Gerolamo Saccheri: Euclide vendicato da ogni neo

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With the rediscovery of the *Euclides Vindicatus*, thanks to Eugenio Beltrami in 1889 (who followed the suggestion of the Jesuit Manganotti), the relevance of the work of Saccheri was firmly established. A few years later, Giovanni Vailati, by pointing out in 1903 the importance of another work of Saccheri, the *Logica demonstrativa*, and its connections with the *Euclides Vindicatus*, laid the basis for the subsequent epistemological investigations of Saccheri's texts.

Interest in Gerolamo Saccheri's work in recent times, at least in the Italian context, has grown considerably. It reaches its summit with this superb edition of Saccheri's *Euclides ab omni naevo vindicatus* by Vincenzo De Risi. Scholars are now given, in two separate volumes, an anastatic reprint of the *Euclides* and a careful Italian translation of the text (achieved by means of explicit, clear and well-founded criteria), preceded by a fine *Introduction* and enriched by extensive notes which follow in detail every proposition of the text.

The notes by De Risi constitute, indeed, a remarkable enriching of the text of Saccheri. They show both what Saccheri inherits (and criticizes) from Clavius, from other Jesuit authors, from Borelli, from the Galilean school and generally from the interpretative tradition of Euclid. De Risi does not hesitate to trace the path that leads to Hilbert's *Grundlagen* (by avoiding every anachronism, of course). These notes, as a whole, constitute a nice conceptual history of non-Euclidean geometry.

The idea of 'nævi' that disfigure the beautiful body of Euclid's *Elements* dates back to Henry Saville (1621) (see the *Introduzione*, p. X) and is part of the rich seventeenth-century discussions related to the Parallel Postulate and to the definitions of Ratio and Proportion (see the Sections 2 and 3 of the *Introduzione*). When Saccheri published his work he was surely behind his times (even if interest in classical mathematics has always had a certain continuity) and this, together with the complex structure of his masterpiece, have led to many simplistic interpretations.

Sometimes the will of vindicating Euclid has been simply juxtaposed to the idea of Saccheri as a 'forerunner of non-Euclidean geometry'. Sometimes he was judged a sort of 'romantic hero' who, after having intuited the possibility of non-Euclidean geometries, was compelled (by whom?) to conceal his discoveries.

Section 5 of the *Introduzione. Scopi e struttura epistemologica dell'*Euclide Vendicato (*Purposes and epistemological structure of the* Euclide Vendicato), on the contrary, offers a serious and well documented investigation of the structure of the work. It clarifies very well the meaning by which a long and well argued geometrical analysis of the Parallel Postulate may coexist with the concept of axiom and how Saccheri's work is really Euclid vindicated, not amended or improved. See particularly the discussion on pages XLIII, XLIV of the *Introduzione*, where the necessity of keeping the distance from "the modern mathematical epistemology, which definitely opposes axioms and theorems" is well explained.

As is well known, Saccheri begins his treatise with the analysis of a simple figure, a quadrilateral with two equal sides perpendicular to the base (Saccheri's quadrilateral), but instead of searching for an immediate contradiction he examines the various possibilities connected to the other two (proved equal) angles of the quadrilateral, by distinguishing three hypotheses, the one of the right angles, the one of the obtuse angles and that of the acute angles. By these hypotheses he distinguishes the three possible geometries (see the Notes on Proposition 3 and 4, pp. 123–126).

The leading idea may well be the one of proving the Parallel Postulate, but it is difficult to deny in Saccheri's work the existence of places where a genuine interest in the three possible geometries is clear. De Risi observes for example that "the Propositions 18 and 19, as the preceding ones 15 and 16 are proved by Saccheri owing to the simplicity and the interest that these results of elementary geometry hold in comparison with the three hypotheses" (p. 138). Actually, Proposition 18, for example, considers a triangle ABC of which the angle at the point B is inscribed in a semicircle of diameter AC. The angle at the point B is proved to be right, obtuse or acute according to the three possible geometries. De Risi rightly observes that "Saccheri discusses these theorems also in the hypotheses of the obtuse angle, that he had already refuted; and besides [. . .] these Propositions 18 and 19 will not find any application in what follows" (pp. 138–39). It is difficult to deny some form of interest in geometrical situations that in the strict view of Saccheri should not exist.

A major interpretative difficulty of the *Euclides Vindicatus* is surely given by Proposition 33, which closes the First Part of Book One, where the impossibility of the hyperbolic geometry should follow from the 'absolute falsity of the hypothesis of the acute angle'. Here, by means of five long Lemmas, Saccheri "carries out a remarkable foundational effort in order to prove and justify the main axioms, related to the straight line, assumed by Euclid" (p. 160). Asymptotic straight lines having a common point at infinity should not exist, because this "is opposed to the nature of the straight line" (p. 161).

De Risi rightly observes that the nature of the error is more metaphysical than mathematical and "that there are no relevant geometrical errors in the first 32 Propositions of the *Euclides Vindicatus*" (p. 161). And consequently, the problem of the 'real meaning' of Saccheri's work surfaces once again.

But the problem becomes even more serious in what follows. In the Second Part of Book One Saccheri wants to prove the impossibility of hyperbolic geometry by using the 'consequentia mirabilis' and by means of the consideration of equidistant lines. And Proposition 37 contains "the fundamental mistake, and may be the single one, of the Euclides Vindicatus" (p. 174). It consists, as De Risi carefully explains in the Notes to this Proposition (pp. 174–179), of a clumsy utilisation of the concepts of infinitesimal analysis. Why a distinguished mathematician should attempt to employ a discipline of which he was not a master is another great problem offered by Saccheri's work.

To conclude this review it may be observed that the part of Saccheri's work devoted to proportions surely has a minor interest. But it is not negligible, and it is a further merit of De Risi to have offered also for this part of Saccheri's work a pertinent and considerable commentary. In the eighteenth century, proportion theory was no longer a vital part of mathematics, but the work of Saccheri, which inherits discussions from the whole of the seventeenth century, is not void of interest.

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