Adjacent Strong Edge Coloring of Graphs

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Abstract—For a graph $G(V, E)$, if a proper $k$-edge coloring $f$ is satisfied with $C(u) \neq C(v)$ for $uv \in E(G)$, where $C(u) = \{f(uv) \mid uv \in E\}$, then $f$ is called $k$-adjacent strong edge coloring of $G$, is abbreviated $k$-ASEC, and

$$\chi'_{as}(G) = \min\{k \mid k \text{-ASEC of } G\}$$

is called the adjacent strong edge chromatic number of $G$. In this paper, we discuss some properties of $\chi'_{as}(G)$, and obtain the $\chi'_{as}(G)$ of some special graphs and present a conjecture: if $G$ are graphs whose order of each component is at least six, then $\chi'_{as}(G) \leq \Delta(G) + 2$, where $\Delta(G)$ is the maximum degree of $G$. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

The coloring problem of graphs is widely applied in practice. In [1], some conditional coloring problems are introduced. Some network problem can be converted to the strong edge coloring and adjacent strong edge coloring.

Definition 1. For a graph $G(V, E)$, if a proper $k$-edge coloring $f$ is satisfied with $C(u) \neq C(v)$ for $uv \in E(G)$, where $C(u) = \{f(uv) \mid uv \in E\}$, then $f$ is called $k$-adjacent strong edge coloring of $G$, is abbreviated $k$-ASEC, and

$$\chi'_{as}(G) = \min\{k \mid k \text{-ASEC of } G\}$$

is called the adjacent strong edge chromatic number of $G$. 

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DEFINITION 2. If $T$ is a tree, $u,v \in V(T)$, if $d(u) = d(v) = \Delta(T)$ and $p = |V(T)| = 2\Delta(T)$, then $T$ is called a double star.

In this paper, we study the adjacent strong edge coloring of graphs.

The existence of $\chi'_{as}(G)$ for simple connected graphs whose order is at least three is obvious, and $\chi'_{as}(G) \geq \Delta(G)$, where $\Delta(G)$ is the maximum degree of graph $G$.

The other terminology can be found in [1-3].

2. MAIN RESULTS

THEOREM 1. If $G$ is composed of $n$ connected components $G_1, G_2, \ldots, G_n$, and $|V(G_i)| \geq 3$ $(i = 1, 2, \ldots, n)$, then $\chi'_{as}(G) = \max \{\chi'_ {as}(G_i) \mid i = 1, 2, \ldots, n\}$.

By this theorem, we can assume that all graphs are connected in following process.

THEOREM 2. Let $G$ be a connected graph with $|V(G)| \geq 3$, if $\chi'_ {as}(G) = \Delta(G)$, then for all $u \in V$ with degree $d(u) = \Delta(G)$, if $uv \in E$, then $d(v) < \Delta(G)$.

Using $N(v)$ denotes the set of adjacent vertices of $v$ in the following.

LEMMA 1. Supposing $G$ is a tree other than a double star. If $G$ has two adjacent maximum degree vertices, then there is $u \in V(G)$, such that $uv, uv_i \in E(G)$ $(i = 1, 2, \ldots, d(u) - 1)$, $d(u) \geq 2$, $d(v_i) = 1$, $(i = 1, 2, \ldots, d(u) - 1)$.

This lemma is obviously true.

THEOREM 3. For a tree $T$ with $p = |V(G)| \geq 3$,

(1) if any two vertices of maximum degree are not adjacent, then $\chi'_ {as}(T) = \Delta(T);

(2) if $T$ has two vertices of maximum degree which are adjacent, then $\chi'_ {as}(T) = \Delta(T) + 1$.

PROOF. Only to prove (2), the proof of (1) is similar to (2). Obviously, $\Delta(T) \geq 2$.

By Theorem 2, we have $\chi'_ {as}(T) \geq \Delta(T) + 1$.

Now using induction on $p = |V(G)|$ to prove the existence of realization function $f$ so that $|\{f(e) \mid e \in E(T)\}| = \Delta(T) + 1$,

when $p = 2\Delta(T)$, the tree $T$ is double star, without loss of generality, let

$$
\begin{align*}
    d(u) &= d(v) = \Delta(T), \quad \text{and} \quad uv_i, vv_i \in E(T) \ (i = 1, 2, \ldots, \Delta(T) - 1), \\
    f(uv) &= 1, \quad f(uv_i) = i + 1 \ (i = 1, 2, \ldots, \Delta(T) - 1), \\
    f(vv_i) &= i + 2 \ (i = 1, 2, \ldots, \Delta(T) - 1).
\end{align*}
$$

By this function $f$, we know that $\chi'_ {as}(T) \leq \Delta(T) + 1$.

Combining with $\chi'_ {as}(T) \geq \Delta(T) + 1$, the conclusion is true.

Supposing the conclusion is true for $p$ $(p \geq 2\Delta(T))$, now we prove it is true for $p + 1$.

Here $T$ is not a double star. By Lemma 1, select $u$ as Lemma 1, $uv \in E(T)$, $d(v) = 1$. Let

$$
T' = T - v,
$$

then $T'$ also has two vertices of maximum degree which are adjacent, and $\Delta(T') = \Delta(T)$, by the induction hypothesis,

$$
\chi'_ {as}(T') = \Delta(T') + 1 = \Delta(T) + 1.
$$

Supposing $f$ is a realization function of $T'$, $C = \{f(e) \mid e \in E(T')\}$, $C_{T'}(z) = \{f(zy) \mid y \in E(T')\}$.

Let $w \in N(u)$, $d(w) \geq 2$. Notice that $\vert C_{T'}(w) \vert \leq \Delta(T)$ and $\vert C_{T'}(u) \vert \leq \Delta(T) - 1$. 

CASE 1. If $C_{T'}(u) \subseteq C_{T'}(w)$, let
\[
 f^*(e) = \begin{cases} 
 f(e), & e \in E(T'), \\
 \alpha, & \alpha \in C \setminus C_{T'}(u), \quad e = uv.
\end{cases}
\]
Obviously, the function $f^*$ is a realization function.

CASE 2. If $C_{T'}(u) \subseteq C_{T'}(w)$, let
\[
 f^*(e) = \begin{cases} 
 f(e), & e \in E(T'), \\
 \alpha, & e = uv, \quad \alpha \in C \setminus C_{T'}(u), \quad \text{and} \quad \alpha \in C_{T'}(w).
\end{cases}
\]
Obviously, $f^*$ is a realization function of $T$.

Combining the above two cases, we know that
\[
\chi_{as}(T) = \Delta(G) + 1.
\]

THEOREM 4. For cycle $C_p$, we have
\[
\chi_{as}(C_p) = \begin{cases} 
 3, & \text{for } p \equiv 0 \pmod{3}, \\
 4, & \text{for } p \not\equiv 0 \pmod{3} \text{ and } p \neq 5, \\
 5, & \text{for } p = 5.
\end{cases}
\]

Theorem 4 is obviously true.

THEOREM 5. For the complete bipartite graph $K_{m,n}$ ($1 \leq m \leq n$), we have
\[
\chi_{as}(K_{m,n}) = \begin{cases} 
 n, & \text{for } m < n, \\
 n + 2, & \text{for } m = n \geq 2.
\end{cases}
\]

PROOF. When $m < n$, the realization function is easy to construct; the proof is omitted. When $m = n \geq 2$, because the combinatorial number of $n$ from $n + 1$ is $n + 1$, it is easy to see that $\chi_{as}(K_{m,n}) \geq n + 2$.

We now prove the existence of realization function $f$ so that $\chi_{as}(K_{m,n}) = n + 2$.

Supposing the independent sets of two parts of $K_{m,n}$ are $\{u_1, u_2, \ldots, u_n\}, \{v_1, v_2, \ldots, v_n\}$, $f$ is defined as
\[
f(u_i v_j) = \begin{cases} 
 i + j - 2, & \text{for } i = 2, \ldots, n, \quad j = 1, 2, \ldots, n, \\
 i + j - 2, & \text{for } i = 1, \quad j = i + 1, i + 2, \ldots, n.
\end{cases}
\]

Obviously, $f$ is a realization function of $K_{m,n}$, so the conclusion is true.

THEOREM 6. For the complete graph $K_p$ ($p \geq 3$), we have
\[
\chi_{as}(K_p) = \begin{cases} 
 p, & \text{for } p \equiv 1 \pmod{2}, \\
 p + 1, & \text{for } p \equiv 0 \pmod{2}.
\end{cases}
\]

PROOF. Suppose $V(K_p) = \{v_1, v_2, \ldots, v_p\}$.

If $p \equiv 1 \pmod{2}$ and $p \geq 3$, obviously, $\chi_{as}(K_p) \geq p \equiv 1 \pmod{2}$.

We now give a $p$-adjacent strong edge coloring $\sigma$ as follows:
\[
\sigma(v_i v_j) = \begin{cases} 
 i + j - 2, \pmod{p} & \text{if } i = 1, 2, \ldots, p - 1, \quad j = i + 1, i + 2, \ldots, p, \\
 0, & \text{if } i + j - 2 \neq p.
\end{cases}
\]

It is easy to see that $\sigma$ is a $p$-strong edge coloring of $K_p$. 

If \( p \equiv 0 \pmod{2} \) and \( p \geq 4 \), we define a \((p + 1)\)-adjacent strong edge coloring \( \sigma \) as follows.

First we prove \( \chi_{as}(K_p) \geq p + 1 \). Otherwise, if \( \chi_{as}(K_p) = p \), then for all \( v \in V(K_p) \), we have \( |C(v)| = p - 1 \) and the colors which occur on distinct vertices are different, that is, each color occurs on \( p - 1 \) vertices so that each color occurs on odd vertices. Obviously, it is impossible, hence, \( \chi_{as}'(K_p) \geq p + 1 \).

We now give a \((p + 1)\)-adjacent strong edge coloring of \( K_p \) as follows:

\[
\sigma(v_iv_j) = i + j - 2, \pmod{p},
\]

\((i = 1, 2, \ldots, p - 1, j = i + 1, i + 2, \ldots, p) \quad \text{and} \quad i + j - 2 \neq p - 1, \]

\[
\sigma(v_iv_j) = p, \quad i + j - 2 = p + 1.
\]

It is easy to see that \( \sigma \) is a \((p + 1)\)-strong edge coloring of \( K_p \).

Hence, the conclusion is true.

**THEOREM 7.** If \( G \) is a graph which has two adjacent maximum degree vertices, then \( \chi_{as}'(G) \geq \Delta(G) + 1 \).

**THEOREM 8.** If \( G \) is a graph, the degree of any two adjacent vertices is different, then \( \chi_{as}'(G) = \Delta(G) \).

Theorem 7 and 8 are obviously true.

If \( H \) is a subgraph of \( G \), it is interesting that \( \chi_{as}'(H) \leq \chi_{as}'(G) \) is not always true.

**Open Problem**

If \( H \) is a proper subgraph of \( G \), when \( \chi_{as}'(H) \leq \chi_{as}'(G) \) is true?

Combining with above conclusion, we present the following conjecture.

**CONJECTURE.** For any connected graph \( G (|V(G)| \geq 6) \), there is \( \chi_{as}'(G) \leq \Delta(G) + 2 \).

**REFERENCES**