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On Averaging, Reduction, and Symmetry in Hamiltonian Systems

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The existence of periodic orbits for Hamiltonian systems at low positive energies can be deduced from the existence of nondegenerate critical points of an averaged Hamiltonian on an associated "reduced space." Alternatively, in classical (kinetic plus potential energy) Hamiltonians the existence of such orbits can often be established by elementary geometrical arguments. The present paper unifies the two approaches by exploiting discrete symmetries, including reversing diffeomorphisms, that occur in a given system. The symmetries are used to locate the periodic orbits in the averaged Hamiltonian, and thence in the original Hamiltonian when the periodic orbits are continued under perturbations admitting the same symmetries. In applications to the Hénon–Heiles Hamiltonian, it is illustrated how "higher order" averaging can sometimes be used to overcome degeneracies encountered at first order.

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INTRODUCTION

In the method of averaging the existence of periodic orbits for Hamiltonian systems is deduced from the existence of nondegenerate critical points of an averaged Hamiltonian on an associated "reduced space" (e.g., see [30; 1, p. 306]). Alternatively, in classical (kinetic plus potential energy) Hamiltonians the existence of such orbits can often be established by elementary geometrical arguments (e.g., see [8]). One goal of this paper is to unify the two approaches by exploiting discrete symmetries, including reversing diffeomorphisms, present in given problems. The symmetries are used to locate periodic orbits in the averaged Hamiltonian, and thence in the original Hamiltonian when the periodic orbits are continued under perturbations admitting the same symmetries.

A second goal is to illustrate how "higher order" averaging can sometimes be used to overcome degeneracies encountered at first order. This is accomplished by examining the Hénon–Heiles Hamiltonian, which by elementary geometrical arguments can be shown to possess eight periodic orbits at low positive energies (see [8, Sect. 3]), but which yields only two such orbits at first order averaging, the rest being "hidden" by a circle of degenerate critical points of the reduced averaged Hamiltonian (see [4, 21]). We show the existence of all eight orbits by using second order averaging, give the stability status in all cases, and through symmetries identify each with those seen by Hénon and Heiles in their original computer experiments [20].

The paper opens with a brief treatment of averaging and normal forms for Hamiltonian systems, and then discusses the effect of normal form conversions on existing symmetries. All this is done in the context of graded Lie algebras, which seems to best reveal the elementary nature of the theory. We remark that the material in Section 1 is folklore. Moreover, some of the material in Section 2 also appears to be known, but as far as the authors are aware is not available in the literature in the format most relevant to this paper.

In Section 3 we review the reduction process in the case of flow-induced free and proper S^1-actions on a symplectic manifold (M, ω). We then show how symmetries of M and the associated averaged Hamiltonians $H : M \to R$ induce corresponding symmetries of the reduced space and reduced Hamiltonians. We concentrate particularly on relative equilibria, i.e., points on periodic orbits of averaged Hamiltonian flows on M which project to equilibria for the reduced Hamiltonian system. Such a periodic orbit is then "symmetric" if and only if the equilibrium point for the reduced Hamiltonian is a fixed point of the induced symmetry on the reduced space (Proposition 3.6). For other research in this area we refer the reader to [1, Chap. 4; 14; 21; 23; 26; 27; 33; 35] and the references therein.

In Section 4 we extend [21] and construct specific models for reduction by an S^1-action in the general two degree of freedom resonance case (see Theorem 4.2). Here the reduced spaces turn out to be simple surfaces of revolution in R^3. The corresponding construction of reduced Hamiltonians becomes a triviality, and by choosing a group of symmetries from $SU(2)$, induced symmetries can be easily identified as rotations of these surfaces. Moreover, in the simplest case the projection to the reduced space is shown to be the standard Hopf fibarion of S^3 over S^2 (see [15] and Theorem 4.2) which motivates the introduction of generalized "Hopf variables." The section concludes with a discussion of some specific symplectic and anti-symplectic involutions on R^4, and describes the induced symmetries on the reduced space needed for the applications.

Section 5 is independent of the preceding material. It concerns the continuation and stability classification for fixed points of one-parameter families of planar symplectic mappings, and collects results in a manner relevant to our applications. In order to apply these results to the examples of Section 7, we explain in Section 6 how to reconstruct the flow of the original Hamiltonian from that of the associated reduced Hamiltonian on the reduced space, and how to then reduce the problem to the study of such a family of mappings. Included is the introduction of coordinates, generalized from [5, 21], which we use to compute the necessary Jacobians, Hessians, and twist coefficients for stability classification of the periodic orbits. The final results are collected in Theorem 6.4, which also examines the effects of symmetries on the continuation (under perturbation) of our periodic orbits.

Section 7 concludes the main body of the paper with three applications: The first views Liapunov's theorem from the standpoint of reduction, and fills a gap in the previous part of the paper concerning "problem" points on the reduced space. The second and third, respectively, detail the results of first and second order averaging and symmetries in the Hénon–Heiles Hamiltonian, and how a further "degeneracy" at second order averaging is overcome. We include a detailed discussion of how discrete symmetries allow us to locate the continued periodic orbits and identify them as the ones formerly constructed by geometric means in [8, Sect. 3]. Contrasting results are then given for two related Hamiltonians.

There are two appendices to the paper. Appendix A sketches the lengthy computations involved in converting the Hénon–Heiles Hamiltonian into normal form through first and second order. For other current discussions of normal form computations we refer to [16, 32]. Appendix B presents the results on the Hénon–Heiles problem in a wider perspective. We discuss what is currently known about the constructed periodic orbits as the energy increases, and describe attendant stability changes.

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1. Normal Forms

Let $\mathscr{L} = \bigoplus_{r=2}^{\infty} \mathscr{L}_r$ be a positively graded real or complex Lie algebra (for definitions see [19, p. 3]). We subscript elements $H_r \in \mathscr{L}_r$, and assume the Lie bracket in $\mathscr{L}$ satisfies

$$[H_s, H_r] \in \mathscr{L}_{r+s-2}. \tag{1.1}$$

If for $F \in \mathscr{L}$ we define the linear mapping $ad_F : \mathscr{L} \to \mathscr{L}$ by $ad_F(H) = [F, H]$, then (1.1) guarantees that $ad_F | \mathscr{L}_r : \mathscr{L}_r \to \mathscr{L}_r$ when $F \in \mathscr{L}_2$.

We define ad_F^0 to be the identity mapping on $\mathscr{L}$, $ad_F^1 = ad_F$, and $ad_F^j = ad_F \circ ad_F^{j-1}$ when $j > 1$. We will say the element $H = \bigoplus_{r=2}^{\infty} H_r$ in $\mathscr{L}$ is in *normal form through terms of orser* $m \geqslant 2$ wrt $F \in \mathscr{L}$ if $ad_F(H_r) = 0$ for $2 \leqslant r \leqslant m$.

As an example let $\mathscr{P}_r$ denote the set of all real-valued homogeneous polynomials of degree $r \geqslant 2$ in the complex variables $z_j = x_j + iy_j$, $\bar{z}_j = x_j - iy_j$, $j = 1,\dots, n$, set $\mathscr{P} = \bigoplus_{r=2}^{\infty} \mathscr{P}_r$, and as bracket use the negative of the usual Poisson bracket, i.e.,

$$[H, G] = -2i \sum_{j=1}^{n} \left(\frac{\partial H}{\partial \bar{z}_j} \frac{\partial G}{\partial z_j} - \frac{\partial H}{\partial z_j} \frac{\partial G}{\partial \bar{z}_j} \right). \tag{1.2}$$

Here $\partial/\partial z_j$ and $\partial/\partial \bar{z}_j$ are the formal derivations on $\mathscr{P}$ define by

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right). \tag{1.3}$$

Returning to generalities, notice from (1.1) that if $K \in \mathscr{L}_s$, then $ad_K^j(H_r) \in \mathscr{L}_{r+j(s-2)}$. However, for $s \geqslant 3$ and fixed n the equality $r + j(s-2) = n$ has only a finite number of solutions in the positive integers r and j. As a consequence, for $K \in \mathscr{L}_s$ with $s \geqslant 3$ we can define a linear operator $\exp(ad_K) : \mathscr{L} \to \mathscr{L}$ by

$$\exp(ad_K)(H) = \sum_{j=0}^{\infty} (1/j!)\, ad_K^j(H).$$

We say that $F \in \mathscr{L}_2$ *splits* $\mathscr{L}$ if for $r \geqslant 2$ we have $\mathscr{L}_r = N_r \oplus R_r$, where $N_r = \ker(ad_F | \mathscr{L}_r)$ and $R_r = \text{range}(ad_F | \mathscr{L}_r)$. When F splits $\mathscr{L}$, then $ad_F | R_r : R_r \to R_r$ is an isomorphism, and we let $\Gamma_r : R_r \to R_r$ denote the inverse.

The following result is well known in the context $\mathscr{L} = \mathscr{P}$ (see, e.g., [1, p. 500]).

Proposition 1.1. *Let* $H = \bigoplus_{r=2}^{\infty} H_r \in \mathscr{L}$ *be in normal form through terms of order* $(m-1) \geqslant 2$ wrt H_2, *and assume* H_2 *splits* $\mathscr{L}$. *Let*

$H_m = F_m + G_m$, where $F_m \in N_m$ and $G_m \in R_m$, and set $K_m = \Gamma_m(G_m)$. Then $\exp(ad_{K_m})(H)$ is in normal form through terms of order m wrt *H_2, agrees with H through terms of order $m-1$, and has F_m as m*th *term.*

Proof. $\exp(ad_{K_m})(H) = H_2 + H_3 + \cdots + H_{m-1} + H_m + ad_{K_m}(H_2) +$ {terms in $\mathscr{L}_j$ with $j \geqslant (m+1)$}. Since $H_m + ad_{K_m}(H_2) = F_m + G_m - ad_{H_2}(K_m) = F_m$, we are done. Q.E.D.

To interpret the proposition for the case of interest in this paper, let $H_2 \in \mathscr{P}_2$ have the form

$$H_2(z, \bar{z}) = \sum_{j=1}^{n} (\gamma_j/2)\,|z_j|^2, \tag{1.4}$$

where the γ_j are real constants and $z_j = x_j + iy_j$. Also, write

$$z^k \bar{z}^l = z_1^{k_1} \cdots z_n^{k_n} \cdot \bar{z}_1^{l_1} \cdots \bar{z}_n^{l_n}, \qquad \langle k - l, \gamma \rangle = \sum_{j=1}^{n} (k_j - l_j)\,\gamma_j. \tag{1.5}$$

By (1.2) we then have

$$ad_{H_2}(z^k \bar{z}^l) = -i \langle k - l, \gamma \rangle\, z^k \bar{z}^l. \tag{1.6}$$

Relation (1.6) implies that H_2 splits $\mathscr{P} = \bigoplus_{r=2}^{\infty} \mathscr{P}_r$, and that if $G_m(z, \bar{z}) = \sum c_{kl} z^k \bar{z}^l$ is in R_m, where $m = \sum_{i=1}^{n} (k_i + l_i)$, then

$$(\Gamma_m \circ G_m)(z, \bar{z}) = i \sum \langle k - l, \gamma \rangle^{-1} c_{kl} z^k \bar{z}^l. \tag{1.7}$$

It remains to explain the meaning of $\exp(ad_K)$ in this context.

PROPOSITION 1.2. *Let H_2 be as in (1.4), and let $H(z, \bar{z}) = \sum_{j=2}^{\infty} H_j(z, \bar{z})$ converge in some neighborhood U of the origin in R^{2n}. Assume H, considered as an element of $\mathscr{P}$, is in normal form with respect to H_2 through terms of order $(m-1) \geqslant 2$. Write $H_m = F_m + G_m$, where $F_m \in N_m$ and $G_m \in R_m$, set $K = K_m = \Gamma_m(G_m)$, and let φ_t be the flow of the Hamiltonian system $\dot{z} = -2i(\partial K/\partial \bar{z})$. Then*

(a) *there is a neighborhood $V \subset U$ of the origin such that φ_t is defined in V for $|t| \leqslant 2$; and*

(b) $\exp(ad_K)(H) = (H \circ \varphi_1)$.

Remark. Since the flow φ_t is canonical for each t, (b) shows that $\exp(ad_K)(H)$ is simply H composed with a canonical transformation converting H into normal form with respect to H_2 through terms of order m. The new series may, of course, have a smaller domain of convergence.

Proof. (a) Since K is homogeneous the origin is a rest point of φ_t. The result follows snce flows have open domains.

(b) We take the Taylor expansion of $F(t) = H \circ \varphi_t$ and note that $F'(t) = (d/dt)(H \circ \varphi_t) = ad_K(H) \circ \varphi_t$, $F''(t) = ad_K^2(H) \circ \varphi_t$, etc. (recall that our bracket $[K, H] = ad_K(H)$ is the negative of the usual Poisson bracket). Evaluating $F(t) = F(0) + F'(0)\, t + (\frac{1}{2!})\ F''(0)\, t^2 + \cdots$, at $t = 1$ gives the result. Q.E.D.

We now explain why F_m in Proposition 1.2 is called the "average" of H_m with respect to the flow generated by H_2, and why the conversion of a Hamiltonian into form is called "averaging."

PROPOSITION 1.3. *With the hypotheses and notation of Proposition* 1.2, *let ρ_t be the flow of the Hamiltonian system $\dot{z} = -2i(\partial H_2/\partial \bar{z})$. Then*

$$F_m(z, \bar{z}) = \lim_{T \to \infty} \frac{1}{T} \int_0^T (H_m \circ \rho_t)(z, \bar{z})\, dt.$$

Proof. $H_m = F_m + G_m$ and $(d/dt)(F_m \circ \rho_t) = ad_{H_2}(F_m) \circ \rho_t = 0$, hence F_m is constant along orbits of ρ_t. Now $K = \Gamma_m(G_m)$ implies $(d/dt)(K \circ \rho_t) = G_m \circ \rho_t$, and hence it suffices to prove that

$$\frac{1}{T} \int_0^T (G_m \circ \rho_t)\, dt = \frac{1}{T} [K \circ \rho_t - K]$$

limits to 0 as $T \to \infty$. Each orbit $\rho_t(z, \bar{z})$ is given by $z_j(t) = z_j e^{-i\gamma_j t}$, $j = 1, 2,..., n$, hence is bounded as the γ_j are real in (1.4). Thus $(K \circ \rho_t)(z, \bar{z})$ is bounded, and the result follows. Q.E.D.

Normal forms are in general not unique. For example, assume $H = \bigoplus_{r=2}^{\infty} H_r \in \mathscr{L}$ is in normal form through terms of order $m \geqslant 2$ wrt H_2, let $f(x)$ be a formal power series in x, and let $G \in \mathscr{L}_s$ with $s \geqslant 3$. Then the Jacobi identity

$$ad_{H_2} \circ ad_G = ad_{(ad_{H_2}(G))} + ad_G \circ ad_{H_2}$$

implies $f(ad_G)(H)$ is in normal form through terms of order m wrt H_2 when $ad_{H_2}(G) = 0$. If H_2 splits $\mathscr{L}$ and $s \leqslant m$ a converse holds: since $(ad_{H_2} \circ ad_G)(H) = -(ad_{H_2} \circ ad_{H_2})(G)$ at the s-level, and ad_{H_2} is an isomorphism on its range, we must have $ad_{H_2}(G) = 0$ for $ad_G(H)$ to be in normal form through terms of order m wrt H_2.

In their paper [14], Cushman and Deprit use an alternate (more general) definition of normal forms (attributed to J. van der Meer). In our context this can be formulated as follows: Suppose $F \in \mathscr{L}$ and a complement $N_r' \subset \mathscr{L}_r$ has been chosen for $R_r = \text{range}(ad_F | \mathscr{L}_r)$ for each $r \geqslant 2$. Then $H = \bigoplus_{r=2}^{\infty} H_r \in \mathscr{L}$ is in *normal form through terms of order* $m \geqslant 2$ wrt F if $H_r \in N_r'$ for $2 \leqslant r \leqslant m$. Proposition 1.1 now holds without the splitting hypothesis on H_2 (which has been incorporated into the choice of complements), but as they show by an example having a degenerate critical point at the origin, this formulation allows a polynomial Hamiltonian $H_2 + H_3$ on R^4 to be in normal form even when the associated differential equations are nonintegrable. Their particular choice of complements is motivated by the representation theory of Lie algebras.

2. Symmetries

As in Section 1 let $\mathscr{L} = \bigoplus_{r=2}^{\infty} \mathscr{L}_r$ be a positively graded Lie algebra with bracket $[\ ,\ ]$. Let $H_2 \in \mathscr{L}_2$ split $\mathscr{L}$ with $\mathscr{L}_r = N_r \oplus R_r$. Assume that a group $\mathscr{G}$ acts linearly (on the right) on $\mathscr{L}$ in the following manner, where $g \in \mathscr{G}$ and $H, F \in \mathscr{L}$:

$$\mathscr{L}_r \cdot g \subset \mathscr{L}_r, \tag{2.1a}$$

$$H_2 \cdot g = H_2, \tag{2.1b}$$

$$[H \cdot g, F \cdot g] = \pm [H, F] \cdot g. \tag{2.1c}$$

Depending on the element $g \in \mathscr{G}$, one chooses the appropriate sign in (2.1c) and fixes that choice throughout this section. In applications to the space $\mathscr{P}$ of formal power series considered in Section 1, $\mathscr{G}$ will be a group of formal diffeomorphisms and the action will be composition of functions. The plus or minus sign in (2.1c) will occur according as $g \in \mathscr{G}$ is symplectic or reversing.

Lemma 2.1. (a) *N_r and R_r both $\mathscr{G}$-invariant.*

(b) *$E \in N_r$ and $F \in R_r$ are both fixed by an element $g \in \mathscr{G}$ if and only if $E + F$ is fixed by g.*

(c) *Let $H = ad_{H_2}(F)$ for $F \in R_r$. Then $H \in R_r$, and H is fixed by $g \in \mathscr{G}$ if and only if $F \cdot g = \pm F$.*

(d) *Let $H \cdot g = H$ and $F \cdot g = \pm F$ for some $g \in \mathscr{G}$. Then $ad_F^j(H)$ is fixed by g for all $j \geqslant 0$.*

(e) *Let $F \cdot g = F$ for some $g \in \mathscr{G}$. Then $H = \bigoplus_{r=2}^{\infty} H_r$ is in normal form through terms of order m with respect to F if and only if $H \cdot g$ has this property.*

Proof. (a) For $E \in N_r$ we have $[H_2, E \cdot g] = [H_2 \cdot g, E \cdot g] = \pm [H_2, E] \cdot g = 0$, and hence $E \cdot g \in N_r$. For $F \in R_r$ there is a $K \in R_r$ such that $ad_{H_2}(K) = F$. Then $F \cdot g = [H_2, K] \cdot g = \pm [H_2 \cdot g, K \cdot g] = [H_2, (\pm K) \cdot g] = ad_{H_2}((\pm K) \cdot g)$ is in R_r.

(b) The result follows from (a) and the uniqueness of direct sum decompositions.

(c) A similar computation to that in (a) gives $H \cdot g = [H_2, (\pm F) \cdot g]$ with $(\pm F) \cdot g \in R_r$. Since $ad_{H_2}|R_r$ is an isomorphism, the result follows.

(d) The result follows by induction; the case $j = 0$ being trivial.

(e) The equalities $[F, H_j] \cdot g = \pm[F, H_j \cdot g]$, $j = 2, 3, \ldots, m$, imply the result. Q.E.D.

THEOREM 2.2. *In addition to the hypotheses of Proposition* 1.1, *assume* (2.1) *and that* $g \in \mathscr{G}$ *fixes* H. *Then* g *also fixes* $\exp(ad_{K_m})(H)$.

Proof. By (2.1a), $H_m = F_m + G_m$ is fixed by g, and hence each of F_m and G_m is fixed by Lemma 2.1(b). Then $G_m = [H_2, K_m]$, where $K_m = \Gamma_m(G_m)$, implies $K_m \cdot g = \pm K_m$ by Lemma 2.1(c). The result now follows from Lemma 2.1(d). Q.E.D.

To summarize, let us say that $H = \bigoplus_{r=2}^{\infty} H_r$ admits a symmetry corresponding to $g \in \mathscr{G}$ if $H \cdot g = H$. Then each H_r admits the same symmetry. Moreover, when H_2 splits $\mathscr{L}$ then each of F_r and G_r also admits the symmetry in the direct sum decomposition $H_r = F_r + G_r$ of Proposition 1.1. Finally, when H is converted into normal form through a certain order using that proposition, then the transformed H will admit the same symmetry.

We remark that if $H = \bigoplus_{r=2}^{\infty} H_r$ is in normal form through terms of order m wrt H_2 and $\mathscr{G}_H \subset \mathscr{G}$ is the isotropy group of H, then the index of $\mathscr{G}_H$ in $\mathscr{G}$ is the cardinality $|S|$ of the orbit $S = \{H \cdot g \mid g \in \mathscr{G}\}$. By Lemma 2.1(e) each $H \cdot g$ is in normal form through terms of order m wrt H_2, and so this index simply counts the number of distinct normal forms obtainable from H by the action of $\mathscr{G}$. If $\mathscr{G}_H$ is normal in $\mathscr{G}$, e.g., if $\mathscr{G}_H$ is in the center $Z(\mathscr{G})$ of $\mathscr{G}$, then $|S| = |\mathscr{G}/\mathscr{G}_H|$.

We refer to [14, 16] for additional discussion of the effect of symmetries on normal form calculations.

3. REDUCED SPACES AND INDUCED SYMMETRIES

Let (M, ω) be a real analytic symplectic manifold of dimension $2n$, and $\mathscr{H} = \mathscr{H}(M)$ the space of real analytic Hamiltonians $H: M \to R$. (The results of this section also hold in the C^∞ category.) Recall that the Hamiltonian vector field X_H on M associated with H is defined by $X_H \lrcorner\, \omega = dH$, where $\lrcorner$ denotes the left interior product. For G, $H \in \mathscr{H}$ we define the Poisson bracket $\{G, H\} = \omega(X_G, X_H)$, we set $[G, H] = -\{G, H\}$, and define $ad_G(H) = [G, H]$ in agreement with our earlier conventions. The local coordinate description of $[\ ,\ ]$ is as in (1.2), with

$$\omega = \tfrac{1}{2} \sum_{j=1}^{n} \operatorname{Im}(d\bar{z}_j \wedge dz_j) = \sum_{j=1}^{n} dx_j \wedge dy_j. \tag{3.1}$$

Now let $F \in \mathscr{H}$ be such that the flow Φ_t of X_F is periodic and defines a free and proper S^1-action on $\Sigma_h = F^{-1}(h) \subset M$, where h is a regular value of F. A *reduced space* for (Φ_t, Σ_h) is then an analytic symplectic manifold (M_R, ω_R) together with an analytic surjective submersion $\pi : \Sigma_h \to M_R = M_R(h)$, called the *(canonical) projection*, such that π maps distinct orbits of Φ_t in Σ_h to distinct points of M_R, and such that

$$\pi^* \omega_R = i^* \omega, \tag{3.2}$$

where $i : \Sigma_h \to M$ is inclusion.

Identifying R with the dual of the Lie algebra of S^1, the Hamiltonian $F : M \to R$ becomes the Ad*-equivariant momentum mapping for the symplectic action Φ_t (for definitions see [1, pp. 276, 279]). The existence of a reduced space for (Φ_t, Σ_h) is then a standard result [1, p. 299], and we can easily show that any model of the reduced space is symplectomorphic to the orbit space Σ_h/S^1 with the induced symplectic form ω_R determined by (3.2).

Let $\mathscr{H}_F = \{H \in \mathscr{H} \mid ad_F(H) = [F, H] = 0\}$. The Jacobi identity implies ad_F is a derivation, hence $\mathscr{H}_F$ is a Lie subalgebra of $\mathscr{H}$. The following result is an adaptation to our context of [1, Theorem 4.3.5, p. 304]:

THEOREM 3.1. *Assume the notation above. Then to each $H \in \mathscr{H}_F$ and $h > 0$ there corresponds an analytic Hamiltonian $K = K_h : M_R \to R$ satisfying $K \circ \pi = H \circ i$, where $i : \Sigma_h \to M$ is inclusion and $\Sigma_h = F^{-1}(h)$. Moreover, $\pi_* X_H = X_K \circ \pi$, where $X_K \lrcorner \omega_R = dK$ and π_* is the tangent mapping. If Ψ_t is the flow of X_H on M, and ρ_t the flow of X_K on M_R, then Ψ_t leaves Σ_h invariant, commutes with the flow Φ_t of X_F, and satisfies $\pi \circ \Psi_t \circ i = \rho_t \circ \pi$.*

Now K is called the *reduced Hamiltonian*, X_K the *reduced vector field*, and ρ_t the *reduced flow*, corresponding to $H \in \mathscr{H}_F$ and $h > 0$.

Let $\mathscr{R}$ be the set of reduced Hamiltonians on M_R. For $K, L \in \mathscr{R}$ define a Poisson bracket by $\{K, L\}_R = \omega_R(X_K, X_L)$, and set $[K, L]_R = -\{K, L\}_R$.

PROPOSITION 3.2. *If $K, L \in \mathscr{R}$ correspond to $H, G \in \mathscr{H}_F$, then*

$$[K, L]_R \circ \pi = [H, G] \circ i. \tag{3.3}$$

Proof. By Theorem 3.1 we have $\pi_* X_H = X_K \circ \pi$ and $\pi_* X_G = X_L \circ \pi$. The result follows by evaluating the relation $\pi^* \omega_R = i^* \omega$ on (X_H, X_G). Q.E.D.

If $\mathscr{G}$ is a group of real analytic diffeomorphisms of (M, ω), then we can regard $\mathscr{G}$ as acting in (M, ω) on the left, and, by composition of functions, as acting on $\mathscr{H}$ on the right. The following result is a variation of [1, Exercise 4.3B, p. 309].

LEMMA 3.3. *In the notation above, assume that each $g \in \mathscr{G}$ fixes $\Sigma_h = F^{-1}(h)$ and carries orbits of the flow Φ_t of X_F with energy h into other such orbits (reparametrizations allowed). Then*

$$\mathscr{G}^* = \{ g \in \mathscr{G} \mid \pi(gx) = \pi(x)\ \forall x \in M \} \tag{3.4}$$

is a normal subgroup of G. The quotient group $\mathscr{G}_R = \mathscr{G}/\mathscr{G}^*$ with canonical projection $\phi : \mathscr{G} \to \mathscr{G}_R$ acts as a group of real analytic diffeomorphisms on M_R by

$$\phi(g) \cdot \pi(x) = \pi(gx). \tag{3.5}$$

Proof. That $\mathscr{G}^*$ is a subgroup of $\mathscr{G}$ is clear. To prove normality first observe that any $f \in \mathscr{G}$ sends the X_F orbit through a point $p \in \Sigma_h$ into the X_F orbit through $f \cdot p$, and that $g \in \mathscr{G}^*$ implies that g sends this latter orbit into itself. But applying $f^{-1} \cdot (g \cdot f \cdot p)$ then gives a point back on the X_F orbit through p. Since $\pi : \Sigma_h \to M_R$ maps the entire X_F orbit through p to a point in M_R, we have $\pi(f^{-1} \cdot g \cdot f \cdot p) = \pi(p)$. Thus $\mathscr{G}^*$ is normal in $\mathscr{G}$.

To verify that (3.5) gives a well-defined action on M_R suppose $\phi(g) = \phi(g')$. Since $\mathscr{G}^*$ is normal there is an $f \in \mathscr{G}^*$ such that $g = f \cdot g'$, and so $\pi(g \cdot p) = \pi(f \cdot g' \cdot p) = \pi(g' \cdot p)$ by definition of $\mathscr{G}^*$. The action $\phi(g)$ of $\mathscr{G}_R$ on M_R is thus well defined; real analyticity follows from formula (3.5).

Q.E.D.

Here $\mathscr{G}_R$ is called the *reduced group* of $\mathscr{G}$.

THEOREM 3.4. *In the notation above assume $g^*\omega = \pm\omega$ and $H \circ g = H$ for some $g \in \mathscr{G}$ and $H \in \mathscr{H}$. Then*

$$g_* X_H = \pm X_H \circ g. \tag{3.6}$$

In particular, if $H = F$ and the above hypotheses hold for each $g \in \mathscr{G}$, then the hypotheses, and hence the conclusion, of Lemma 3.3 *hold. In this case we also have:*

(a) $\phi(g)^* \omega_R = \pm\omega_R$.

(b) *Let $H \in \mathscr{H}_F$ with $K : M_R \to R$ the reduced Hamiltonian. Then $(H \circ i) \circ g = H \circ i$ if and only if $K \circ \phi(g) = K$ and $\phi(g)_* X_K = \pm X_K \circ \phi(g)$ hold.*

Proof. Taking into account $\pm$, (3.6) is just Theorem 3.3.19 of [1, p. 194]. For $H = F$ the flow Φ_t of X_F then satisfies $g \circ \Phi_t = \Phi_{\pm t} \circ g$ and Lemma 3.3 follows.

(a) $\phi(g) \circ \pi = \pi \circ g$, $i^*\omega = \pi^*\omega_R$, and $i \circ g = g \circ i$ imply $\pi^*(\phi(g)^* \omega_R) = g^*(\pi^*\omega_R) = g^*(i^*\omega) = i^*(g^*\omega) = \pm i^*\omega = \pm\pi^*\omega_R$. Since π is surjective the result follows.

(b) If $(H \circ i) \circ g = H \circ i = K \circ \pi$, then $K \circ \pi = K \circ \pi \circ g = K \circ \phi(g) \circ \pi$. Since π is surjective, this implies $\phi(g)$ fixes K, and the relation for the tangent map on vectorfields follows as for (3.6). The converse follows easily.
Q.E.D.

The reader will note that in the case $g^*\omega = -\omega$ we have never assumed that g is an involution. This will be the case, however, in our applications. Further results on involutions and reduced spaces can be found in [33].

If $H \in \mathscr{H}_F$, then $p \in \Sigma_h = F^{-1}(h)$ is a *relative equilibrium* for X_H if $\pi(p) \in M_R$ is an equilibrium point for the reduced vector field X_K on M_R, where $K \circ \pi = H \circ i$. If Ψ_t is the flow of X_H, then p is a relative equilibrium if and only if there is a smooth reparametrization $g(t)$ of the S^1-orbit of the flow Φ_t of X_F through p such that $\Psi_t(p) = \Phi_{g(t)}(p)$ [1, p. 306].

PROPOSITION 3.5. *Let $\mathscr{G}$ act as in Theorem* 3.4, *and let $H \in \mathscr{H}_F$ satisfy $H \circ g = H$ for all $g \in \mathscr{G}$. If $p \in \Sigma_h$ is a relative equilibrium for X_H, then so are all points in the orbit $\mathscr{G} \cdot p$.*

Proof. By Lemma 3.3 the reduced group is defined. But $\pi_* X_H = X_K \circ \pi$ and $\pi \circ g = \phi(g) \circ \pi$ then imply $\pm X_K(\pi \circ g)(p) = \pm \pi_* X_H(g \cdot p) = \pi_* g_* X_H(p) = \phi(g)_* \pi_* X_H(p) = \phi(g)_* (X_K b\, \pi)(p) = 0$, where we have used formula (3.6). The result now follows. Q.E.D.

Note that if $\Psi_t(p)$ is a periodic orbit of X_H corresponding to a relative equilibrium p and $H \circ g = H$, then Proposition 3.5 guarantees that $g \cdot \Psi_t(p)$ must also have this property.

We call a set $A \subset M$ *symmetric* wrt $g \in \mathscr{G}$ if $g \cdot A = A$. In particular, a periodic orbit of a flow on M is symmetric wrt g if and only if it is symmetric wrt g as a point set. The following result is immediate from Proposition 3.5:

PROPOSITION 3.6. *Assume the hypotheses of Proposition* 3.5, *and let $p \in \Sigma_h$ be a relative equilibrium for X_H. If Ψ_t is the flow of X_H, then the orbit $\Psi_t(p)$ is symmetric* wrt *$g \in \mathscr{G}$ if and only if $\pi(p) \in M_R$ is a fixed point of $\phi(g) \in \mathscr{G}_R$.*

4. EXAMPLES OF REDUCED SPACES AND INDUCED SYMMETRIES

Let $R^4 \simeq \mathbb{C} \times \mathbb{C}$ have global coordinates $x = (x_1, x_2, y_1, y_2) \simeq (z_1, z_2) = z$, where $z_j = x_j + iy_j$, $j = 1, 2$, and let ω be given by (3.1) with $n = 2$. Then (R^4, ω) is a symplectic space, and for any Hamiltonian function $H: R^4 \to R$

Hamilton's equations can be written as $\dot{z}_j = -2i(\partial H/\partial \bar{z}_j)$, $j = 1, 2$. We will consider Hamiltonians of the form $H = \bigoplus_{r=2}^{\infty} H_r$ with

$$H_2(z, \bar{z}) = (\alpha/2)\,|z_1|^2 + (\beta/2)\,|z_2|^2, \qquad \alpha \quad \text{and} \quad \beta > 0, \tag{4.1}$$

where either (α/β) is irrational (the *nonresonance* case), or w.l.o.g. α and β are both one or are relatively prime integers (the *resonance* case). In the second instance we speak of "$\alpha - \beta$ resonance." Recall that H is in normal form wrt H_2 provided $ad_H(H) = 0$, and that H_2 as in (4.1) splits. The following result is well known:

LEMMA 4.1. *Let H_2 be as in (4.1), and let $H : R^4 \to R$ be an arbitrary polynomial in $(z_1, z_2, \bar{z}_1, \bar{z}_2) = (z, \bar{z})$.*

(a) *In the resonance case $ad_{H_2}(H) = 0$ if and only if H can be written as a polynomial in the "Hopf variables" $W_j = W_j(z)$ (see [28, p. 102]), where*

$$\begin{aligned} W_1 &= 2\,\mathrm{Re}(z_1^{\beta} \bar{z}_2^{\alpha}), & W_2 &= 2\,\mathrm{Im}(z_1^{\beta} \bar{z}_2^{\alpha}), \\ W_3 &= \alpha\,|z_1|^2 - \beta\,|z_2|^2, & W_4 &= \alpha\,|z_1|^2 + \beta\,|z_2|^2, \end{aligned} \tag{4.2}$$

and

$$\frac{1}{4}(W_1^2 + W_2^2) = \left(\frac{W_4 + W_3}{2\alpha}\right)^{\beta} \left(\frac{W_4 - W_3}{2\beta}\right)^{\alpha}. \tag{4.3}$$

(b) *In the case of nonresonance $ad_{H_2}(H) = 0$ if and only if H can be written as a polynomial in the Hopf variables W_3 and W_4, i.e., as a polynomial in $|z_1|^2$ and $|z_2|^2$.*

Proof. Using the notational conventions of (1.5) it suffices to assume $H(z, \bar{z}) = z^k \bar{z}^l$, and in this case (1.6) with $\gamma = (\alpha, \beta)$ gives

$$ad_{H_2}(z^k \bar{z}^l) = -i\langle k - l, \gamma\rangle\, z^k \bar{z}^l = -i[\alpha(k_1 - l_1) + \beta(k_2 - l_2)]\, z^k \bar{z}^l. \tag{4.4}$$

(a) If α and β are relatively prime (or both one), then there exist integers u and v such that $\alpha u + \beta v = 1$. If (4.4) vanishes, then set $m = (k_1 - l_1)\,v - (k_2 - l_2)\,u$, and note that $(k_1 - l_1) = m\beta$ and $(k_2 - l_2) = -m\alpha$. We then have

$$\begin{aligned} z^k \bar{z}^l = z_1^{k_1} z_2^{k_2} \bar{z}_1^{l_1} \bar{z}_2^{l_2} &= |z_1|^{2l_1} \cdot |z_2|^{2k_2} \cdot (z_1^{\beta} \bar{z}_2^{\alpha})^m, && \text{for} \quad m \geqslant 0, \\ &= |z_1|^{2k_1} \cdot |z_2|^{2l_2} \cdot (\bar{z}_1^{\beta} z_2^{\alpha})^{-m}, && \text{for} \quad m < 0. \end{aligned} \tag{4.5}$$

The forward implication is now clear since each of the factors in (4.5) can be expressed in terms of the Hopf variables (4.2). The reverse implication follows since $ad_{H_2}(W_j) = 0$, $j = 1, 2, 3, 4$. Identity (4.3) follows by direct computation.

(b) If (α/β) is irrational, then (4.4) vanishes if and only if each $k_j = l_j$, $j = 1, 2$, giving the result. Q.E.D.

For the remainder of the section we consider (4.1) in the resonance case, and assume w.l.o.g. that $\beta \geqslant \alpha \geqslant 1$. Letting P_j be the z_j plane, $j = 1, 2$, we define

$$\begin{aligned} M &= R^4, && \text{if} \quad \beta = \alpha = 1, \\ &= R^4 - P_2, && \text{if} \quad \beta > \alpha = 1, \\ &= R^4 - (P_1 \cup P_2), && \text{if} \quad \beta > \alpha > 1, \end{aligned} \tag{4.6}$$

and we define $F : M \to R$ by $F = H_2 | M$. The flow Φ_t of F is then given by

$$\Phi_t(z_1, z_2) = (e^{-i\alpha t} \cdot z_1, e^{-i\beta t} \cdot z_2), \tag{4.7}$$

which for $0 \leqslant t < 2\pi$ defines a free and proper S^1-action on any $\Sigma_h = F^{-1}(h) \subset M$ for $h > 0$. We are thus in the reduction framework of the previous section.

Let $R^3 \simeq \mathbb{C} \times R$ have global coordinates $w = (w_1, w_2, w_2) \simeq (w_1 + iw_2, w_3)$, and let $\langle \, , \rangle$ be the standard inner product and $\times$ the standard cross product. Restricting $|w_3| \leqslant 2h$ and noting (4.3) we define

$$Q(w) = \frac{1}{2}(w_1^2 + w_2^2) - 2\left(\frac{2h + w_3}{2\alpha}\right)^\beta \left(\frac{2h - w_3}{2\beta}\right)^\alpha. \tag{4.8}$$

Now let $SP = (0, 0, -2h)$, $NP = (0, 0, 2h)$, and let

$$\begin{aligned} M_R = M_R(h) &= Q^{-1}(0), && \text{if} \quad \beta = \alpha = 1, \\ &= Q^{-1}(0) - \{SP\}, && \text{if} \quad \beta > \alpha = 1, \\ &= Q^{-1}(0) - \{SP, NP\}, && \text{if} \quad \beta > \alpha > 1 \end{aligned} \tag{4.9}$$

(see Fig. 1).

THEOREM 4.2. *For each of the three cases of α and β in* (4.6) *and* (4.9), *the space (M_R, ω_R) is the reduced space for (Φ_t, Σ_h). The projection $\pi : \Sigma_h \to M_R$ is given by the first three Hopf variables*

$$\pi(z) = (W_1(z), W_2(z), W_3(z)) \simeq (2z_1^\beta \bar{z}_2^\alpha, \alpha |z_1|^2 - \beta |z_2|^2), \tag{4.10}$$

and the symplectic form ω_R is defined for $d, f \in T_w M_R$ by

$$\omega_R(d, f)(w) = -(4\alpha\beta)^{-1} |\nabla Q(w)|^{-2} \cdot \langle \nabla Q(w), d \times f \rangle. \tag{4.11}$$

Moreover, the reduced Hamiltonian $K = K_h$ *corresponding to* $H \in \mathscr{H}_F$ *is obtained in the following way: write* $H = H(W_1, W_2, W_3, W_4)$ *using Lemma* 4.1(a), *define* $K^e = K_h^e(w_1, w_2, w_3) = H(w_1, w_2, w_3, 2h)$, *and set* $K = K^e \,|\, M_R$. *The reduced vector field on* M_R *is then given by*

$$X_K = 4\alpha\beta(\nabla Q) \times (\nabla K^e). \tag{4.12}$$

In particular, $z_0 \in \Sigma_h$ *is a relative equilibrium for* X_H *if and only if* $\nabla K^e(\pi(z_0))$ *is normal to* M_R *at* $\pi(z_0)$.

Proof. Now $\nabla Q \,|\, M_R \neq 0$ implies M_R is an analytic manifold, while (4.2) and (4.3) imply $(Q \circ \pi)(z) = 0$. It is then easy to check that $\pi : \Sigma_h \to M_R$ is an analytic surjective submersion, and (4.7) and (4.10) imply $(\pi \circ \Phi_t)(z) = \pi(z)$. Of course we still must show that each fibre $\pi^{-1}(\pi(z))$ is a single Φ_t orbit in Σ_h; that is, $\pi(z) = \pi(z')$ implies that $z' = \Phi_t(z)$ for some t. Let $z = (z_1, z_2)$ and $z' = (z_1', z_2')$ with $z_j = r_j e^{i\theta j}$ and $z_j' = r_j' e^{i\theta_j'}$. Then $W_3(z) = W_3(z')$ and $W_4(z) = 2h = W_4(z')$ imply $r_j = r_j'$, $j = 1, 2$, and $(W_1 + iW_2)(z) = (W_1 + iW_2)(z')$ implies

$$(\beta\theta_1 - \alpha\theta_2) = (\beta\theta_1' - \alpha\theta_2') + 2k\pi$$

for some integer k. Since α and β are relatively prime, there is an integer n so that $n\beta + k = 0 \pmod{\alpha}$. Setting

$$t = \beta^{-1}(\theta_2 - \theta_2') + (\alpha\beta)^{-1}\,(2k\pi) + \alpha^{-1}(2n\pi), \tag{4.13}$$

we calculate $\Phi_t(z) = z'$ as asserted.

A direct calculation using the fact that ∇Q is normal to M_R shows that the ω_R of (4.11) and the X_K of (4.12) satisfy $dK = X_K \lrcorner\, \omega_R$. Moreover, by Lemma 4.1(a) the Hopf variables W_i are in $\mathscr{H}_F$, $i = 1, 2, 3, 4$, thus giving a sufficient number of independent $H \in \mathscr{H}_F$ to conclude $\pi^*\omega_R = i^*\omega$ provided we can show $\pi_* X_H = X_K \circ \pi$ for the X_K of (4.12).

Let $\{\,,\}$ be the Poisson bracket defined by ω on R^4. For vector-valued functions $D = (D_1, ..., D_n)$ and $E = (E_1, ..., E_m)$ on R^4 we define $\{D, E\} = \text{matrix}\{D_i, E_j\}$. We regard vector fields V on R^l as partial differential operators, and for the vector-valued function $G = (G_1, ..., G_n)$ on R^l define $V(G) = (V(G_1), ..., V(G_n))$. Notice $V = V(\text{Id})$, where Id is the identity map on R^l.

For K and $K^e = K_h^e$ as defined, we obviously have $K \circ \pi = K^e \circ \pi = H \circ i$ as required, and this identity implies

$$(\nabla K^e) \circ \pi = (\nabla_W H) \circ i, \tag{4.14}$$

where $\nabla_W H = (\partial H/\partial W_1, \partial H/\partial W_2, \partial H/\partial W_3)$. If ψ_t^j is the flow of X_{W_j}, then we calculate (see [1, pp. 192–193])

$$\begin{aligned}\{W_j, H\} &= -\{H, W_j\} = -(d/dt)(H \circ \psi_t^j)|_{t=0} \\ &= -\sum_{i=1}^{4} (\partial H/\partial W_i) \frac{d}{dt} (W_i \circ \psi_t^j)|_{t=0} \\ &= \sum_{i=1}^{3} \{W_j, W_i\}(\partial H/\partial W_i),\end{aligned} \tag{4.15}$$

since $\{W_j, W_4\} = 0$ for $j = 1, 2, 3, 4$; hence

$$\{\pi, H\} = \{\pi, \pi\} \cdot (\nabla_W H). \tag{4.16}$$

For $B \in R^3$ a direct calculation (using the negative of (1.2)) shows that

$$\{\pi, \pi\} \cdot B = 4\alpha\beta[(\nabla Q) \circ \pi] \times B, \tag{4.17}$$

and hence

$$\pi_* X_H = (\pi_* X_H)(\mathrm{Id}) = X_H(\mathrm{Id} \circ \pi) = \{\pi, H\} = X_K \circ \pi, \tag{4.18}$$

with X_K as in (4.12), and where we have used (4.14), (4.16), and (4.17).

Q.E.D.

We sketch the reduced spaces M_R for the various cases of α and β in Fig. 1. Notice that in case (a) M_R is simply the 2-sphere of radius $2h$, and that when $2h = 1$ the projection $\pi : S^3 \to S^2$ is the usual Hopf fibration; hence the name "Hopf variables." The "pinching" in (c) and (d) reflects the tangential approach of M_R to the w_3 axis.

Now define the α and β *normal modes* of H_2 as in (4.1) to be the periodic orbits of X_{H_2} with energy h in the z_1 and z_2 planes, respectively. Then π sends the α normal mode to *NP*, and the β normal mode to *SP*. In cases (b)–(d) we thus see that our reduction process may ignore (at most two) periodic orbits of X_{H_2} on $H_2^{-1}(h)$. This is remedied by an application of Liapunov's theorem, a point we return to in Sections 6 and 7.

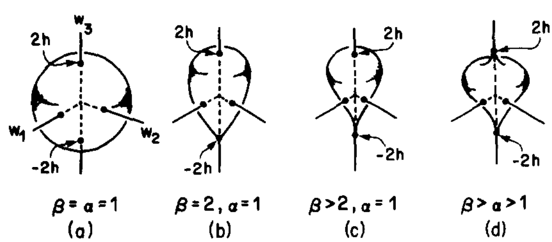

FIGURE 1

We now consider linear symplectic actions on $R^4 \simeq \mathbb{C} \times \mathbb{C}$ given by elements of discrete and Lie subgroups of $SU(2)$. Let $(C + iD) \in SU(2)$ and recall that $(z_1, z_2) = z \simeq x = (x_1, x_2, y_1, y_2)$, where $z_j = x_j + iy_j$, $j = 1, 2$. Then

$$(C + iD)\, z \simeq \begin{pmatrix} C & -D \\ D & C \end{pmatrix} x \tag{4.19}$$

defines a linear symplectic action of $SU(2)$ on (R^4, ω), and the map

$$\psi : SU(2) \to Sp(2, R) \cap SO(4, R), \psi(C + iD) = \begin{pmatrix} C & -D \\ D & C \end{pmatrix}, \tag{4.20}$$

is a Lie group isomorphism of $SU(2)$ onto its image.

The Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.21}$$

give a basis $\sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ for the Lie algebra $su(2)$. Relative to this basis the adjoint mapping $\mathrm{Ad}_V : su(2) \to su(2)$ of $V \in SU(2)$ is represented by a 3×3 matrix $B = (b_{kj})$, i.e.,

$$\mathrm{Ad}_V(\sigma_j) = V\sigma_j V^{-1} = \sum_{k=1}^{3} b_{kj}\sigma_k. \tag{4.22}$$

We claim $B \in SO(3)$. Indeed, letting * denote the conjugate transpose of a matrix, the relations

$$[\mathrm{Ad}_V(\sigma_j)]^* = \mathrm{Ad}_V(\sigma_j), \qquad \mathrm{Ad}_V \circ \mathrm{Ad}_{V^{-1}} = \mathrm{Id}, \tag{4.23}$$

imply B is real and invertible. Moreover, from (4.22) we have

$$\sum_{k=1}^{3} (b_{kj})^2 = -\det(V\sigma_j V^{-1}) = -\det(\sigma_j) = +1, \tag{4.24}$$

hence the columns of B are of unit length. Now note the Lie bracket relation

$$[\sigma_1, \sigma_2] = \sigma_1\sigma_2 - \sigma_2\sigma_3 = -2i\sigma_3, \tag{4.25}$$

with corresponding identities on cyclic permutation of the indices. Then

$$(-2i)\,\mathrm{Ad}_V(\sigma_3) = \mathrm{Ad}_V[\sigma_1, \sigma_2] = [\mathrm{Ad}_V(\sigma_1), \mathrm{Ad}_V(\sigma_2)], \tag{4.26}$$

when expanded using (4.22), shows that the third column of B is the standard cross product in R^3 of the first two columns; hence $B \in SO(3)$ as asserted. We now easily check that the 2 to 1 map

$$\phi : SU(2) \to SO(3), \qquad \phi(\pm V) = B, \tag{4.27}$$

where V and B are related by (4.22), is a Lie group homomorphism with $\mathrm{Ker}(\phi) = \{\pm I_2\}$.

Since $SU(2)$ and $SO(3)$ are compact and connected, a standard result [1, Ex. 4.4D, p. 338] implies that the exponential map of their respective Lie algebras is surjective. In particular, for each $V \in SU(2)$ there is an $a = (a_1, a_2, a_3) \in R^3$ such that $V = \exp[-i(1/2)(a \cdot \sigma)]$, where $a \cdot \sigma = \sum_{i=1}^{3} a_i \sigma_i$. Now let

$$A = \begin{matrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{matrix} \in so(3),$$

set $V(t) = \exp[-i(t/2)(a \cdot \sigma)]$ in (4.22), thus defining $B(t) = (b_{kj}(t)) = \exp(Pt)$ with $P \in so(3)$, and differentiate at $t = 0$. Using relations (4.25) we conclude that $P = A$, and hence

$$B = \exp(A) = \phi(V). \tag{4.28}$$

The linear action on R^3 given by $w \to Bw$ is thus a *clockwise* rotation of w about the axis a through an angle $|a|$ (the eigenvalues of B are $\exp(0) = 1$ and $\exp(\pm i\,|a|)$, and $Aw = w \times a$). In particular,

$$\phi(\exp[-i(t/2)\,\sigma_1]) = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{array}\right), \tag{4.29a}$$

$$\phi(\exp[-i(t/2)\,\sigma_2]) = \begin{pmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{pmatrix}, \tag{4.29b}$$

$$\phi(\exp[-i(t/2)\,\sigma_3]) = \left(\begin{array}{cc|c} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ \hline 0 & 0 & 1 \end{array}\right). \tag{4.29c}$$

THEOREM 4.3. (a) *Suppose in* (4.6) *and* (4.9) *we take* $\beta = \alpha = 1$, *and let* $\mathscr{G} = SU(2)$ *act as in* (4.19). *Then Lemma* 3.3 *holds, with* $\mathscr{G}^* = \{\pm I_2\}$, *and* $\mathscr{G}_R = (\mathscr{G}/\mathscr{G}^*) \simeq SO(3)$. *Moreover, the actions of* $\mathscr{G}$ *on the* 3-*sphere* $H_2^{-1}(h) \subset R^4$ *of radius* $(2h)^{1/2}$, *and of* $\mathscr{G}_R$ *on the* 2-*sphere* $M_R \subset R^3$ *of radius*

$(2h)$, are transitive. Finally, for $V \in \mathscr{G}$ and $\phi(V) = B \in \mathscr{G}_R$ given by (4.27) *we have*

$$\pi(V \cdot z) = B \cdot \pi(z), \tag{4.30}$$

where $\pi : \Sigma_h \to M_R$ is the projection (4.10).

(b) *Suppose in* (4.6) *and* (4.9) *we take $\beta \geqslant \alpha \geqslant 1$, we let $\mathscr{G}$ be the one-parameter subgroup $V(t) = \exp[-i(t/2)\,\sigma_3]$ of $SU(2)$, and we let $\mathscr{G}$ act as in* (4.19). *Then Lemma* 3.3 *holds, with $\mathscr{G}^* = \{\pm I_2\}$, and $\mathscr{G}_R = (\mathscr{G}/\mathscr{G}^*) \subset SO(3)$. Moreover, the action of $\mathscr{G}$ restricts to each "ellipsoid" $H_2^{-1}(h)$, and for $V(t) \in \mathscr{G}$ and $\phi(V(t)) = B(t) \in \mathscr{G}_R$ given by* (4.27) *we have*

$$\phi(V(t)) = B(t) = \left(\begin{array}{cc|c} \cos[t(\alpha+\beta)/2] & \sin[t(\alpha+\beta)/2] & 0 \\ -\sin[t(\alpha+\beta)/2] & \cos[t(\alpha+\beta)/2] & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \tag{4.31}$$

reflecting the rotational symmetry of M_R about the w_3 axis in Fig. 1.

For the proof a simple observation will be useful. If

$$\sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then for $\beta = \alpha = 1$ the Hopf variables (4.2) can be written

$$W_j = \langle \sigma_j z, z \rangle_{\mathbb{C}}, \qquad j = 1, 2, 3, 4, \tag{4.32}$$

where $\langle\ ,\ \rangle_{\mathbb{C}}$ is the usual Hermitian inner product on $\mathbb{C} \times \mathbb{C}$, i.e.,

$$\langle z, u \rangle_{\mathbb{C}} = \sum_{j=1}^{2} z_j \bar{u}_j, \qquad z = (z_1, z_2), \quad u = (u_1, u_2). \tag{4.33}$$

Proof. Since the linear action on R^4 given by (4.19) is symplectic, the induced action on M_R will be symplectic by Theorem 3.4. The remaining results in the proposition will follow readily once we establish formula (3.5).

(a) For $\beta = \alpha = 1$, the Hopf variables are given by (4.32). Also, $V \in SU(2)$ implies $V^* = V^{-1}$ and $B^{-1} = B'$ in (4.22). Thus for $j = 1, 2, 3, 4$, we have

$$W_j(V \cdot z) = \langle \sigma_j V \cdot z, V \cdot z \rangle_{\mathbb{C}} = \langle V^* \sigma_j V \cdot z, z \rangle_{\mathbb{C}} = \sum_{k=1}^{3} b_{jk} \langle \sigma_k z, z \rangle_{\mathbb{C}}. \tag{4.34}$$

which implies (4.30).

(b) For $\beta \geqslant \alpha \geqslant 1$ and $V(t) = \exp[-i(t/2)\,\sigma_3]$, the action (4.19) in (4.2) implies

$$\begin{aligned} (W_1 + iW_2)(V \cdot z) &= \exp[-it(\alpha+\beta)/2] \cdot (W_1 + iW_2)(z), \\ W_3(V \cdot z) &= W_3(z). \end{aligned} \tag{4.35}$$

Hence the projection $\pi : \Sigma_h \to M_R$ satisfies $\pi(V(t) \cdot z) = B(t) \cdot \pi(z)$ with $B(t)$ given by (4.31). Q.E.D.

In addition to $SU(2)$ symmetries, we will also study the following involutions on (R^4, ω):

$$R_1(z_1, z_2) = (\bar{z}_1, \bar{z}_2), \qquad R_2(z_1, z_2) = (\bar{z}_1, -\bar{z}_2), \qquad S(z_1, z_2) = (z_1, -z_2). \tag{4.36}$$

Notice these satisfy the identities

$$R_1 = R_2 \circ S = S \circ R_2, \qquad R_2 = R_1 \circ S = S \circ R_1, \qquad S = R_1 \circ R_2 = R_2 \circ R_1, \tag{4.37}$$

and

$$R_j^* \omega = -\omega, \qquad j = 1, 2, \qquad S^* \omega = \omega. \tag{4.38}$$

R_1 is called "time-reversing" since in kinetic plus potential energy Hamiltonians it reverses time. As in computations (4.34) and (4.35), use of the defining relations (4.36) in (4.2) yields the following result:

THEOREM 4.4. *For each of the cases of β and α in* (4.6) *and* (4.9), *let the group $\mathscr{G} = \{I_4, R_1, R_2, S\} \subset GL(4, R)$. Then the hypotheses of Lemma* 3.3 *are satisfied and $\mathscr{G}^* = \{I_4\}$. On both M and M_R the respective actions of $g \in \mathscr{G}$ and $\phi(g) \in \mathscr{G}_R$ are linear symplectic or anti-symplectic, and the map $\phi : \mathscr{G} \to \mathscr{G}_R \subset GL(3, R)$ is given by*

$$\phi(I_4) = I_3, \tag{4.3a}$$

$$\phi(R_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.3b}$$

$$\phi(R_2) = \begin{pmatrix} (-1)^{\alpha} & 0 & 0 \\ 0 & (-1)^{\alpha-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.3c}$$

$$\phi(S) = \phi(R_1) \cdot \phi(R_2) = \begin{pmatrix} (-1)^{\alpha} & 0 & 0 \\ 0 & (-1)^{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.3d}$$

Remark a). We define the $\mathscr{G}$-meridian of M_R to be the intersection of that set with the (w_1, w_3) plane in R^3; it is the fixed point set of $\phi(R_1)$. In the context of kinetic plus potential energy Hamiltonians, points on the $\mathscr{G}$-meridian correspond to the time-symmetric orbits.

Remark b). The only points in M_R left fixed by both $\phi(R_1)$ and $\phi(R_2)$ are those poles $\{SP, NP\}$ that lie in M_R (see Fig. 1). Recall that these poles correspond to the β and α normal modes in $H_2^{-1}(h)$.

5. Symplectic Planar Mappings

Our applications of reduced space techniques will employ some technical results concerning symplectic planar mappings. For the convenience of the reader these are assembled in this section.

We consider an open neighborhood $V \subset \mathbb{C}$ of the origin and an analytic symplectic mapping $M : V \to \mathbb{C}$ of the form

$$M : u_1 = A_1 u + A_2 \bar{u} + A_3 u^2 + A_4 u\bar{u} + A_5 \bar{u}^2 + A_6 u^3 + A_7 u^2 \bar{u} + A_8 u \bar{u}^2 + A_9 \bar{u}^3 + O_4(u, \bar{u}). \tag{5.1}$$

Note that the origin is a fixed point.

Lemma 5.1. *For the symplectic map* (5.1) *we have*

(a) $|A_1|^2 - |A_2|^2 = 1$,

(b) $[2\bar{A}_1 A_3 + A_1 \bar{A}_4 - \bar{A}_2 A_4 - 2A_2 \bar{A}_5] = 0$,

(c) $[3\bar{A}_1 A_6 + 2A_3 \bar{A}_4 + A_1 \bar{A}_8 - \bar{A}_2 A_7 - 2A_4 \bar{A}_5 - 3A_2 \bar{A}_9] = 0$,

(d) $[\mathrm{Re}(A_1 \bar{A}_7) - \mathrm{Re}(\bar{A}_2 A_8) + |A_3|^2 - |A_5|^2] = 0$.

In particular, a linear map $u_1 = A_1 u + A_2 \bar{u}$ is symplectic if and only if (a) *holds.*

Proof. M symplectic is equivalent to the power series identity $\partial(u_1, \bar{u}_1)/\partial(u, \bar{u}) \equiv 1$; now compare coefficients. Q.E.D.

Lemma 5.2. *If in* (5.1) *we write $A_1 = a + ib$, then the eigenvalues of the fixed point* 0 *are $v = a \pm \sqrt{a^2 - 1}$, hence that point is:*

(a) *elliptic if and only if* $|a| < 1$,

(b) *parabolic if and only if* $|a| = 1$, *and*

(c) *hyperbolic if and only if* $|a| > 1$.

Proof. With $A_2 = c + id$ the mapping M in real coordinates is given by

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} a+c & d-b \\ d+b & a-c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + O_2(x, y).$$

Using Lemma 5.1(a), the result follows by direct computation. Q.E.D.

In the elliptic case suppose $A_2 \neq 0$ in (5.1). Then a simple calculation with the eigenvalue $\nu = a + i(\operatorname{sgn}(b))\sqrt{1-a^2}$ shows that $[1 - \operatorname{Re}(\nu A_1)] > 0$. If we define a real number α by the requirements $\alpha^{-2} = 2[1 - \operatorname{Re}(\nu A_1)]$ and $\operatorname{sgn}(\alpha) = \operatorname{sgn}(b)$, then transforming (5.1) by the canonical transformation $u = i\alpha[(\bar{A}_1 - \nu)\, w + A_2 \bar{w}]$ converts (5.1) to the form

$$w_1 = \nu w + O_2(w, \bar{w}). \tag{5.2}$$

Conversely, Lemma 5.2(a) shows that (5.2) must be elliptic unless $\nu = \pm 1$. For the rest of this section, when we are in the elliptic case we will assume that the mapping (5.1) already has the form

$$\begin{aligned} M : u_1 = \nu u + A_3 u^2 + A_4 u\bar{u} + A_5 \bar{u}^2 + A_6 u^3 + A_7 u^2 \bar{u} \\ + A_8 u\bar{u}^2 + A_9 \bar{u}^3 + O_4(u, \bar{u}). \end{aligned} \tag{5.3}$$

THEOREM 5.3. *If $\nu^j \neq 1$ for $j = 1, 2, 3, 4$ in (5.3), then the origin is stable provided the "first twist coefficient"*

$$\gamma = -i \left\{ i \operatorname{Im}(\bar{\nu} A_7) + 3\,|A_3|^2 \left(\frac{\nu + 1}{\nu - 1}\right) + |A_5|^2 \left(\frac{\nu^3 + 1}{\nu^3 - 1}\right) \right\} \tag{5.4}$$

is non zero (notice γ is real).

Remarks. More precisely, Theorem 5.3 states that the origin is surrounded by M-invariant analytic closed curves shrinking sown upon that point, and that collectively these curves form a set of positive measure. Moreover, the restriction of M to any such curve is conjugate to an irrational rotation of S^1. The neighborhood of the origin for these invariant curves will in general be smaller than the original neighborhood V in (5.1).

The case $\gamma = 0$ does not rule out stability. In fact there is an infinite sequence of twist coefficients, any one of which being nonzero guarantees stability. The condition $\nu^j \neq 1$ for $j = 1, 2, 3, 4$, can also be relaxed, but not eliminated entirely. Indeed, for each qth root of unity a counterexample to stability is constructed in [37, pp. 222–224].

Proof. From [31, Theorem 2.12, p. 55] there is a real analytic symplectic mapping $C : u = w + O_2(w, \bar{w})$ such that the mapping $N = C^{-1} \circ M \circ C$ has the form $N : w_1 = \nu w[1 + i\gamma\,|w|^2] + O_4(w, \bar{w})$. Comparing coefficients in the power series identity $C \circ N = M \circ C$, and using the identities in Lemma 5.1, we find γ given by (5.4). The result then follows from [31, Theorem 2.13, p. 56]. Q.E.D.

THEOREM 5.4. *Let $V \subset \mathbb{C}$ be an open neighborhood of the origin, and consider a one-parameter family $M_\varepsilon : V \to \mathbb{C}$ of symplectic planar mappings of the form*

$$M_\varepsilon : u_1 = u + \varepsilon^n f(u, \bar{u}; \varepsilon) \tag{5.5}$$

with

$$f(u, \bar{u}; \varepsilon) = B_1 u + B_2 \bar{u} + B_3 u^2 + B_4 u\bar{u} + B_5 \bar{u}^2 + B_6 u^3 + B_7 u^2 \bar{u} + B_8 u\bar{u}^2 + B_9 \bar{u}^3 + O_4(u, \bar{u}; \varepsilon) + O(\varepsilon). \tag{5.6}$$

Here each $B_j = B_j(\varepsilon)$ is analytic in ε, $O_4(u, \bar{u}; \varepsilon)$ represents terms of degree at least 4 in u, $\bar{u}$, with coefficients analytic in ε, and $O(\varepsilon)$ is a function of $(u, \bar{u}; \varepsilon)$ for $|\varepsilon|$ small. In particular, $f(0, 0; 0) = 0$. Now set

$$J = \left. \frac{\partial(f, \bar{f})}{\partial(u, \bar{u})} \right|_{u=\varepsilon=0} = |B_1(0)|^2 - |B_2(0)|^2. \tag{5.7}$$

(a) *If $J \neq 0$, then for $|\varepsilon|$ small M_ε admits a unique fixed point u_ε near 0, and $u_\varepsilon \to 0$ as $\varepsilon \to 0$.*

(b) *For sufficiently small $\varepsilon > 0$ the point u_ε is elliptic or hyperbolic according as $J > 0$ or $J < 0$.*

(c) *If $J > 0$, $B_2 \equiv 0$, and*

$$i \operatorname{Im}(B_7(0)) + 6B_1^{-1}(0)\,|B_3(0)|^2 + \tfrac{2}{3}B_1^{-1}(0)\,|B_5(0)|^2 \neq 0, \tag{5.8}$$

then for small $\varepsilon > 0$ the fixed point u_ε is stable in the sense of Theorem 5.3 *(note from* (5.7) *that $J > 0$ implies $B_1(0) \neq 0$ in* (5.8)).

Proof. (a) By the implicit function theorem there is a unique u_ε, analytic in ε for ε sufficiently small, such that $f(u_\varepsilon, \bar{u}_\varepsilon; \varepsilon) \equiv 0$ with $u_\varepsilon \to 0$ as $\varepsilon \to 0$. Obviously u_ε is a fixed point of M_ε.

(b) Expand (5.5) about u_ε and let $w = (u - u_\varepsilon)$, obtaining

$$w_1 = (1 + \varepsilon^n B_1(\varepsilon))\, w + \varepsilon^n B_2(\varepsilon)\, \bar{w} + O_2(w, \bar{w}; \varepsilon); \tag{5.9}$$

this moves the fixed point to the origin. By Lemma 5.1(a) we have $|1 + \varepsilon^n B_1(\varepsilon)|^2 - |\varepsilon^n B_2(\varepsilon)|^2 \equiv 1$, i.e.,

$$2 \operatorname{Re}(B_1(\varepsilon)) \equiv -\varepsilon^n (|B_1(\varepsilon)|^2 - |B_2(\varepsilon)|^2) \approx -\varepsilon^n J,$$

hence $\operatorname{Re}(B_1(\varepsilon))$ has the opposite sign as J for small $\varepsilon > 0$. But by Lemma 5.2 the fixed point (which is the origin in the new coordinates) is elliptic, parabolic, or hyperbolic according as $1 + \varepsilon^n \operatorname{Re}(B_1(\varepsilon))$ is less than, equal, or greater than one, and (b) follows.

(c) When $B_2(\varepsilon) \equiv 0$ (5.9) becomes

$$w_1 = (1 + \varepsilon^n B_1(\varepsilon))\, w + \varepsilon^n \{B_3(\varepsilon)\, w^2 + \cdots + B_9(\varepsilon)\, \bar{w}^3 + O_4(w, \bar{w}; \varepsilon)\}$$

which has form (5.3) with $\nu = 1 + \varepsilon^n B_1(\varepsilon)$. Using

$$\left(\frac{\nu+1}{\nu-1}\right) = 1 + 2\varepsilon^{-n} B_1^{-1}(\varepsilon),$$

$$\left(\frac{\nu^3+1}{\nu^3-1}\right) = 1 + 2\varepsilon^{-n} B_1^{-1}(\varepsilon)\{3 + 3\varepsilon^n B_1(\varepsilon) + \varepsilon^{2n} B_1^2(\varepsilon)\}^{-1},$$

we compute expression (5.4) in this case to be

$$\begin{aligned}
&-i\{i\,\mathrm{Im}((1+\varepsilon^n \bar{B}_1(\varepsilon))\,\varepsilon^n B_7(\varepsilon)) + 3\varepsilon^{2n}\,|B_3(\varepsilon)|^2\,(1 + 2\varepsilon^{-n}B_1^{-1}(\varepsilon)) \\
&\quad + \varepsilon^{2n}\,|B_5(\varepsilon)|^2\,(1 + 2\varepsilon^{-n} B_1^{-1}(\varepsilon)[3 + 3\varepsilon^n B_1(\varepsilon) + \varepsilon^{2n} B_1^2(\varepsilon)]^{-1})\} \\
&= -i\{\varepsilon^n[i\,\mathrm{Im}(B_7(0)) + 6B_1^{-1}(0)\,|B_3(0)|^2 + \tfrac{2}{3}B_1^{-1}(0)\,|B_5(0)|^2] + O(\varepsilon^{n+1})\}.
\end{aligned}$$

This gives (c) and completes the proof. Q.E.D.

While Theorem 5.4 covers analytic symplectic perturbations of the identity mapping, we will also have occasion to study analytic symplectic perturbations of pure rotations.

THEOREM 5.5. *Let $V \subset \mathbb{C}$ be an open neighborhood of the origin, and consider a one-parameter family of symplectic planar mappings $M_\varepsilon : V \to \mathbb{C}$ of the form*

$$M_\varepsilon : u_1 = \nu u + \varepsilon^n f(u, \bar{u}; \varepsilon), \tag{5.10}$$

with f as in (5.6), and $|\nu| = 1$, $\nu \neq 1$. Then for $\varepsilon > 0$ small, M_ε admits a unique fixed point u_ε with $u_\varepsilon \to 0$ as $\varepsilon \to 0$. Moreover, u_ε is elliptic if $\nu \neq \pm 1$, and elliptic stable (in the sense of Theorem 5.3) provided $\nu^j \neq 1$, $j = 1, 2, 3, 4$, and

$$B_7(0) \neq 0. \tag{5.11}$$

As the proof is similar to that of Theorem 5.4, it will only be sketched.

Proof. Setting $g(u, \bar{u}; \varepsilon) = (\nu - 1)\,u + \varepsilon^n f(u, \bar{u}; \varepsilon)$, the condition that u_ε be a fixed point of (5.10) is then $g(u_\varepsilon, \bar{u}_\varepsilon; \varepsilon) = 0$. The relevant Jacobian is $\partial(g, \bar{g})/\partial(u, \bar{u}) = (\nu - 1)(\bar{\nu} - 1) + O(\varepsilon^n)$, which for $\nu \neq 1$ will not vanish for $\varepsilon > 0$ small. Thus u_ε exists. Moreover, when $\nu \neq \pm 1$ the eigenvalues in (5.10) vary analytically with ε (for small $|\varepsilon|$); hence u_ε is elliptic.

Expanding (5.10) about u_ε and setting $\omega = (u - u_\varepsilon)$, we obtain a one-parameter family of symplectic planar mappings

$$\begin{aligned}
w_1 = \nu w + \varepsilon^n[&B_1(0)\,w + B_2(0)\,\bar{w} + B_3(0)\,w^2 + \cdots \\
&+ B_7(0)\,w^2\bar{w} + \cdots + O_4(w, \bar{w}; \varepsilon)] + O(\varepsilon^{n+1})
\end{aligned} \tag{5.12}$$

having the origin as common fixed point. As in the proof of Theorem 5.3, the condition $\nu^j \neq 1$ for $j = 1, 2, 3, 4$ guarantees we can convert (5.12) to the form $N_\varepsilon : z_1 = \eta z[1 + i\gamma |z|^2] + O_4(z, \bar{z}; \varepsilon)$, where η and γ depend on ε, by means of a symplectic planar mapping $C_\varepsilon : z = u + \varepsilon^n(C_1 u + C_2 \bar{u} + \cdots) + O(\varepsilon^{n+1})$ which is analytic in ε (for the analyticity one must examine [37, Sect. 23]). A straightforward comparison of coefficients then shows $\gamma = -i\varepsilon^n \bar{\nu} B_7(0) + O(\varepsilon^{n+1})$, and the result follows. Q.E.D.

In our main applications the symplectic planar mappings will arise by integrating a periodic time-dependent Hamiltonian flow through one period. Specifically, in Section 6 we will consider analytic Hamiltonians of the form

$$G(u, \bar{u}, \lambda; \varepsilon) = \varepsilon^n [E(u, \bar{u}) + O(\varepsilon)], \tag{5.13}$$

where u varies in an open neighborhood of the origin in $\mathbb{C}$, G is periodic of period $T > 0$ in λ, ε is small, and $O(\varepsilon)$ denotes a function of $(u, \bar{u}, \lambda; \varepsilon)$. The associated equations are

$$u' = -2i(\partial G/\partial \bar{u}) = -2i\varepsilon^n(\partial E/\partial \bar{u}) + O(\varepsilon^{n+1}), \qquad ' = d/d\lambda. \tag{5.14}$$

Integrating (5.14) from 0 to T gives a family of symplectic planar mappings

$$M_\varepsilon : u_1 = u - \varepsilon^n 2iT \left(\frac{\partial E}{\partial \bar{u}} (u, \bar{u}) \right) + O(\varepsilon^{n+1}) \tag{3.15}$$

which is of the form (5.5).

In this paper E will be real analytic and always have the form

$$E(u, \bar{u}) = \tfrac{1}{2}(A\xi^2 + B\eta^2) + O_3(u, \bar{u}), \qquad u = \xi + i\eta, \tag{5.16}$$

with A and B constant. In this case the Jacobian J of (5.7) is simply $J = ABT^2$. If $J > 0$ (the elliptic case), then the canonical substitution of $(B/A)^{1/4} \cdot \xi$ and $(A/B)^{1/4} \cdot \eta$ for ξ and η converts (5.16) to the form

$$\begin{aligned} E(u, \bar{u}) = {} & \tfrac{1}{2}(\operatorname{sgn} A)(AB)^{1/2} |u|^2 + E_3 u^3 + E_4 u^2 \bar{u} + E_5 u \bar{u}^2 + E_6 \bar{u}^3 + E_7 u^4 \\ & + E_8 u^3 \bar{u} + E_9 u^2 \bar{u}^2 + E_{10} u \bar{u}^3 + E_{11} \bar{u}^4 + O_5(u, \bar{u}), \end{aligned} \tag{5.17}$$

where the coefficients E_j are constant and satisfy the reality conditions imposed by $E = \bar{E}$. Mapping (5.15) then becomes

$$\begin{aligned} M_\varepsilon : u_1 = u + \varepsilon^n [& -i(\operatorname{sgn} A)\, T(AB)^{1/2} u - 2iTE_4 u^2 - 4iTE_5 u\bar{u} - 6iTE_6 \bar{u}^2 \\ & - 2iTE_8 u^3 - 4iTE_9 u^2 \bar{u} - 6iTE_{10} u \bar{u}^2 - 8iTE_{11} \bar{u}^3 \\ & + O_4(u, \bar{u})] + O(\varepsilon^{n+1}). \end{aligned} \tag{5.18}$$

This is in the format given in (5.5) and (5.6) with $B_2 = 0$, and here a short computation shows that the stability condition (5.8) in this case becomes, on using the reality conditions $E_6 = \bar{E}_3$ and $E_9 = \bar{E}_9$,

$$4iT(\operatorname{sgn} A)(AB)^{-1/2}\,[-(\operatorname{sgn} A)(AB)^{1/2}\,E_9 + 6(|E_3|^2 + |E_4|^2)] \neq 0. \tag{5.19}$$

We thus arrive at the following consequence of Theorem 5.4:

THEOREM 5.6. *Let $V \subset \mathbb{C}$ be an open neighborhood of the origin, and consider a one-parameter family $M_\varepsilon : V \to \mathbb{C}$ of symplectic planar mappings arising as in* (5.12) *with E as in* (5.13). *Also consider the Hamiltonian system*

$$u' = -2i(\partial G/\partial \bar{u}), \qquad ' = d/d\lambda, \tag{5.20}$$

associated with (5.13).

(a) *If $AB \neq 0$, then for small $\varepsilon > 0$ there is a unique periodic orbit Π_ε of* (5.20) *of period $T(\varepsilon)$ such that Π_ε approaches the λ axis and $T(\varepsilon) \to T$ as $\varepsilon \to 0$.*

(b) *For $\varepsilon > 0$ sufficiently small, Π_ε is elliptic or hyperbolic according as $AB > 0$ or $AB < 0$.*

(c) *If $AB > 0$ and E is converted to form* (5.17) *by the canonical substitution of $(B/A)^{1/4}\,\xi$ and $(A/B)^{1/4}\,\eta$ for ξ and η, where $\xi + i\eta = u$, then Π_ε is stable for sufficiently small $\varepsilon > 0$ provided* (5.19) *holds.*

Remark a. "Stable" in (c) means that Π_ε is encased in invariant tori that shrink down upon that orbit, and which collectively form a set of positive measure.

Remark b. An alternative to (c) for proving stability is the following: Use Proposition 1.1 to convert E to normal form through higher order, and mapping (5.15) will then have the "normal form" N of the proof of Theorem 5.3. Then γ can be "read off" as a single coefficient of the converted E.

Remark c. A corresponding stability criterion in Theorem 5.6(c) could be given in the case that the quadratic terms of $E(u, \bar{u})$ possess a cross-product term involving $\xi \cdot \eta$. As we will see in the discussion preceding Proposition 6.3, such cross-product terms can, however, be eliminated by appropriate coordinate transformations.

Remark d. In Section 7.A we will consider analytic Hamiltonians of the form

$$G(u, \bar{u}, \lambda; \varepsilon) = (\gamma/2)\,|u|^2 + O(\varepsilon^n) \tag{5.21}$$

in place of (5.13); here again u varies in an open neighborhood of the origin in $\mathbb{C}$, G is periodic of period $T > 0$ in λ, ε is small, and $O(\varepsilon^n)$ denotes a function of $(u, \bar{u}, \lambda; \varepsilon)$. In place of (5.14) the associated equations now become

$$u' = -i\gamma u + O(\varepsilon^n), \qquad ' = d/d\lambda, \tag{5.22}$$

and integrating from $\lambda = 0$ to $\lambda = T$ gives a family of symplectic planar mappings

$$M_\varepsilon : u_1 = e^{-i\gamma T} \cdot u + O(\varepsilon^n) = \nu u + O(\varepsilon^n). \tag{5.23}$$

This is of the form (5.10), and continuation and stability criteria can be obtained from Theorem 5.5.

6. Periodic Orbits and Their Continuation

Let $z = (z_1, z_2)$ and

$$H_2(z, \bar{z}) = (\alpha/2)\,|z_1|^2 + (\beta/2)\,|z_2|^2, \qquad \alpha \quad \text{and} \quad \beta > 0, \tag{6.1}$$

where $\beta = \alpha = 1$ or $\beta > \alpha \geqslant 1$ are relatively prime integers (the resonance case). Also, let M be defined as in (4.6), let $F = H_2 | M$, and let $H \in \mathscr{H}_F$ be as in Theorem 4.2. As before, h is assumed to be a regular value of F, and with $\Sigma_h = F^{-1}(h)$ we let $i : \Sigma_h \to M$ be inclusion. Finally, $\pi : M \to M_R$ is the orbit projection, and $K : M_R \to R$ the reduced Hamiltonian corresponding to H.

Recall that a relative equilibrium for H is a point $p \in \Sigma_h$ such that $X_K(\pi(p)) = 0$; by (4.12) this is a point for which $\nabla K^e(\pi(p))$ is normal to M_R. In the notation of Theorem 3.1, the orbit $\Psi_t(p)$ of X_H is then a reparametrized orbit $\Phi_{g(t)}(p)$ of X_F, and thus is periodic provided $(dg/dt) \neq 0$ along the curve. We wish to use Theorem 5.6 to continue such a periodic orbit to a perturbation of H, and for this purpose we must first reconstruct the flow of X_H on M from the flow of X_K on M_R. A great deal of the following construction also applies in the nonresonance case, but we restrict attention to the resonance case as we wish to emphasize correspondences with the reduced space M_R. An alternate reconstruction of the flow of X_H on M from that of X_K on M_R can be found in [1, pp. 303–304].

For any $h > 0$, the level surface $H_2^{-1}(h)$ of the quadratic Hamiltonian (6.1) is an ellipsoid with $|z_1|^2 \leqslant (2h/\alpha)$ and $|z_2|^2 \leqslant (2h/\beta)$. If we write $z_1 = |z_1|\, e^{i\theta}$ for $z_1 \neq 0$ (i.e., $|z_2|^2 \neq (2h/\beta)$), then we can use (z_2, θ) as local coordinates on $H_2^{-1}(h)$ provided we identify $(z_2, \pm\pi)$. With these conventions consider Fig. 2, in which the vertical axis is parametrized by θ for $-\pi \leqslant \theta \leqslant \pi$. The point p, assumed to be interior to the solid cone, is coor-

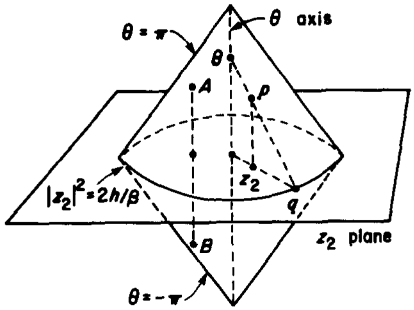

FIGURE 2

dinatized by (z_2, θ) as follows: z_2 is simply the planar projection of p, q is obtained as the intersection of the "outer circle" $|z_2|^2 = (2h/\beta)$ with the place through p and the vertial axis, and θ is the obtained as the intersection with the vertical axis of the line through q and p. Of course, if p is already on the vertical axis at a point θ, then the coordinates of p are $(0, \theta)$. We note that these coordinates are not valid on the "outer circle" $|z_2|^2 = (2h/\beta)$. In keeping with the identification $(z_2, \pm\pi)$, we regard two points on the "boundary" of the solid cone as being identical if they have the same z_2 coordinate, e.g., A and B in Fig. 2. Thus all vertical line segments running from boundary to boundary are actually circles; in particular this is the case for the vertical axis which we call the "inner circle."

The flow

$$\Phi_t(z) = (e^{-i\alpha t} \cdot z_1, e^{-i\beta t} \cdot z_2) \tag{6.2}$$

of the quadratic Hamiltonian (6.1) is easily visualized in these terms as in Fig. 3. Indeed, the normal modes $(e^{-i\alpha t} \cdot z_1, 0)$ and $(0, e^{-i\beta t} \cdot z_2)$ run down the inner and around the outer circles, respectively, and all other orbits wind downward around invariant tori. Moreover, on these tori the flow gives a free

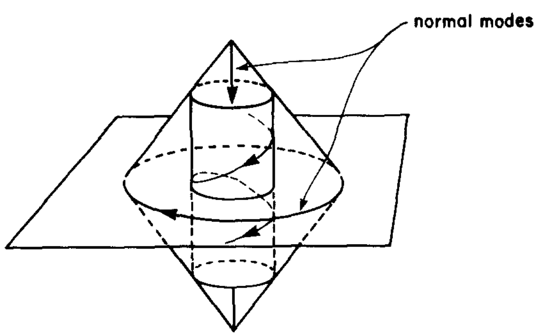

FIGURE 3

and proper S^1-action in the resonance case. Indeed, Fig. 3, after removal of the appropriate normal mode(s) as in (4.6) can be regarded as a picture of the S^1-action on the surface Σ_h of Sections 3 and 4.

It is important to realize that Fig. 3 can be untwisted (and the flow direction reversed) by a canonical transformation. To this end write the coordinates on $R^4 \simeq R^2 \times \mathbb{C}$ as (λ, L, u), $u = \xi + i\eta$, let

$$U_\alpha = \{(\lambda, L, u) \in R^4 \mid L > 0, |u^2| < (2L/\beta)\}, \tag{6.3}$$

and consider U_α as a symplectic space with two-form

$$\omega_\alpha = d\lambda \wedge dL + d\xi \wedge d\eta. \tag{6.4}$$

Now define $\rho_\alpha : U_\alpha \to R^4 \simeq \mathbb{C} \times \mathbb{C}$ by

$$\rho_\alpha \begin{cases} z_1 = \alpha^{-1/2}(2L - \beta|u|^2)^{1/2}\, ie^{-i\alpha\lambda}, \\ z_2 = ue^{-i\beta\lambda}, \end{cases} \tag{6.5}$$

and let

$$\omega = \tfrac{1}{2} \sum_{j=1}^{2} \operatorname{Im}(d\bar{z}_j \wedge dz_j) = \sum_{j=1}^{2} dx_j \wedge dy_j \tag{6.6}$$

be the standard symplectic form on R^4 as in (3.1). Then a direct calculation shows that

$$\rho_\alpha^*(\omega) = \omega_\alpha \tag{6.7}$$

and

$$(H_2 \circ \rho_\alpha)(\lambda, L, u) = L. \tag{6.8}$$

Thus ρ_α is canonical and maps the domain

$$V_\alpha = \{(\lambda, L, u) \in U_\alpha \mid -(\pi/\alpha) < \lambda \leqslant (\pi/\alpha), L = h\} \tag{6.9}$$

in a 1–1 manner onto $H_2^{-1}(h)$ less the outer circle in Fig. 3 (since $|z_2|^2 = |u|^2 < (2L/\beta) = (2h/\beta)$). Notice that the differential equations associated with the new Hamiltonian (6.8) are trivial, i.e.,

$$\dot{\lambda} = 1, \qquad \dot{L} = 0, \qquad \dot{u} = 0. \tag{6.10}$$

We can visualize V_α as in Fig. 2 except the vertical axis now runs from $-(\pi/\alpha)$ to (π/α), and to be consistent with previous identifications the point A on the "top boundary" must be identified with B on the bottom if the projection of A to the horizontal plane, when rotated by $(2\pi\beta/\alpha)$, agrees with the projection of B. This is just the usual identification for an (α, β) lens space (see [36, pp. 217–218; 38, pp. 256–257]). The flow of (6.10) can be visualized as in Fig. 4, wherein once an orbit reaches the top it re-enters the bottom after a twist around the vertical axis of $(2\pi/\alpha)$.

The treatment above favors the α normal mode $(e^{-i\alpha t} \cdot z_1, 0)$ since the coordinates are not valid at the other mode. However, this is easily rectified by replacing U_α by

$$U_\beta = \{(\lambda, L, u) \in R^4 \mid L > 0, |u|^2 < (2L/\alpha)\},$$

letting $\omega_\beta = \omega_\alpha$ in (6.4), and replacing ρ_α by $\rho_\beta : U_\beta \to R^4 \simeq \mathbb{C} \times \mathbb{C}$,

$$\rho_\beta \begin{cases} z_1 = ue^{-i\alpha\lambda} \\ z_2 = \beta^{-1/2}(2L - \alpha |u|^2)^{1/2}\, ie^{-i\beta\lambda}. \end{cases} \tag{6.11}$$

The preceding discussion then holds with α replaced by β, except that in Fig. 4 the "inner circle" now represents the β normal mode $(0, e^{-i\beta t} \cdot z_2)$.

Set $\tilde{\rho}_\alpha = \rho_\alpha | V_\alpha$ and define the open 2-disc

$$D_\alpha = \{u \mid |u|^2 < (2h/\beta)\} = \{u \mid (0, h, u) \in V_\alpha\}. \tag{6.12}$$

Then we have a sequence of maps

$$\begin{tikzcd} D_\alpha \arrow[r, "j"] \arrow[rr, bend right, "p_\alpha"'] & V_\alpha \arrow[r, "\tilde{\rho}_\alpha"] & H_2^{-1}(h) \arrow[r, "i"] & R^4, \end{tikzcd} \tag{6.13}$$

where $j(u) = (0, h, u)$, and i is inclusion (we will also continue to use i to denote the inclusion $i : \Sigma_h \to M$). We set $p_\alpha = \tilde{\rho}_\alpha \circ j$, and note that $j(D_\alpha)$ is the open 2-disc in the horizontal plane of Fig. 4 which is interior to the solid cone and transverse to the flow of (6.10). Similar definitions hold for the

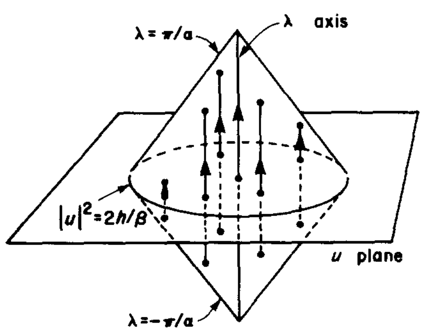

FIGURE 4

map ρ_β and set V_β, where we again use j to denote the map $j : D_\beta \to V_\beta$, $j(u) = (0, h, u)$. Then (6.4) and (6.7) imply

$$p_\alpha^*(i^*\omega) = d\xi \wedge d\eta = p_\beta^*(i^*\omega), \tag{6.14}$$

where $d\xi \wedge d\eta$ is a symplectic form on D_α and D_β.

PROPOSITION 6.1. *Extend the map* π *to* $\pi : H_2^{-1}(h) \to R^2$ *by* (4.10). *Then* $\pi \circ p_\alpha$ *is a symplectic* α*-sheeted covering of* (*see* Fig. 1)

(a) $M_R - \{SP\}$ *by* D_α *when* $\beta = \alpha = 1$,

(b) M_R *by* D_α *when* $\beta > \alpha = 1$,

(c) M_R *by* $D_\alpha - \{0\}$ *when* $\beta > \alpha > 1$,

where $(\pi \circ p_\alpha)(0) = NP$. *Replace* α *by* β, *a similar conclusion holds with* $(\pi \circ p_\beta)(0) = SP$.

Proof. From (4.2) and (6.5) the map $\pi \circ p_\alpha$ is given in coordinates (w_1, w_2, w_3) on R^3 by

$$w_1 + iw_2 = 2(i)^\beta \alpha^{-\beta/2}(2h - \beta\,|u|^2)^{\beta/2}\,(\bar{u})^\alpha, \qquad w_3 = 2(h - \beta\,|u|^2), \tag{6.15}$$

where $|u|^2 < (2h/\beta)$ from (6.3) implies $-2h < w_3 \leqslant 2h$. The fact that the covering is α-sheeted follows from the first equation in (6.16), and the remainder of the proposition follows from the placement of the pinch points in Fig. 1 and the definition of M_R in (4.9). The proof for $\pi \circ p_\beta$ is similar.

Q.E.D.

Now assume for $H = H_2 + N$ we have $ad_{H_2}(N) = 0$, hence by Lemma 4.1(a) that N is written in terms of the Hopf variables (4.2). By (6.8) we have $(W_4 \circ \rho_\alpha) = 2(H_2 \circ \rho a) = 2L$, and so (6.16), with h replaced by L, implies

$$H \circ \rho_\alpha = L + N \circ \rho_\alpha \tag{6.16}$$

is independent of λ. Since ρ_α is canonical the transformed differential equations are

$$\dot{\lambda} = 1 + \partial(N \circ \rho_\alpha)/\partial L, \tag{6.17a}$$

$$\dot{L} = 0, \tag{6.17b}$$

$$\dot{u} = -2i\,\partial(N \circ \rho_\alpha)/\partial\bar{u}. \tag{6.17c}$$

From (6.17b) we see that L is an integral, and hence on any surface of constant $L = h$ we have represented the flow of X_H on $H_2^{-1}(h)$ in the new coordinates (λ, L, u) as a product flow on V_α, governed on the disc $j(D_\alpha)$ by (6.17c), and in the vertical direction by (6.17a). Note that $j(D_\alpha)$ is a local cross section for the flow provided $\dot{\lambda} \neq 0$ on $j(D_\alpha)$.

We now define a function E on the disc D_α by

$$E(u, \bar{u}) = (N \circ \rho_\alpha)(\lambda, h, u). \tag{6.18}$$

PROPOSITION 6.2. (a) *In the notation above, assume there is a $\delta > 0$ so that $\dot{\lambda} > \delta$ on V_α. Then for $u_0 \neq 0$ the point $\rho_\alpha(\lambda, h, u_0)$ is a relative equilibrium for X_H if and only if u_0 is an equilibrium point of X_E on D_α.*

(b) *If $\alpha = 1$, then* (a) *also holds for $u_0 = 0$.*

Proof. (6.15) implies

$$(\pi \circ p_\alpha)(u) = (\pi \circ \rho_\alpha)(\lambda, h, u). \tag{6.19}$$

Since $\pi^* \omega_R = i^* \omega$ and (6.14) holds, the map $(\pi \circ p_\alpha) : D_\alpha \to M_R$ is a local symplectomorphism. Let K be the reduced Hamiltonian corresponding to N. Then in a suitable neighborhood of any point on $M_R - \{SP, NP\}$ we have

$$(K \circ \pi \circ p_\alpha)(u) = (K \circ \pi \circ \rho_\alpha)(\lambda, h, u) = (N \circ i \circ \rho_\alpha)(\lambda, h, u) = E(u, \bar{u}); \tag{6.20}$$

hence the (local) pull-back satisfies

$$(\pi \circ p_\alpha)^* X_K = X_E. \tag{6.21}$$

(a) If $\rho_\alpha(\lambda, h, u_0)$ is a relative equilibrium for X_H (equivalently, X_N), then (6.21) states that the vanishing of X_K at $(\pi \circ p_\alpha)(u_0)$ implies $X_E(u_0) = 0$. Conversely, assume $X_E(u_0) = 0$. Then Eqs. (6.17) reduce at $u = u_0$ to

$$\dot{\lambda} = \text{const} > 0, \tag{6.22a}$$

$$\dot{L} = 0, \tag{6.22b}$$

$$\dot{u} = 0. \tag{6.22c}$$

Thus the point (λ, h, u_0) is on the periodic orbit through $(0, h, u_0)$. Restricting ρ_α to this periodic orbit then gives a reparametrization of an X_{H_2} orbit, hence $\rho_\alpha(\lambda, h, u_0)$ is a relative equilibrium for X_H.

(b) This follows from the above reasoning applied to $u_0 = 0$ since for $\alpha = 1$ the point $(\pi \circ p_\alpha)(0) = NP \in M_R$. Q.E.D.

Remark a. A completely analogous result to Proposition 6.2 holds for points $\rho_\beta(\lambda, h, u_0)$ on taking account of the absence of various poles for M_R in Fig. 1.

Remark b. The condition $\dot{\lambda} > \delta$ in Proposition 6.2 will in practice be satisfied after stretching variables.

Remark c. By (6.21) we have transferred the critical points of K on M_R to critical points of E on D_α. Moreover, the transference (6.21) allows us to

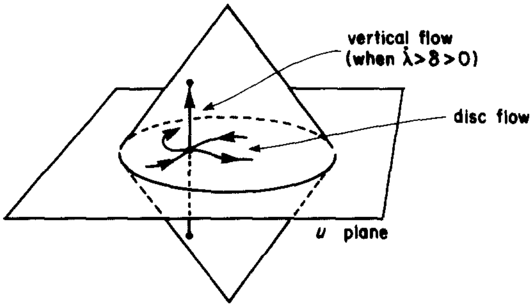

FIGURE 5

reconstruct the flow of X_H on M from the flow of X_K on M_R via (6.17) on V_α. Figure 5 shows a typical reconstructed flow on V_α under the assumptions of Proposition 6.2 (assuming the identifications for an (α, β) lens space).

To extend results from a truncated system in normal form to a full Hamiltonian we first need to recall a standard reduction technique. Let $H(x, y)$ be defined on a neighborhood $U \subset R^{2n}$ of the origin, and assume at a point $(x_0, y_0) \in U$ we have

$$\frac{\partial H}{\partial y_1}(x_0, y_0) \neq 0. \tag{6.23}$$

If $h_0 = H(x_0, y_0)$, then the implicit function theorem guarantees that in some neighborhood of (x_0, y_0, h_0) we can solve for y_1 as a function of the remaining variables, say

$$y_1 = -S(x, \hat{y}, h) \tag{6.24}$$

(the minus sign is a convenience), where $\hat{y} = (y_2, \ldots, y_n)$. It follows that

$$H(x, -S(x, \hat{y}, h), \hat{y}) = h, \tag{6.25}$$

and differentiating this identity gives

$$S_{x_j} = (H_{x_j}/H_{y_1}), \qquad S_{y_j} = (H_{y_j}/H_{y_1}), \qquad j = 2, 3, \ldots, n. \tag{6.26}$$

Since $\dot{x}_1 = H_{y_1} \neq 0$ by (6.23), we can replace the time t by the variable x_1 along orbit segments sufficiently close to (x_0, y_0, h_0). Letting $' = (d/dx_1)$, (6.26) then shows for $j = 2, 3, \ldots, n$ that

$$\begin{aligned} x_j' &= (\dot{x}_j/H_{y_1}) = (H_{y_j}/H_{y_1}) &&= S_{y_j} \\ y_j' &= (\dot{y}_j/H_{y_1}) = -(H_{x_j}/H_{y_1}) &&= -S_{x_j}, \end{aligned} \tag{6.27}$$

and thus we obtain a time-dependent system with time variable x_1.

In other words, solutions of (6.27) at fixed energy h_0 will give local reparametrizations of solutions of energy h_0 of the original system $\dot{x} = H_y$, $\dot{y} = -H_x$, where y_1 is reconstructed from (6.24). The process of replacing these latter equations for H by (6.27) is an example of *isoenergetic reduction*, which in general refers to any technique involving a change of the time variable for the purpose of studying solutions of a Hamiltonian system at a fixed energy.

Now consider a parameter-dependent Hamiltonian of the form

$$H(x, y; \varepsilon) = y_1 + \varepsilon^{r-2} N(\hat{x}; y) + O(\varepsilon^{r-1}), \qquad r \geqslant 3, \tag{6.28}$$

defined n a neighborhood $U \times (-\varepsilon_0, \varepsilon_0) \subset R^{2n} \times R$ containing the origin, where $\hat{x} = (x_2, ..., x_n)$ and the absence of x_1 from N is intentional (recall the absence of λ in $N \circ \rho_\alpha$ of (6.16)). Then $H_{y_1} = 1 + \varepsilon^{r-2} N_{y_1} + O(\varepsilon^{r-1})$, and so (6.23) holds near any point in U for sufficiently small ε. A straightforward calculation then shows

$$S = -h + \varepsilon^{r-2} N(\hat{x}; h, \hat{y}) + O(\varepsilon^{r-1}). \tag{6.29}$$

In this instance (6.27) becomes, with $' = (d/dx_1)$ and $j = 2, 3, ..., n$,

$$x_i' = \varepsilon^{r-2} N_{y_j} + O(\varepsilon^{r-1}), \qquad y_j' = -\varepsilon^{r-2} N_{x_j} + O(\varepsilon^{r-1}). \tag{6.30}$$

To apply these ideas in the case of two degrees of freedom consider

$$H(z, \bar{z}) = H_2(z, \bar{z}) + H_r(z, \bar{z}) + O_{r+1}(z, \bar{z}), \tag{6.31}$$

where $z = (z_1, z_2) \in \mathbb{C} \times \mathbb{C} \simeq R^4$, H_2 is given by (6.1), and $H_r \in \mathscr{P}_r$. If in the associated differential equations we scale variables by $z_j \to \varepsilon z_j$, then we obtain a new Hamiltonian system with Hamiltonian

$$H_\varepsilon(z, \bar{z}) = \varepsilon^{-2} H(\varepsilon z, \varepsilon \bar{z}) = H_2(z, \bar{z}) + \varepsilon^{r-2} H_r(z, \bar{z}) + O(\varepsilon^{r-1}). \tag{6.32}$$

Moreover, for $h \neq 0$ the mapping $z \to \varepsilon z$ is a conjugacy between the flow of (6.31) at energy $\varepsilon^2 h$ and that of (6.32) at energy h. In short, statements concerning (6.32) for fixed $h > 0$ and $\varepsilon > 0$ sufficiently small reflect the behavior of (6.31) at small positive energies.

Now assume the H_ε of (6.32) has the form

$$H_\varepsilon = H_2 + \varepsilon^n N + O(\varepsilon^{n+1}). \tag{6.33}$$

where $ad_{H_2}(N) = 0$ and N is expressed in terms of the Hopf variables (4.2). The analog of (6.16) for (6.33) is then

$$H_\varepsilon \circ \rho_\alpha = L + \varepsilon^n (N \circ \rho_\alpha) + O(\varepsilon^{n+1}), \tag{6.34}$$

where $H_\varepsilon = h$ corresponds to $(H_\varepsilon \circ \rho_\alpha) = h$, and $(N \circ \rho_\alpha)$ is independent of λ. Note that (6.34) is in the form (6.28) with $(x_1, x_2, y_1, y_2) = (\lambda, \xi, L, \eta)$, where $u = \xi + i\eta$. Then by isoenergetic reduction the study of the flow on $(H_\varepsilon \circ \rho_\alpha) = h > 0$ reduces (locally) to examining the analog of (6.30), namely,

$$u' = -2i\varepsilon^n(\partial E/\partial \bar{u}) + O(\varepsilon^{n+1}), \qquad ' = (d/d\lambda), \tag{6.35}$$

where

$$E(u, \bar{u}) = (N \circ \rho_\alpha)(\lambda, h, u). \tag{6.36}$$

Moreover, along a periodic orbit these "local" representations can be pieced together (by the uniqueness assertion in the implicit function theorem) so as to have (6.35) hold in a neighborhood of that orbit.

Now (6.35) is precisely of the form (5.14). Moreover, dropping the $O(\varepsilon^n)$ terms puts us in the context of Proposition 6.2 since $\dot{\lambda} = [1 + \varepsilon^n \partial(N \circ \rho_\alpha)/\partial L] > \delta > 0$ for small $\varepsilon \geqslant 0$. Thus, if $\rho_\alpha(\lambda, h, u_0)$ is a relative equilibrium for X_N (and hence for X_H), Proposition 6.2 implies $X_E(u_0) = 0 = X_K(\pi \circ \rho_\alpha)(u_0)$, where $K \circ \pi = N \circ i$ (we could also define K by $K \circ \pi = (H_2 + N) \circ i$). We can always translate u_0 to the origin and rotate in the disc D_α (these are local canonical transformations) so that the Hessian matrix of E is diagonal. In this case for $X_E(0) = 0$ we can assume (on ignoring constants) that E has the form

$$E(u, \bar{u}) = \tfrac{1}{2}(A\xi^2 + B\eta^2) + O_3(u, \bar{u}), \qquad u = \xi + i\eta, \tag{6.37}$$

as in (5.16), where the new variables are again denoted by u.

We note that when $\beta = \alpha = 1$ the placement of u_0 at the origin of D_α can be effected by a linear canonical transformation of R^4 by some $V \in SU(2)$. Indeed, as in Theorem 4.3(a), V induces a rotation of M_R by $B \in SO(3)$, and (4.30) applied to $z_0 = \rho_\alpha(u_0)$ results in $\pi(V \cdot z_0) = B \cdot \pi(z_0) = NP = (\pi \circ \rho_\alpha)(0)$. Proposition 6.3 then shows how a further rotation of M_R about the NP will provide new local coordinates (again denoted by $u = \xi + i\eta$) about the origin in D_α in which E has the form (6.37).

In the statement of Proposition 6.3 we let (w_1, w_2) be local coordinates about the NP of M_R, where $w_3 = f(w_1, w_2)$ is given in terms of (w_1, w_2) via the constraint (4.3), which in the case $\beta = \alpha = 1$ is simply

$$w_1^2 + w_2^2 + w_3^2 = (2h)^2. \tag{6.38}$$

The reduced Hamiltonian K on M_R is then related to the extended function K^e on $R^3 - \{0\}$ defined in Section 4 by

$$K(w_1, w_2) = K^e(w_1, w_2, f(w_1, w_2)). \tag{6.39}$$

where

$$w_3 = f(w_1, w_2) = [(2h)^2 - w_1^2 - w_2^2]^{1/2}. \tag{6.40}$$

An analog of this proposition can be found in [21, p. 58].

PROPOSITION 6.3. *In the notation above, assume $\beta = \alpha = 1$ and that $u_0 = 0$ corresponds to a relative equilibrium of the system in Proposition 6.2. Then*

$$\begin{aligned}\begin{pmatrix} \dfrac{\partial^2 E}{\partial \xi^2} & \dfrac{\partial^2 E}{\partial \xi\, \partial \eta} \\ \dfrac{\partial^2 E}{\partial \eta\, \partial \xi} & \dfrac{\partial^2 E}{\partial \eta^2} \end{pmatrix}_0 &= 8h \begin{pmatrix} \dfrac{\partial^2 K}{\partial w_2^2} & \dfrac{\partial^2 K}{\partial w_2\, \partial w_1} \\ \dfrac{\partial^2 K}{\partial w_1\, \partial w_2} & \dfrac{\partial^2 K}{\partial w_1^2} \end{pmatrix}_{NP} \\ &= 8h \begin{pmatrix} \dfrac{\partial^2 K^e}{\partial w_2^2} - \mu & \dfrac{\partial^2 K^e}{\partial w_2\, \partial w_1} \\ \dfrac{\partial^2 K^e}{\partial w_1\, \partial w_2} & \dfrac{\partial^2 K^e}{\partial w_1^2} - \mu \end{pmatrix}_{NP},\end{aligned} \tag{6.41}$$

where $\nabla K^e(NP) = (0, 0, \mu)$. Thus there is an induced rotation of M_R about the NP by $B \in SO(3)$, where B is in the form 4.29(c), *such that the corresponding induced rotation in the (w_1, w_2) plane and (ξ, η) plane diagonalize the Hessian matrices in* (6.41); *hence E has form* (6.37) *in the new coordinates.*

Proof. The Hopf variables $(W_j \circ \pi \circ p_\alpha)$ evaluated on $u = \xi$ are

$$W_1 = 0, \qquad W_2 = 2(2h - \xi^2)^{1/2} \cdot \xi, \qquad W_3 = 2(h - \xi^2), \tag{6.42}$$

and evaluated on $u = i\eta$ are

$$W_1 = 2(2h - \eta^2)^{1/2} \cdot \eta, \qquad W_2 = 0, \qquad W_3 = 2(h - \eta^2). \tag{6.43}$$

A direct computation then gives

$$\begin{aligned}(\pi \circ p_\alpha)_* \, (\partial/\partial\xi)_0 &= 2\sqrt{2h}(\partial/\partial w_2)_{NP}, \\ (\pi \circ p_\alpha)_* \, (\partial/\partial\eta)_0 &= 2\sqrt{2h}(\partial/\partial w_1)_{NP}.\end{aligned} \tag{6.44}$$

A further direct computation shows

$$\begin{aligned}\frac{\partial^2 K}{\partial w_1^2}(NP) &= \frac{\partial^2 K^e}{\partial w_1^2}(NP) - \mu, \\ \frac{\partial^2 K}{\partial w_1\, \partial w_2}(NP) &= \frac{\partial^2 K^e}{\partial w_1\, \partial w_2}(NP), \\ \frac{\partial^2 K}{\partial w_2^2}(NP) &= \frac{\partial^2 K^e}{\partial w_2^2}(NP) - \mu.\end{aligned} \tag{6.45}$$

Since $K \circ \pi \circ p_\alpha = E$ by (6.20), relations (6.44) and (6.45) imply (6.41). The remainder of the result then follows since real symmetric matrices can be diagonalized by rotation matrices. Q.E.D.

Remark. The switch in the order of the variables in (6.44) reflects the fact that $d\xi \wedge d\eta$ is the symplectic form on D_α, and ω_R given by (4.11) has a minus sign in its definition. This is also consistent with the presence of the factor $\bar{u}$ in the calculation of $\pi \circ p_\alpha$ in (6.15).

Denote by Hess $[K(q)]$ the determinant of the 2×2 Hessian matrix of K at $q \in M_R$. Theorem 5.6 now gives (a)–(c) of the following result (we no longer assume $\beta = \alpha = 1$):

THEOREM 6.4. *Let $ad_{H_2}(N) = 0$ and assume $\rho_\alpha(\lambda, h, u_0) = z_0$ is a relative equilibrium for N (hence for $H_2 + N$). Assume $w_0 = \pi(z_0)$ is a nondegenerate critical point of K on M_R, where K is the reduced Hamiltonian corresponding to N. Then for sufficiently small $\varepsilon > 0$ we have the following results:*

(a) *System* (6.32) *has a unique periodic orbit Π_ε with energy h through the point $z_0(\varepsilon)$, with period $T(\varepsilon)$, such that $z_0(\varepsilon) \to z_0$ and $T(\varepsilon) \to 2\pi$ as $\varepsilon \downarrow 0$. When $N = H_r$ and $n = (r-2)$ in* (6.33), *then $P_\varepsilon = \varepsilon \cdot \Pi_\varepsilon$ is a periodic orbit of* (6.31) *through the point $\varepsilon \cdot z_0(\varepsilon)$ with energy $\varepsilon^2 \cdot h$ and period $T(\varepsilon)$.*

(b) *Π_ε and P_ε are elliptic or hyperbolic according as* Hess$[K(w_0)]$ *is positive or negative.*

(c) *Assume Π_ε (and hence P_ε) is elliptic and that the function E of* (6.36) *has form* (6.37) *after u_0 has been shifted to the origin of the disc D_α (the new local coordinates about the origin are again denoted by u). Then on conversion of E from form* (5.16) *to* (5.17), *the periodic orbit Π_ε (and hence P_ε) is stable provided condition* (5.19) *holds.*

(d) *Let $\mathscr{G}$ be a group of transformations of (R^4, ω) as in Proposition* 3.4 *with reduced group $\mathscr{G}_R$ acting on (M_R, ω_R). Assume the Hamiltonian H_ε of* (6.33) *is fixed by some $g \in \mathscr{G}$ with $g^*\omega = \pm\omega$, and let Ψ_t^ε be the flow of X_{H_ε}. Then $\phi(g) \cdot w_0$ is a nondegenerte critical point of K on M_R, and*

$$g \cdot \Pi_\varepsilon(t; z_0(\varepsilon)) = \Psi^\varepsilon_{\pm t}(g \cdot z_0(\varepsilon)) \tag{6.46}$$

is a periodic orbit of X_{H_ε} through the point $g \cdot z_0(\varepsilon)$ having the same stability status as Π_ε. Moreover, Π_ε will be symmetric wrt *g if and only if w_0 is fixed by $\phi(g)$. Similar statements hold for P_ε in the case that H of* (6.31) *(and hence H_ε of* (6.32)) *is fixed by g.*

Proof. Results (a)–(c) follow directly from Theorem 5.6 on noting that the signum of Hess$[K(\omega_0)]$ is independent of coordinates. Therefore, when u_0

has been translated to the origin of D_α and E has been put into form (6.37), we will have $\operatorname{sgn}\{\operatorname{Hess}[K(w_0)]\} = \operatorname{sgn}(AB)$.

(d) Equation (6.46) and the assertions concerning nondegeneracy and stability type are clear. Now Proposition 3.6 implies that Π_0 is symmetric wrt g. If Π_ε were not symmetric wrt g, then $g^*\omega = \pm\omega$ implies by (3.6) that (6.46) gives another family of periodic orbits (for the same Hamiltonian flow) emanating from $\Pi_0(t; z_0)$ as ε increases from zero. This, however, contradicts the uniqueness assertion of the implicit function theorem as applied in the proof of Theorem 5.6(a), since the new periodic orbits will also intersect the disc $p_\alpha(D_\alpha)$ transversely in a curve emanating from $p_\alpha(D_\alpha) \cap \Pi_0$. Q.E.D.

Remark a. When g is an antisymplectic involution and Π_ε is symmetric wrt g in Theorem 6.4(d), it is traditional to choose the initial points $z_0(\varepsilon)$ in the fixed point set of g, which is a Lagrangian submanifold of R^4 (see [17, 27]).

Remark b. For $\beta = \alpha = 1$ a classification of the possible critical points of a polynomial K of degree 2 on the 2-sphere M_R is given in [21, pp. 67–71]. See also [22].

Remark c. The "pinched" points of Fig. 1 are poles that do not lie in M_R but correspond to the α and β normal modes of H_2. Theorem 6.4(a) cannot be applied in these cases to continue the normal modes to a full Hamiltonian such as (6.31). However, Liapunov's theorem does apply, and in Section 7.A we will present a reduction version of this result with corresponding stability criteria. In all other cases in Fig. 1 we have $\alpha = 1$ at the *NP* (hence $\beta/\alpha =$ integer), and $\beta = \alpha = 1$ at the *SP* (hence $\alpha/\beta =$ integer), hence Liapunov's theorem does not apply but Theorem 6.4(a) does. To summarize, for the continuation problem, if $u_0 = 0$ we can work with $(\pi \circ p_\alpha)(0) = NP$ for $\alpha = 1$, and use Liapunov's theorem if $\alpha > 1$.

Remark d. The proof of Theorem 6.4(b) shows that the linear stability classification of periodic orbits can be made either on M_R or on the discs D_α or D_β. Questions of elliptic stability, however, are best posed in terms of the coordinates on D_α or D_β as we have done in Section 5, since the reconstructed and perturbed flows are easy to obtain in these coordinates (see Theorem 6.4(c) and Sections 7.B and 7.C. Theoretically this could also be done by the alternate flow reconstruction in [1, pp. 304–305].

7. Applications

A.

As discussed in Remark c at the end of Section 6, when $\beta > \alpha \geqslant 1$ the problem of continuing the normal modes corresponding to those poles not contained in M_R (see Fig. 1) cannot be handled by reduction wrt the S^1-action generated by the flow (4.7). Liapunov's theorem, however, does apply in these other cases, and the following review of the proof (see [1, pp. 498–499]) shows that this result can be viewed as involving reduction by a different S^1-action.

Let $H(x, y)$ be of class C^k on (R^{2n}, ω), with $k \geqslant 2$ and ω as in (3.1). Assume H has a critical point at the origin with characteristic exponents $\{\nu_1, ..., \nu_n, -\nu_1, ..., -\nu_n\}$, where ν_1 and $-\nu_1 = \bar{\nu}_1$ are purely imaginary and (ν_j/ν_1) is not an integer for $j = 2, 3, ..., n$. Writing $\nu_1 = i\gamma$ with $\gamma \neq 0$, by a linear canonical change of variables we can then assume H has the form

$$H(x, y) = L_2(x_1, y_1) + \hat{H}_2(\hat{x}, \hat{y}) + O_3(x, y), \tag{7.1}$$

where $\hat{x} = (x_2, ..., x_n)$, $\hat{y} = (y_2, ..., y_n)$, and

$$L_2(x_1, y_1) = (\gamma/2)(x_1^2 + y_1^2). \tag{7.2}$$

Moreover, by reversing time (if necessary) we can assume $\gamma > 0$. Now observe that X_{L_2} generates a flow giving a free and proper S^1-action on $\{R^2 - (0,0)\} \times R^{2n-2}$ for $0 \leqslant t < (2\pi/\gamma)$. With $L_2^{-1}(l) = \Sigma_l \simeq S^1 \times R^{2n-2}$, the reduced space M_R is $(\Sigma_l/S^1) \simeq R^{2n-2}$, with reduced symplectic form $\omega_R = \sum_{j=2}^n dx_j \wedge dy_j$. Also, $ad_{L_2}(\hat{H}_2) = 0$, and the reduced Hamiltonian associated to $\hat{H}_2$ is just $K = \hat{H}_2$ itself. The normal mode in the (x_1, y_1) plane for the quadratic Hamiltonian $H_2 = L_2 + \hat{H}_2$ projects to the origin of $M_R \simeq R^{2n-2}$, and this origin is a nondegenerate critical point for $K = \hat{H}_2$ since $\nu_j \neq 0$. As we now indicate, this critical point is continued in Liapunov's theorem in essentially the same manner as that in Theorem 6.4.

Here the analog of the canonical transformation ρ_α of (6.5) is

$$\rho : x_1 + iy_1 = (2L/\gamma)^{1/2}\, ie^{-i\gamma\lambda}; \qquad \text{all other variables are fixed.} \tag{7.3}$$

In this case (7.1) becomes, after stretching variables (as in the derivation of (6.32)),

$$(H_\varepsilon \circ \rho)(\lambda, L, \hat{x}, \hat{y}) = L + \hat{H}_2 + O(\varepsilon). \tag{7.4}$$

Notice that (7.4) is periodic in λ with period $(2\pi/\gamma)$, and the associated differential equations are

$$\dot{\lambda} = 1 + O(\varepsilon), \qquad \dot{L} = O(\varepsilon), \qquad (d/dt)\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} = A\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} + O(\varepsilon), \tag{7.5}$$

where A is a $(2n-2) \times (2n-2)$ constant matrix. If we employ isoenergetic reduction on (7.4) with λ as the new time variable, then on integrating wrt λ from 0 to $(2\pi/\gamma)$ we obtain a mapping

$$\begin{pmatrix}\hat{x}_1\\ \hat{y}_1\end{pmatrix} = \exp[(2\pi/\gamma)A]\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} + O(\varepsilon). \tag{7.6}$$

The hypothesis $(\nu_j/\nu_1) \neq$ integer for $j = 2, 3, \ldots, n$, guarantees that the Jacobian matrix of (7.6) at $\varepsilon = 0$ does not admit one as an eigenvalue, hence the origin can be continued as a fixed point $(\hat{x}_\varepsilon, \hat{y}_\varepsilon)$ of the mapping (7.6) by an argument similar to that in the proof of Theorem 5.5. In the case $n = 2$, if $H_2 = L_2 + \hat{H}_2$ is in form (6.1), then the fixed point is elliptic and Remark d following Theorem 5.6 applies to provide a stability criterion. For a concrete application see [6].

From the above discussion we see that Liapunov's theorem can be viewed as a case of 0th order averaging in the sense of Section 1; that is, the Hamiltonian (7.1) is already in prepared form for a continuation argument analogue to that in Theorem 6.4.

B.

In this and the following section we study a 1–1 resonance example, the Hénon–Heiles Hamiltonian, using first and second order averaging. Here we review first order methods contained in [4, 21], and extend those results by giving a simple geometrical description of the resulting orbits.

The Hénon–Heiles Hamiltonian H_A is given by

$$H_A(x, y) = \tfrac{1}{2}\,|y|^2 + V(x), \tag{7.7}$$

where $x = (x_1, x_2)$, $y = (y_1, y_2) \in R^2$, and where the potential V is given by

$$V(x) = \tfrac{1}{2}\,|x|^2 + \tfrac{1}{3}x_1^3 - x_1 x_2^2. \tag{7.8}$$

(This is equivalent to the Hamiltonian originally studied by Hénon and Heiles [20] through a trivial canonical transformation that renames variables.) If we regard $x \simeq x_1 + ix_2$ and $y \simeq y_1 + iy_2$ as being in $\mathbb{C}$, then (7.7) remains the same, but (7.8) simplifies to

$$V(x) = \tfrac{1}{2}\,|x|^2 + \tfrac{1}{3}\,\mathrm{Re}(x)^3. \tag{7.9}$$

From (7.9) the rotational symmetry

$$\Omega : x \to e^{2i\pi/3} \cdot x \tag{7.10}$$

of the potential in R^2 becomes obvious, and the lifted action

$$g : (x, y) \to (\Omega x, \Omega y) \tag{7.11}$$

is a linear canonical transformation on R^4 leaving (7.7) invariant. Notice that the Hamiltonian is also invariant under the group $\mathscr{G}$ of Theorem 4.4.

For fixed $h \in R$ the x-plane projection of the energy surface $H^{-1}(h) \subset \mathbb{C} \times \mathbb{C}$ is called *Hill's region corresponding to h*, and simply *Hill's region* when h is understood. For $0 < h < (\frac{1}{6})$ the region has four components, and

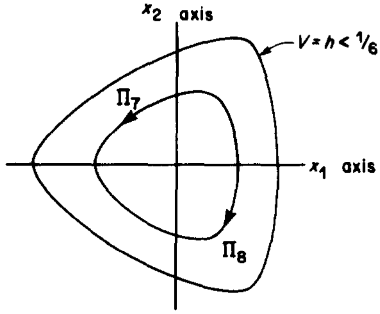

FIGURE 6

the one of interest, that containing the origin $x = 0$, is sketched in Fig. 6. This shrinks to the origin as $h \downarrow 0$, and as $h \uparrow \frac{1}{6}$ limits to an equilateral triangle whose vertices are critical points of V. We will show that at first order averaging (7.7) yields two periodic orbits with continuations at low positive energies (see Theorem 6.4) projecting to the curves Π_7 and Π_8 of Fig. 6. Moreover, we will show these orbits are elliptic stable at these energies.

In Appendix A we use successive applications of Proposition 1.1 to calculate the normal form of (7.7) through terms of order 6 wrt H_2. The result from (A23) is

$$H = H_2 + H_4 + H_6 + O_7, \tag{7.12a}$$

where in terms of the Hopf variables (4.2) we have $\beta = \alpha = 1$ and

$$\begin{aligned} H_2 &= \tfrac{1}{2} W_4, \\ H_4 &= \tfrac{1}{48} [7 W_2^2 - 5 W_4^2], \\ H_6 &= \tfrac{1}{64} [-\tfrac{67}{54} W_4^3 - \tfrac{7}{18} W_2^2 W_4 - \tfrac{28}{9} \operatorname{Re}(W_3 + i W_1)^3]. \end{aligned} \tag{7.12b}$$

We refer to $H_2 + H_4$ as the *first order average* of (7.7), to $H_2 + H_4 + H_6$ as the *second order average*, and to H as the *full Hamiltonian*.

Using $\beta = \alpha = 1$ in Theorem 4.2, the reduced Hamiltonians K_1 and K_2 corresponding to H_4 and H_6 are seen to be $K_j = K_j^e | M_R$, $j = 1, 2$, where

$$\begin{aligned} K_1^e &= \tfrac{1}{48} [7 w_2^2 - 5(2h)^2], \\ K_2^e &= \tfrac{1}{64} [-\tfrac{67}{54}(2h)^3 - \tfrac{7}{18}(2h) w_2^2 - \tfrac{28}{9} w_3^3 + \tfrac{28}{3} w_1^2 w_3]. \end{aligned} \tag{7.13}$$

We set $K^e = K_1^e + K_2^e$ and $K = K_1 + K_2$.

To apply Theorems 4.2 and 6.4 in the case of first order averaging we must find all points on the 2-sphere M_R at which

$$\nabla K_1^e = \tfrac{7}{24}(0, w_2, 0) \tag{7.14}$$

is a normal vector. But this set obviously consists of the two points $(0, \pm 2h, 0)$, together with the circle of points on M_R, where $w_2 = 0$ (this is exactly the "$\mathscr{S}$-meridian" introduced in the remarks following Theorem 4.4).

We begin with an analysis of $w_0 = (0, 2h, 0)$, which we first rotate to NP. Recall from Section 4 and the discussion preceding Proposition 6.3 that this can be accomplished by a linear canonical change of variables on R^4 generated by $V \in SU(2)$, with induced transformation $B \in SO(3)$ on M_R given by (4.29a) at $t = -(\pi/2)$. In a neighborhood of NP the expression for K_1 then becomes

$$\tilde{K}_1 = \tfrac{1}{48}[7w_3^2 - 5(2h)^2] = \tfrac{1}{48}[2(2h)^2 - 7(w_1^2 + w_2^2)], \tag{7.15}$$

where as in (6.40) we have

$$w_3 = f(w_1, w_2) = [(2h)^2 - (w_1^2 + w_2^2)]^{1/2}. \tag{7.16}$$

A simple calculation then shows

$$\operatorname{Hess}[\tilde{K}_1(NP)] > 0,$$

hence Theorem 6.4(a) and (b) gives the existence of a family $\Pi_7(t) = \Pi_7(t; \varepsilon^2 h)$ of elliptic periodic solutions of (7.7) at low positive energies $\varepsilon^2 h$, with periods $T(\varepsilon) \to 2\pi$ as $\varepsilon \downarrow 0$. Notice that the projection $\pi\colon \Sigma_h \to M_R$ sends $z_0 = (\sqrt{h}, 0, 0, \sqrt{h})$ to $w_0 = (0, 2h, 0)$, and to $\Pi_7(t)$ can be regarded as a continuation of the Φ_t-orbit of X_{H_2} through $\varepsilon \cdot z_0$; for this reason we also write $\Pi_7(t) = \Pi_7(t; \varepsilon \cdot z_0(\varepsilon))$, where $z_0(\varepsilon) \to z_0$ as $\varepsilon \downarrow 0$.

For each sufficiently small value of the parameter ε the orbit $\Pi_7(t; \varepsilon^2 h)$ is stable. To see this we use (7.15), (6.15), and (6.20) to calculate the function E of (5.13) obtaining (up to an additive constant)

$$E(u, \bar{u}) = \tfrac{1}{2}(-\tfrac{7}{3})\, h\, |u|^2 + \tfrac{7}{12} u^2 \bar{u}^2. \tag{7.17}$$

But then (5.19) is nonzero since $E_3 = E_4 = 0$ and $E_9 = \frac{7}{12}$; hence Theorem 5.6(c) applies to give the elliptic stability of this family of periodic orbits.

From Theorem 6.4(d) we immediately conclude the existence of a second family $\Pi_8(t) = \Pi_8(t; \varepsilon^2 h) = \Pi_8(t, \varepsilon R_1(z_0))$ of elliptic stable periodic orbits, related to the first by

$$\Pi_8(t; \varepsilon R_1(z_0)) = R_1 \Pi_7(-t; \varepsilon z_0), \tag{7.18}$$

where R_1 is as in (4.36). Indeed, we have $R_1(z_0) = (\sqrt{h}, 0, 0, -\sqrt{h})$ and $\pi(R_1(z_0)) = (0, -2h, 0)$; hence $\Pi_8(t)$ is associated with the nondegenerate critical point $\phi(R_1) \cdot w_0 = (0, -2h, 0)$ of K_1 on M_R, where $\phi(R_1)$ is given by (4.39b).

Turning to the positioning of the projections of Π_7 and Π_8 in Hill's region (Fig. 6), first note from (7.18) and the definition of R_1 that these x-plane projections must coincide (as sets). Next observe from (4.36) that R_2 fixes $z_0 = (\sqrt{h}, 0, 0, \sqrt{h})$, and from (4.39c) that $\phi(R_2)$ fixes $(0, 2h, 0)$. Since $R_2 \circ R_1 = R_1 \circ R_2$, Theorem 6.4(d) implies that both Π_7 and Π_8 are symmetric wrt R_2, and, as we observed in Remark a following Theorem 6.4, their respective "initial points" $\varepsilon \cdot z_0(\varepsilon)$ and $\varepsilon \cdot R_1(z_0(\varepsilon))$ may be chosen so as to be fixed by R_2. Hence these orbits must have x-plane projections crossing the x_1 axis perpendicularly, in agreement with Fig. 6. Moreover, there can be only two such crossings in one period, since Π_7 and Π_8 are unique continuations of periodic orbits which have this property (consider Hill's region of the Hamiltonian H_2, which coincides with (6.32) at $\varepsilon = 0$). Finally, since (7.7) is invariant under the rotation $g = \exp[-i(-2\pi/3)\,\sigma_2]$ of (7.11), and since (4.29b) implies

$$\phi(g) = \begin{pmatrix} \cos(4\pi/3) & 0 & \sin(4\pi/3) \\ 0 & 1 & 0 \\ -\sin(4\pi/3) & 0 & \cos(4\pi/3) \end{pmatrix} \tag{7.19}$$

(which we note fixes both w_0 and $\phi(R_1) \cdot w_0$), we see by Theorem 6.4(d) that the orbits $\Pi_7(t)$ and $\Pi_8(t)$ are both symmetric wrt g. Hence their x-plane projections must be symmetric wrt the rotation Ω of (7.10) as depicted in Fig. 6. (The signs in (7.19) are reversed from (4.29b) since Ω in (7.10) is a counterclockwise rotation.)

It remains to show that the projections of Π_7 and Π_8 do not intersect the boundary of our Hill's region. However, if this were the case, then at such a point the energy relation $\frac{1}{2}|y|^2 + V(x) = h$ would force the velocity y to vanish (hence the name *zero-velocity curve* for the boundary). But the equations are time-reversible (being equivalent to $\ddot{x} = -V_x$), and so orbits with projections touching the boundary are precisely the orbits symmetric wrt R_1. Since this is not the case with Π_7 and Π_8 (recall that w_0 was not fixed by $\phi(R_1)$), our picture is complete.

As previously noted, the remaining critical points of K_1 comprise the $\mathscr{G}$-meridian of M_R. However, these are all degenerate, and an analysis as above is doomed to fail. In Section C we show how these can be handled using the second order average $(H_2 + H_4 + H_6)$ of (7.7).

In view of Fig. 6, we refer to Π_7 and Π_8 as the "central periodic orbits" of the Hénon–Heiles Hamiltonian. We now give a brief discussion of the corresponding orbits in two additional Hamiltonians

$$H_B(x, y) = \tfrac{1}{2}|y|^2 + \tfrac{1}{2}|x|^2 - x_1 x_2^2, \tag{7.20}$$

$$H_C(x, y) = \tfrac{1}{2}|y|^2 + \tfrac{1}{2}|x|^2 - \tfrac{1}{2}x_1^2 x_2^2. \tag{7.21}$$

Numerical work on the Hamiltonian H_B in [11] shows the existence of two elliptic stable central periodic orbits at low positive energies corresponding to Π_7 and Π_8. The elliptic stability was verified in [21], and the methods of the present paper show that the orbit projections cross the x_1 axis perpendicularly and stay away from the zero velocity curve.

The Hamiltonian H_C was briefly discussed in [20], and central periodic orbits at all positive energies were constructed by geometrical methods in [8, Sect. 8]. Using the results of this paper we can show that at low positive energies these orbits are hyperbolic, in contrast with H_A and H_B. However, the orbits will still have x-plane projections intersecting the x_1 axis perpendicularly, will stay away from the zero velocity curve, and will be invariant under rotations through an angle $(\pi/2)$.

As with H_C, the central periodic orbits of H_A can be shown to exist at all positive energies by simple geometrical arguments [8]. However, numerical evidence (see [24]) indicates they become hyperbolic as h increases. The existence of these orbits in H_B for all positive energies has yet to be established.

C.

We now study the Hénon–Heiles Hamiltonian (7.7) with regard to second order averaging. From (7.13) we have

$$\nabla K^e = \nabla(K_1^e + K_2^e) = \tfrac{1}{64}\,\tfrac{28}{3}(2w_1 w_3,\ [2-(h/6)]\,w_2,\ w_1^2 - w_3^2), \quad (7.22)$$

which is normal to $M_R = M_R(h)$ precisely at the points

$$(0, \pm 2h, 0) \quad (7.23a)$$

$$e_1 = (0, 0, 2h), \qquad e_2 = \phi(g)\,e_1, \qquad e_3 = \phi(g)\,e_2 \quad (7.23b)$$

$$h_4 = (0, 0, -2h), \qquad h_5 = \phi(g)\,h_4, \qquad h_6 = \phi(g)\,h_5, \quad (7.23c)$$

where $\phi(g)$ is given by (7.19). Since the critical points (7.23a) were handled in Section B, we concentrate on those in (7.23b) and (7.23c); these lie on the

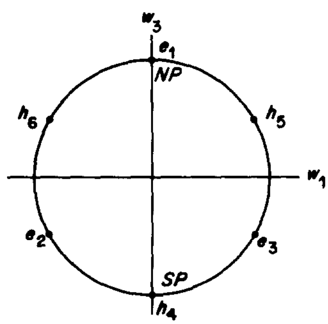

FIGURE 7

$\mathscr{G}$-meridian as shown in Fig. 7. In view of the symmetry of (7.7) wrt g, by Theorem 6.4(d) is suffices to consider only $e_1 = NP$ and $h_4 = SP$. We will show these respectively generate elliptic stable and hyperbolic families of periodic orbits. We remark that our subscripting has been chosen so as to correspond with [8].

Stretching variables by $z \to \varepsilon z$, let

$$H_\varepsilon(z) = \varepsilon^{-2} H(\varepsilon z) = H_2 + \varepsilon^2 [H_4 + \varepsilon^2 H_4] + O(\varepsilon^5) \tag{7.24}$$

and $K_\varepsilon = K_1 + \varepsilon^2 K_2$. Using (6.41) and (7.22) we calculate

$$\mathrm{Hess}[K_\varepsilon(NP)] = (\tfrac{1}{64})^2 (\tfrac{28}{3})^2 \tfrac{1}{6} \{12 + \varepsilon^2(-h + 24h^2)\} \cdot \{4\varepsilon^2(h + h^2)\} > 0 \tag{7.25}$$

for $\varepsilon > 0$ sufficiently small; hence NP is elliptic.

To study SP we rotate that point to NP using (4.29b) at $t = \pi$, obtaining a new Hamiltonian $\tilde{K}_\varepsilon$ with

$$\mathrm{Hess}[\tilde{K}_\varepsilon(NP)] = (\tfrac{1}{64})^2 (\tfrac{28}{3})^2 \tfrac{1}{6} \{12 - \varepsilon^2(h + 24h^2)\} \cdot \{-4\varepsilon^2(h + h^2)\}, \tag{7.26}$$

hence the SP for K_ε is a hyperbolic point for $\varepsilon > 0$ sufficiently small. By symmetry $\{e_1, e_2, e_3\}$ are elliptic points, and $\{h_4, h_5, h_6\}$ are hyperbolic points for K_ε on M_R when $\varepsilon > 0$ is sufficiently small.

The critical points (7.23) of K_ε on M_R correspond to periodic orbits of $(H_2 + \varepsilon^2 H_4 + \varepsilon^4 H_6)$; we now continue these orbits to the full Hamiltonian (7.7) at low positive energies. Applying isoenergetic reduction at energy h to $(H_\varepsilon \circ \rho_\alpha)$, where H_ε is given by (7.24), we obtain the time-dependent one degree of freedom Hamiltonian

$$\tilde{G} = \{\varepsilon^2 B_1 + \varepsilon^4 [B_2 - B_1(\partial B_1/\partial L)] + O(\varepsilon^5)\}_{L=h}, \tag{7.27}$$

where the irrelevant constant has been dropped, and where

$$B_1 = (H_4 \circ \rho_\alpha) = \tfrac{1}{12}[14L\xi^2 - 7\xi^4 - 7\xi^2\eta^2 - 5L^2], \qquad B_2 = (H_6 \circ \rho_\alpha). \tag{7.28}$$

Note that at $L = h$ the B_1 term is degenerate at the origin, reflecting the associated degeneracy of K_1 at NP. To circumvent this difficulty we scale $\xi \to \varepsilon\xi$ in the equations associated with (7.27). Since the Hopf variables $(W_j \circ \rho_\alpha)$ are given by (6.15) with $\beta = \alpha = 1$ and $(W_4 \circ \rho_\alpha) = 2L$, it is easy to compute the effect of this scaling; in fact we obtain an equivalent Hamiltonian system with Hamiltonian $G(\xi, \eta; \lambda) = \varepsilon^{-1} \tilde{G}(\varepsilon\xi, \eta; \lambda)$, i.e.,

$$G = \varepsilon^3 [\tfrac{7}{6} h\xi^2 + \tfrac{7}{2} h^2\eta^2 - \tfrac{14}{3} h\eta^4 - \tfrac{7}{12}\xi^2\eta^2 + \tfrac{14}{9}\eta^6] + O(\varepsilon^4). \tag{7.29}$$

For the periodic orbit assocítated to the critical point of K_ε at SP, we again rotate SP to NP by (4.29b) at $t = \pi$, which changes W_1 and W_3 to their negatives in (7.12b). Performing the above calculations in this case and again stretching variables $\xi \to \varepsilon\xi$, the analog of (7.29) becomes

$$G^* = \varepsilon^3[\tfrac{7}{6}h\xi^2 - \tfrac{7}{2}h^2\eta^2 + \tfrac{14}{3}h\eta^4 - \tfrac{7}{12}\xi^2\eta^2 - \tfrac{14}{9}\eta^6] + O(\varepsilon^4). \quad (7.30)$$

We do not appeal directly to Theorem 6.4(a) and (b) due to the additional scaling $\xi \to \varepsilon\xi$, although with appropriate arguments this could be done. However, by Theorem 5.6(a) and (b) we obtain one family each of elliptic (from (7.29)) and hyperbolic (from (7.30)) periodic orbits at low positive energies for the Hénon–Heiles Hamiltonian (7.7), respectively associated to the nondegenerate critical points NP and SP of K_ε on M_R.

For the elliptic (NP) case, the substitution $\xi \to (B/A)^{1/4} \cdot \xi$ and $\eta \to (A/B)^{1/4} \cdot \eta$ with $A = \frac{7}{3}h$ and $B = 7h^2$ in (7.29) gives a new Hamiltonian of the form (5.13) with E in the form (5.17) given by

$$E(u, \bar{u}) = \tfrac{7}{2}(h^3/3)^{1/2}|u|^2 + \tfrac{1}{16}[-\tfrac{35}{36}u^4 + \tfrac{56}{9}u^3\bar{u} - \tfrac{21}{2}u^2\bar{u}^2 + \tfrac{56}{9}u\bar{u}^3 - \tfrac{35}{36}u^4] + O_5(u, \bar{u}), \quad (7.31)$$

where $u = \xi + i\eta$. In terms of (5.19) we have $E_3 = E_4 = 0$ and $E_9 = -\frac{21}{32}$; hence for small $\varepsilon > 0$ Theorem 5.6(c) shows that the above elliptic orbits associated with NP are stable for (7.7) (the flow of (7.31) is conjugate to the flow of (7.7) via the above series of "stretching" transformations).

We now concentrate on the placement of these continued orbits in Hill's region for (7.7). First note that e_1 and h_4 are the only points in Fig. 7 fixed by both $\phi(R_1)$ and $\phi(R_2)$ of (4.39). These points correspond to the normal modes of H_2 as discussed in Remark d following Theorem 4.4. All the points $\{e_1, e_2, e_3, h_4, h_5, h_6\}$ are fixed by $\phi(R_1)$, hence correspond to time-symmetric periodic orbits of (7.7) whose x-plane projections touch the level curve $V = \varepsilon^2 h$ twice (*brake orbits* in the sense of [39]). We also have $\phi(R_2)\, h_5 = h_6$ and $\phi(R_2)\, e_2 = e_3$.

Recalling that our coordinates are $(x_1, x_2, y_1, y_2) \cong (z_1, z_2) = z$ with $z_j = x_j + iy_j$, for

$$Z_1 = (\sqrt{2h}, 0, 0, 0), \qquad Z_2 = (0, \sqrt{2h}, 0, 0), \qquad Z_3 = (0, 0, 0, \sqrt{2h}), \quad (7.32)$$

we have

$$NP = \pi(Z_1), \qquad SP = \pi(Z_2) = \pi(Z_3), \quad (7.33)$$

where $\{Z_1, Z_2\}$ are fixed by R_1, and $\{Z_1, Z_3\}$ are fixed by R_2. The continuations to (7.7) at energies $\varepsilon^2 h$ of the periodic orbits for $(H_2 + \varepsilon^2 H_4 + \varepsilon^4 H_6)$ through these points are then symmetric wrt the corresponding action

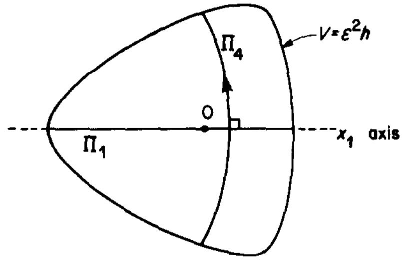

FIGURE 8

R_1 or R_2 by Theorem 6.4(d). By Remark a following Theorem 6.4, we may assume initial conditions

$$\begin{aligned} Z_1(\varepsilon) &= (a(\varepsilon), 0, 0, 0), \\ Z_2(\varepsilon) &= (b_1(\varepsilon), b_2(\varepsilon), 0, 0), \\ Z_3(\varepsilon) &= (c_1(\varepsilon), 0, 0, c_2(\varepsilon)), \end{aligned} \tag{7.34}$$

for our periodic orbits with $\{Z_1(\varepsilon), Z_2(\varepsilon)\}$ fixed by R_1, and $\{Z_2(\varepsilon), Z_3(\varepsilon)\}$ fixed by R_2. Since we have stretched variables, the entries of $Z_j(\varepsilon)$ are $O(\varepsilon)$ and $\varepsilon^{-1} \cdot Z_j(\varepsilon) \to Z_j$ as $\varepsilon \downarrow 0$. Note that the x_2 component of $Z_1(\varepsilon)$ has been set equal to zero since Z_1 is fixed by both R_1 and R_2.

Thus for (7.7) at energies $\varepsilon^2 h$ the x-plane projection of the elliptic stable periodic orbit Π_1 associated to e_1 and the hyperbolic orbit Π_4 associated to h_4 must appear as in Fig. 8 (where the projections are also denoted by Π_1 and Π_4). Now Π_1 projects to a gradient line of potential (7.8), and Π_4 projects to a brake orbit perpendicular to the x_1 axis. In the Appendix to [8, Sect. 10] it is shown that $c_1(\varepsilon) > 0$ for $\varepsilon > 0$ and that Π_4, when traversed with $\dot{x}_2 = y_2 > 0$, has positive curvature as a curve in the x plane (except at its endpoints on $V = \varepsilon^2 h$ where the curvature is undefined); this explains the "bow" shape of Π_4.

Using the symmetries g and g^2 of (7.11), we see that at energies $\varepsilon^2 h$ with $\varepsilon > 0$ small (7.7) admits elliptic stable periodic orbits Π_2 and Π_3, respec-

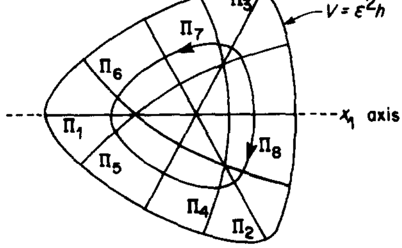

FIGURE 9

tively associated to e_2 and e_3, and hyperbolic periodic orbits Π_5 and Π_6, respectively associated to h_5 and h_6. Using the rotations Ω and Ω^2 of (7.10), the x-plane projections of these periodic orbits are sketched in Fig. 9. The periodic orbits Π_j, $j = 1, 2,..., 8$, can thus be identified with the periodic orbits for (7.7) that were geometrically constructed in [8, Sect. 3], except that the labelling for Π_5 and Π_6 has been interchanged.

The Hamiltonians H_B and H_C of (7.20) and (7.21) also have "normal mode" periodic orbits analogous to Π_1 and Π_4 in Fig. 8, but of different stability types. In fact, for H_B both orbits are hyperbolic at low positive energies (see [11, 21]). This can be shown using first order averaging, or, in the case of the continuation of the α normal mode in the z_1 plane, by a direct computation of the eigenvalues of the linearized Poincaré mapping along the orbit (see also [8, Sect. 8]). For H_C the two analogs of Π_1 and Π_4 are elliptic (stable) at low positive energies [10], and thus H_A, H_B, and H_C provide examples of all three possible types of stability behavior for the continuation of normal modes in two degree of freedom Hamiltonians. We should note here that generically Hamiltonians of form (6.31) will admit two elliptic periodic orbits at low positive energies [3].

For information on the continuation of the periodic orbits of (7.7) to higher energies we refer to [8, Appendix B].

APPENDIX A: THE NORMAL FORM COMPUTATIONS

We outline the computations involved in converting the Hénon–Heiles Hamiltonian (7.7) to normal form through terms of order 6. However, it will be useful to begin at a more general level, assuming as in Section 1 a positively graded Lie algebra $\mathscr{L} = \bigoplus_{r=2}^{\infty} \mathscr{L}_r$ with an element

$$H = \bigoplus_{r=2}^{\infty} H_r \tag{A1}$$

in $\mathscr{L}$ having the property that H_2 splits. As in Section 1 we subscript $H_r \in \mathscr{L}_r$, but now we write $H_r = \hat{H}_r + \tilde{H}_r$, where $\hat{H}_r \in N_r$ and $\tilde{H}_r \in R_r$. It will also be notationally convenient to replace $ad_K^j(G)$ by $[K^j, G]$, but we warn the reader that this notation does not respect the Jacobi identity. We also indicate $[K^j, G] \in \mathscr{L}_r$ by subscripting $[K^j, G]_r$.

If (A1) is in normal form wrt H_2 through terms of order $(m-1) \geqslant 2$, where $m \leqslant 6$, and

$$K_m = \Gamma \tilde{H}_m = (ad_{H_2} \mid R_m)^{-1} (\tilde{H}_m), \tag{A2}$$

then elaborating on the proof of Proposition 1.1 we obtain

$$
\begin{aligned}
&\exp(ad_{K_m})(H)\\
&= H_2 \qquad + H_3 \qquad + H_4 \qquad + H_5 \qquad + H_6\\
&\quad + [K_m, H_2]_m \qquad + [K_m, H_3]_{m+1} \qquad + [K_m, H_4]_{m+2} + [K_m, H_5]_{m+3} +\\
&\quad + \tfrac{1}{2}[K_m^2, H_2]_{2m-2} + \tfrac{1}{2}[K_m^2, H_3]_{2m-1} + \tfrac{1}{2}[K_m^2, H_4]_{2m}\\
&\quad + \tfrac{1}{6}[K_m^3, H_2]_{3m-4} + \tfrac{1}{6}[K_m^3, H_3]_{3m-3}\\
&\quad + \tfrac{1}{24}[K_m^4, H_2]_{4m-6}\\
&\quad + \{\text{terms in } \mathscr{L}_j, j \geqslant 7\}.
\end{aligned}
$$

However, by (A2) we have $[K_m, H_2]_m = -\tilde{H}_m$, and a simple induction argument gives $[K_m^j, H_2] = -[K_m^{j-1}, \tilde{H}_m]$. This simplifies the formula to

$$
\begin{aligned}
&\exp(ad_{K_m})(H)\\
&= H_2 \qquad + H_3 \qquad + H_4 \qquad + H_5 \qquad + H_6\\
&\quad - \tilde{H}_m \qquad + [K_m, H_3]_{m+1} \qquad + [K_m, H_4]_{m+2} + [K_m, H_5]_{m+3}\\
&\quad - \tfrac{1}{2}[K_m, \tilde{H}_m]_{2m-2} + \tfrac{1}{2}[K_m^2, H_3]_{2m-1} + \tfrac{1}{2}[K_m^2, H_4]_{2m}\\
&\quad - \tfrac{1}{6}[K_m^2, \tilde{H}_m]_{3m-4} + 2\tfrac{1}{6}[K_m^3, H_3]_{3m-3}\\
&\quad - \tfrac{1}{24}[K_m^3, \tilde{H}_m]_{4m-6}\\
&\quad + \{\text{terms in } \mathscr{L}_j, j \geqslant 7\}.
\end{aligned}
\tag{A3}
$$

Applying (A2) and (A3) to (A1) in the case $m = 3$, we conclude that the normal form of (A1) wrt H_2, through terms of order three (with all terms through sixth order included) is

$$H^{(3)} = \exp(ad_{K_3})(H) = H_2 + \hat{H}_3 + H_4^{(3)} + H_5^{(3)} + H_6^{(3)} + \cdots,$$

where

$$
\begin{aligned}
H_4^{(3)} &= H_4 + \tfrac{1}{2}[K_3, \tilde{H}_3]_4 + [K_3\ \hat{H}_3]_4\\
H_5^{(3)} &= H_5 + [K_3, H_4]_5 + \tfrac{1}{2}[K_3^2, \hat{H}_3]_5 + \tfrac{1}{3}[K_3^2, \tilde{H}_3]_5\\
H_6^{(3)} &= H_6 + [K_3, H_5]_6 + \tfrac{1}{2}[K_3^2, H_4]_6 + \tfrac{1}{6}[K_3^2, \hat{H}_3]_6 + \tfrac{1}{8}[K_3^3, \tilde{H}_3]_6.
\end{aligned}
\tag{A4}
$$

For the Hénon–Heiles Hamiltonian (7.7) this simplifies considerably. Indeed, in that case we have

$$H_3 = \tilde{H}_3, \qquad \hat{H}_3 = 0, \qquad H_j = 0, \qquad j \geqslant 4, \qquad K_3 = \Gamma H_3, \tag{A.5}$$

and with these assumptions (A4) collapses to

$$H^{(3)} = \exp(ad_{K_3})(H) = H_2 + H_4^{(3)} + H_5^{(3)} + H_6^{(3)} + \cdots,$$

where

$$\begin{aligned} H_4^{(3)} &= \tfrac{1}{2}[K_3, \tilde{H}_3]_4 \\ H_5^{(3)} &= \tfrac{1}{3}[K_3^2, \tilde{H}_3]_5 \\ H_6^{(3)} &= \tfrac{1}{8}[K_3^3, \tilde{H}_3]_6. \end{aligned} \tag{A6}$$

We next apply (A2) and (A3) to (A6) in the case $m = 4$, hence

$$K_4 = \Gamma \tilde{H}_4^{(3)} = \tfrac{1}{2}\Gamma[K_3, H_3]_4^{\sim}. \tag{A7}$$

We conclude that the normal form of (A1) wrt H_2 through terms of order four (with all terms through sixth order included) in the case (A5) is

$$H^{(4)} = \exp(ad_{k_4})(H^{(3)}) = H_2 + \hat{H}_4^{(3)} + H_5^{(4)} + H_6^{(4)} + \cdots,$$

where

$$\begin{aligned} \hat{H}_4^{(3)} &= \tfrac{1}{2}[K_3, H_3]_4^{\wedge} \\ H_5^{(4)} &= H_5^{(3)} \\ H_6^{(4)} &= H_6^{(3)} + [K_4, \hat{H}_4^{(3)}] + \tfrac{1}{2}[K_4, \tilde{H}_4^{(3)}]. \end{aligned} \tag{A8}$$

But observe that Lemma 4.1 in the case of 1–1 resonance implies

$$H_5^{(4)} = \tilde{H}_5^{(4)}, \qquad \hat{H}_5^{(4)} = 0. \tag{A9}$$

Thus

$$K_5 = \Gamma \tilde{H}_5^{(4)} = \Gamma H_5^{(4)}, \tag{A10}$$

and applying (A2) and (A3) to (A8) at $m = 5$ gives the normal form of (A1) wrt H_2 through terms of order five in the case (A5) (with all terms through sixth order displayed) as

$$H^{(5)} = \exp(ad_{K_5})(H^{(4)}) = H_2 + \hat{H}_4^{(3)} + H_6^{(5)} + \cdots, \qquad \text{with} \quad H_6^{(5)} = H_6^{(4)}, \tag{A.11}$$

where we have used the equality $H_4^{(4)} = \hat{H}_4^{(3)}$ of (A8). However, using (A8) and (A11) we have

$$H_6^{(5)} = H_6^{(4)} = H_6^{(3)} + [K_4, \hat{H}_4^{(3)}]_6 + \tfrac{1}{2}[K_4, \tilde{H}_4^{(3)}]_6,$$

and we can easily check that $[K_4, \hat{H}_4^{(3)}]_6^{\wedge} = [\hat{K}_4, \hat{H}_4^{(3)}]_6 = 0$. We can therefore summarize our calculations as follows: The normal form we want is given in terms of (A6) by

$$H^{(6)} = H_2 + \hat{H}_4^{(3)} + \hat{H}_6^{(4)} + \cdots, \qquad \text{where} \quad \hat{H}_6^{(4)} = \hat{H}_6^{(3)} + \tfrac{1}{2}[K_4, \tilde{H}_4^{(3)}]_6^{\wedge}. \tag{A12}$$

Notice that although K_5 is needed to arrive at this form, it plays no role in the expressions we want, and will therefore not be computed.

To actually perform the computations introduce

$$z = x_1 + ix_2, \qquad w = y_1 + iy_2 \tag{A13}$$

and their associated derivations

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), & \frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right), \\ \frac{\partial}{\partial w} &= \frac{1}{2}\left(\frac{\partial}{\partial y_1} - i\frac{\partial}{\partial y_2}\right), & \frac{\partial}{\partial \bar{w}} &= \frac{1}{2}\left(\frac{\partial}{\partial y_1} + i\frac{\partial}{\partial y_2}\right). \end{aligned} \tag{A14}$$

In this framework we check that

$$H_2 = \tfrac{1}{2}(|z|^2 + |w|^2), \tag{A15}$$

that

$$[f, g] = 2\left\{\frac{\partial f}{\partial \bar{w}} \cdot \frac{\partial g}{\partial z} + \frac{\partial f}{\partial w} \cdot \frac{\partial g}{\partial \bar{z}} - \frac{\partial f}{\partial z} \cdot \frac{\partial g}{\partial \bar{w}} - \frac{\partial f}{\partial \bar{z}} \cdot \frac{\partial g}{\partial w}\right\}, \tag{A16}$$

and hence that

$$ad_{H_2}(g) = \left\{w\frac{\partial g}{\partial z} + \bar{w}\frac{\partial g}{\partial \bar{z}} - \bar{z}\frac{\partial g}{\partial \bar{w}} - z\frac{\partial g}{\partial w}\right\}. \tag{A17}$$

Also, notice that the Hopf variables (4.2) become

$$\begin{aligned} W_1 &= \mathrm{Im}(z^2) + \mathrm{Im}(w^2), & W_2 &= 2\,\mathrm{Im}(z\bar{w}), \\ W_3 &= \mathrm{Re}(z^2) + \mathrm{Re}(w^2), & W_4 &= |z|^2 + |w|^2. \end{aligned} \tag{A18}$$

Now write the Hénon–Heiles Hamiltonian (7.7) as

$$H = \tfrac{1}{2}W_4 + \mathrm{Re}(f), \qquad f = \tfrac{1}{3}z^3, \tag{A19}$$

and use (A17) to check that

$$K_3 = -\tfrac{1}{3}\mathrm{Re}(g), \qquad g = (z^2 w + \tfrac{2}{3}w^3). \tag{A20}$$

A straightforward computation then gives the terms of (A6), decomposed as needed in (A12), as

$$\begin{aligned}
\hat{H}_4^{(3)} &= \tfrac{1}{48}(7W_2^2 - 5W_4^2),\\
\tilde{H}_4^{(3)} &= -\tfrac{1}{6}\{\tfrac{3}{8}|z|^4 - \tfrac{5}{8}|w|^4 + \tfrac{1}{2}|z|^2|w|^2 + \tfrac{1}{8}((z\bar{w})^2 + (\bar{z}w)^2)\},\\
H_5^{(3)} &= \tfrac{4}{27}\operatorname{Re}\{(z^2+2w^2)(z^2+w^2)\bar{z} - 2z^3|w|^2\} = \tilde{H}_5^{(3)},\\
\hat{H}_6^{(3)} &= -\tfrac{7}{9}\operatorname{Re}(W_4 + iW_1)^3 - \tfrac{1}{216}W_4^3 - \tfrac{1}{72}W_1^2W_4.
\end{aligned} \tag{A21}$$

Next, using (A7) we can check that

$$K_4 = \tfrac{1}{6}\operatorname{Re}(k), \qquad k = (\tfrac{3}{8}\bar{z}w|z|^2 + \tfrac{5}{8}z\bar{w}|w|^2). \tag{A22}$$

From (A21) and (A22) we then compute (A12), arriving at

$$\begin{aligned}
\hat{H}_4^{(3)} &= \tfrac{1}{48}(7W_2^2 - 5W_4^2),\\
\hat{H}_6^{(4)} &= \tfrac{1}{64}(-\tfrac{67}{54}W_4^3 - \tfrac{28}{9}W_3^3 + \tfrac{28}{3}W_1^2W_3 - \tfrac{7}{18}W_2^2W_4).
\end{aligned} \tag{A23}$$

APPENDIX B: Remarks on the Hénon–Heiles Hamiltonian

The periodic orbits $\Pi_1,\dots,\Pi_8$ of the Hénon–Heiles Hamiltonian (7.7) sketched in Figs. 6 and 9 can be shown to exist at all energies $0 < h < \frac{1}{6}$ by simple geometrical arguments [8, Sect. 3]; the present paper has shown their stability status at low energies. In Appendix B we wish to tie in these facts with other known results, both analytic and numerical.

In their paper [20] introducing (7.7), Hénon and Heiles used the intersection of the (x_1, y_1) plane with the energy manifold $H = \varepsilon^2 h$ as a cross section of the flow and numerically integrated along orbits to give the Poincaré map on this cross section. The pictures they obtained in [20] have been well publicized (e.g. [29, pp. 16–19; [2, p. 92; 1, pp. 611–612]), and at low energies can be explained in terms of the results of this paper. Indeed, at any sufficiently low positive energy they found Fig. 10 in the (x_1, y_1) plane (which we have adjusted to our coordinate system), which has Π_1 as boundary. We note four elliptic points and three hyperbolic points, and with Π_1 added this gives precisely the eight periodic orbits found in Section 7.B and 7.C as we have shown by the use of symmetries.

As the energy is increased, the simple picture in Fig. 10 disintegrates, and single orbits repeatedly intersect the (x_1, y_1) plane so as to cover nearly the entire picture. This is no doubt caused initially by the transversal intersection of the stable and unstable manifolds of the hyperbolic periodic orbits Π_4, Π_5, and Π_6, but this fact awaits proof (see [24] for numerical verification of this phenomenon). What has been established rigorously is that Π_1, Π_2, and

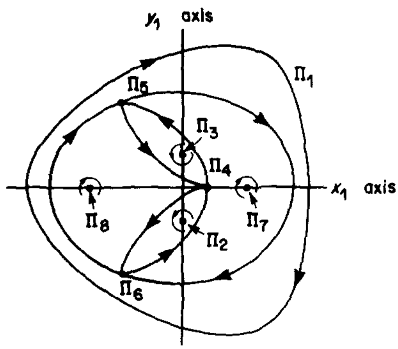

FIGURE 10

Π_3 pass through infinitely many transitions between ellipticity and hyperbolicity as $h \uparrow \frac{1}{6}$ [9], and limit to orbits homoclinic to equilibrium points at $h = \frac{1}{6}$. (The techniques of [9, Sect. 5] sometimes apply for a stability analysis as $h \downarrow 0$, but they failed to prove ellipticity of Π_1 for small positive energies due to a degeneracy similar to the one encountered at NP for first order averaging in Section 7.B. Numerical evidence of [24] (also private communication of John Greene) suggests that Π_7 and Π_8 switch to hyperbolic status much below energy $\frac{1}{6}$. Similar phenomena appear in the Hamiltonians H_B and H_C of (7.20) and (7.21).

For $h > \frac{1}{6}$ the periodic orbits Π_1, Π_2, and Π_3 do not exist, but are replaced by orbits escaping to infinity. However, in [8] it was shown that Π_7 and Π_8 exist at all energies $h > 0$. More recently [24] has observed that Π_4, Π_5, and Π_6 also exist at all energies $h > 0$, and it is conjectured from numerical evidence of [24] that they remain hyperbolic. We now show the existence of Π_4, Π_5, Π_6 at energies $h > \frac{1}{6}$, recalling that [8] has already established their existence for $0 < h \leqslant \frac{1}{6}$. Using symmetriy it is enough to work with Π_6.

At energies $h > \frac{1}{6}$ the level curve $V = h$ has three branches arranged symmetrically as in a monkey saddle (see Fig. 11). The projected periodic orbits Π_j, $j = 1, 2, 3$, at energies $0 < h < \frac{1}{6}$ lie along gradient lines G_j of the potential V, where G_1 is the x_1 axis and G_2 intersects $V = h$ at p. (These gradient lines are the projections of orbits with energy $h > \frac{1}{6}$.) In [7, 34] it is shown that there is a hyperbolic periodic orbit Π whose x-plane projection lies to the left of G_2 and G_3 and intersects two branches of $V = h$ at r and r' as in Fig. 11. Let M be the open region in Fig. 11 bounded by these two branches of $V = h$, Π, G_2, and G_3. Now consider orbits (i.e., solutions $x(t)$ of $\ddot{x} = -V_x$, where $x = (x_1, x_2)$) dropping from that branch of $V = h$ above the x_1 axis from points in the open interval (r, p) on $V = h$. Such orbits start out with zero velocity on $V = h$, and in [7, Sect. 5] it is shown that they leave the region M by crossing G_2 or G_3, never crossing Π in this time interval. We claim that after leaving $V = h$ such orbits have *first* intersection

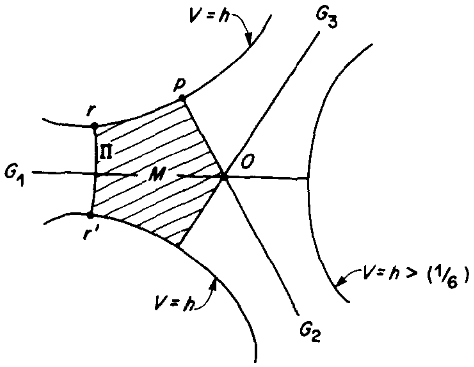

FIGURE 11

with the x_1 axis at points q' with $x_1(q') < 0$. The argument is similar to the one in the appendix of [8, Sect. 10], and so will only be sketched.

First recall that orbits dropping from $V = h$ are time reversible, and so if our claim is false, then some such orbit as in Fig. 12 can be regarded as entering the region M at a point s' on G_2 with $x_2(s') \geqslant 0$ as shown, and then rising up to a point s on $V = h$ in the interval (r, p). We first assume $x_2(s') > 0$. But then with

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

the curvature $k(t) = \langle V_x, J\dot{x}\rangle\, |\dot{x}|^{-3}$ (see [7, Sect. 2]) of the orbit along this segment must be positive at s'. By standard arguments in [7] this curvature must be negative when the orbit crosses the equilateral triangle defined by the level curves $V = \frac{1}{6}$, since otherwise the velocity vector $\dot{x} = y$ could not rotate clockwise in the region above $V = \frac{1}{6}$ so as to allow the orbit to meet

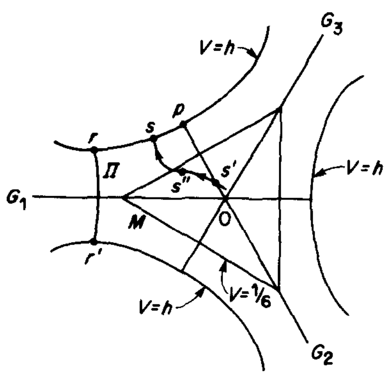

FIGURE 12

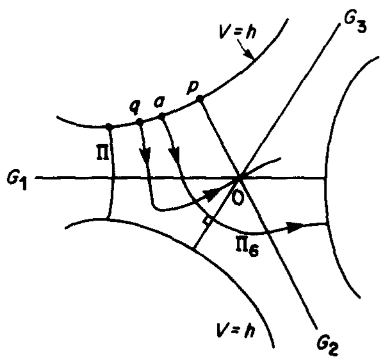

FIGURE 13

$V = h$ orthogonally at s. It follows that as the orbit rises from s' there must be a first point s'' below $V = \frac{1}{6}$ at which the curvature passes from positive to negative (or zero) values. But using [8, Formula (10.3), p. 129], which in this region gives an inequality >0, we can show that at s'' the curvature must pass from negative to positive values as the orbit rises, hence we have a contradiction. If the orbit rises from the origin to a point in the interval (r, p) on $V = h$, then some nearby orbit through a point s' on G_2 with $x_2(s') > 0$ must come near $V = h$ and go through the above curvature changes, and hence this case is also impossible (see [8, Sect. 10] for a similar argument). Thus our claim has been established.

Since Π is periodic, orbits dropping from points in (r, p) on $V = h$ near r cross the x_1 axis many times before leaving the region M. Hence there is a maximal open interval (q, p) of points on $V = h$ (where q is to the right of r in Fig. 13) having the property that orbits dropped from this interval cross G_1 to the left of the origin only once before leaving the region M. Then the orbit falling from q must pass through the origin as in Fig. 13. Measuring angles counterclockwise from G_3, note that the orbit along G_2 dropping from p has first intersection with G_3 in an acute angle, whereas that from q results in an obtuse angle. By continuity in initial conditions, there is an orbit starting at a point a on $V = h$ between q and p that intersects G_3 in a right angle, and by symmetry about G_3 this must be Π_6. This argument is valid at all energies $h > \frac{1}{6}$.

For a more complete discussion of the Hénon–Heiles Hamiltonian and comparisons with the Hamiltonians H_B and H_C of (7.20) and (7.21) see [8].

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