A method for improving the performance of the WENO5 scheme near discontinuities

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WENO5 uses a convex combination of the polynomials reconstructed on the three stencils of ENO3 in order to achieve higher accuracy on smooth profiles. However, in some cases WENO5 generates oscillations or smears near discontinuities due to the time scheme used. Here, we present a method to reduce those oscillations without damping and this yields a sharper approximation. Our technique uses smoothness indicators to identify severe shocks and switches from WENO5 to ENO3. Numerical tests show that the behaviour of WENO5 is improved near discontinuities while preserving high accuracy on smooth profiles.

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1. Introduction

We present a modification of the fifth-order WENO [1] scheme for approximating the solution of hyperbolic conservation laws of the form

\[ u_t + f(u)_x = 0, \quad u \in \mathbb{R}^d, \quad d \geq 1, \]

with the initial data \( u(x, 0) = u_0(x) \). In recent years, there have been various works to improve the ENO [2] and WENO schemes; see [3–6]. In [1], it was observed that in the presence of discontinuities, WENO5 is not more accurate than third order and in [7], it was reported that a fixed value of \( \epsilon \) biases the stencil of WENO.

An outline of this work is as follows. We improve the resolution of WENO5 near discontinuities and corners in Section 2. We use WENO5 reconstruction in smooth regions and select the smoothest ENO3 stencil in the presence of severe shocks. These are identified by comparing the smoothness indicators of all the candidate stencils and in case one of them is much greater, then we use ENO3. We also adapt the parameter \( \epsilon \) on non-smooth regions. In Section 3 we test the new scheme on scalar and gas dynamics problems. We show that the main advantage of the improved scheme with respect to WENO5 is the reduction of oscillations near discontinuities.

2. An improved fifth-order WENO scheme

We consider a semi-discretisation of (1) on a uniformly spaced grid where each cell \( l_j = [x_{j-1/2}, x_{j+1/2}] \) has a width \( h \). We denote the cell centre by \( x_j = \frac{1}{2}(x_{j-1/2} + x_{j+1/2}) \) and the uniform time step by \( \Delta t = t^{n+1} - t^n \). Integrating (1) over the cell \( l_j \) leads to the semi-discrete equation that samples solutions at cell centres

\[ \frac{d\hat{u}_j(t)}{dt} = -\frac{1}{h} \left( f\left(u(x_{j+\frac{1}{2}}, t)\right) - f\left(u(x_{j-\frac{1}{2}}, t)\right)\right), \]

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where $\bar{u}_j \equiv \frac{1}{h} \int_{j}^{j+1} u(x, t) \, dx$ is the cell average of $u$ on $l_j$. The ENO3 approximations to the point $u(x_{j+\frac{1}{2}}, t)$ are based on interpolating polynomials, $p_r(x)$, where $x \in l_j$ and $r = 0, 1$ and 2 represent the right, central, and left stencils $S_r$, respectively. Then those grid points are obtained from the least discontinuous interpolations of the three candidate stencils (refer to [21]).

In [1], the procedure of selecting only one stencil was avoided by using the convex combination $u_{j+\frac{1}{2}} = \sum_{r=0}^{2} w_r p_r(x_{j+\frac{1}{2}})$. The nonlinear weights $w_r$ are given by

$$w_r = \frac{\alpha_r}{\alpha_0 + \alpha_1 + \alpha_2}, \quad r = 0, 1, 2,$$

where $\alpha_r = \frac{d_r h^{2s}}{1 - d_i + d_r h^{2s}}$, $d_r$ are linear weights and $\epsilon$, used to avoid divisions by zero can range from $10^{-5}$ to $10^{-7}$, [1]. However, in [7] and [6], it is mentioned that different values of $\epsilon$ changes the order of convergence of the scheme and yields different sharpness near discontinuities. Large values of $\epsilon$, say $10^{-6}$, causes the stencil to be biased toward central differencing. Smaller values of $\epsilon$ degrades the order of the predictions towards ENO3. The smoothness indicator $\beta_l$ is obtained from $L^2$ norm of the derivatives of $p_r(x)$ (refer to (3.1) of [1]). The smoothness indicators of smooth and discontinuous stencils yield $\beta_r = O(h^2)$ and $\beta_r = O(1)$ respectively.

We consider the case when the stencil $S_l$ contains a discontinuity, that is, $\beta_l$ is much greater than $\beta_r$ for $r \neq l$. Here, we pay attention to the case

$$\beta_l > \frac{1}{h^s} \beta_r,$$

where $s$ is yet to be determined. Then from (2), we get

$$w_r = \frac{d_r h^{2s}}{1 - d_i + d_r h^{2s}} \quad \text{if } r = l,$$

$$\frac{1}{1 - d_i} \quad \text{if } r \neq l.$$  

(4)

Since choosing more than one stencil on discontinuous profiles will not improve the accuracy any further and may generate oscillations due to the time-stepping scheme employed, see [8,9], we improve the resolution of WENO5 by considering only the non-discontinuous stencils $S_l$ where $w_r = O(1)$ and neglecting those where $w_r = O(h^2)$. This leads to $s = 1$ in (4). We use (3) to identify the presence of shocks and then we approximate the flux at the grid points by ENO3.

The test case 5 of [10] shows that WENO5 with $\epsilon = 10^{-6}$ is sharper than ENO3 and WENO5 with $\epsilon = 10^{-9}$. However, the combination of various stencils by WENO5 with $\epsilon = 10^{-6}$ generates oscillations near discontinuities which are avoided by the other two schemes. Moreover, in [6], it is showed that fixed small $\epsilon$, $10^{-40}$ in that case, combined with WENO5 drops the rate of convergence for some problems. Following [7], we propose a new adaptive $\epsilon$ which depends on the smoothness of the stencils and the difference between the $\beta$s, and $10^{-99}$ is used in order to avoid division by zero. In smooth regions we choose $\epsilon \sim 10^{-6}$ and near shocks $\epsilon \sim 10^{-99}$.

$$\epsilon = 10^{-6} \min \left(1, \frac{\min \beta_r}{\max \beta_r - \min \beta_r + 10^{-99}} \right) + 10^{-99}.$$  

(5)

We summarize the improved scheme in the following algorithm.

if $\beta_l > \beta_r / h$, \quad $l \neq r$, \quad $r = 0, 1, 2$,

use ENO3,

else

use WENO5 with $\epsilon$ given by (5),

end

Here, we advance in time using the fifth-order linear multistep method TVB$_0(5, 5)$ due to [11], which is stable up to CFL 0.377052834833475. The first time steps are obtained from the third-order optimal SSPRK method [8]. The stability of many high-order time methods, including TVB$_0(5, 5)$, relies on the that of the forward Euler. However, in [9], it is shown that WENO5 is in some sense unstable with forward Euler. Nevertheless, ENO does not necessarily suffers from this instability with a biased choice of the stencils [12]. As illustrated in the next section, our proposed scheme, which combines WENO5 on smooth regions with only non-discontinuous ENO stencils, reduces the oscillations compared to WENO5.

3. Numerical experiments

We first describe the results of numerical experiments using some scalar problems with periodic boundary conditions over the domain $[-1, 1]$ and CFL 0.2.

Test 3.1: We test our scheme on linear advection case 1 of [10] with the initial condition $u(x, 0) = \sin(\pi x)$ at $T = 1$. The results obtained are fifth-order accurate and similar to WENO5 on smooth profiles (results omitted due to space constraints).
Test 3.2: Here, we solve the linear advection test case 2 of [10]. The results on 100 cells and at time $T = 2$ are shown in Fig. 1. We observe that WENO5 with $\epsilon = 10^{-6}$ generates oscillations of magnitude $10^{-4}$ near corners, whereas WENO5 with $\epsilon = 10^{-99}$ is smeared. However, our scheme is sharp while maintaining a non-oscillatory profile. For comparison, we run WENO5 with (5) and observe that it still generates oscillations (results omitted due to space constraint).

Test 3.3: Next we solve the Burgers' equation test case 3 of [10] for $N = 200$. The results in Fig. 2 are up to time $T = 0.64$, before leading and trailing discontinuities coalesce. We observe that WENO5 with $\epsilon = 10^{-6}$ generates oscillations of magnitude $10^{-4}$ near discontinuities. Our scheme reduces the oscillations, while giving a sharper approximation of the rarefaction wave than WENO5 with $\epsilon = 10^{-99}$.

Next, we extend our scheme to solve (1) with $d > 1$. We consider the Euler equations of gas dynamics for a polytropic gas, where $u = (\rho, \rho q, E)^T, f(u) = (\rho q, \rho q^2 + p, q(E + p))^T$ and $p = (\gamma - 1)(E - \frac{1}{2} \rho q^2)$. Here $\rho, q, p$ and $E$ are the density, velocity, pressure and total energy respectively, and $\gamma = 1.4$. The numerical schemes are extended componentwise and we solve the Euler equations before the perturbations in the solutions reach the boundary of the domain $[-5, 5]$.

Shock-Entropy Test: We consider the schemes' performance in smooth regions and their ability to capture shocks on the moving Mach 3 shock of [13]. Fig. 3 shows the computed pressure for $N = 200$ against the reference solution computed by WENO5 with $\epsilon = 10^{-6}$ for $N = 4000$ at $T = 1.8$. We observe in Fig. 3(b) that the new scheme reduces the oscillations which are generated by WENO5 with $\epsilon = 10^{-6}$ and is sharper than WENO5 with $\epsilon = 10^{-99}$. Moreover, in the reference solution, WENO5 still oscillates near discontinuities even for refined grids. We make similar observations on the velocity profile and Sod's problems [14].

Conclusion

We developed a new technique for combining WENO5 with ENO3, and an adaptive $\epsilon$ in the weights was presented. In the regions of shocks, only the least discontinuous stencil is used instead of a combination which may generate oscillations. Experiments showed that our scheme improved the approximation of discontinuities while giving high accuracy in smooth regions. However, the observed contributions were less than $1e - 4$ for quantities that are $O(1)$ in size. Lastly, though it is
recognised that the choice of $\epsilon$ influences the results, a more thorough theoretical analysis shall be a valuable contribution to the limited available literature on this topic.

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References