Travelling waves in a convection–diffusion equation

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\begin{abstract}
We study existence and stability of travelling waves for nonlinear convection–diffusion equations in the 1-D Euclidean space. The diffusion coefficient depends on the gradient in analogy with the \( p \)-Laplacian and may be degenerate. Unconditional stability is established with respect to initial data perturbations in \( L^1(\mathbb{R}) \).
\end{abstract}

\section{1. Introduction}

We wish to establish the existence and stability of travelling waves for a convection–diffusion equation in the whole real line \( \mathbb{R} \),

\begin{equation}
\begin{cases}
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[ \varphi \left( \frac{\partial u}{\partial x} \right) + f(u) \right] & \text{in } \mathbb{R} \times (0, \infty); \\
u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}.
\end{cases}
\end{equation}

Here \( \varphi : \mathbb{R} \to \mathbb{R} \) is a function of class \( C^{2+\alpha}(\mathbb{R} \setminus \{0\}) \cap C^0(\mathbb{R}) \) for some \( \alpha \in (0, 1) \), and \( f \in C^2(\mathbb{R}) \).

More specifically, we assume that \( \varphi \) is an odd function satisfying

\begin{equation}
0 < d_1 |z|^{p-2} \leq \varphi'(z) \leq d_2 |z|^{p-2} \quad \text{for all } 0 < |z| < 1,
\end{equation}

where \( d_1, d_2 \) are positive constants and \( p > 1 \). For example, one may take \( \varphi(\partial_x u) = \psi_p(\partial_x u) \triangleq |\partial_x u|^{p-2} \partial_x u \) to model a 1-D analogue of the \( p \)-Laplace, \( 1 < p < \infty \).

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Our goal is to establish the existence and stability of travelling waves – solutions in the form $u(x, t) = v(x - ct)$ – with respect to the $L^1$-norm on $\mathbb{R}$. This means that, for any initial condition $u_0$ within the set of permissible data, such that

$$\|u_0 - v\|_{L^1(\mathbb{R})} = \int_\mathbb{R} |u_0 - v| \, dx < \infty,$$

we show (see Theorems 6.1 and 7.1) that

$$\|u(x, t) - v(x - ct)\|_{L^1(\mathbb{R})} = \int_\mathbb{R} |u(x, t) - v(x - ct)| \, dx \to 0 \quad \text{as } t \to \infty.$$

Results of this type for porous-medium like equations were obtained in the seminal paper by S. Osher and J. Ralston [14] under the additional hypothesis

$$\inf_{y \in \mathbb{R}} v(y) \leq u_0(x) \leq \sup_{y \in \mathbb{R}} v(y) \quad \text{for all } x \in \mathbb{R}. \quad (1.3)$$


Similarly to [5], we focus on the case when the travelling wave solution $v$ is non-degenerate, specifically,

$$\inf_{y \in \mathbb{R}} v(y) < v(x) < \sup_{y \in \mathbb{R}} v(y) \quad \text{for all } x \in \mathbb{R}, \quad (1.4)$$

see Section 2. The key ingredient of the proof of stability of travelling waves is strong contractivity property of solutions to problem (1.1) established in Section 5 by means of a flux identity, a new substitute for the well-known lap number – the number of non-degenerate intersections of two different solutions. With this property, the problem fits the abstract framework introduced by S. Osher and J. Ralston [14], in particular, stability of travelling waves holds under the extra hypothesis (1.3). Finally, exploiting some dispersive estimates, we remove (1.3), see Section 7.

2. Travelling waves

In this section we study the travelling waves $v : \mathbb{R} \to \mathbb{R}$ for the convection–diffusion equation (1.1). We substitute $u(x, t) = v(x - ct)$ into Eq. (1.1), thus arriving at

$$-cv_x = \partial_x \left( \varphi(v_x) + f(v) \right) \quad \text{in } \mathbb{R},$$

or equivalently

$$\partial_x (\varphi(v_x) + f(v) + cv) = 0 \quad \text{in } \mathbb{R}. \quad (2.1)$$

This equation is again equivalent with

$$\varphi(v_x) + f(v) + cv = K \equiv \text{const} \quad \text{in } \mathbb{R}. \quad (2.2)$$

We abbreviate $f_{c,K}(v) \overset{\text{def}}{=} f(v) + cv - K$ for $v \in \mathbb{R}$. 
We assume that the convection function \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function with the following properties:

**Hypotheses.**

1. There are two numbers \( w_+ < w_- \) such that \( f_{c,K}(w_+) = f_{c,K}(w_-) = 0 \) and \( f_{c,K}(v) = f(v) + cv - K > 0 \) for all \( v \in (w_+, w_-) \).

2. For \( w_0 \in (w_+, w_-) \), the following integrals diverge,

\[
\int_{w_0}^{w_+} \frac{dv}{\varphi^{-1}(f_{c,K}(v))} = \infty \quad \text{and} \quad \int_{w_0}^{w_-} \frac{dv}{\varphi^{-1}(f_{c,K}(v))} = \infty. \tag{2.3}
\]

From these hypotheses we can easily deduce the existence and also uniqueness of certain type of travelling waves:

**Proposition 2.1.** Under Hypotheses **(f1)** and **(f2)**, every solution \( v : \mathbb{R} \to (w_+, w_-) \) of Eq. (2.2) takes the following form: There is a unique \( y \in \mathbb{R} \) such that

\[
v(x) = w_+^y(x) \overset{\text{def}}{=} w(x - y) \quad \text{for all} \quad x \in \mathbb{R}, \tag{2.4}
\]

where the function \( w : \mathbb{R} \to (w_+, w_-) \) is determined from the equation

\[
\int_{w_0}^{w} \frac{dv}{\varphi^{-1}(f_{c,K}(v))} = -x \quad \text{for all} \quad x \in \mathbb{R}. \tag{2.5}
\]

Notice that \( w' < 0 \) in \( \mathbb{R} \) and \( \lim_{x \to -\infty} w(x) = w_- > w_0, \lim_{x \to +\infty} w(x) = w_+ < w_0. \)

Below we give a simple example of \( \varphi \) and \( f \) that satisfy all hypotheses stated above.

**Example 2.2.** We take \( \varphi(z) = |z|^{p-2}z \) for all \( z \in \mathbb{R} \), which corresponds to the well-known case of the \( p \)-Laplacian with \( p \in (1, \infty) \). Hence, the inverse function to \( \varphi \) is given by \( \varphi^{-1}(z) = |z|^{p-2}z \) for all \( z \in \mathbb{R} \), where \( p' = p/(p-1) \in (1, \infty) \). Now let \( c > 0 \) and \( K \in \mathbb{R} \). We define \( f : (w_+, w_-) \to \mathbb{R} \) by

\[
f(z) = \begin{cases} 
K - cz + a_-(w_- - z)^{q_-} & \text{for } w_- - \delta < z < w_-; \\
K - cz + a_+(z - w_+)^{q_+} & \text{for } w_+ < z < w_+ + \delta,
\end{cases} \tag{2.6}
\]

where \( \delta \) is small enough, such that \( 0 < \delta < \min\{-w_+, w_-\} \), and extend it on the complement

\[
(w_+, w_-) \setminus [(w_+, w_+ + \delta) \cup (w_-, w_- - \delta)] = [w_+ + \delta, w_- - \delta]
\]
to a continuous function on the whole of \((w_+, w_-)\), such that \( f(z) > K - cz \) for all \( z \in (w_+, w_-) \). The coefficients \( a_- \) and \( a_+ \) are arbitrary positive numbers, and \( q_- \) and \( q_+ \) are positive constants satisfying \((p' - 1)q_\pm \geq 1\), respectively, that is, \( q_\pm \geq p - 1 > 0 \). This choice guarantees conditions (2.3). Of course, the positivity of the continuous function \( f_{c,K}(v) = f(v) + cv - K > 0 \) for all \( v \in (w_+, w_-) \) yields \( \varphi^{-1}(f_{c,K}(v)) > 0 \) together with

\[
\int_{w_- + \delta}^{w_-} \frac{dv}{\varphi^{-1}(f_{c,K}(v))} < \infty \quad \text{for all} \quad x \in \mathbb{R}.
\]
To provide a more general example of the functions \(\varphi, f\), we may take \(\varphi\) satisfying (1.2) and replace the definition (2.6) of \(f\) by the following weaker conditions – inequalities

\[
K - cz < f(z) \leq \begin{cases} 
K - cz + a_-(w_- - z)^q^- & \text{for } w_- - \delta < z < w_-, \\
K - cz + a_+(z - w_+)^q^+ & \text{for } w_+ < z < w_+ + \delta,
\end{cases}
\]

with \(q^\pm \geq p - 1 > 1\). Note that (2.7) is satisfied if \(f : \mathbb{R} \to \mathbb{R}\) is of class \(C^k(\mathbb{R})\) for some integer \(k \geq \min\{2, p - 1\}\) and all derivatives \(f^{(m)}\) of order \(m\) satisfying \(2 \leq m \leq p - 2\) vanish at \(w_\pm\), i.e., \(f^{(m)}(w_\pm) = 0\).

### 3. General regularity results

In this section we apply the standard regularity results for degenerate or singular parabolic problems, see G.M. Lieberman [13], E. DiBenedetto [2] or E. DiBenedetto and A. Friedman [3,4], and for classical regular parabolic problems, see A. Friedman [7,8] or O.A. Ladyzhenskaya, N.N. Uraltseva, and V.A. Solonnikov [10]. In particular, it follows that any (essentially) bounded weak solution \(u : \mathbb{R} \times (0, T) \to \mathbb{R}\) to problem (1.1), with \(u_x\) bounded, is of class \(C^{1+\beta, \frac{1}{2}(1+\beta)}(\mathbb{R} \times (T', T))\) whenever \(0 < T' < T < \infty\). More specifically, if \(u\) satisfies

\[
\|u\|_{L^1(\mathbb{R} \times (0, T))} \leq L \quad \text{and} \quad \|\partial_x u\|_{L^\infty(\mathbb{R} \times (0, T))} < \infty,
\]

then there are positive constants \(\epsilon\) and \(\beta \in (0, 1)\), depending only on \(T', L\), and the structural properties of \(\varphi\) and \(f\), such that

\[
\left|u_x(t_1, x_1) - u_x(t_2, x_2)\right| \leq \epsilon(|x_1 - x_2|^\beta + |t_1 - t_2|^{\beta/2})
\]

whenever \(x_1, x_2 \in \mathbb{R}\) and \(t_1, t_2 \in [T', T]\). These results extend to \(T' = 0\) provided \(u_0 \in C^{1+\alpha}(\mathbb{R})\), \(0 < \alpha < 1\), specifically

\[
\partial_x u_0 \in L^\infty(\mathbb{R}) \quad \text{and} \quad \left|\partial_x u_0(x_1) - \partial_x u_0(x_2)\right| \leq \epsilon_0|x_1 - x_2|^{\alpha}.
\]

In this case, the constants \(\epsilon\) and \(\beta\) in (3.1) depend also on \(c_0\) and \(\alpha\). We point out that the estimates are uniform on the whole line \(\mathbb{R}\) and reflect the regularising properties of the parabolic operator. In particular, as any solution remains bounded by virtue of the maximum principle, the estimates are uniform for \(T \to \infty\).

Furthermore, if \(u_x(y, s) \neq 0\) for some \((y, s) \in \mathbb{R} \times (0, T)\), then there is some \(\delta > 0\) such that \(u_x(x, t) \neq 0\) holds for all \((x, t) \in \mathbb{R} \times (0, T)\) satisfying \(\max(|x - y|, |t - s|^{1/2}) < \delta\). Consequently, we may formally differentiate in \(\text{Eq. (1.1)}\) to get

\[
\partial_t u = \varphi'(u_x)u_{xx} + f'(u)u_x \quad \text{in} \quad (y - \delta, y + \delta) \times (t_1, t_2),
\]

where \(t_1 = \max\{s - \delta^2, 0\}\) and \(t_2 = \min\{s + \delta^2, T\}\). We apply the classical regularity theory for parabolic problems to conclude that \(u\) is actually of class

\[
C^{2+\beta, 1+\frac{1}{2}\beta}(y - \delta, y + \delta) \times (t_1, t_2).
\]

Now, if \(v : \mathbb{R} \times (0, T) \to \mathbb{R}\) is another solution to problem (1.1), with a possibly different initial condition, that is also of class (3.3), then we have in \((y - \delta, y + \delta) \times (t_1, t_2)\) (cf. Eq. (1.1)):
\[\partial_t (u - v) = \partial_x \left[ \int_0^1 \varphi'((1 - \theta)u_x + \theta v_x) \, d\theta \right] (u_x - v_x) + \left( \int_0^1 f'((1 - \theta)u + \theta v) \, d\theta \right) (u - v)\]

\[= \left( \int_0^1 \varphi''((1 - \theta)u_x + \theta v_x) ((1 - \theta)u_{xx} + \theta v_{xx}) \, d\theta \right) (u_x - v_x)\]

\[+ \left( \int_0^1 \varphi'((1 - \theta)u_x + \theta v_x) \, d\theta \right) (u_{xx} - v_{xx}) + \left( \int_0^1 f''((1 - \theta)u + \theta v) ((1 - \theta)u_x + \theta v_x) \, d\theta \right) (u - v)\]

\[+ \left( \int_0^1 f'((1 - \theta)u + \theta v) \, d\theta \right) (u_x - v_x). \tag{3.4}\]

Abbreviating the coefficients in front of \(u - v, (u_x - v_x),\) and \((u_{xx} - v_{xx})\) on the right-hand side, we can rewrite Eq. (3.4) in the following standard form:

\[\partial_t (u - v) = a(x, t) \partial_{xx}(u - v) + b(x, t) \partial_x (u - v) + c(x, t) (u - v) \tag{3.5}\]

in \((y - \delta, y + \delta) \times (t_1, t_2).\) It is obvious that all coefficients \(a, b,\) and \(c\) are Hölder-continuous with \(a(x, t) \geq \text{const} > 0.\)

4. Existence and uniqueness in \(C^{1+\alpha}(\mathbb{R}) + L^1(\mathbb{R})\)

The question of existence and uniqueness for problem (1.1) for a general non-integrable initial data \(u_0\) can be resolved by constructing a suitable solution semigroup. This can be done by first regularising the functions \(\psi\) and \(f,\) and taking smooth, compactly supported initial data \(u_0.\) The existence and uniqueness of a classical solution to such a regularised problem is obtained from well-known theory in A. Friedman [7] or O.A. Ladyzhenskaya, N.N. Ural’tseva, and V.A. Solonnikov [10].

In the second step, we take the initial data bounded in \(C^{1+\alpha}(\mathbb{R})\) for some \(\alpha \in (0, 1),\) specifically, \(\sup_{x \in \mathbb{R}}(|u_0(x)| + |\partial_x u_0(x)|) \leq c\) and \(|\partial_x u_0(x_1) - \partial_x u_0(x_2)| \leq c|x_1 - x_2|^\alpha\) for all \(x_1, x_2 \in \mathbb{R} \tag{4.1}\)

For this class of initial data, the a priori bounds established in G.M. Lieberman [12] (see (3.1)) guarantee uniform boundedness of \(u_x\) in \(\mathbb{R} \times [0, T];\) therefore, the above mentioned approximation procedure yields the viscosity solution to problem (1.1) satisfying the weak comparison principle

\[u(t, x) \leq v(t, x) \quad \text{for all } x \in \mathbb{R}, \ t \geq 0, \tag{4.2}\]

provided \(u\) and \(v,\) respectively, are sub- and super-solutions of problem (1.1) together with
see Y. Giga et al. [9]. Indeed, since both $u$ and $\partial_x u$ are bounded in terms of the initial data and the function $f$ is twice continuously differentiable, the term $f'(u)u_x$ may be viewed to be Lipschitz continuous in $u$; therefore, the differential operator $\partial_x \varphi(u_x) + \partial_x f(u)$ fits well in the class considered in Y. Giga et al. [9]. In particular, the viscosity solutions to (1.1) emanating from the initial data (4.1) are unique in the class $u, u_x \in L^\infty(\mathbb{R} \times (0, T))$.

We denote by $S \equiv \{ S(t); \ t \geq 0 \}$ this solution semigroup, that is, the solution $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ of problem (1.1) is given by $u(\cdot, t) = S(t)u_0$ for all $t \geq 0$, where $u_0$ belongs to the class (4.1). The following properties of $S$ can be verified by a density argument:

**Contraction:**

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^1(\mathbb{R})} = \|S(t)u_0 - S(t)v_0\|_{L^1(\mathbb{R})} \leq \|u_0 - v_0\|_{L^1(\mathbb{R})}; \quad (4.3)$$

whenever $u_0, v_0 \in C^{1+\alpha}(\mathbb{R})$ satisfy $u_0 - v_0 \in L^1(\mathbb{R})$.

**Translation Invariance:**

$$S(t)u_0(\cdot - y) = u(t, \cdot - y) \quad \text{where} \ u = S(t)u_0; \quad (4.4)$$

**Conservation:**

$$\int_{\mathbb{R}} \left( S(t)u_0 - S(t)v_0 \right) \, dx = \int_{\mathbb{R}} \left( u_0 - v_0 \right) \, dx \quad (4.5)$$

whenever $u_0, v_0 \in C^{1+\alpha}(\mathbb{R})$ satisfy $u_0 - v_0 \in L^1(\mathbb{R})$.

Using the contractivity property (4.3), we can extend the solution semigroup to the class of initial data $C^{1+\alpha}(\mathbb{R}) + L^1(\mathbb{R})$ in a unique canonical way.

5. **Strict contractivity in $L^1(\mathbb{R})$**

In our proof of stabilisation of solutions $u(\cdot, t) = S(t)u_0 \ (t \geq 0)$ we take advantage of the abstract framework developed by S. Osher and J. Ralston [14]. To this end, we use the strict inequality in (4.3) for certain pairs $u$ and $v$ and for all times $t \geq t_0 > 0$.

To begin with, we derive a priori bounds that ensure local integrability of all terms in Eq. (1.1). To this end, we consider a weight function

$$\varrho = \varrho(x) = \exp(-|x|), \quad x \in \mathbb{R}. \quad (5.1)$$

It is easy to check that $\varrho$ is Lipschitz continuous in $\mathbb{R}$:

$$\left| \varrho_x(x) \right| = \varrho(x) \leq 1 \quad \text{for a.e.} \ x \in \mathbb{R}. \quad (5.1)$$

Multiplying Eq. (1.1) by $\varrho \partial_t u$ and integrating by parts, we formally deduce that

$$\int_{\mathbb{R}} \varrho |\partial_t u|^2 \, dx + \frac{d}{dt} \int_{\mathbb{R}} \varrho \Phi(u_x) \, dx = \int_{\mathbb{R}} \left( \varrho f'(u)u_x \partial_t u + \varrho_x \varphi(u_x) \partial_t u \right) \, dx,$$

where we have denoted $\Phi(v) = \int_0^v \varphi(s) \, ds$ for $v \in \mathbb{R}$. Thus, using (5.1) we deduce that
\[ \partial_t u \in L^2_{\text{loc}}(\mathbb{R} \times [0, T]). \]

Clearly, this bound can be justified for \( u = S(t)u_0 \) provided \( u_0 \in C^{1+\alpha}(\mathbb{R}) \).

Next, we establish the following flux identity, which seems to be new:

**Proposition 5.1.** Assume that \( u, v : \mathbb{R} \times \mathbb{R}_{+} \to \mathbb{R} \) are solutions of problem (1.1) with the initial data \( u_0, v_0 \in C^{1+\alpha}, u_0 - v_0 \in L^1(\mathbb{R}), \) and \( u(\cdot, t) = S(t)u_0, v(\cdot, t) = S(t)v_0 \) for \( t \geq 0 \).

Then we have the flux identity

\[
\int_{\mathbb{R}} (u(x, t_2) - v(x, t_2))^+ \, dx - \int_{\mathbb{R}} (u(x, t_1) - v(x, t_1))^+ \, dx \\
= -\int_{t_1}^{t_2} \left( \sum_{[x \in \mathbb{R}: \ u(x, t) = v(x, t), \ u_x(x, t) \neq v_x(x, t)]} |\varphi(u_x(x, t)) - \varphi(v_x(x, t))| \right) \, dt
\]

for all \( t_2 \geq t_1 \geq 0. \) The same identity holds also for the negative part \((u - v)^-\) of \( u - v\).

**Remark 5.2.** We remark that the sum on the right-hand side is at most countable. This claim follows from the fact that for every fixed \( t \in [0, \infty), \) every point \( x_0 \in \mathbb{R} \) at which \( u(x_0, t) = v(x_0, t) \) and \( \varphi(u_x(x_0, t)) \neq \varphi(v_x(x_0, t)) \) (i.e., \( u_x(x_0, t) \neq v_x(x_0, t) \)) is isolated, by the strict monotonicity of \((u - v)(\cdot, t)\) near \( x_0.\)

**Proof of Proposition 5.1.** We calculate, starting with the chain rule [16, Theorem 2.1.11, p. 48] and using Eq. (1.1) for \( u \) and \( v \) in \( \mathbb{R} \times (0, \infty):\)

\[
\int_{-R}^{R} (u(t_2, x) - v(t_2, x))^+ \, dx - \int_{-R}^{R} (u(t_1, x) - v(t_1, x))^+ \, dx \\
= \int_{t_1}^{t_2} \int_{\{x \in (-R, R): \ u(x, s) - v(x, s) > 0\}} \partial_x \left[ \varphi(u_x(x, t)) + f(u(x, t)) - \varphi(v_x(x, t)) - f(v(x, t)) \right] \, dx \, dt,
\]

where we have used the local integrability of \( \partial_t u \) established in (5.2).

Since \( \varphi(u_x)_x \) is locally integrable, it follows that for a.e. \( t \in (t_1, t_2), \) the function \( x \mapsto \varphi(u_x(x, t)) \) is absolutely continuous with the derivative \( \varphi(u_x(x, t))_x. \) For such a time \( t, \) we consider the set \( \mathcal{O}_+(t) = \{ x \in \mathbb{R}: u(x, t) - v(x, t) > 0 \} \) which is open in \( \mathbb{R} \) and, therefore, admits the representation as the disjoint union of open intervals \( J_i = (a_i, b_i) \subset \mathbb{R} \) indexed by \( i \in J, \) \( -\infty \leq a_i < b_i \leq \infty. \) Since each of these (pairwise disjoint) open intervals contains at least one rational number, the index set \( J \) is at most countable.

We also remark that, for any fixed \( t, \) the set \( \mathcal{N}_0 = \{ x \in \mathbb{R}: u(x, t) - v(x, t) = 0 \} \) of all nodal points of the function \( u(\cdot, t) - v(\cdot, t) \) satisfies \( \bigcup_{i \in J} [a_i, b_i] \subset \mathcal{N}_0. \) Moreover, the set \( \mathcal{N} = \{ x \in \mathcal{N}_0: \ \partial_x(u(x, t) - v(x, t)) \neq 0 \} \) of all non-degenerate (i.e., regular) nodal points contains only points that are isolated in \( \mathbb{R}, \) and we have

\[
\mathcal{N} \subset \bigcup_{i \in J} [a_i, b_i] \subset \mathcal{N}_0.
\]

Finally, we may embed \( J \) into the ordered set of all integers \( \mathbb{Z} = \{ 0, \pm 1, \pm 2, \ldots \} \) using the index ordering \( i < j \) in \( J \) if and only if \( b_i < a_j. \)
Now we are ready to evaluate the integral on the right-hand side of (5.4) over each interval $J_i \cap (-R, R)$. Choosing $R > 0$ large enough, we need to be concerned with the following three cases only:

**Case 1.** Let $i \in I$ be such that $J_i \subset (-R, R)$, i.e., $-R \leq a_i < b_i \leq R$. Then we have

$$
\int_{J_i \cap (-R, R)} \partial_x [\varphi(u_x(x, t)) + f(u(x, t)) - \varphi(v_x(x, t)) - f(v(x, t))] \, dx
$$

by virtue of $u(x, t) - v(x, t) = 0$ at both endpoints $x = a_i$ and $x = b_i$, and by $u_x(a_i, t) - v_x(a_i, t) \geq 0$ and $u_x(b_i, t) - v_x(b_i, t) \leq 0$. Recall that $\varphi' > 0$ in $\mathbb{R}$.

**Case 2.** Let $i \in I$ be such that $-R \in J_i$, i.e., $a_i < -R < b_i \leq R$. Then we have

$$
\int_{J_i \cap (-R, R)} \partial_x [\varphi(u_x(x, t)) + f(u(x, t)) - \varphi(v_x(x, t)) - f(v(x, t))] \, dx
$$

by virtue of $u(b_i, t) - v(b_i, t) = 0$ and $u_x(b_i, t) - v_x(b_i, t) \leq 0$.

**Case 3.** Let $i \in I$ be such that $R \in J_i$, i.e., $-R \leq a_i < R < b_i$. Then we have

$$
\int_{J_i \cap (-R, R)} \partial_x [\varphi(u_x(x, t)) + f(u(x, t)) - \varphi(v_x(x, t)) - f(v(x, t))] \, dx
$$

by virtue of $u(a_i, t) - v(a_i, t) = 0$ and $u_x(a_i, t) - v_x(a_i, t) \geq 0$.

Taking the limit as $R \to \infty$ in Eq. (5.4) and using the fact that $u(x, t) - v(x, t) \to 0$ and $u_x(x, t) - v_x(x, t) \to 0$ as $x \to \pm \infty$, for every $t \geq t_0$, we obtain the desired flux identity (5.3). □

From Proposition 5.1 we derive a strong contraction property of the semigroup $S$. Consider the initial data

$$u_0 \in C^{1+\alpha}(\mathbb{R}) \quad \text{and} \quad \nu_0(x) = w(x - y) \quad \text{for some} \ y \in \mathbb{R},$$

(5.8)
where \( w \) is the travelling wave connecting the constant states \( w_+ \) and \( w_- \) specified in Proposition 2.1. In addition, assume that there are points \( x_1, x_2 \in \mathbb{R} \) such that

\[
x_1 < x_2 \quad \text{and} \quad (u_0(x_1) - v_0(x_1))(u_0(x_2) - v_0(x_2)) < 0.
\] (5.9)

Under these circumstances we claim that

\[
\int_{\mathbb{R}} (u(x, t) - v(x, t))^+ \, dx < \int_{\mathbb{R}} (u_0(x) - v_0(x))^+ \, dx \quad \text{for all } t > 0.
\] (5.10)

Indeed, by virtue of (5.9), there exists \( x_0 \in (x_1, x_2) \) such that \( u_0(x_0) = v_0(x_0) \). Then either \( \partial_x u_0(x_0) \neq \partial_x v_0(x_0) \) and the desired conclusion (5.10) follows directly from Proposition 5.1, or else \( \partial_x u_0(x_0) = \partial_x v_0(x_0) \) in which case the equation is non-degenerate, thanks to \( v'_0(x) = w'(x - y) < 0 \) for all \( x \in \mathbb{R} \), by Proposition 2.1, and one may apply (3.5) to the difference \( u - v \) to conclude that there is \( x_t \in (x_1, x_2) \) such that

\[
u(t, x_t) = v(t, x_t) \quad \text{and} \quad u_x(t, x_t) \neq v_x(t, x_t)
\]

for any \( t > 0 \) small enough; hence, (5.10) follows again from Proposition 5.1.

6. Stability of travelling waves

We adopt the general approach proposed by S. Osher and J. Ralston [14]. To this end, it is more convenient to work in the moving coordinate frame attached to the travelling wave. Accordingly, we transform our original problem (1.1) into the corresponding problem in moving coordinates:

\[
\begin{cases}
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[ \varphi \left( \frac{\partial u}{\partial x} \right) + f_{c,K}(u) \right] & \text{in } \mathbb{R} \times (0, \infty); \\
u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}.
\end{cases}
\] (6.1)

Here, \( w \) constructed in Proposition 2.1 becomes a stationary solution.

Let us consider the initial data \( u_0 \in C^{1+\alpha}(\mathbb{R}) \) such that

\[
w_+ \leq w(x + N) \leq u_0(x) \leq w(x - N) \leq w_-,
\] (6.2)

for some \( N > 0 \),

\[
u_0 - w(\cdot - y) \in L^1(\mathbb{R}) \quad \text{and} \quad \int_{\mathbb{R}} (u_0(x) - w(x - y)) \, dx = 0
\] (6.3)

for some \( y \in \mathbb{R} \).

Using Proposition 5.1 we deduce that

\[
\int_{\mathbb{R}} \left| u(t, x) - w(x - y) \right| \, dx \searrow L \quad \text{as } t \to \infty, \quad L \geq 0.
\] (6.4)

Arguing by contradiction, let us suppose that \( L > 0 \). By virtue of (6.2) and regularity of \( S(t) \) established in (3.1), there is a sequence of times \( t_n \to \infty \) such that
\[ S(t_n)u_0 - w(\cdot - y) \to h \quad \text{in } L^1(\mathbb{R}) \text{ where } h \in C^{1+\alpha}(\mathbb{R}) \text{ satisfies } \int h \, dx = 0. \]

Indeed note that \( w(\cdot - N) - w(\cdot + N) \) is an integrable function for any fixed \( N \); hence, (6.2) guarantees that

\[
\int_{-X}^{-\infty} \left| S(t)u_0 - w(\cdot + y) \right| \, dx + \int_{X}^{\infty} \left| S(t)u_0 - w(\cdot + y) \right| \, dx \to 0 \quad \text{as } X \to \infty,
\]

uniformly for \( t \to \infty \).

Thus, taking \( v_0 = h + w(\cdot - y) \) as new initial data, we can repeat the previous argument based on Proposition 5.1 to show that

\[
\int_{\mathbb{R}} \left| v(t, x) - w(x - y) \right| \, dx < L \quad \text{for all } t > 0,
\]

in contrast with (6.4).

Using the density of \( C^{1+\alpha}(\mathbb{R}) \) functions satisfying (6.2), we infer the following stability theorem:

**Theorem 6.1.** Let \( \varphi \) satisfy (1.2), and let \( f_{c,K} \in C^2(\mathbb{R}) \) obey Hypotheses (f1) and (f2) for some constants \( w_+ < w_- \). Let \( w(\cdot - y) \) be a stationary solution connecting \( w_+, w_- \), and let \( u_0 : \mathbb{R} \to \mathbb{R} \) be a Lebesgue measurable function, such that

\[
w_+ \leq u_0(x) \leq w_-, \quad u_0 - w(\cdot - y) \in L^1(\mathbb{R}), \quad \text{and} \quad \int_{\mathbb{R}} (u_0(x) - w(x - y)) \, dx = 0,
\]

for some \( y \in \mathbb{R} \).

Then

\[
\| S(t)u_0 - w(\cdot - y - ct) \|_{L^1(\mathbb{R})} \to 0 \quad \text{as } t \to \infty,
\]

where \( u(t) = S(t)u_0 \) is the solution of problem (6.1).

**7. Stability of constant states**

Our ultimate goal is to remove the hypothesis

\[
w_+ \leq u_0 \leq w_- \quad \text{in } \mathbb{R}
\]

in Theorem 6.1. Our argument is based on certain stability of the constant states \( w_- \) and \( w_+ \) that follows from the dispersive estimates proved in the next paragraph. In our treatment we find it more convenient to transform the travelling waves described in Proposition 2.1 into stationary states by replacing function \( f \) in problem (1.1) by \( f_{c,K}(v) = f(v) + cv - K \) for all \( v \in \mathbb{R} \). Namely, this function satisfies \( f_{c,K}(v) > 0 \) for all \( v \in (w_+, w_-) \), by Hypothesis (f1).
7.1. Dispersive estimates

We start with a simple version of the dispersive estimates proved in full generality by M. Bonforte and G. Grillo [1]. Although all steps indicated below are formal, they can be justified by the approximation procedure discussed in Section 4.

To begin with, observe that for any \( v \in L^1(\mathbb{R}) \) we have

\[
\|v\|_{L^\infty(\mathbb{R})} \leq c \|v_x\|_{L^p(\mathbb{R})} \|v\|_{L^2(\mathbb{R})}^{\alpha - 1}, \quad \alpha = 3 - \frac{2}{p}, \tag{7.1}
\]

by the Gagliardo–Nirenberg inequality with a constant \( c \in (0, \infty) \), and, obviously,

\[
\|v\|_{L^2(\mathbb{R})} \leq \|v\|_{L^1(\mathbb{R})}^{1/2} \|v\|_{L^\infty(\mathbb{R})}^{1/2}, \tag{7.2}
\]

Let us consider \( u_0 \in C^{1+\alpha}(\mathbb{R}) \) such that \( u_0 - w_\pm \in L^1(\mathbb{R}) \), where \( w_\pm \) stands for either of the constants \( w_+ \) or \( w_- \). Multiplying Eq. (1.1) by \( u - w_\pm \) and integrating by parts, we deduce the following energy inequality:

\[
\frac{1}{2} \frac{d}{dt} \|u - w_\pm\|_{L^2(\mathbb{R})}^2 + \frac{c}{2} \|u_x\|_{L^p(\mathbb{R})}^p \leq - \int_{-\infty}^{\infty} f_{c,K}(u)u_x \, dx
\]

\[
= -F_{c,K}(u(x)) \bigg|_{x=\infty}^{x=-\infty} = -F_{c,K}(w_-) + F_{c,K}(w_+) = 0,
\]

with a positive constant \( c \) and \( F'_{c,K} = f_{c,K} \), that is,

\[
\frac{d}{dt} \|u - w_\pm\|_{L^2(\mathbb{R})}^2 + c \|u_x\|_{L^p(\mathbb{R})}^p \leq 0. \tag{7.3}
\]

Here, in accordance with (7.1) and (7.2),

\[
\|u - w_\pm\|_{L^2(\mathbb{R})}^{2(2p-1)} \leq c_1 \|u_x\|_{L^p(\mathbb{R})}^p \|u - w_\pm\|_{L^1(\mathbb{R})}^{3p-2} \leq c_2 \|u_x\|_{L^p(\mathbb{R})}^p \|u_0 - w_\pm\|_{L^1(\mathbb{R})}^{3p-2}. \tag{7.4}
\]

Combining (7.3) and (7.4) we conclude that

\[
\|u(\cdot, t) - w_\pm\|_{L^2(\mathbb{R})}^{2(2p-1)} \leq \frac{c \|u_0 - w_\pm\|_{L^1(\mathbb{R})}^{3p-2}}{t}. \tag{7.5}
\]

Finally, using the energy inequality (7.3), together with (7.5), maximum principle, and the regularity of \( L^\infty \)-solutions specified in Section 3, we conclude that for any \( t_0 > 0 \) there is a positive function \( h: [t_0, \infty) \rightarrow \mathbb{R}^+ \), \( h(t) \rightarrow 0 \) as \( t \rightarrow \infty \), such that

\[
\|u(\cdot, t) - w_\pm\|_{W^{1,\infty}(\mathbb{R})} \leq h(t) \quad \text{for every } t \geq t_0 \tag{7.6}
\]

holds true for any solution \( u \) of (1.1) with \( u_0 \in C^{1+\alpha}(\mathbb{R}) \) and \( u_0 - w_\pm \in L^1(\mathbb{R}) \). The specific form of \( h \) depends solely on \( \|u_0 - w_\pm\|_{L^1(\mathbb{R})} \) and the structural properties of \( \varphi \).
7.2. Convergence

In order to show unconditional stability of travelling waves, we need to slightly strengthen Hypotheses (f1) and (f2):

**Hypotheses.**

(f3) Function \( f_{c,K} \) satisfies

\[
f_{c,K}(z)(z - w_+) \geq c|z - w_+|^{q+1} \quad \text{and} \quad f_{c,K}(z)(w_- - z) \geq c|z - w_-|^{q+1}
\]

for all \( z \) from an open interval containing \([w_+, w_-]\), where \( c > 0 \) and \( q \) are some constants, \( 0 < q < 2(p - 1) \).

(f4) \[
\left| f_{c,K}(z) \right| \leq C|z - w_+|^r \quad \text{and} \quad \left| f_{c,K}(z) \right| \leq C|z - w_-|^r
\]

for all \( z \) from an open interval containing \([w_+, w_-]\), where \( C > 0 \) and \( r \) are some constants, \( r \geq p - 1 \).

Hypotheses (f3) and (f4) are similar to (2.7). Note that (f3) guarantees sufficiently fast decay of \( w \) to the stationary states \( w_- \) and \( w_+ \), whereas (f4) ensures the non-degeneracy property \( w_+ < w(x) < w_- \) for all \( x \in \mathbb{R} \).

In accordance with Hypotheses (f3) and (f4), there is a stationary solution \( \overline{w} \) to problem (6.1), such that

\[
\begin{align*}
\overline{w}(x) &> w_- \quad \text{for all } x \in \mathbb{R}, \quad (\overline{w} - w_-) \in L^1(-\infty, 0) \quad \text{and} \\
\lim_{x \to -\infty} \overline{w}(x) &= w_-, \quad \liminf_{x \to \infty} \overline{w}(x) > w_-.
\end{align*}
\]  

(7.7)

Similarly, there is a stationary solution \( w \) to problem (6.1), such that

\[
\begin{align*}
w(x) &< w_+ \quad \text{for all } x \in \mathbb{R}, \quad (w_+ - w) \in L^1[0, \infty) \quad \text{and} \\
\lim_{x \to \infty} w(x) &= w_+, \quad \limsup_{x \to -\infty} w(x) < w_+.
\end{align*}
\]  

(7.8)

Now, consider the initial data \( u_0 \in C^{1+\alpha}(\mathbb{R}) \) such that, for some \( y \in \mathbb{R} \),

\[
u_0 - w(\cdot + y) \in L^1(\mathbb{R}) \quad \text{and} \quad \int_{\mathbb{R}} (u_0(x) - w(x + y)) \, dx = 0.
\]  

(7.9)

Unlike in Theorem 6.1, here we do not assume \( w_+ \leq u_0(x) \leq w_- \) for every \( x \in \mathbb{R} \). Our goal is to show that

\[
\|S(t)u_0 - w(\cdot + y)\|_{L^1(\mathbb{R})} \to 0 \quad \text{as } t \to \infty.
\]  

(7.10)

In view of the dispersive estimates (7.6), we can furthermore assume, without loss of generality, that

\[
\overline{w}(x - y) \leq u_0(x) \leq \overline{w}(x - y) \quad \text{for all } x \in \mathbb{R}.
\]  

(7.11)
where \( y, \bar{y} \in \mathbb{R} \). Indeed, it follows from (7.6) that there exists \( t_0 > 0 \) such that for any \( L > 0 \) there is \( y_0 = y_0(L) \in \mathbb{R} \) such that

\[
[S(t_0)u_0](x) \leq \overline{w}(x - y) \quad \text{for all} \ x > -L, \ y < y_0.
\]

Therefore, given \( \varepsilon > 0 \) we find \( L(\varepsilon) > 0 \) such that

\[
\left| \int_{-\infty}^{-L} |S(t_0)u_0 - w_-| \, dx \right| < \varepsilon,
\]

and consider the new initial data

\[
U_0(x) = \begin{cases} 
  w_- & \text{for} \ x < -L, \\
  S(t_0)u_0 & \text{for} \ x \geq -L.
\end{cases}
\]

Accordingly,

\[
\| U_0 - S(t_0)u_0 \|_{L^1(\mathbb{R})} < \varepsilon,
\]

and the inequality on the right-hand side in (7.11) holds with \( S(t_0)u_0 \) in place of \( u_0 \) and for a suitable \( \bar{y} \in \mathbb{R} \). The same argument can be used to get the lower bound in (7.11).

The weak comparison principle guarantees also

\[
\overline{w}(x - y) \leq S(t)u_0(x) \leq \overline{w}(x - \bar{y}) \quad \text{for all} \ (x, t) \in \mathbb{R} \times \mathbb{R}_+.
\]

(7.12)

By virtue of Proposition 6.1, the desired conclusion (7.10) will follow as soon as we show that

\[
\left\| \left[ S(t_0)u_0 - w_- \right]^+ \right\|_{L^1(\mathbb{R})} \to 0 \quad \text{and} \quad \left\| \left[ w_+ - S(t)u_0 \right]^+ \right\|_{L^1(\mathbb{R})} \to 0 \quad \text{as} \ t \to \infty.
\]

(7.13)

In order to see (7.13), we choose \( Y > 0 \) such that, for some \( \delta > 0 \), we have

\[
\overline{w}(x) - w(x + y) > \delta > 0 \quad \text{for all} \ x > Y.
\]

(7.14)

If the first limit in (7.13) is false, that is,

\[
k \overset{\text{def}}{=} \limsup_{t \to \infty} \left\| \left[ S(t_0)u_0 - w_- \right]^+ \right\|_{L^1(\mathbb{R})} > 0,
\]

there is a sequence of times \( t_n \to \infty (n \to \infty) \) such that

\[
\left\| \left[ S(t_n)u_0 - w_- \right]^+ \right\|_{L^1(\mathbb{R})} \geq k/2 > 0 \quad \text{for each} \ n = 1, 2, \ldots.
\]

(7.15)

We claim that this fact forces

\[
\left| \left\{ x > L \colon S(t_n)u_0 > w_- \right\} \right| \to \infty \quad \text{as} \ n \to \infty,
\]

(7.16)

for any fixed \( 0 < L < \infty \).
Indeed, we invoke (7.11) to get
\[
\int_{-\infty}^{L} \left| S(t_n)u_0 - w_+ \right|^+ \, dx \leq \int_{-\infty}^{-M} (w(x) - w_-) \, dx + \int_{-M}^{L} \left| S(t_n)u_0 - w_- \right|^+ \, dx \\
\leq \delta(M) + \sqrt{L + M} \cdot \| S(t_n)u_0 - w_- \|_{L^2(\mathbb{R})}^{1/2},
\]
where, by virtue of (7.7), \( \delta(M) \to 0 \) as \( M \to -\infty \). Thus, (7.15) forces
\[
0 < \frac{k}{2} \leq \liminf_{t_n \to \infty} \int_{L}^{\infty} \left| S(t_n)u_0 - w_- \right|^+ \, dx
\leq \left| \{ x > L : S(t_n)u_0 > w_- \} \right|^{1/2} \cdot \| S(t_n)u_0 - w_- \|_{L^2(\mathbb{R})}^{1/2},
\]
whence (7.16) follows from (7.5).
Now, relation (7.16) together with (7.14) entail
\[
\| S(t_n)u_0 - w(\cdot + y) \|_{L^1(\mathbb{R})} \to \infty \quad \text{as } n \to \infty,
\]
thus contradicting (7.9). Indeed, we have
\[
\int_{L}^{\infty} \left| S(t_n)u_0 - w(\cdot + y) \right| \, dx
\geq \left| \{ x > L : S(t_n)u_0 > w_- \} \right| \cdot \inf_{x > L} \left| w_- - w(x + y) \right| \to \infty.
\]
The second claim in (7.13) follows by similar arguments.
We have thus proved the following result where we use the original function \( f \) in problem (1.1), not \( f_{c,K} \), although we prefer to employ Hypotheses (f3) and (f4) imposed directly on \( f_{c,K} \).

**Theorem 7.1.** Assume that \( \varphi \) satisfies (1.2) and let \( f_{c,K} \in C^2(\mathbb{R}) \) obey Hypotheses (f3) and (f4) with some constants \(-\infty < w_+ < w_- < \infty \). Let \( w(\cdot - y) \) be a stationary solution connecting \( w_+ \) and \( w_- \), where \( y \in \mathbb{R} \) complies with
\[
u_0 - w(\cdot - y) \in L^1(\mathbb{R}) \quad \text{and} \quad \int_{\mathbb{R}} (u_0(x) - w(x - y)) \, dx = 0.
\]
Then we have
\[
\| S(t)u_0 - w(\cdot - y - ct) \|_{L^1(\mathbb{R})} \to 0 \quad \text{as } t \to \infty,
\]
where \( u(t) = S(t)u_0 \) is the semigroup solution of problem (6.1) specified in Section 4.
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