.

THEOREM 2. *Let G be a connected graph with $|V(G)| \geq 3$, if $\chi'_{as}(G) = \Delta(G)$, then for all $u \in V$ with degree $d(u) = \Delta(G)$, if $uv \in E$, then $d(v) < \Delta(G)$.*

Using $N(v)$ denotes the set of adjacent vertices of v in the following.

LEMMA 1. *Supposing G is a tree other than a double star. If G has two adjacent maximum degree vertices, then there is $u \in V(G)$, such that $uv, uv_i \in E(G)$ $(i = 1, 2, \ldots, d(u) - 1)$, $d(u) \geq 2$, $d(v_i) = 1$, $(i = 1, 2, \ldots, d(u) - 1)$.*

This lemma is obviously true.

THEOREM 3. *For a tree T with $p = |V(G)| \geq 3$,*

(1) *if any two vertices of maximum degree are not adjacent, then $\chi'_{as}(T) = \Delta(T)$;*
(2) *if T has two vertices of maximum degree which are adjacent, then $\chi'_{as}(T) = \Delta(T) + 1$.*

PROOF. Only to prove (2), the proof of (1) is similar to (2). Obviously, $\Delta(T) \geq 2$.

By Theorem 2, we have $\chi'_{as}(T) \geq \Delta(T) + 1$.

Now using induction on $p = |(G)|$ to prove the existence of realization function f so that

$$|\{f(e) \mid e \in E(T)\}| = \Delta(T) + 1,$$

when $p = 2\Delta(T)$, the tree T is double star, without loss of generality, let

$$\begin{gathered} d(u) = d(v) = \Delta(T), \quad \text{and} \quad uu_i, vv_i \in E(T)\ (i = 1, 2, \ldots, \Delta(T) - 1), \\ f(uv) = 1, \qquad f(uu_i) = i + 1\ (i = 1, 2, \ldots, \Delta(T) - 1), \\ f(vv_i) = i + 2\ (i = 1, 2, \ldots, \Delta(T) - 1). \end{gathered}$$

By this function f, we know that $\chi'_{as}(T) \leq \Delta(T) + 1$.

Combining with $\chi'_{as}(T) \geq \Delta(T) + 1$, the conclusion is true.

Supposing the conclusion is true for p $(p \geq 2\Delta(T))$, now we prove it is true for $p + 1$.

Here T is not a double star. By Lemma 1, select u as Lemma 1, $uv \in E(T)$, $d(v) = 1$. Let

$$T' = T - v,$$

then T' also has two vertices of maximum degree which are adjacent, and $\Delta(T') = \Delta(T)$, by the induction hypothesis,

$$\chi'_{as}(T') = \Delta(T') + 1 = \Delta(T) + 1.$$

Supposing f is a realization function of T', $C = \{f(e) \mid e \in E(T')\}$, $C_{T'}(z) = \{f(zy) \mid y \in E(T')\}$.

Let $w \in N(u)$, $d(w) \geq 2$. Notice that $|C_{T'}(w)| \leq \Delta(T)$ and $|C_{T'}(u)| \leq \Delta(T) - 1$.

CASE 1. If $C_{T'}(w) \subset C_{T'}(u)$, let

$$f^*(e) = \begin{cases} f(e), & e \in E(T'), \\ \alpha, & \alpha \in C \setminus C_{T'}(u), \quad e = uv. \end{cases}$$

Obviously, the function f^* is a realization function.

CASE 2. If $C_{T'}(u) \subset C_{T'}(w)$, let

$$f^*(e) = \begin{cases} f(e), & e \in E(T'), \\ \alpha, & e = uv, \quad \alpha \in C \setminus C_{T'}(u), \text{ and } \alpha \overline{\in} C_{T'}(w). \end{cases}$$

Obviously, f^* is a realization function of T.

Combining the above two cases, we know that

$$\chi'_{as}(T) = \Delta(G) + 1.$$

THEOREM 4. *For cycle C_p, we have*

$$\chi'_{as}(C_p) = \begin{cases} 3, & \text{for } p \equiv 0 \ (\text{mod}3), \\ 4, & \text{for } p \not\equiv 0 \ (\text{mod}3) \text{ and } p \neq 5, \\ 5, & \text{for } p = 5. \end{cases}$$

Theorem 4 is obviously true.

THEOREM 5. *For the complete bipartite graph $K_{m,n}$ $(1 \leq m \leq n)$, we have*

$$\chi'_{as}(K_{m,n}) = \begin{cases} n, & \text{for } m < n, \\ n+2, & \text{for } m = n \geq 2. \end{cases}$$

PROOF. When $m < n$, the realization function is easy to construct; the proof is omitted. When $m = n \geq 2$, because the combinatorial number of n from $n+1$ is $n+1$, it is easy to see that $\chi'_{as}(K_{n,n}) \geq n+2$.

We now prove the existence of realization function f so that $\chi'_{as}(K_{n,n}) = n+2$.

Supposing the independent sets of two parts of $K_{n,n}$ are $\{u_1, u_2, \ldots, u_n\}$, $\{v_1, v_2, \ldots, v_n\}$, f is defined as

$$\begin{aligned} &f(u_1 v_j) = j, \qquad (i = 2, \ldots, n, \ j = 1, 2, \ldots, n), \\ &f(u_i v_j) = i + j \ (\text{mod}(n+2)). \end{aligned}$$

Obviously, f is a realization function of $K_{n,n}$, so the conclusion is true.

THEOREM 6. *For the complete graph K_p $(p \geq 3)$, we have*

$$\chi'_{as}(K_p) = \begin{cases} p, & \text{for } (p \equiv 1 \ (\text{mod}2)), \\ p+1, & \text{for } (p \equiv 0 \ (\text{mod}2)). \end{cases}$$

PROOF. Suppose $V(K_p) = \{v_1, v_2, \ldots, v_p\}$.

If $p \equiv 1 \ (\text{mod } 2)$ and $p \geq 3$, obviously, $\chi'_{as}(K_p) \geq p$ $(p \equiv 1 \ (\text{mod } 2))$.

We now give a p-adjacent strong edge coloring σ as follows:

$$\begin{aligned} &\sigma(v_i v_j) = i + j - 2, (\text{mod } p)(i = 1, 2, \ldots, p-1, \ j = i+1, i+2, \ldots, p) \quad \text{and} \quad i + j - 2 \neq p, \\ &\sigma(v_i v_j) = p, \qquad i + j - 2 = p. \end{aligned}$$

It is easy to see that σ is a p-strong edge coloring of K_p.

If $p \equiv 0 \pmod 2$ and $p \geq 4$, we define a $(p+1)$-adjacent strongt edge coloring σ as follows.

First we prove $\chi'_{as}(K_p) \geq p+1$. Otherwise, if $\chi'_{as}(K_p) = p$, then for all $v \in V(K_p)$, we have $|C(v)| = p - 1$ and the colors which occur on distinct vertices are different, that is, each color occurs on $p-1$ vertices so that each color occurs on odd vertices. Obviously, it is impossible, hence, $\chi'_{as}(K_p) \geq p+1$.

We now give a $(p+1)$-adjacent strong edge coloring of K_p as follows:

$$\sigma(v_i v_j) = i + j - 2, (\text{mod p}),$$
$$(i = 1, 2, \ldots, p-1,\ j = i+1,\ i+2, \ldots, p) \quad \text{and} \quad i + j - 2 \neq p - 1,$$
$$\sigma(v_i v_j) = p, \qquad i + j - 2 = p + 1.$$

It is easy to see that σ is a $(p+1)$-strong edge coloring of K_p.

Hence, the conclusion is true.

THEOREM 7. *If G is a graph which has two adjacent maximum degree vertices, then $\chi'_{as}(G) \geq \Delta(G) + 1$.*

THEOREM 8. *If G is a graph, the degree of any two adjacent vertices is different, then $\chi'_{as}(G) = \Delta(G)$.*

Theorem 7 and 8 are obviously true.

If H is a subgraph of G, it is interesting that $\chi'_{as}(H) \leq \chi'_{as}(G)$ is not always true.

Open Problem

If H is a proper subgraph of G, when $\chi'_{as}(H) \leq \chi'_{as}(G)$ is true?

Combining with above conclusion, we present the following conjecture.

CONJECTURE. *For any connected graph G ($|V(G)| \geq 6$), there is $\chi'_{as}(G) \leq \Delta(G) + 2$.*

REFERENCES

1. F. Harary, Conditional colorability in graphs, In *Graphs and Applications, Proc. First Colorado Symp. Graph Theory*, John Wiley & Sons, New York, (1985).
2. R. Nelson and R.J. Wilson, Graph colorings, In *Pitman Research Notes in Mathematics Series*, p. 218, Pitman, London, (1990).
3. G. Chartrand and L. Lesniak-Foster, *Graphs and Digraphs*, Ind. Edition, Wadsworth Brooks/Cole, Monterey, CA, (1986).
4. S. Fiorini and R.J. Wilson, Edge-coloring of graphs, In *Research Notes in Mathematics, Volume 16*, Pitman, London, (1977).