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Approximate solution for a class of hypersingular integral equations

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Abstract

A simple approximate method for solving a general hypersingular integral equation of the first kind with its kernel consisting of a hypersingular part and a regular part is developed here. The method is illustrated by considering some simple examples.

Keywords: Integral equations; Hypersingular kernels; Approximate solution

1. Introduction

A general hypersingular integral equation of the first kind, over a finite interval, can be represented by

$$\int\!\!\!\!\!\!\!\!\times_{-1}^{1} \phi(t)\left[\frac{k(t,x)}{(t-x)^2}+L(t,x)\right]\mathrm{d}t = f(x), \quad -1 \le x \le 1, \tag{1.1}$$

with $\phi(\pm 1) = 0$, where $K(t,x)$ and $L(t,x)$ are regular square-integrable functions of t and x, and $K(x,x) \neq 0$. In (1.1), the hypersingular integral denoted by $\int\!\!\!\!\!\!\times_{-1}^{1} \frac{\phi_1(t)}{(t-x)^2}\mathrm{d}t$, $(-1 \le x \le 1)$ where $\phi_1(t) = \phi(t)K(t,.)$ is defined as

$$\int\!\!\!\!\!\!\!\!\times_{-1}^{1} \frac{\phi_1(t)}{(t-x)^2}\mathrm{d}t = \lim_{\epsilon\to 0^+}\left[\int_{-1}^{x-\varepsilon}\frac{\phi_1(t)}{(t-x)^2}\mathrm{d}t + \int_{x+\varepsilon}^{1}\frac{\phi_1(t)}{(t-x)^2}\mathrm{d}t - \frac{\phi_1(x+\varepsilon)+\phi_1(x-\varepsilon)}{\varepsilon}\right], \quad -1 \le x \le 1, \tag{1.2}$$

and is understood in the sense of the Hadamard finite part. The solution of the *simple* hypersingular integral equation

$$\int\!\!\!\!\!\!\!\!\times_{-1}^{1} \frac{\phi(t)}{(t-x)^2}\mathrm{d}t = f(x), \quad -1 \le x \le 1, \tag{1.3}$$

with $\phi(\pm 1) = 0$, is well known (cf. [1,2]) and is given by

$$\phi(t) = \frac{1}{\pi^2}\int_{-1}^{1} f(x)\ln\left|\frac{(x-t)}{1-xt+\{(1-x^2)(1-t^2)\}^{\frac{1}{2}}}\right|\mathrm{d}x, \quad -1 \le t \le 1. \tag{1.4}$$

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A somewhat less general form of a first-kind hypersingular integral equation given by

$$\oint_{-1}^{1} \phi(t)\left[\frac{1}{(t-x)^2} + L(t,x)\right]\mathrm{d}t = f(x), \quad -1 \le x \le 1, \tag{1.5}$$

with $\phi(\pm 1) = 0$, arises in a variety of mixed boundary value problems in mathematical physics such as water wave scattering and radiation problems involving thin submerged plates (cf. [3–9]), and fracture mechanics [10]. Eq. (1.5) is usually solved approximately by an expansion–collocation method, the expansion being in terms of a finite series involving Chebyshev polynomials $U_i(t)$ of the second kind. In particular, $\phi(t)$ in (1.5) is approximated as

$$\phi(t) \approx (1-t^2)^{\frac{1}{2}} \sum_{i=0}^{n} a_i U_i(t) \tag{1.6}$$

where a_i $(i = 0, 1, \ldots, n)$ are unknown constants. Substitution of (1.6) in (1.5) produces (cf. [3])

$$\sum_{i=0}^{n} a_i A_i(x) = f(x), \quad -1 \le x \le 1 \tag{1.7}$$

where

$$A_i(x) = -\pi(i+1)U_i(x) + \int_{-1}^{1} (1-t^2)^{\frac{1}{2}} L(t,x) U_i(t)\mathrm{d}t. \tag{1.8}$$

To find the unknown constants a_i $(i = 0, 1, \ldots, n)$, we put $x = x_j$ $(j = 0, 1, \ldots, n)$ where the x_j's are suitable collocation points such that $-1 \le x_j \le 1$. This produces the linear systems

$$\sum_{i=0}^{n} a_i A_{ij} = f_j, \quad j = 0, 1, \ldots, n \tag{1.9}$$

with $A_{ij} = A_i(x_j)$ and $f_j = f(x_j)$. These can be solved by standard methods. The collocation points are usually chosen to be the zeros of $U_{n+1}(x)$ or $T_{n+1}(x)$ (Chebyshev polynomials of the first kind). The method described above becomes somewhat unsuitable for solving the general hypersingular integral equation (1.1) due to the presence of the factor $K(t,x)$ with $(t-x)^{-2}$. Here we develop a modified method for approximately solving Eq. (1.1). This method stems from recent work of Chakrabarti and Vanden Berghe [11] wherein an approximate method has been developed to solve a general type of first-kind singular integral equation with a Cauchy-type kernel, given by

$$⨍_{-1}^{1} \phi(t)\left[\frac{K(t,x)}{(t-x)} + L(t,x)\right]\mathrm{d}t = f(x), \quad -1 < x < 1, \tag{1.10}$$

$\phi(t)$ satisfying appropriate conditions at the end points, and the integral involving $\frac{1}{t-x}$ is in the sense of the Cauchy principal value $(K(x,x) \ne 0)$. The approximate method developed below appears to be quite appropriate for solving the most general type of first-kind hypersingular integral equation (1.1) assuming of course that $K(t,x)$ and $L(t,x)$ can be approximated as in [11]. Some simple examples are given to illustrate the method.

2. Method of solution

The unknown function $\phi(x)$ satisfying $\phi(\pm 1) = 0$ can be represented in the form

$$\phi(x) = (1-x^2)^{\frac{1}{2}} \psi(x), \quad -1 \le x \le 1 \tag{2.1}$$

where $\psi(x)$ is a well-behaved unknown function of $x \epsilon [-1, 1]$. Approximating $\psi(x)$ by means of a polynomial of degree n, given by

$$\psi(x) \approx \sum_{j=0}^{n} c_j x^j \tag{2.2}$$

where c_j's $(j = 0, 1, \ldots, n)$ are unknown constants, the original integral equation (1.1) produces

$$\sum_{j=0}^{n} c_j \left[\int\!\!\!\!\!\!= \frac{(1-t^2)^{\frac{1}{2}} K(t,x) t^j}{(t-x)^2} \mathrm{d}t + \int_{-1}^{1} (1-t^2)^{\frac{1}{2}} L(t,x) t^j \,\mathrm{d}t \right] = f(x), \quad -1 \le x \le 1. \tag{2.3}$$

The functions $K(t, x)$ and $L(t, x)$ can be approximated as (for fixed x; cf. [11])

$$K(t,x) \approx \sum_{p=0}^{m} K_p(x) t^p, \quad L(t,x) \approx \sum_{q=0}^{s} L_q(x) t^q \tag{2.4}$$

with known expressions for $K_p(x)$ and $L_q(x)$. Then (2.3) gives

$$\sum_{j=0}^{n} c_j \alpha_j(x) = f(x), \quad -1 \le x \le 1 \tag{2.5}$$

where

$$\alpha_j(x) = \sum_{p=0}^{m} K_p(x) u_{p+j}(x) + \sum_{q=0}^{s} \gamma_{q+j} L_q(x) \tag{2.6}$$

with

$$u_{p+j}(x) = \int\!\!\!\!\!\!=_{-1}^{1} \frac{(1-t^2)^{1/2} t^{p+j}}{(t-x)^2} \mathrm{d}t, \quad -1 \le x \le 1, \tag{2.7}$$

$$\gamma_{q+j} = \int_{-1}^{1} (1-t^2)^{\frac{1}{2}} t^{q+j} \,\mathrm{d}t, \tag{2.8}$$

which can be easily evaluated. The unknown constants $c_j (j = 0, 1, \ldots, n)$ are now obtained by putting $x = x_l (l = 0, 1, \ldots, n)$ in (2.5), where $-1 \le x_l \le 1$ and are to be chosen suitably. Thus we obtain a system of $(n + 1)$ linear equations, given by

$$\sum_{j=0}^{n} c_j \alpha_{jl} = f_l, \quad l = 0, 1, \ldots, n \tag{2.9}$$

where

$$\alpha_{jl} = \alpha_j(x_l), \quad f_l = f(x_l), \tag{2.10}$$

for the determination of the $(n+1)$ unknowns c_j $(j = 0, 1, \ldots, n)$. This completes the description of the approximate method for solving (1.1). Below we give some simple examples to illustrate the method.

3. Illustrative examples

Example 1. If we consider $K(t, x) \equiv 1$, $L(t, x) \equiv 0$, then Eq. (1.1) reduces to Eq. (1.3) whose solution is given by (1.4). However, we use the method developed above to obtain the solution for the particular forcing function $f(x) = 1$. For this case, $K_p(x)$ and $L_q(x)$ in (2.4) are given by

$$K_0(x) = 1, \quad K_p(x) = 0 \ (p > 0) \quad \text{and} \quad L_q(x) = 0 \ (q \ge 0). \tag{3.1.1}$$

Hence we find that relation (2.5) produces

$$\sum_{j=0}^{n} c_j u_j(x) = 1, \quad -1 \le x \le 1 \tag{3.1.2}$$

where

$$u_0(x) = -\pi, \quad u_1(x) = -2\pi x, \quad u_2(x) = \pi\left(\frac{1}{2} - 3x^2\right),$$
$$u_3(x) = \pi(x - 4x^3), \quad u_4(x) = \pi\left(\frac{1}{8} + \frac{3}{2}x^2 - 5x^4\right), \ldots. \tag{3.1.3}$$

Substituting (3.1.3) in (3.1.2) and comparing the coefficients on both sides, we obtain

$$c_0 = -\frac{1}{\pi}, \quad c_1 = c_2 = \cdots = 0 \tag{3.1.4}$$

so that

$$\phi(x) = -\frac{1}{\pi}(1 - x^2)^{\frac{1}{2}}$$

which is in fact the exact solution of (1.3) for $f(x) = 1$ obtained by using the relation (1.4).

Examples 2. Next we consider the equation

$$\oint_{-1}^{1} \left[\frac{1}{(t - x)^2} + (t + x)\right]\phi(t)\mathrm{d}t = f(x), \quad -1 \le x \le 1 \tag{3.2.1}$$

with $\phi(\pm 1) = 0$. This corresponds to $K(t, x) \equiv 1$ and $L(t, x) \equiv t + x$. This, however, is also of the form (1.5), and in view of this, we use the method developed in Section 3 and the method described in Section 1 to obtain approximate solutions of (3.2.1) for the purpose of comparison. Now, here, $K_p(x)$ and $L_q(x)$ are given by

$$K_0 = 1, \quad K_p(x) = 0\ (p \ge 1) \quad \text{and} \quad L_0(x) = x, \quad L_1(x) = 1, \quad L_q(x) = 0\ (q \ge 2). \tag{3.2.2}$$

Thus (2.6) gives

$$\alpha_j(x) = u_j(x) + \gamma_j x + \gamma_{j+1}, \quad j = 0, 1, \ldots \tag{3.2.3}$$

where $u_j(x)$ $(j = 0, 1, \ldots)$ are the same as these given in (3.1.3) and

$$\gamma_{2j+1} = 0, \quad \gamma_{2j} = \frac{\pi^{\frac{1}{2}}\Gamma(j + \frac{1}{2})}{2(j + 1)!}, \quad j = 0, 1, \ldots \tag{3.2.4}$$

so that $\alpha_0(x), \alpha_1(x), \ldots$ etc are obtained in closed forms.

For simplicity, if we choose the forcing function $f(x)$ to be of the form $f(x) = b_0 + b_1 x$ where b_0 and b_1 are known constants, then we can determine the unknown constants $c_0, c_1, \ldots$ directly by comparing the coefficients of various powers of x on the two sides of (2.5), as both sides are now polynomials. This produces

$$c_0 = -\frac{2}{31\pi}(16b_0 + b_1), \quad c_1 = -\frac{16}{31\pi}\left(\frac{b_0}{2} + b_1\right), \quad c_j = 0\ (j \ge 2) \tag{3.2.5}$$

and the solution of (3.2.1) in this case is obtained as

$$\phi(x) = (c_0 + c_1 x)(1 - x^2)^{\frac{1}{2}}. \tag{3.2.6}$$

However, if we use the expansion of $\phi(x)$ in terms of Chebyshev polynomials given by (1.6), then in this case the functions A_i $(i = 0, 1, \ldots)$ are obtained as

$$A_0(x) = -\pi + \frac{\pi}{2}x, \quad A_1(x) = \frac{\pi}{4} - 4\pi x, \quad A_2(x) = 3\pi - 16\pi x^2, \quad A_3(x) = 16\pi x^2 - 32\pi x^3, \ldots \tag{3.2.7}$$

and comparing the coefficients of both sides of the relation (1.7) we obtain

$$a_0 = -\frac{2}{31\pi}(16b_0 + b_1), \quad a_1 = -\frac{8}{31\pi}\left(\frac{b_0}{2} + b_1\right), \quad a_i = 0\ (i \ge 2). \tag{3.2.8}$$

Noting that $U_0(x) = 1$ and $U_1 = 2x$, we find from (1.6) that $\phi(x)$ is exactly the same as that given in (3.2.6). It may be noted that the collocation method used to obtain the unknown constants c_i $(i = 0, 1, \ldots)$ in (2.2) and a_i $(i = 0, 1, \ldots)$

in (1.6) for this problem can be used if for simplicity we choose $f(x) = 1 + 2x$ so that $b_0 = 1$ and $b_1 = 2$ above. Choosing $n = 3$ in the expansion (2.2), the unknown constants c_0, c_1, c_2, c_3 are determined from the linear system

$$\sum_{j=0}^{3} c_j \alpha_{jl} = f_l, \quad l = 0, 1, 2, 3. \tag{3.2.9}$$

If we choose the collocation points as $x_0 = -1, x_1 = -\frac{1}{3}, x_2 = \frac{1}{3}, x_3 = 1$, then the linear equation (3.2.9) produces

$$c_0 = -0.3696501, \quad c_1 = -0.4107224, \quad c_2 \approx 0, \quad c_3 \approx 0 \tag{3.2.10}$$

which are almost the same as those given in (3.2.5). Similarly, choosing $n = 3$ in (1.6), we see that the unknown constants a_0, a_1, a_2, a_3 are to be found by solving the linear system

$$\sum_{i=0}^{3} a_i A_{ij} = f_j, \quad j = 0, 1, 2, 3. \tag{3.2.11}$$

Choosing the same set of collocation points, $-1, -\frac{1}{3}, \frac{1}{3}, 1$, we find that the linear (3.2.11) when solved produces

$$a_0 = -0.3696500, \quad a_1 = -0.2053610, \quad a_2 \approx 0, \quad a_3 \approx 0 \tag{3.2.12}$$

which are again almost the same as those given in (3.2.8). It may be noted that on increasing n, the same results as above are obtained for both the methods.

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References

[1] P.A. Martin, Exact solution of a simple hypersingular integral equation, J. Integral. Equations Appl. 4 (1992) 197–204.
[2] A. Chakrabarti, B.N. Mandal, Derivation of the solution of a simple hypersingular integral equation, Int. J. Math. Educ. Sci. Technol. 29 (1998) 47–53.
[3] N.F. Parsons, P.A. Martin, Scattering of water waves by submerged plates using hypersingular integral equations, Appl. Ocean Res. 14 (1992) 313–321.
[4] N.F. Parsons, P.A. Martin, Scattering of water waves by submerged curved plates and by surface-piercing flat plates, Appl. Ocean Res. 16 (1994) 129–139.
[5] P.A. Martin, N.F. Parsons, L. Farina, Interaction of water waves with thin plates, in: B.N. Mandal (Ed.), Advances in Fluid Mechanics: Mathematical Techniques for Water Waves, Computational Mechanics Publications, Southampton, 1997, pp. 197–229.
[6] B.N. Mandal, S. Banerjea, D.P. Dolai, Interface wave diffraction by a thin vertical barrier submerged in the lower fluid, Appl. Ocean Res. 17 (1995) 93–102.
[7] S. Banerjea, D.P. Dolai, B.N. Mandal, On waves due to rolling of a plate submerged in finite depth water, J. Appl. Mech. Engrg. 2 (1996) 285–297.
[8] B.N. Mandal, R. Gayen (Chowdhury), Water wave scattering by two symmetric circular arc shaped thin plates, J. Engrg. Math. 44 (2002) 297–303.
[9] M. Kanoria, B.N. Mandal, Water wave scattering by a submerged circular-arc-shaped plate, Fluid Dynam. 31 (2002) 317–331.
[10] Y.-S. Chan, A.C. Fannjiang, G.H. Paulino, Integral equations with hypersingular kernels—theory and applications to fracture mechanics, Internat. J. Engrg. Sci. 41 (2003) 683–720.
[11] A. Chakrabarti, G. Vanden Berghe, Approximate solution of singular integral equations, Appl. Math. Lett. 17 (2004) 553–559.