Approximate solution for a class of hypersingular integral equations

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Abstract

A simple approximate method for solving a general hypersingular integral equation of the first kind with its kernel consisting of a hypersingular part and a regular part is developed here. The method is illustrated by considering some simple examples.

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1. Introduction

A general hypersingular integral equation of the first kind, over a finite interval, can be represented by

$$\int_{-1}^{1} \phi(t) \left[ \frac{k(t, x)}{(t-x)^2} + L(t, x) \right] dt = f(x), \quad -1 \leq x \leq 1,$$  \hspace{1cm} (1.1)

with \( \phi(\pm 1) = 0 \), where \( K(t, x) \) and \( L(t, x) \) are regular square-integrable functions of \( t \) and \( x \), and \( K(x, x) \neq 0 \). In (1.1), the hypersingular integral denoted by \( \int_{-1}^{1} \frac{\phi_1(t)}{(t-x)^2} dt, \quad (-1 \leq x \leq 1) \) where \( \phi_1(t) = \phi(t)K(t, .) \) is defined as

$$\int_{-1}^{1} \frac{\phi_1(t)}{(t-x)^2} dt = \lim_{\epsilon \to 0^+} \left[ \int_{-1}^{1-x} \frac{\phi_1(t)}{(t-x)^2} dt + \int_{1+x}^{1} \frac{\phi_1(t)}{(t-x)^2} dt - \frac{\phi_1(x+\epsilon) + \phi_1(x-\epsilon)}{\epsilon} \right], \quad -1 \leq x \leq 1,$$ \hspace{1cm} (1.2)

and is understood in the sense of the Hadamard finite part. The solution of the simple hypersingular integral equation

$$\int_{-1}^{1} \frac{\phi(t)}{(t-x)^2} dt = f(x), \quad -1 \leq x \leq 1,$$  \hspace{1cm} (1.3)

with \( \phi(\pm 1) = 0 \), is well known (cf. [1,2]) and is given by

$$\phi(t) = \frac{1}{\pi^2} \int_{-1}^{1} f(x) \ln \left| \frac{(x-t)}{1-xt + [(1-x^2)(1-t^2)]^{1/2}} \right| dx, \quad -1 \leq t \leq 1.$$ \hspace{1cm} (1.4)

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A somewhat less general form of a first-kind hypersingular integral equation given by
\[
\int_{-1}^{1} \phi(t) \left[ \frac{1}{(t-x)^2} + L(t,x) \right] \, dt = f(x), \quad -1 \leq x \leq 1, \tag{1.5}
\]
with \( \phi(\pm1) = 0 \), arises in a variety of mixed boundary value problems in mathematical physics such as water wave scattering and radiation problems involving thin submerged plates (cf. [3–9]), and fracture mechanics [10]. Eq. (1.5) is usually solved approximately by an expansion–collocation method, the expansion being in terms of a finite series involving Chebyshev polynomials \( U_i(t) \) of the second kind. In particular, \( \phi(t) \) in (1.5) is approximated as
\[
\phi(t) \approx (1 - t^2)^{\frac{1}{2}} \sum_{i=0}^{n} a_i U_i(t) \tag{1.6}
\]
where \( a_i \ (i = 0, 1, \ldots, n) \) are unknown constants. Substitution of (1.6) in (1.5) produces (cf. [3])
\[
\sum_{i=0}^{n} a_i A_i(x) = f(x), \quad -1 \leq x \leq 1 \tag{1.7}
\]
where
\[
A_i(x) = -\pi (i+1) U_i(x) + \int_{-1}^{1} (1 - t^2)^{\frac{1}{2}} L(t,x) U_i(t) \, dt. \tag{1.8}
\]
To find the unknown constants \( a_i \ (i = 0, 1, \ldots, n) \), we put \( x = x_j \ (j = 0, 1, \ldots, n) \) where the \( x_j \)'s are suitable collocation points such that \( -1 \leq x_j \leq 1 \). This produces the linear systems
\[
\sum_{i=0}^{n} a_i A_{ij} = f_j, \quad j = 0, 1, \ldots, n \tag{1.9}
\]
with \( A_{ij} = A_i(x_j) \) and \( f_j = f(x_j) \). These can be solved by standard methods. The collocation points are usually chosen to be the zeros of \( U_{n+1}(x) \) or \( T_{n+1}(x) \) (Chebyshev polynomials of the first kind). The method described above becomes somewhat unsuitable for solving the general hypersingular integral equation (1.1) due to the presence of the factor \( K(t,x) \) with \( (t-x)^{-2} \). Here we develop a modified method for approximately solving Eq. (1.1). This method stems from recent work of Chakrabarti and Vanden Berghe [11] wherein an approximate method has been developed to solve a general type of first-kind singular integral equation with a Cauchy-type kernel, given by
\[
\int_{-1}^{1} \phi(t) \left[ \frac{K(t,x)}{(t-x)} + L(t,x) \right] \, dt = f(x), \quad -1 < x < 1, \tag{1.10}
\]
\( \phi(t) \) satisfying appropriate conditions at the end points, and the integral involving \( \frac{1}{t-x} \) is in the sense of the Cauchy principal value (\( K(x,x) \neq 0 \)). The approximate method developed below appears to be quite appropriate for solving the most general type of first-kind hypersingular integral equation (1.1) assuming of course that \( K(t,x) \) and \( L(t,x) \) can be approximated as in [11]. Some simple examples are given to illustrate the method.

2. Method of solution

The unknown function \( \phi(x) \) satisfying \( \phi(\pm1) = 0 \) can be represented in the form
\[
\phi(x) = (1 - x^2)^{\frac{1}{2}} \psi(x), \quad -1 \leq x \leq 1 \tag{2.1}
\]
where \( \psi(x) \) is a well-behaved unknown function of \( x \in [-1, 1] \). Approximating \( \psi(x) \) by means of a polynomial of degree \( n \), given by
\[
\psi(x) \approx \sum_{j=0}^{n} c_j x^j \tag{2.2}
\]
where \( c_j \)’s \((j = 0, 1, \ldots, n)\) are unknown constants, the original integral equation (1.1) produces

\[
\sum_{j=0}^{n} c_j \left[ \int_{-1}^{1} \frac{(1-t^2)^{1/2} K(t, x) t^j}{(t-x)^2} dt + \int_{-1}^{1} (1-t^2)^{1/2} L(t, x) t^j dt \right] = f(x), \quad -1 \leq x \leq 1.
\]

(2.3)

The functions \( K(t, x) \) and \( L(t, x) \) can be approximated as (for fixed \( x \); cf. [11])

\[
K(t, x) \approx \sum_{p=0}^{m} K_p(x) t^p, \quad L(t, x) \approx \sum_{q=0}^{s} L_q(x) t^q
\]

(2.4)

with known expressions for \( K_p(x) \) and \( L_q(x) \). Then (2.3) gives

\[
\sum_{j=0}^{n} c_j \alpha_j(x) = f(x), \quad -1 \leq x \leq 1
\]

(2.5)

where

\[
\alpha_j(x) = \sum_{p=0}^{m} K_p(x) u_{p+j}(x) + \sum_{q=0}^{s} \gamma_{q+j} L_q(x)
\]

(2.6)

with

\[
u_{p+j}(x) = \int_{-1}^{1} \frac{(1-t^2)^{1/2} t^{p+j}}{(t-x)^2} dt, \quad -1 \leq x \leq 1,
\]

(2.7)

\[
\gamma_{q+j} = \int_{-1}^{1} (1-t^2)^{1/2} t^{q+j} dt,
\]

(2.8)

which can be easily evaluated. The unknown constants \( c_j \) \((j = 0, 1, \ldots, n)\) are now obtained by putting \( x = x_l(l = 0, 1, \ldots, n) \) in (2.5), where \(-1 \leq x_l \leq 1\) and are to be chosen suitably. Thus we obtain a system of \((n+1)\) linear equations, given by

\[
\sum_{j=0}^{n} c_j \alpha_{jl} = f_l, \quad l = 0, 1, \ldots, n
\]

(2.9)

where

\[
\alpha_{jl} = \alpha_j(x_l), \quad f_l = f(x_l),
\]

(2.10)

for the determination of the \((n+1)\) unknowns \( c_j \) \((j = 0, 1, \ldots, n)\). This completes the description of the approximate method for solving (1.1). Below we give some simple examples to illustrate the method.

3. Illustrative examples

**Example 1.** If we consider \( K(t, x) \equiv 1, L(t, x) \equiv 0 \), then Eq. (1.1) reduces to Eq. (1.3) whose solution is given by (1.4). However, we use the method developed above to obtain the solution for the particular forcing function \( f(x) = 1 \). For this case, \( K_p(x) \) and \( L_q(x) \) in (2.4) are given by

\[
K_0(x) = 1, \quad K_p(x) = 0 (p > 0) \quad \text{and} \quad L_q(x) = 0 (q \geq 0).
\]

(3.1.1)

Hence we find that relation (2.5) produces

\[
\sum_{j=0}^{n} c_j u_j(x) = 1, \quad -1 \leq x \leq 1
\]

(3.1.2)
where

\[ u_0(x) = -\pi, \quad u_1(x) = -2\pi x, \quad u_2(x) = \pi \left( \frac{1}{2} - 3x^2 \right), \]
\[ u_3(x) = \pi(x - 4x^3), \quad u_4(x) = \pi \left( \frac{1}{8} + \frac{3}{2}x^2 - 5x^4 \right), \ldots \]  

(3.1.3)

Substituting (3.1.3) in (3.1.2) and comparing the coefficients on both sides, we obtain
\[ c_0 = -\frac{1}{\pi}, \quad c_1 = c_2 = \cdots = 0 \]  

(3.1.4)

so that
\[ \phi(x) = -\frac{1}{\pi}(1 - x^2)^{\frac{3}{2}} \]

which is in fact the exact solution of (1.3) for \( f(x) = 1 \) obtained by using the relation (1.4).

**Examples 2.** Next we consider the equation

\[ \int_{-1}^{1} \left[ \frac{1}{(t-x)^2} + (t+x) \right] \phi(t) dt = f(x), \quad -1 \leq x \leq 1 \]  

(3.2.1)

with \( \phi(\pm 1) = 0 \). This corresponds to \( K(t,x) \equiv 1 \) and \( L(t,x) \equiv t + x \). This, however, is also of the form (1.5), and in view of this, we use the method developed in Section 3 and the method described in Section 1 to obtain approximate solutions of (3.2.1) for the purpose of comparison. Now, here, \( K_p(x) \) and \( L_q(x) \) are given by
\[ K_0 = 1, \quad K_p(x) = 0 (p \geq 1) \quad \text{and} \quad L_0(x) = x, \quad L_1(x) = 1, \quad L_q(x) = 0 (q \geq 2). \]  

(3.2.2)

Thus (2.6) gives
\[ \alpha_j(x) = u_j(x) + \gamma_j x + \gamma_{j+1}, \quad j = 0, 1, \ldots \]  

(3.2.3)

where \( u_j(x) \) (\( j = 0, 1, \ldots \)) are the same as those given in (3.1.3) and
\[ \gamma_{2j+1} = 0, \quad \gamma_{2j} = \frac{\pi^{\frac{1}{2}} \Gamma(j + \frac{1}{2})}{2(j+1)!}, \quad j = 0, 1, \ldots \]  

(3.2.4)

so that \( a_0(x), a_1(x), \ldots \) etc are obtained in closed forms.

For simplicity, if we choose the forcing function \( f(x) \) to be of the form \( f(x) = b_0 + b_1x \) where \( b_0 \) and \( b_1 \) are known constants, then we can determine the unknown constants \( c_0, c_1, \ldots \) directly by comparing the coefficients of various powers of \( x \) on the two sides of (2.5), as both sides are now polynomials. This produces
\[ c_0 = -\frac{2}{31\pi}(16b_0 + b_1), \quad c_1 = -\frac{16}{31\pi} \left( \frac{b_0}{2} + b_1 \right), \quad c_j = 0 \ (j \geq 2) \]  

(3.2.5)

and the solution of (3.2.1) in this case is obtained as
\[ \phi(x) = (c_0 + c_1 x)(1 - x^2)^{\frac{3}{2}}. \]  

(3.2.6)

However, if we use the expansion of \( \phi(x) \) in terms of Chebyshev polynomials given by (1.6), then in this case the functions \( A_i \) (\( i = 0, 1, \ldots \)) are obtained as
\[ A_0(x) = -\pi + \frac{\pi}{2} x, \quad A_1(x) = \frac{\pi}{4} - 4\pi x, \quad A_2(x) = 3\pi - 16\pi x^2, \quad A_3(x) = 16\pi x^2 - 32\pi x^3, \ldots \]  

(3.2.7)

and comparing the coefficients of both sides of the relation (1.7) we obtain
\[ a_0 = -\frac{2}{31\pi}(16b_0 + b_1), \quad a_1 = -\frac{8}{31\pi} \left( \frac{b_0}{2} + b_1 \right), \quad a_i = 0 \ (i \geq 2). \]  

(3.2.8)

Noting that \( U_0(x) = 1 \) and \( U_1 = 2x \), we find from (1.6) that \( \phi(x) \) is exactly the same as that given in (3.2.6). It may be noted that the collocation method used to obtain the unknown constants \( c_j \) (\( j = 0, 1, \ldots \)) in (2.2) and \( a_i \) (\( i = 0, 1, \ldots \))
in (1.6) for this problem can be used if for simplicity we choose \( f(x) = 1 + 2x \) so that \( b_0 = 1 \) and \( b_1 = 2 \) above. Choosing \( n = 3 \) in the expansion (2.2), the unknown constants \( c_0, c_1, c_2, c_3 \) are determined from the linear system

\[
\sum_{j=0}^{3} c_j \alpha_{jl} = f_l, \quad l = 0, 1, 2, 3.
\]

(3.2.9)

If we choose the collocation points as \( x_0 = -1, x_1 = -\frac{1}{3}, x_2 = \frac{1}{3}, x_3 = 1 \), then the linear equation (3.2.9) produces

\[
c_0 = -0.3696501, \quad c_1 = -0.4107224, \quad c_2 \approx 0, \quad c_3 \approx 0
\]

(3.2.10)

which are almost the same as those given in (3.2.5). Similarly, choosing \( n = 3 \) in (1.6), we see that the unknown constants \( a_0, a_1, a_2, a_3 \) are to be found by solving the linear system

\[
\sum_{i=0}^{3} a_i A_{ij} = f_j, \quad j = 0, 1, 2, 3.
\]

(3.2.11)

Choosing the same set of collocation points, \( -1, -\frac{1}{3}, \frac{1}{3}, 1 \), we find that the linear (3.2.11) when solved produces

\[
a_0 = -0.3696500, \quad a_1 = -0.2053610, \quad a_2 \approx 0, \quad a_3 \approx 0
\]

(3.2.12)

which are again almost the same as those given in (3.2.8). It may be noted that on increasing \( n \), the same results as above are obtained for both the methods.

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References