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Nonsmooth differential geometry and algebras of generalized functions

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Dedicated to John Horváth on the occasion of his 80th birthday. With special thanks for all his support over the years

Abstract

Algebras of generalized functions offer possibilities beyond the purely distributional approach in modelling singular quantities in nonsmooth differential geometry. This article presents an introductory survey of recent developments in this field and highlights some applications in mathematical physics.

Keywords: Algebras of generalized functions; Colombeau algebras; Nonsmooth differential geometry; Generalized pseudo-Riemannian geometry

1. Introduction

Nonsmooth differential geometry provides an important tool in a variety of applications, in particular in mathematical physics. As examples we mention nonsmooth Hamiltonian mechanics [25,26] and the analysis of singular spacetimes in general relativity (cf., e.g., [2,11,34] and [35] for a recent survey). Linear distributional geometry [9,25,30] is only of limited use in a genuinely nonlinear context, as, e.g., in general relativity, where the nonlinearity of the Einstein field equations and the interest in curvature quantities introduces requirements on the underlying theory of generalized functions which distribution theory is unable to meet. A nonlinear extension of linear distributional geometry displaying promising capabilities for overcoming these conceptual problems has been developed

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over the past years based on Colombeau's theory of generalized functions. It is the aim of the present paper to provide an introduction to this field and some of its applications.

In the remainder of this section we fix some notation and terminology from differential geometry and distribution theory. Section 2 gives a quick introduction to some of the fundamental ideas of Colombeau theory both in the local and in the manifold setting. In Section 3 we consider generalized functions taking values in a differentiable manifold, a construction which has no analogue in distribution theory yet is of central importance for nonlinear distributional geometry as it allows to formulate a functorial theory of generalized functions in a global context. In particular, it allows to introduce notions like flows of generalized vector fields or geodesics of generalized metrics. Finally, in Section 4 we develop a generalized pseudo-Riemannian geometry in this setting and give some applications of the resulting theory in general relativity.

In what follows, X and Y always mean paracompact, smooth Hausdorff manifolds of dimension n, respectively, m. We denote vector bundles with base space X by (E, X, π_X) or $E \to X$ for short and write a vector bundle chart over the chart (V, ψ) of X as (V, Ψ). For vector bundles $E \to X$ and $F \to Y$, by $\mathrm{Hom}(E, F)$ we mean the space of vector bundle homomorphisms from E to F. Given $f \in \mathrm{Hom}(E, F)$ the unique smooth map from X to Y satisfying $\pi_Y \circ f = \underline{f} \circ \pi_X$ is denoted by $\underline{f}$. For vector bundle charts (V, Φ) of E and (W, Ψ) of F we write the local vector bundle homomorphism

$$f_{\Psi\Phi} := \Psi \circ f \circ \Phi^{-1} : \varphi\bigl(V \cap \underline{f}^{-1}(W)\bigr) \times \mathbb{K}^{n'} \to \phi(W) \times \mathbb{K}^{m'}$$

in the form

$$f_{\Psi\Phi}(x, \xi) = \bigl(f^{(1)}_{\Psi\Phi}(x), f^{(2)}_{\Psi\Phi}(x) \cdot \xi\bigr).$$

The space of smooth sections of a vector bundle $E \to X$ is denoted by $\Gamma(X, E)$. $T^r_s(X)$ is the (r, s)-tensor bundle over X and we use the following notation for spaces of tensor fields $\mathcal{T}^r_s(X) := \Gamma(X, T^r_s(X))$, $\mathfrak{X}(X) := \Gamma(X, TX)$ and $\Omega^1(X) := \Gamma(X, T^*X)$, where TX and T^*X denote the tangent and cotangent bundle of X, respectively. $\mathcal{P}(X, E)$ is the space of linear differential operators $\Gamma(X, E) \to \Gamma(X, E)$. For $E = X \times \mathbb{R}$ we write $\mathcal{P}(X)$ instead of $\mathcal{P}(X, E)$.

We denote by $\mathrm{Vol}(X)$ the volume bundle over X, its smooth sections are called one-densities. The space $\mathcal{D}'(X, E)$ of E-valued distributions on X is defined as the topological dual of the space of compactly supported sections of the bundle $E^* \otimes \mathrm{Vol}(X)$,

$$\mathcal{D}'(X, E) := \bigl[\Gamma_c\bigl(X, E^* \otimes \mathrm{Vol}(X)\bigr)\bigr]'.$$

For $E = X \times \mathbb{R}$ we obtain $\mathcal{D}'(X) := \mathcal{D}'(X, E)$, the space of distributions on X. The isomorphism of $\mathcal{C}^\infty(X)$-modules

$$\mathcal{D}'(X, E) \cong \mathcal{D}'(X) \otimes_{\mathcal{C}^\infty(X)} \Gamma(X, E)$$

shows that E-valued distributions can be viewed as sections with distributional coefficients.

2. Colombeau generalized functions on differentiable manifolds

When trying to extend linear distribution theory to a nonlinear theory of generalized functions one is faced with certain fundamental obstacles. To give a simple example, let $vp(1/x)$ be the Cauchy principal value of $1/x$ on $\mathbb{R}$. Then since

$$0 = \bigl(\delta(x) \cdot x\bigr) \cdot vp\frac{1}{x} \neq \delta(x) \cdot \left(x \cdot vp\frac{1}{x}\right) = \delta(x),$$

it follows that the usual multiplication on $C^\infty \times \mathcal{D}'$ cannot be extended to an associative and commutative multiplication on $\mathcal{D}' \times \mathcal{D}'$. Similarly, it can be shown that $\mathcal{D}'$ cannot be endowed with the structure of an associative commutative algebra compatible with the usual product in L^∞: with H the Heaviside function, the fact that $H^2 = H$ would by the Leibniz rule entail $(H^2)' = 2HH'$, $(H^3)' = 3H^2H'$, so $2HH' = H' = 3HH'$. But then $\delta = H' = 0$, a contradiction. For a comprehensive analysis of the problem of multiplication of distributions see [27].

Apart from nonlinear analysis on certain (function-)subalgebras of $\mathcal{D}'$ (Sobolev spaces) the second main option therefore consists in embedding the space of distributions into an appropriate (associative and commutative) algebra $\mathcal{G}$ of generalized functions, the aim being to retain as many of the standard features of distribution theory as possible. In particular, we want $\mathcal{G}$ to be a differential algebra with unit $f(x) \equiv 1$ and derivation operators extending those on $\mathcal{D}'$. Our previous example demonstrates that under these assumptions the product in $\mathcal{G}$ cannot extend the pointwise product of functions in L^∞_{loc}. Furthermore, by a celebrated result of L. Schwartz [32], it cannot extend the pointwise product of $\mathcal{C}^k$-functions for any $k \in \mathbb{N}_0$ either. Due to these differential–algebraic constraints the maximal possible compatibility of the product $\cdot$ in $\mathcal{G}$ is that $\cdot|_{\mathcal{C}^\infty \times \mathcal{C}^\infty}$ coincide with the usual pointwise product of functions.

Differential algebras satisfying this maximal set of requirements were first constructed by J.F. Colombeau in the early 1980s [3–7]. The basic principles underlying his approach are regularization through convolution and asymptotic estimates in terms of a regularization parameter. In the so-called special version of the construction, $\mathcal{D}'(\mathbb{R}^n)$ is embedded into a certain subalgebra $\mathcal{E}_M(\mathbb{R}^n)$ of $\mathcal{C}^\infty(\mathbb{R}^n)^I$ (with $I := (0, 1]$) through convolution

$$\mathcal{D}'(\mathbb{R}^n) \ni w \mapsto (w * \rho_\varepsilon)_{\varepsilon \in I}.$$

Here ρ is a Schwartz function with $\int \rho = 1$ and $\rho_\varepsilon(x) = 1/\varepsilon^n \rho(x/\varepsilon)$. $\mathcal{C}^\infty(\mathbb{R}^n)^I$ is a differential algebra with operations defined componentwise and the above map is obviously linear and commutes with partial derivatives. On the other hand, a natural way of embedding $\mathcal{C}^\infty(\mathbb{R}^n)$ into $\mathcal{C}^\infty(\mathbb{R}^n)^I$ is the diagonal embedding

$$\mathcal{C}^\infty(\mathbb{R}^n) \ni f \mapsto (f)_{\varepsilon \in I}.$$

Clearly this map preserves the pointwise product of smooth functions. The idea, therefore, is to factor $\mathcal{E}_M(\mathbb{R}^n)$ by an ideal $\mathcal{N}(\mathbb{R}^n)$ containing $(f * \rho_\varepsilon - f)_\varepsilon$ for each $f \in \mathcal{C}^\infty(\mathbb{R}^n)$. The resulting quotient algebra would then satisfy the above maximal set of requirements on a differential algebra containing the space of distributions. Now (assuming $n = 1$ for the moment), Taylor's theorem gives

$$
\begin{aligned}
(f * \rho_\varepsilon - f)(x) &= \int \big(f(x-y) - f(x)\big)\rho_\varepsilon(y)\,dy \\
&= \int \sum_{k=1}^{m} \frac{(-\varepsilon y)^k}{k!} f^{(k)}(x)\rho(y)\,dy \\
&\quad + \int \frac{(-\varepsilon y)^{m+1}}{(m+1)!} f^{(m+1)}(x - \theta\varepsilon y)\rho(y)\,dy.
\end{aligned}
$$

If we additionally suppose that $\int \rho(x)x^k\,dx = 0$ for all $k \geqslant 1$ then this expression converges to zero, faster than any power of ε, uniformly on each compact set, in each derivative. The natural candidate for $\mathcal{N}(\mathbb{R}^n)$ therefore is

$$
\mathcal{N}(\mathbb{R}^n) = \Big\{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n)^I \mid \forall K \Subset \mathbb{R}^n,\ \forall \alpha \in \mathbb{N}_0^n,\ \forall m \in \mathbb{N}: \sup_{x\in K}\big|\partial^\alpha u_\varepsilon(x)\big| = O(\varepsilon^m) \text{ as } \varepsilon \to 0\Big\}.
$$

Elements of $\mathcal{N}(\mathbb{R}^n)$ are called *negligible*. The definition of $\mathcal{N}(\mathbb{R}^n)$ in turn fixes the maximal subalgebra $\mathcal{E}_M(X)$ (the algebra of *moderate* nets) of $\mathcal{C}^\infty(\mathbb{R}^n)^I$ in which $\mathcal{N}(\mathbb{R}^n)$ is an ideal as

$$
\mathcal{E}_M(\mathbb{R}^n) = \Big\{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n)^I \mid \forall K \Subset \mathbb{R}^n,\ \forall \alpha \in \mathbb{N}_0^n,\ \exists N \in \mathbb{N} \text{ with } \sup_{x\in K}\big|\partial^\alpha u_\varepsilon(x)\big| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\Big\}.
$$

The (special) Colombeau algebra on $\mathbb{R}^n$ is then defined as the factor algebra $\mathcal{G}(\mathbb{R}^n) = \mathcal{E}_M(\mathbb{R}^n)/\mathcal{N}(\mathbb{R}^n)$. As indicated above, the map $\iota: \mathcal{D}'(\mathbb{R}^n) \to \mathcal{G}(\mathbb{R}^n)$, $\iota(w) =$ [class of $(w * \rho_\varepsilon)_\varepsilon$] provides a linear embedding which coincides with the diagonal embedding σ : $\mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{G}(\mathbb{R}^n)$, $\sigma(f) =$ [class of $(f)_\varepsilon$] on $\mathcal{C}^\infty(\mathbb{R}^n)$, hence verifies all the requirements made above. From here one may proceed, using partitions of unity and suitable cut-off functions to construct embeddings $\mathcal{D}'(\Omega) \hookrightarrow \mathcal{G}(\Omega)$ for any open subset Ω of $\mathbb{R}^n$. Instead, we turn directly to the manifold case [1,10,15,20]. The basic features of the following definition are in close correspondence to the Euclidean case discussed above.

Definition 2.1. Let X be a smooth, paracompact Hausdorff manifold and set $\mathcal{E}(X) := (\mathcal{C}^\infty(X))^I$. The Colombeau algebra $\mathcal{G}(X)$ on X is defined as the quotient $\mathcal{E}_M(X)/\mathcal{N}(X)$, where

$$
\mathcal{E}_M(X) = \Big\{(u_\varepsilon)_\varepsilon \in \mathcal{E}(X) \mid \forall K \Subset X,\ \forall P \in \mathcal{P}(X),\ \exists N \in \mathbb{N}: \sup_{p\in K}\big|Pu_\varepsilon(p)\big| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\Big\},
$$

$$
\mathcal{N}(X) = \Big\{(u_\varepsilon)_\varepsilon \in \mathcal{E}(X) \mid \forall K \Subset X,\ \forall P \in \mathcal{P}(X),\ \forall m \in \mathbb{N}: \sup_{p\in K}\big|Pu_\varepsilon(p)\big| = O(\varepsilon^{m}) \text{ as } \varepsilon \to 0\Big\}.
$$

We write $u = [(u_\varepsilon)_\varepsilon]$ for the class of $(u_\varepsilon)_\varepsilon$ in $\mathcal{G}(X)$. Restrictions of elements of $\mathcal{G}(X)$ to open subsets of X are defined componentwise on representatives and $\mathcal{G}(\_)$ is seen to be a fine and supple (but not flabby) sheaf of differential algebras [8,10,29].

Our first fundamental observation concerning the structure of $\mathcal{G}(X)$ is that $\mathcal{N}(X)$ can be characterized as a subspace of $\mathcal{E}_M(X)$ without resorting to derivatives ([12, Theorem 13.1], [20, Section 4]),

$$\mathcal{N}(X) = \Big\{(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(X) \mid \forall K \Subset X,\ \forall m \in \mathbb{N}\colon \sup_{p\in K}\big|u_\varepsilon(p)\big| = O(\varepsilon^m)\Big\}. \tag{1}$$

This characterization is a very convenient means both within Colombeau theory (as we shall see shortly) and in applications to partial differential equations (where it considerably simplifies uniqueness proofs).

An important feature distinguishing Colombeau algebras from spaces of distributions is the availability of a point value description of Colombeau functions. Componentwise insertion of points of X into elements of $\mathcal{G}(X)$ yields well-defined *generalized numbers*, i.e., elements of the ring of constants $\mathcal{K} := \mathcal{E}_M/\mathcal{N}$ (with $\mathcal{K} = \mathcal{R}$ or $\mathcal{K} = \mathcal{C}$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), where

$$\mathcal{E}_M = \big\{(r_\varepsilon)_\varepsilon \in \mathbb{K}^I \mid \exists N \in \mathbb{N}\colon |r_\varepsilon| = O(\varepsilon^{-N})\big\},$$
$$\mathcal{N} = \big\{(r_\varepsilon)_\varepsilon \in \mathbb{K}^I \mid \forall m \in \mathbb{N}\colon |r_\varepsilon| = O(\varepsilon^{m})\big\}.$$

Example 2.2. Let $\varphi \in \mathcal{D}(\mathbb{R})$, $\int \varphi = 1$, $\varphi_\varepsilon(x) := \varepsilon^{-1}\varphi(x/\varepsilon)$ and set $u_\varepsilon(x) := \varphi_\varepsilon(x - \varepsilon)$. Then $u_\varepsilon \to \delta$ in $\mathcal{D}'(\mathbb{R})$, so $u := [(u_\varepsilon)_\varepsilon]$ is not 0 in $\mathcal{G}(\mathbb{R})$. Nevertheless, it is easily seen that every point value of every derivative of u is zero in $\mathcal{K}$.

Thus point values on "classical" points $p \in X$ do not characterize elements of $\mathcal{G}(X)$. As can be seen in the above example, the reason for this failure is that Colombeau functions are capable of modelling infinitesimal quantities which standard points are unable to detect. Borrowing an idea from nonstandard analysis, the plan is therefore to introduce "nonstandard points" which themselves may move around in the manifold in order to keep track of the infinitesimal behavior of elements of $\mathcal{G}(X)$. To this end we define an equivalence relation $\sim$ on the space $X_c := \{(p_\varepsilon)_\varepsilon \in X^I \mid \exists K \Subset X,\ \exists \varepsilon_0 > 0 \text{ s.t. } p_\varepsilon \in K,\ \forall \varepsilon < \varepsilon_0\}$ as follows: for any Riemannian metric h on X with distance function d_h, two nets $(p_\varepsilon)_\varepsilon$, $(q_\varepsilon)_\varepsilon$ are called equivalent, $(p_\varepsilon)_\varepsilon \sim (q_\varepsilon)_\varepsilon$ if $d_h(p_\varepsilon, q_\varepsilon) = O(\varepsilon^m)$ for each $m \in \mathbb{N}$. We call $\tilde{X}_c := X_c/\sim$ the space of compactly supported generalized points. Obviously this definition does not depend on the specific Riemannian metric h. Then we have

Theorem 2.3. *Let $u \in \mathcal{G}(X)$ and $\tilde{p} = [(p_\varepsilon)_\varepsilon] \in \tilde{X}_c$. Then $u(\tilde{p}) := [(u_\varepsilon(p_\varepsilon))_\varepsilon]$ is a well-defined element of $\mathcal{K}$. Moreover, $u = 0$ if and only if $u(\tilde{p}) = 0$ in $\mathcal{K}$ for all $\tilde{p}$ in $\tilde{X}_c$.*

For the proof, see [20,28]. To give an idea of the argument, let us have a look at the case $X = \mathbb{R}^n$ (following [29, Proposition 3.1]). If $u = 0 \in \mathcal{G}(\mathbb{R}^n)$ and $p_\varepsilon \in K \Subset \mathbb{R}^n$ for ε small then it is immediate from the definition of $\mathcal{N}(\mathbb{R}^n)$ that $(u_\varepsilon(p_\varepsilon))_\varepsilon \in \mathcal{N}$, i.e., $u(\tilde{p}) = 0 \in \mathcal{K}$. Conversely, suppose that $u(\tilde{p}) = 0$ for all $\tilde{p} \in \tilde{\mathbb{R}}^n_c$ and let $K \Subset \mathbb{R}^n$. For each $\varepsilon \in I$ denote by

p_ε the point in K where $|u_\varepsilon|$ attains its maximum. Since $\tilde{p} = [(p_\varepsilon)_\varepsilon] \in \tilde{\mathbb{R}}^n_c$, the negligibility estimates of order 0 for $(u_\varepsilon)_\varepsilon$ on K follow from $(u_\varepsilon(p_\varepsilon))_\varepsilon \in \mathcal{N}$. But then $u = 0$ due to (1).

Note that in Example 2.2, $u(\tilde{p}) \neq 0$ for $\tilde{p} = [(\varepsilon)_\varepsilon]$ if $\varphi(0) \neq 0$.

There are essentially two ways of connecting linear distribution spaces with Colombeau algebras. Firstly, one can construct injective sheaf morphisms $\iota : \mathcal{D}'(\_) \hookrightarrow \mathcal{G}(\_)$. This can be done either using de Rham regularizations or, which basically amounts to the same, directly by convolution with a fixed mollifier in charts (cf. [10,20]). The resulting embedding is noncanonical, i.e., it depends on the ingredients of the construction (partition of unity, mollifier, cut-off functions, etc.). The main field of application of the special version of Colombeau algebras therefore lies in areas where a regularization procedure for the singular quantities to be modelled suggests itself by the nature of the problem (cf. [10,13,27]). For so-called full variants of Colombeau algebras on manifolds, allowing for a *canonical* embedding of the space of distributions we refer to [12,14].

The second link to linear distribution theory is the concept of association: two elements u, v of $\mathcal{G}(X)$ are called *associated*, $u \approx v$ if $u_\varepsilon - v_\varepsilon \to 0$ in $\mathcal{D}'(X)$. If $\int u_\varepsilon \mu \to \langle w, \mu \rangle$ for some $w \in \mathcal{D}'(X)$ and each compactly supported one density μ, i.e., if $u_\varepsilon \to w$ in $\mathcal{D}'(X)$ then w is called associated distribution to u. Clearly these definitions do not depend on the chosen representatives. Besides this concept of "equality in the sense of distributions" one may also introduce more restrictive equivalence relations on $\mathcal{G}(X)$. In particular, we mention the concept of $\mathcal{C}^k$-association: $u, v \in \mathcal{G}(X)$ are called $\mathcal{C}^k$-associated, $u \approx_k v$ if for all $l \leqslant k$ and all $\xi_1, \ldots, \xi_l \in \mathfrak{X}(X)$, $L_{\xi_1} \ldots L_{\xi_l}(u_\varepsilon - v_\varepsilon) \to 0$, uniformly on compact sets. In applications it is often the case that modelling of singular quantities and analytical treatment of the problem at hand (e.g., solution of a nonlinear PDE) is carried out in $\mathcal{G}$, while a distributional interpretation of the result is effected through the notion of association. Concerning the examples inspected at the beginning of this section we note that, in $\mathcal{G}(\mathbb{R})$, $x \cdot \delta$ is associated but not equal to 0 and $H^m \neq H$, but $H^m \approx H$ for all $m \in \mathbb{N}$. This complies with the intuitive feeling that over and above the distributional picture, modelling in $\mathcal{G}$ allows to fix the "microstructure" of singular quantities, reflected in a notion of equality which is more restrictive than equality in the distributional sense. It can also be viewed as a further nonstandard aspect of the theory (cf. [27, §10], for an in-depth discussion).

For a vector bundle $E \to X$ we define the spaces of moderate, respectively, negligible sections as

$$\Gamma_{\mathcal{E}_M}(X,E) = \Big\{ (s_\varepsilon)_{\varepsilon \in I} \in \Gamma(X,E)^I \mid \forall P \in \mathcal{P}(X,E),\ \forall K \Subset X,\ \exists N \in \mathbb{N}: \sup_{p \in K} \big\| P u_\varepsilon(p) \big\| = O(\varepsilon^{-N}) \Big\},$$

$$\Gamma_{\mathcal{N}}(X,E) = \Big\{ (s_\varepsilon)_{\varepsilon \in I} \in \Gamma(X,E)^I \mid \forall P \in \mathcal{P}(X,E),\ \forall K \Subset X,\ \forall m \in \mathbb{N}: \sup_{p \in K} \big\| P u_\varepsilon(p) \big\| = O(\varepsilon^{m}) \Big\},$$

where $\|\cdot\|$ denotes the norm induced on the fibers of E by any Riemannian metric. $\Gamma_{\mathcal{E}_M}(X,E)$ is a $\mathcal{G}(X)$-module with submodule $\Gamma_{\mathcal{N}}(X,E)$ and we define the $\mathcal{G}(X)$-module $\Gamma_{\mathcal{G}}(X,E)$ of generalized sections of the bundle $E \to X$ as the quotient $\Gamma_{\mathcal{E}_M}(X,E)/\Gamma_{\mathcal{N}}(X,E)$. As in the scalar case we may omit all differential operators from the definition of $\Gamma_{\mathcal{N}}(X,E)$ if we suppose the $(s_\varepsilon)_\varepsilon$ to be moderate. Important special cases are the space $\mathcal{G}^r_s(X)$ of generalized (r,s)-tensor fields and the space $\bigwedge^k_{\mathcal{G}}(X)$ of generalized k-forms, corresponding to $E = T^r_s(X)$ and $E = \bigwedge^k T^*X$, respectively.

$\Gamma_{\mathcal{G}}(\_, E)$ is a fine sheaf of $\mathcal{G}(\_)$-modules. Its algebraic structure is clarified by the following theorem [20, Section 6].

Theorem 2.4. *The $\mathcal{G}(X)$-module $\Gamma_{\mathcal{G}}(X,E)$ is projective and finitely generated. Moreover, the following isomorphisms of $\mathcal{C}^\infty(X)$-modules hold*:

$$\Gamma_{\mathcal{G}}(X,E) \cong \mathcal{G}(X) \otimes_{\mathcal{C}^\infty(X)} \Gamma(X,E) \cong L_{\mathcal{C}^\infty(X)}\big(\Gamma(X,E^*), \mathcal{G}(X)\big).$$

In particular, this implies that generalized sections may be viewed as smooth sections with generalized coefficients (in complete analogy to the distributional case). In addition, for spaces of generalized tensor fields we have

$$\mathcal{G}^r_s(X) \cong L_{\mathcal{G}(X)}\big(\mathcal{G}^0_1(X)^r, \mathcal{G}^1_0(X)^s; \mathcal{G}(X)\big) \quad \text{as } \mathcal{G}(X)\text{-module},$$
$$\mathcal{G}^r_s(X) \cong L_{\mathcal{C}^\infty(X)}\big(\Omega^1(X)^r, \mathfrak{X}(X)^s; \mathcal{G}(X)\big) \quad \text{as } \mathcal{C}^\infty(X)\text{-module}.$$

Contrary to the purely distributional picture where ill-defined products of distributions have to be avoided carefully, our current setting allows unrestricted application of multilinear operations like tensor product, wedge product, Lie derivatives w.r.t. generalized vector fields, Poisson brackets, etc.

The relationship to the distributional setting is again governed by the notion of association: a generalized section $s \in \Gamma_{\mathcal{G}}(X,E)$ is called *associated to* $w \in \mathcal{D}'(X,E)$, $s \approx w$, if for all $\mu \in \Gamma_c(X, E^* \otimes \mathrm{Vol}(X))$ and one (hence every) representative $(s_\varepsilon)_\varepsilon$ of s,

$$\lim_{\varepsilon\to 0} \int_X (s_\varepsilon|\mu) = \langle w, \mu\rangle.$$

Here, $(\cdot|\cdot)$ denotes the natural pairing

$$\mathrm{tr}_E \otimes \mathrm{id} : (E \otimes E^*) \otimes \mathrm{Vol}(X) \to (X \times \mathbb{C}) \otimes \mathrm{Vol}(X) = \mathrm{Vol}(X).$$

Stronger notions of association like $\approx_k$ are defined analogously to the scalar case. Typically, multilinear operations on generalized sections display compatibility properties with their distributional counterparts expressible in terms of association relations. For example, if $\xi \in \mathcal{G}^1_0(X)$ and $\xi \approx \eta \in \mathcal{D}'^1_0(X)$, $t \in \mathcal{G}^r_s(X)$, $t \approx_\infty u \in \mathcal{T}^r_s(X)$, then $L_\xi(t) \approx L_\eta(u)$.

Furthermore, classical theorems of smooth and distributional analysis (cf. [25]) like the Poincaré lemma, Stokes' theorem, or the characterization of generalized vector fields as derivations on generalized functions can be extended to the Colombeau setting [13,20].

3. Manifold-valued generalized functions

When applying generalized function techniques to problems of global analysis one inevitably encounters situations where a concept of generalized functions defined on a manifold X and taking values in another manifold is needed. Examples include flows of generalized vector fields or geodesics of distributional spacetime metrics. Within classical distribution theory, clearly no such concept is available. Colombeau algebras on the other hand put more emphasis on the function-character of the generalized functions (as opposed to the description as linear functionals on spaces of test functions in the $\mathcal{D}'$-setting), which allows to develop an appropriate theory in this framework. One main requirement with respect to such a construction is that it be functorial. In particular, it must allow for unrestricted composition of generalized functions. In the local case, the problem of composition of Colombeau functions was first addressed in [1]. The construction suggested there formed the basis for the manifold case presented in [16,22]. Since Colombeau functions by construction are localized on compact subsets of their domain (in the sense that they are completely determined by the behavior of their representatives on such sets, for small values of the regularization parameter), in order to satisfy this requirement we have to single out representatives $(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(X,Y)^I$ which are *compactly bounded* (or c-bounded) in the following sense:

$$\forall K \Subset X,\ \exists \varepsilon_0 > 0,\ \exists K' \Subset Y,\ \forall \varepsilon < \varepsilon_0\colon\ u_\varepsilon(K) \subseteq K'.$$

Moderateness of nets $(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(X,Y)^I$, on the other hand, is formulated using local charts. We thus arrive at the following definition.

Definition 3.1. The space $\mathcal{E}_M[X,Y]$ of compactly bounded (c-bounded) moderate maps from X to Y is defined as the set of all $(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(X,Y)^I$ such that

(i) $(u_\varepsilon)_\varepsilon$ is c-bounded.
(ii) $\forall k \in \mathbb{N}$, for each chart (V,φ) in X, each chart (W,ψ) in Y, each $L \Subset V$ and each $L' \Subset W$ there exists $N \in \mathbb{N}$ with

$$\sup_{p \in L \cap u_\varepsilon^{-1}(L')} \bigl\| D^{(k)}(\psi \circ u_\varepsilon \circ \varphi^{-1})\bigl(\varphi(p)\bigr)\bigr\| = O(\varepsilon^{-N}).$$

Note that the "safety compact sets" L and L' in this definition are needed in order to control the potentially arbitrarily fast growth of chart diffeomorphisms towards the boundary of their domains.

In the absence of a linear structure on the target space Y, we have to introduce an equivalence relation in $\mathcal{E}_M[X,Y]$ which precisely reduces to negligibility of differences of representatives in the case $Y = \mathbb{R}^m$. We do this in a two step process. First, we assure that the distance between representatives as measured in any Riemannian metric on Y goes to zero. Growth conditions on derivatives are then formulated in local charts:

Definition 3.2. Two elements $(u_\varepsilon)_\varepsilon, (v_\varepsilon)_\varepsilon$ of $\mathcal{E}_M[X,Y]$ are called equivalent, $(u_\varepsilon)_\varepsilon \sim (v_\varepsilon)_\varepsilon$, if the following conditions are satisfied:

(i) For all $K \Subset X$, $\sup_{p\in K} d_h(u_\varepsilon(p), v_\varepsilon(p)) \to 0$ $(\varepsilon \to 0)$ for some (hence every) Riemannian metric h on Y.

(ii) $\forall k \in \mathbb{N}_0$, $\forall m \in \mathbb{N}$, for each chart (V, φ) in X, each chart (W, ψ) in Y, each $L \Subset V$ and each $L' \Subset W$:

$$\sup_{p \in L \cap u_\varepsilon^{-1}(L') \cap v_\varepsilon^{-1}(L')} \left\| D^{(k)}(\psi \circ u_\varepsilon \circ \varphi^{-1} - \psi \circ v_\varepsilon \circ \varphi^{-1})\big(\varphi(p)\big) \right\| = O(\varepsilon^m).$$

Finally, we define the space of Colombeau generalized functions defined on X and taking values in Y as $\mathcal{G}[X, Y] := \mathcal{E}_M[X, Y]/\sim$. Elements of $\mathcal{G}[X, Y]$ typically model jump discontinuities, whereas delta-type singularities are excluded by the c-boundedness of representatives (on the other hand, it seems unclear anyways what a delta-type singularity should be in a manifold without additional structure).

In analogy to (1) one would expect that condition (ii) in Definition 3.2 need only hold for $k = 0$ in case $(u_\varepsilon)_\varepsilon$ is assumed to be moderate. It turns out, however, that a proof of this fact cannot be carried along the lines of the local result (based in turn on a classical argument by Landau [24]). Similarly, one would hope for a point value characterization of elements of $\mathcal{G}[X, Y]$. However, in the absence of an analogue to (1) this seems difficult to obtain.

The remedy for both problems lies in a nonlocal characterization of c-boundedness, moderateness and equivalence [22, Section 3]. The key idea is to replace composition with charts in the target space by composition with globally defined smooth functions.

Proposition 3.3. *Let $(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(X, Y)^I$. The following conditions are equivalent*:

(i) *$(u_\varepsilon)_\varepsilon$ is c-bounded.*
(ii) *$(f \circ u_\varepsilon)_\varepsilon$ is c-bounded for all $f \in \mathcal{C}^\infty(Y)$.*
(iii) *$(f \circ u_\varepsilon)_\varepsilon$ is moderate of order zero for all $f \in \mathcal{C}^\infty(Y)$, i.e.,*

$$\forall K \Subset X,\ \exists N \in \mathbb{N}\colon\ \sup_{p \in K} \left| f \circ u_\varepsilon(p) \right| = O(\varepsilon^{-N})$$

for all $f \in \mathcal{C}^\infty(Y)$.
(iv) *$(u_\varepsilon(x_\varepsilon))_\varepsilon \in Y_c$ for all $(x_\varepsilon)_\varepsilon \in X_c$.*

Based on this result, moderateness can be characterized as follows.

Proposition 3.4. *Let $(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(X, Y)^I$. Then $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M[X, Y]$ if and only if $(f \circ u_\varepsilon)_\varepsilon \in \mathcal{E}_M(X)$ for all $f \in \mathcal{C}^\infty(Y)$.*

Finally, concerning the equivalence relation $\sim$ on $\mathcal{E}_M[X, Y]$ we obtain

Theorem 3.5. *Let $(u_\varepsilon)_\varepsilon, (v_\varepsilon)_\varepsilon \in \mathcal{E}_M[X, Y]$. The following statements are equivalent*:

(i) *$(u_\varepsilon)_\varepsilon \sim (v_\varepsilon)_\varepsilon$.*
(ii) *For every Riemannian metric h on Y, every $m \in \mathbb{N}$ and every $K \Subset X$,*

$$\sup_{p \in K} d_h\big(u_\varepsilon(p), v_\varepsilon(p)\big) = O(\varepsilon^m) \quad (\varepsilon \to 0).$$

(iii) *$(f \circ u_\varepsilon - f \circ v_\varepsilon)_\varepsilon \in \mathcal{N}(X)$ for all $f \in \mathcal{C}^\infty(Y)$.*

Since by [16, Theorem 2.14], condition (ii) in Theorem 3.5 is equivalent with conditions 3.2(i) and (ii) with $k = 0$, we obtain the desired characterization of $\sim$. This in turn provides the key building block in the proof of the following point value description of manifold-valued generalized functions.

Theorem 3.6. *Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}[X, Y]$ and $\tilde{p} = [(p_\varepsilon)_\varepsilon] \in \tilde{X}_c$. Then $u(\tilde{p}) := [(u_\varepsilon(p_\varepsilon))_\varepsilon]$ is a well-defined element of $\tilde{Y}_c$. Moreover, $u, v \in \mathcal{G}[X, Y]$ are equal if and only if their point values in each generalized point agree.*

Once this point value characterization is established, also the problem of composition of generalized functions can be resolved ([16, Theorem 2.16], and [22, Theorem 3.6]):

Theorem 3.7. *Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}[X, Y]$, $v = [(v_\varepsilon)_\varepsilon] \in \mathcal{G}[Y, Z]$. Then $v \circ u := [(v_\varepsilon \circ u_\varepsilon)_\varepsilon]$ is a well-defined element of $\mathcal{G}[X, Z]$.*

Although by the c-boundedness of representatives the "worst" singularities that can be modelled by elements of $\mathcal{G}[X, Y]$ are jump discontinuities it is to be expected that derivatives (i.e., tangent maps) of such generalized maps will behave δ-like. We must therefore provide for a concept of generalized vector bundle homomorphisms (containing such tangent maps as special cases) with substantially less restrictive growth conditions in the vector components.

Definition 3.8. For $E \to X$, $F \to Y$ vector bundles, $\mathcal{E}_M^{\mathrm{VB}}[E, F]$ is the set of all $(u_\varepsilon)_\varepsilon \in \mathrm{Hom}(E, F)^I$ satisfying

(i) $(\underline{u}_\varepsilon)_\varepsilon \in \mathcal{E}_M[X, Y]$.
(ii) $\forall k \in \mathbb{N}_0$, $\forall (V, \Phi)$ vector bundle chart in E, $\forall (W, \Psi)$ vector bundle chart in F, $\forall L \Subset V$, $\forall L' \Subset W$, $\exists N \in \mathbb{N}$, $\exists \varepsilon_1 > 0$, $\exists C > 0$ with

$$\left\| D^{(k)}\left(u_{\varepsilon \Psi \Phi}^{(2)}\left(\varphi(p)\right)\right) \right\| \leqslant C\varepsilon^{-N}$$

for all $\varepsilon < \varepsilon_1$ and all $p \in L \cap \underline{u}_\varepsilon^{-1}(L')$, with $\|\cdot\|$ any matrix norm.

Definition 3.9. $(u_\varepsilon)_\varepsilon, (v_\varepsilon)_\varepsilon \in \mathcal{E}_M^{\mathrm{VB}}[E, F]$ are called vb-equivalent, $((u_\varepsilon)_\varepsilon \sim_{\mathrm{vb}} (v_\varepsilon)_\varepsilon)$ if

(i) $(\underline{u}_\varepsilon)_\varepsilon \sim (\underline{v}_\varepsilon)_\varepsilon$ in $\mathcal{E}_M[X, Y]$.
(ii) $\forall k \in \mathbb{N}_0$, $\forall m \in \mathbb{N}$, $\forall (V, \Phi)$ vector bundle chart in E, $\forall (W, \Psi)$ vector bundle chart in F, $\forall L \Subset V$, $\forall L' \Subset W$, $\exists \varepsilon_1 > 0$, $\exists C > 0$ such that

$$\left\| D^{(k)}\left(u_{\varepsilon \Psi \Phi}^{(2)} - v_{\varepsilon \Psi \Phi}^{(2)}\right)\left(\varphi(p)\right) \right\| \leqslant C\varepsilon^{m}$$

for all $\varepsilon < \varepsilon_1$ and all $p \in L \cap \underline{u}_\varepsilon^{-1}(L') \cap \underline{v}_\varepsilon^{-1}(L')$.

We now set $\mathrm{Hom}_{\mathcal{G}}[E, F] := \mathcal{E}_M^{\mathrm{VB}}[E, F]/\sim_{\mathrm{vb}}$. For $u \in \mathrm{Hom}_{\mathcal{G}}[E, F]$, $\underline{u} := [(\underline{u}_\varepsilon)_\varepsilon]$ is a well-defined element of $\mathcal{G}[X, Y]$ uniquely characterized by $\underline{u} \circ \pi_X = \pi_Y \circ u$. The tangent map $Tu := [(Tu_\varepsilon)_\varepsilon]$ of any $u \in \mathcal{G}[X, Y]$ is then a well-defined element of $\mathrm{Hom}_{\mathcal{G}}[TX, TY]$.

Also in the context of generalized vector bundle homomorphisms a global characterization of moderateness is available:

Proposition 3.10. *Let* $(u_\varepsilon)_\varepsilon \in \mathrm{Hom}(E,F)^I$. *Then* $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M^{\mathrm{VB}}[E,F]$ *if and only if* $(\hat{f} \circ u_\varepsilon)_\varepsilon \in \mathcal{E}_M^{\mathrm{VB}}(E, \mathbb{R} \times \mathbb{R}^{m'})$ *for all* $\hat{f} \in \mathrm{Hom}(F, \mathbb{R} \times \mathbb{R}^{m'})$.

A similar statement holds for $\sim_{\mathrm{vb}}$ [22, Proposition 4.1 and Theorem 4.2]. Based on these results, appropriate point value descriptions of elements of $\mathrm{Hom}_{\mathcal{G}}[TX, TY]$ can be derived. As a final ingredient, in Theorem 3.12 below we shall make use of the *hybrid* space $\mathcal{G}^h[X,F]$ whose elements are defined on X and take values in F, c-bounded in the base component and moderate in the vector component [21,22]. All of the above constructions are functorial (with compositions defined unrestrictedly). We do not go into the details here (cf. [21,22]) but instead turn to another concept which is of relevance in applications to nonsmooth pseudo-Riemannian geometry (cf. Section 4). Denote by

$$\mathrm{Hom}_u(E,F) := \left\{ v \in \mathrm{Hom}(E,F) \mid \underline{v} = u \right\}$$

the space of generalized vector bundle homomorphisms over the generalized map u. While in the smooth setting the corresponding space can trivially be endowed with a vector space structure, the main obstruction in extending this property to the present context is that, a priori, representatives $(v_\varepsilon)_\varepsilon, (v'_\varepsilon)_\varepsilon$ of elements v, v' of $\mathrm{Hom}_u(E,F)$ need not project onto the same representative $(u_\varepsilon)_\varepsilon$ of $u = \underline{v} = \underline{v}' \in \mathcal{G}[X,Y]$, so that simple fiberwise addition is in general not possible. The following result [22, Proposition 5.7 and Corollary 5.8] remedies this problem.

Proposition 3.11. *Let* $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}[X,Y]$ *and* $v \in \mathrm{Hom}_u(E,F)$. *Then there exists a representative* $(v_\varepsilon)_\varepsilon$ *of* v *such that* $\underline{v}_\varepsilon = u_\varepsilon$ *for all* $\varepsilon \in I$. *Consequently,* $\mathrm{Hom}_u(E,F)$ *is a vector space.*

To conclude this section let us have a look at the problem of determining the flow of a generalized vector field $\xi \in \mathcal{G}_0^1(X)$. We first note that in the distributional setting already the notion of the flow of a distributional vector field ζ is problematic, as it would have to denote a "manifold-valued distribution." In [25], a regularization approach is used to cope with this problem, by introducing a c-bounded sequence of smooth vector fields ξ_ε approximating ζ. Each ξ_ε has a classical flow Φ^ε and under certain assumptions the assignment $\Psi = \lim_{\varepsilon \to 0} \Phi^\varepsilon$ allows to associate a measurable flow Ψ to the distributional vector field ζ. This approach is naturally related to the Colombeau picture, where any $\xi = (\xi_\varepsilon)_\varepsilon \approx \zeta$ can be viewed as a regularization of the distributional vector field ζ. We first give a basic existence and uniqueness result for flows of generalized vector fields [17, Theorem 3.6].

Theorem 3.12. *Let* (X,h) *be a complete Riemannian manifold and suppose that* $\xi \in \mathcal{G}_0^1(X)$ *satisfies*

(i) $\xi = [(\xi_\varepsilon)]$ *with each* ξ_ε *globally bounded with respect to* h.

(ii) *For each differential operator* $P \in \mathcal{P}(X, TX)$ *of first order and each* $K \Subset X$, $\sup_{p\in K} \|(P\xi_\varepsilon)|_p\|_h \leqslant C|\log\varepsilon|$ (*with h any Riemannian metric*).

Then there exists a unique generalized function $\Phi \in \mathcal{G}[\mathbb{R} \times X, X]$, *the generalized flow of* ξ, *such that*

$$\frac{d}{dt}\Phi(t,x) = \xi\bigl(\Phi(t,x)\bigr) \quad \text{in } \mathcal{G}^h[\mathbb{R}\times X, TX],$$
$$\Phi(0,.) = \mathrm{id}_X \quad \text{in } \mathcal{G}[X,X],$$
$$\Phi(t+s,.) = \Phi\bigl(t, \Phi(s,.)\bigr) \quad \text{in } \mathcal{G}[\mathbb{R}^2\times X, X].$$

Example 3.13. Let $X = T^2 = S^1 \times S^1$ and $\xi = [(\xi_\varepsilon)_\varepsilon] \in \mathcal{G}^1_0(X)$ with

$$\xi_\varepsilon(e^{i\alpha}, e^{i\beta}) = \bigl(e^{i\alpha}, e^{i\beta}; 1, 1-\rho_{\sigma(\varepsilon)}(\alpha)\bigr).$$

Here, ρ is a test function with unit integral and $\sigma(\varepsilon) = |\log(\varepsilon)|^{-1}$. Then since X is compact, each ξ_ε possesses a global flow Φ^ε and $\Phi := [(\Phi^\varepsilon)_\varepsilon] \in \mathcal{G}[\mathbb{R}\times X, X]$ is the unique generalized flow of ξ. Φ possesses a discontinuous pointwise limit Ψ, namely

$$\Phi^\varepsilon(t; e^{i\alpha}, e^{i\beta}) = \begin{pmatrix} e^{i(\alpha+t)} \\ e^{i(\beta+t-\int_\alpha^{\alpha+t}\rho_{\sigma(\varepsilon)}(\gamma)\,d\gamma)} \end{pmatrix} \to \begin{pmatrix} e^{i(\alpha+t)} \\ e^{i(\beta+t-H(\alpha+t)+H(\alpha))} \end{pmatrix},$$

which satisfies the flow property $\Psi_{s+t} = \Psi_s \circ \Psi_t$ for all $s, t \in \mathbb{R}$.

In general the question whether the unique generalized flow of a generalized vector field possesses a limiting (measurable) flow is quite involved, cf. [17,25].

4. Generalized connections and nonsmooth Riemannian geometry

Applications in general relativity have constituted one of the main driving forces behind the development of nonsmooth differential geometry in the setting of Colombeau generalized functions (see [34]). As an example, we consider so called *impulsive pp-waves* (i.e., impulsive gravitational waves with parallel rays, cf. [2,33]). These are described by a distributional pseudo-Riemannian metric with line-element

$$ds^2 = f(x,y)\delta(u)\,du^2 - du\,dv + dx^2 + dy^2. \tag{2}$$

To extract physically relevant information from this spacetime metric one has to be able to calculate curvature quantities and find solutions of the corresponding geodesic equations (determining the trajectories of particles in the spacetime at hand). However, all of these operations are undefined within linear distribution theory: the former due to the nonlinear operations involved in their calculation, the latter due to the lack of a concept of manifold-valued distributions. On the other hand, as we have seen in the previous sections, algebras of generalized functions make available all the necessary tools to address these issues.

The following result forms the basis for the description of singular pseudo-Riemannian metrics in the Colombeau framework [21, Theorem 3.1].

Theorem 4.1. *For any generalized* $(0,2)$*-tensor* $g \in \mathcal{G}^0_2(X)$, *the following are equivalent*:

(i) *For each chart* (V_α, ψ_α) *and each* $\tilde{p} \in (\psi_\alpha(V_\alpha))^\sim_c$ *the map* $g_\alpha(\tilde{p}) : \mathcal{K}^n \times \mathcal{K}^n \to \mathcal{K}^n$ *is symmetric and nondegenerate.*
(ii) $g : \mathcal{G}^1_0(X) \times \mathcal{G}^1_0(X) \to \mathcal{G}(X)$ *is symmetric and* $\det(g)$ *is invertible in* $\mathcal{G}(X)$.
(iii) $\det(g)$ *is invertible in* $\mathcal{G}(X)$ *and for each relatively compact open set* $V \subseteq X$ *there exists a representative* $(g_\varepsilon)_\varepsilon$ *of* g *and an* $\varepsilon_0 > 0$ *such that* $g_\varepsilon|_V$ *is a smooth pseudo-Riemannian metric for all* $\varepsilon < \varepsilon_0$.

Definition 4.2. Let $g \in \mathcal{G}^0_2(X)$ satisfy the conditions in Theorem 4.1. If, in addition, there exists $j \in \mathbb{N}_0$ such that the index of the g_ε as in Theorem 4.1(iii) equals j, we call g a generalized pseudo-Riemannian metric of index j and (X, g) a generalized pseudo-Riemannian manifold. If $j = 1$ or $j = n - 1$, (X, g) is called a generalized spacetime.

It follows from finite-dimensional perturbation theory that the index so defined does not depend on the chosen representative $(g_\varepsilon)_\varepsilon$ of g. With respect to applications, the most important characterization in Theorem 4.1 is (iii), as it guarantees that locally any generalized metric has a representative consisting entirely of smooth pseudo-Riemannian metrics.

We note first that the above way of modelling singular metrics is considerably more flexible than the purely distributional approach: In [25], a distributional $(0,2)$-tensor field $g \in \mathcal{D}'^0_2(X)$ is called nondegenerate if $g(\xi, \eta) = 0$ for all $\eta \in \mathfrak{X}(X)$ implies $\xi = 0 \in \mathfrak{X}(X)$, while in [30], g is called nondegenerate if it is nondegenerate (in the classical sense) off its singular support. The drawback of the first definition is its "nonlocality," which is too weak to reproduce the classical notion: e.g., $ds^2 = x^2\,dx^2$ is nondegenerate in this sense although it is clearly singular at $x = 0$. The second notion, on the other hand, does not provide any restrictions on g at its points of singularity.

Since $\mathcal{G}(X)$ is an algebra, all curvature quantities (Riemann tensor, Ricci and Einstein tensor...) of a generalized metric can be calculated unrestrictedly. Moreover, in parallel to the smooth setting, we may develop a generalized pseudo-Riemannian geometry based on the above notions. Our first basic result towards that goal is the following [21, Proposition 3.9].

Proposition 4.3. *Let* (X, g) *be a generalized pseudo-Riemannian manifold.*

(i) g *is nondegenerate in the following sense*: *if* $\xi \in \mathcal{G}^1_0(X)$*and* $g(\xi, \eta) = 0$, $\forall \eta \in \mathcal{G}^1_0(X)$, *then* $\xi = 0$.
(ii) g *induces a* $\mathcal{G}(X)$*-linear isomorphism* $\mathcal{G}^1_0(X) \to \mathcal{G}^0_1(X)$ *by* $\xi \mapsto g(\xi, \cdot)$.

The isomorphism in (ii) can naturally be extended to higher order tensor fields, so that, as in the smooth case, generalized metrics can be used to raise and lower indices.

Definition 4.4. A generalized connection $\hat{D}$ on X is a map $\mathcal{G}^1_0(X) \times \mathcal{G}^1_0(X) \to \mathcal{G}^1_0(X)$ satisfying

(D1) $\hat{D}_\xi \eta$ is $\mathcal{R}$-linear in η.

(D2) $\hat{D}_\xi \eta$ is $\mathcal{G}(X)$-linear in ξ.
(D3) $\hat{D}_\xi(u\eta) = u\hat{D}_\xi \eta + \xi(u)\eta$ for all $u \in \mathcal{G}(X)$.

With this notion we have the following *fundamental lemma of pseudo-Riemannian geometry* [21, Theorem 5.2]).

Theorem 4.5. *On each generalized pseudo-Riemannian manifold* (X, g) *there exists a unique generalized Levi-Civita connection* $\hat{D}$ *such that for all* ξ, η, ζ *in* $\mathcal{G}^1_0(X)$:

(D4) $[\xi, \eta] = \hat{D}_\xi \eta - \hat{D}_\eta \xi$ *and*
(D5) $\xi g(\eta, \zeta) = g(\hat{D}_\xi \eta, \zeta) + g(\eta, \hat{D}_\xi \zeta)$.

Suppose now that $\gamma \in \mathcal{G}[J, X]$ is a generalized curve in X defined on some interval $J \subseteq \mathbb{R}$. Using a representative $(g_\varepsilon)_\varepsilon$ as in Theorem 4.1(iii) we may componentwise define an induced covariant derivative $\xi \mapsto \xi'$ on the space $\mathfrak{X}_\mathcal{G}(u) := \{\xi \in \mathcal{G}^h[X, TY] \mid \underline{\xi} = u\}$ of generalized vector fields on γ. Its basic properties are summarized in the following result [21, Proposition 5.6] and [22, Section 5].

Theorem 4.6. *Let* (X, g) *be a generalized pseudo-Riemannian manifold and let* $\gamma \in \mathcal{G}[J, X]$. *Then*

(i) $(\tilde{r}\xi_1 + \tilde{s}\xi_2)' = \tilde{r}\xi_1' + \tilde{s}\xi_2'$ $(\tilde{r}, \tilde{s} \in \mathcal{K}, \xi_1, \xi_2 \in \mathfrak{X}_\mathcal{G}(\gamma))$.
(ii) $(u\xi)' = (du/dt)\xi + u\xi'$ $(u \in \mathcal{G}(J), \xi \in \mathfrak{X}_\mathcal{G}(\gamma))$.
(iii) $(\xi \circ \gamma)' = \hat{D}_{\gamma'(.)}\xi$ *in* $\mathfrak{X}_\mathcal{G}(\gamma)$ $(\xi \in \mathcal{G}^1_0(X))$.
(iv) $(d/dt)g(\xi, \eta) = g(\xi', \eta) + g(\xi, \eta')$ $(\xi, \eta \in \mathfrak{X}_\mathcal{G}(\gamma))$.

Note in particular that property (iv) only makes sense due to Proposition 3.11. Now that we have induced covariant derivatives at our disposal we may as in the smooth case (and contrary to the distributional setting) give the following definition.

Definition 4.7. A curve $\gamma \in \mathcal{G}[J, X]$ in a generalized pseudo-Riemannian manifold is called geodesic if $\gamma'' = 0$. Here γ'' is the induced covariant derivative of the velocity vector field γ' of γ.

Locally, therefore, the determination of the geodesics of a given singular metric amounts to the solution of a system of ordinary differential equations in the Colombeau setting. This program has been carried out for our first example (2) in [19,33]. Using a generic regularization procedure for the delta-term in (2), the resulting system is uniquely solvable in $\mathcal{G}[\mathbb{R}, X]$. Moreover, for $\varepsilon \to 0$ (i.e., in the sense of association) this unique solution displays the physically expected behavior of broken, refracted straight lines as geodesics.

As a further aspect of the spacetime (2) we note that its analysis naturally leads to the concept of manifold-valued generalized functions: In [31], R. Penrose introduced a discontinuous coordinate transformation T that formally transforms the distributional metric (2) into a continuous form. Although the two forms of the metric are physically equivalent (in the sense that they have the same geodesics), the transformation relating them is clearly ill-defined in the distributional picture. In [18], however, T was identified as an element $[(T_\varepsilon)_\varepsilon]$ of $\mathcal{G}[X, X]$ with each T_ε a diffeomorphism. In this sense T itself may be considered a "discontinuous diffeomorphism."

Recently, generalized pseudo-Riemannian geometry in the sense of the present section has been identified as a special case of an encompassing theory of generalized connections on fiber bundles. For this theory as well as for first applications to singular solutions of Yang–Mills equations we refer to [23].

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