NOTE

A Note on the Stability of Iteration Procedures for Strong Pseudocontractions and Strongly Accretive Type Equations

M. O. Osilike

Department of Mathematics, University of Nigeria, Nsukka, Nigeria

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We show that recent theorems of Hai-Yun Zhou (1999, J. Math. Anal. Appl. 230, 1–10) concerning stable iteration procedures for strong pseudocontractions and nonlinear equations involving strongly accretive operators without Lipschitz assumptions are false.

Suppose $E$ is a real Banach space and $T$ is a selfmap of $E$. Suppose $x_0 \in E$ and $x_{n+1} = f(T, x_n)$ defines an iteration procedure which yields a sequence of points $(x_n)_{n=0}^\infty$ in $E$, for example, the function iteration, $x_{n+1} = f(T, x_n) = Tx_n$. Suppose $F(T) = \{x \in E : Tx = x\} \neq \emptyset$ and that $(x_n)$ converges strongly to $x^* \in F(T)$. Suppose $(y_n)_{n=0}^\infty$ is a sequence in $E$ and $(\varepsilon_n)_{n=0}^\infty$ is a sequence in $[0, \infty)$ given by $\varepsilon_n = ||y_{n+1} - f(T, y_n)||$. If $\lim_{n \to \infty} \varepsilon_n = 0$ implies that $\lim_{n \to \infty} y_n = x^*$, then the iteration procedure defined by $x_{n+1} = f(T, x_n)$ is said to be $T$-stable or stable with respect to $T$ (see, for example, [1–4, 6, 7, 11–14]).

Let $J$ denote the normalized duality mapping from $E$ into $2^{E^*}$ given by

$$J(x) = \{f \in E^* : \langle x, f \rangle = ||x||^2 = ||f||^2\},$$

where $E^*$ denotes the dual space of $E$ and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if $E^*$ is strictly convex then $J$ is single-valued. In the sequel we shall denote the single-valued normalized duality mapping by $j$.

1 Regular Associate of the Abdus Salam ICTP, Trieste, Italy.
An operator $T$ with domain $D(T)$ and range $R(T)$ in $E$ is called a strong pseudocontraction if for all $x, y \in D(T)$ there exist $j(x - y) \in J(x - y)$ and $t > 1$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \frac{1}{t} \|x - y\|^2.$$  

$T$ is called strongly accretive if $(I - T)$ is strongly pseudocontractive.

In [4] the author proved that certain Mann and Ishikawa iteration procedures are stable with respect to Lipschitz strong pseudocontractions in real $q$-uniformly smooth Banach spaces, $E$. As a consequence of our results we proved that certain Mann and Ishikawa iteration procedures for approximating the solution of the equation $Tx = f$, $f \in E$ when $T : E \to E$ is a Lipschitz strongly accretive operator are stable. In [6] we extended the results of [4] to arbitrary real Banach spaces.

Recently, Hai-Yun Zhou [14] studied, in real uniformly smooth Banach spaces, stable iteration methods for strong pseudocontractions and solutions of nonlinear strongly accretive operator equations $Tx = f$ without Lipschitz assumption on the operators. He employed the following well known lemma of Reich [10].

**Lemma R (Reich [10]).** Let $E$ be a real uniformly smooth Banach space. Then there exists a nondecreasing continuous function $b : [0, \infty) \to [0, \infty)$ such that

(i) $b(ct) \leq cb(t)$, $\forall c \geq 1$

(ii) $b(0) = 0$

(iii) $\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, j(x) \rangle + \max\{\|x\|, 1\}\|y\|b(\|y\|)$  

(1)

for all $x, y \in E$.

In [14] Zhou proved the following theorem:

**Theorem HZ [14, p. 4].** Let $T : E \to E$ be a continuous strong pseudo-contracting operator with a bounded range. Let $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ be two real sequences in $(0, 1)$ satisfying the conditions:

(i) $0 < \alpha < \alpha_n$ for all $n \geq 0$ and for some constant $\alpha$,

(ii) $\lim_{n \to \infty} \beta_n = 0$,

(iii) $\sum_{n=0}^\infty \alpha_n = \infty$, and

(iv) $\lim_{n \to \infty} b(\alpha_n) = 0$, where $b$ is the function in Lemma R.

Let $\{x_n\}_{n=0}^\infty$ be the sequence generated from an arbitrary $x_0 \in E$ by

$$z_n = (1 - \beta_n)x_n + \beta_nTx_n, \quad n \geq 0,$$

and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTz_n, \quad n \geq 0.$$  

(2) \hspace{1cm} (3)
Let \( y_n \) be any sequence in \( E \) and define \( \{e_n\}_{n=0}^{\infty} \subseteq \mathbb{R}^+ \) by
\[
\omega_n = (1 - \beta_n) y_n + \beta_n T y_n, \quad n \geq 0,
\]
\[
e_n = \|y_{n+1} - (1 - \alpha_n) y_n - \alpha_n T \omega_n\|, \quad n \geq 0.
\]

Then \( \{y_n\} \) converges strongly to the fixed point of \( T \) so that \( \{x_n\} \) is \( T \)-stable.

Similar theorems were proved for the iterative approximation of the solution of the operator equation \( Tx = f \) when \( T : E \to E \) is a continuous strongly accretive operator, and the equation \( x + Tx = f \) when \( T : E \to E \) is a continuous accretive operator.

It is our purpose in this note to first show that conditions (i) and (iv) of the hypothesis of Theorem HZ are contradictory. We shall then show by a simple example that the theorem is false under conditions (ii), (iii), and (iv) for which the sequence \( \{x_n\} \) is known to converge to the fixed point of \( T \).

It is clear that condition (i) implies condition (iii). The following lemma shows that conditions (i) and (iv) are contradictory.

**Lemma.** Let \( E \) be a real uniformly smooth Banach space and let \( b : [0, \infty) \to [0, \infty) \) be the function in Lemma R. Let \( \{\alpha_n\}_{n=0}^{\infty} \) be a sequence of nonnegative numbers. Then \( \lim_{n \to \infty} b(\alpha_n) = 0 \) if and only if \( \lim_{n \to \infty} \alpha_n = 0 \).

**Proof.** Suppose \( \lim_{n \to \infty} b(\alpha_n) = 0 \). Let \( z \neq 0 \) be an arbitrary element of \( E \), and let \( t \in (0, \infty) \) be arbitrary. Letting \( x = 0 \) and \( y = \frac{t}{1 + t} \) in inequality (1) of Lemma R we obtain \( t^2 \leq b(t) \), so that \( t \leq b(t) \). Since \( b(0) = 0 \), it follows that \( t \leq b(t) \), \( \forall t \in [0, \infty) \), from which it follows that \( 0 \leq \alpha_n \leq b(\alpha_n) \), \( \forall n \geq 0 \). Hence \( \lim_{n \to \infty} \alpha_n = 0 \).

Conversely, if \( \lim_{n \to \infty} \alpha_n = 0 \), then since \( b(0) = 0 \), it follows from the continuity of \( b \) that \( \lim_{n \to \infty} b(\alpha_n) = 0 \), completing the proof of the lemma.

If condition (iv) is assumed (in which case condition (i) is dropped), it follows from several well known results (see, for example, [9]) that the sequence \( \{x_n\}_{n=0}^{\infty} \) generated from an arbitrary \( x_0 \in E \) by (2) and (3) converges strongly to the fixed point of \( T \). However, the following example shows that \( \{x_n\}_{n=0}^{\infty} \) is not \( T \)-stable even when \( T \) is a strict contraction.

**Example.** Let \( \mathbb{R} \) denote the reals with the usual norm and define \( T : \mathbb{R} \to \mathbb{R} \) by \( Tx = \frac{\sin x}{x} \). Then \( T \) is a strict contraction (hence, a strong pseudocontraction) and has 0 as the unique fixed point. Thus it follows from [9, Corollary 3] that the sequence \( \{x_n\} \) generated from an arbitrary \( x_0 \in \mathbb{R} \) by (2) and (3) with \( \{\alpha_n\} \) and \( \{\beta_n\} \) satisfying conditions (ii), (iii), and (iv) of Theorem HZ converges strongly to 0.
We show that \( \{x_n\} \) is not \( T \)-stable. Let \( y_n = \frac{n}{n+1}, n \geq 0 \). Then

\[
\epsilon_n = |y_{n+1} - (1 - \alpha_n)y_n - \alpha_n T \omega_n| = |y_{n+1} - y_n + \alpha_n (y_n - T \omega_n)|
\]

\[
\leq |y_{n+1} - y_n| + \alpha_n |y_n - \frac{1}{2} \sin \omega_n|
\]

\[
\leq \frac{1}{(n+1)(n+2)} + \alpha_n \left[ \frac{n}{n+1} + \frac{1}{2} \right]
\]

\[
\leq \frac{1}{(n+1)(n+2)} + \frac{3\alpha_n}{2} \to 0 \quad \text{as } n \to \infty,
\]

so that \( \lim \epsilon_n = 0 \). However, \( \lim_{n \to \infty} y_n = 1 \neq 0 = \lim_{n \to \infty} x_n \) is the unique fixed point of \( T \).

Since the operator \( S \) in Theorems 3.2 and 3.3 of Zhou [14] is a continuous strong pseudocontraction with a bounded range, it also follows from the above example that the theorems are false under conditions (ii), (iii), and (iv).

If condition (i) is assumed (in which case conditions (iii) and (iv) are dropped), it is not known whether or not either \( \{x_n\} \) or \( \{y_n\} \) converges to the fixed point of \( T \).

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REFERENCES


