Appl. Math. Lett. Vol. 3, No. 2, pp. 69–71, 1990
Printed in Great Britain.
0893-9659/90 $3.00 + 0.00

Maximal Elements of Condensing Preference Maps

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Abstract. We use the methods of nonliner analysis [1–4] to prove the existence of a maximal element for a class of preference maps defined on a closed, bounded, and convex, but not necessarily compact, subset of a Banach space.

1. INTRODUCTION

Suppose that K is a subset of a topological vector space E. Then each binary relation P on K gives rise to a multivalued map $T : K \to 2^K$ as follows: if $x \in K$, then $T(x) = \{y \in K : (x, y) \in P\}$. We say that a multivalued map $T : K \to 2^K$ is a preference map if it is generated from a binary relation P on K so that $y \in T(x)$ for $x \in K$ if and only if $(x, y) \in P$. The binary relation P may be interpreted as a preference relation on a set K of alternatives. A point x in K is said to be a maximal element of the preference map T if $T(x) = \phi$.

A standard theorem in mathematical economics [7, Theorem 2.1] on the existence of maximal elements states that an irreflexive preference map with convex values and open inverse images has a maximal element in any compact convex subset of a Hausdorff topological vector space.

The objective of this note is to remove altogether the compactness assumption on the domain and to prove the existence of a maximal element in a noncompact subset of a Banach space by using a nonexpansive or Lipschitz condition on the preference map. This condition may be interpreted in the following way. Let P be a preference map and D a subset of commodity bundles that is large in a sense to be made precise below. Then it is intuitively reasonable to expect that because of "diminishing returns" the size of the upper contour set $P(D)$ is smaller than that of D. D.A. A preference map satisfying this condition is said to be a condensing map. Using the methods of nonlinear analysis [1–4], we prove that a condensing preference map satisfying the other usual conditions has a maximal element in a closed bounded and convex, but not necessarily compact, subset of a Banach space.

2. PRELIMINARIES

We shall use the following notation. If K is a subset of a Banach space, then int K denotes the topological interior of K, *co* K denotes the convex hull of K, and $\overline{co}\, K$ denotes the closed convex hull of K.

Let X be a Banach space and S a bounded subset of X. Then the Kuratowski *measure of noncompactness* of S, $\alpha(S)$ is defined by $\alpha(S) = \inf\{\varepsilon > 0 : S$ can be covered by a finite number of sets with diameter no larger than $, \varepsilon\}$.

Observe that $\alpha(S) = 0$ if and only if S is relatively compact. Hence, a closed bounded set S has positive Kuratowski measure if and only if S is noncompact.

The proof of the following important theorem may be found in Lloyd [3, Chapter 6].

Some of the work for this paper was done while visiting the London School of Economics in 1988. I should like to thank Professor M. Desai, Professor L. Foldes, and Dr. A. Horsley for their hospitality.

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THEOREM (DARBO). *If S is a bounded subset of a Banach space, then $\alpha(S) = \alpha(\overline{co}\, S)$.*

Let Y_1, Y_2 be metric spaces. Then a multivalued map $T : Y_1 \to 2^{Y_2}$ is said to be *condensing* if for each bounded subset D such that $\alpha(D) > 0$, $T(D)$ is bounded and $\alpha(T(D)) < \alpha(D)$.

Let K be a subset of a Banach space and $T : K \to 2^K$ a multivalued preference map. Then $x_0 \in K$ is said to be a *maximal element* of T if $T(x_0) = \phi$.

3. MAXIMAL ELEMENTS

We first prove the following lemma. The proof is an adaptation of the argument of Martin [4, Chapter 4] to the multivalued case.

LEMMA. *Let E be a Banach space and D a nonempty closed, bounded and convex subset of E. Suppose that $T : D \to 2^D$ is a condensing map. Then there exists a compact convex subset K of D such that T is a multivalued map of K into 2^K.*

PROOF: Let x_0 be an element of D and consider the family $\mathcal{F}$ of all closed convex subsets C of D such that $x_0 \in C$ and $T : C \to 2^C$. Clearly, $\mathcal{F}$ is nonempty. Let $C_0 = \cap_{C \in \mathcal{F}} C$. Then C_0 is closed and convex and $x_0 \in C_0$. If $x \in C_0$, $T(x) \subset C$ for all C so that $T : C_0 \to 2^{C_0}$.

It remains to be proved that C_0 is compact. If C_0 is not compact $\alpha(C_0) > 0$. Since T is a condensing map we have $\alpha(T(C_0)) < \alpha(C_0)$. Let $C_1 = \overline{co}(\{x_0\} \cup T(C_0))$. Then $C_1 \subset C_0$, which implies that $T(C_1) \subset T(C_0) \subset C_1$. Hence, $C_1 \in \mathcal{F}$ and $C_0 \subset C_1$. Therefore $C_0 = C_1$, a contradiction because $\alpha(C_1) = \alpha[\overline{co}(\{x_0\} \cup T(C_0))] = \alpha(T(C_0)) < \alpha(C_0)$ where the second equality holds because of Darbo's theorem. This contradiction proves the lemma. ∎

We are now ready to prove the following theorem on the existence of maximal elements.

THEOREM. *Let E be a Banach space and D a closed bounded and convex subset of E. Suppose that $P : D \to 2^D$ is a multivalued preference map such that the following conditions are satisfied:*

(i) *for each $x \in D$, $x \notin co\, P(x)$;*
(ii) *for each $x \in D$ such that $P(x) \neq \emptyset$ there exists $y \in D$ such that $x \in int\ P^{-1}(y)$;*
(iii) *P is condensing.*

Then there exists a maximal element of P in D.

PROOF: Suppose, per absurdum, that there is no maximal element. Then $P(x) \neq \emptyset$ for each $x \in D$.

The lemma implies that there exists a compact convex subset K of D such that $P : K \to 2^K$. Define $Q : K \to 2^K$ by $Q(x) = co\, P(x)$ for each $x \in K$. Clearly, for each $x \in K$, $Q(x)$ is nonempty and convex.

We prove next that for each $x \in K$ there exists $y \in K$ such that $x \in$ inte $Q^{-1}(y)$ in the relative topology of K. Let $x \in K$. Then since $P(x) \neq \emptyset$, condition (ii) implies that there exists $y \in D$ with $x \in$ int $P^{-1}(y)$. Since $P(x) \subset Q(x) = co\, P(x)$ for each $x \in K$, int $P^{-1}(y) \subset$ int $Q^{-1}(y)$ where the interiors are taken in D. Hence $x \in Q^{-1}(y)$, which implies that $y \in Q(x) \subset K$ so that $y \in K$. Therefore, for each $x \in K$ there exists $y \in K$ such that $x \in$ int $Q^{-1}(y)$ where the interior is in D. It follows that there is an open neighbourhood U in D such that $x \in U \subset Q^{-1}(y)$. This implies that $U \cap K$ is open in K and $x \in U \cap K \subset$ relative interior of $Q^{-1}(y)$ in K. This proves that for each $x \in K$ there exists $y \in K$ such that $x \in$ int $Q^{-1}(y)$ in the relative topology of K.

Hence, all the conditions of the fixed point theorem of Tarafdar [6] are satisfied and we may conclude that there exists a point x_0 such that $x_0 \in Q(x_0) = co\, P(x_0)$ contradicting condition (i). This contradiction proves the theorem. ∎

REMARK 1. *The method used here can also be applied to weaken the compactness assumption in the other theorems on the existence of maximal elements. The reader is referred to Mehta [5, Chapter 7] for further discussion.*

REMARK 2. *I do not know if the theorem proved above can be extended to a topological vector space. This would be an interesting generalization.*

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