ELSEVIER

Discrete Mathematics 146 (1995) 159–167

DISCRETE MATHEMATICS

The modular n-queens problem in higher dimensions

Scott P. Nudelman*

Department of Mathematics, Washington University in St. Louis, St. Louis, MO 63130, USA

Received 24 August 1993; revised 18 January 1994

Abstract

Let $M(n,d)$ denote the maximum number of queens on a d-dimensional modular chessboard such that no two attack each other. We show that if $\gcd(n, (2^d-1)!) = 1$ then $M(n,d) = n$. We also prove that if $\gcd(n,(2^d-1)!) > 1$ then there are no complete linear solutions, and if $\gcd(n,(2d-1)!) > 1$ then $M(n,d) < n$. Moreover, if $n \leqslant 2^d - 1$ we show $M(n,d) = 1$.

1. Introduction

Combinatorial problems on chessboards have a long history. One of the best known problems is to determine the maximum number of non-attacking queens for an $n \times n$ chessboard. Ahrens [1] showed that for $n \geqslant 4$, n queens can be placed on a $n \times n$ chessboard. However, in the case of modular chessboards the results are not quite so straightforward. The results of the modular n-queen problem are summarized by the following theorem [5].

Theorem 1. (i) $M(n,2) = n$ *if* $\gcd(n,6) = 1$,
(ii) $M(n,2) = n - 1$ *if* $\gcd(n,12) = 2$,
(iii) $M(n,2) = n - 2$ *otherwise.*

In this paper we explore how the modular conditions on n change as we generalize the chessboard problem to higher dimensions. Let $M(n,d)$ denote the maximum number of queens on a d-dimensional modular chessboard such that no two attack each other. Then $M(n,2)$ is the well-known two dimensional case, which has already been studied in detail [2–4, 5]. After generalizing the concept of chessboards and queens to higher dimensions, we will proceed to first generate a class of solutions

* Correspondence address: 5819 Back Bay Lane, Austin, TX 78739, USA.

SSDI 0012-365X(94)00161-8

for certain values of n, then explore restrictions on complete solutions for other values of n.

2. Preliminaries

In the original modular n-queen problem two queens, located at (x_1, x_2) and (y_1, y_2) respectively, attack each other if one of the following conditions is satisfied:

$$x_1 \equiv y_1 \pmod n, \tag{1}$$

$$x_2 \equiv y_2 \pmod n, \tag{2}$$

$$x_1 + x_2 \equiv y_1 + y_2 \pmod n, \tag{3}$$

$$x_1 - x_2 \equiv y_1 - y_2 \pmod n. \tag{4}$$

In other words, the two queens attack each other if they are colinear on one of the four standard lines. We can summarize these four equations by saying a queen located at (x_1, x_2) attacks (y_1, y_2) if for some choice of $\varepsilon_1, \varepsilon_2 \in \{-1, 0, 1\}$, where not both are zero, we have the following:

$$\varepsilon_1 x_1 + \varepsilon_2 x_2 \equiv \varepsilon_1 y_1 + \varepsilon_2 y_2 \pmod n. \tag{5}$$

This compact formulation leads to a useful generalization of the non-attacking queens problems in higher dimensions. We say two queens *attack* each other in d-dimensions if they lie on a common modular hyperplane. In other words the queens located at $\boldsymbol{x} = (x_1, \ldots, x_d)$ and $\boldsymbol{y} = (y_1, \ldots, y_d)$ attack each other if there exists some non-zero $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_d)$ with $\varepsilon_i \in \{-1, 0, 1\}$, such that

$$\boldsymbol{\varepsilon} \cdot \boldsymbol{x} \equiv \boldsymbol{\varepsilon} \cdot \boldsymbol{y} \pmod n, \tag{6}$$

where

$$\boldsymbol{\varepsilon} \cdot \boldsymbol{x} = \sum_{i=1}^{d} \varepsilon_i x_i. \tag{7}$$

From this definition we see that the number of hyperplanes a queen can attack in increases as the number of dimensions increases. In particular, we have the following result.

Proposition 2. *A d-dimensional queen attacks in $\frac{1}{2}(3^d - 1)$ hyperplanes.*

Proof. Consider all $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_d)$, there are $3^d - 1$ non-zero possibilities. However, both $\boldsymbol{\varepsilon}$ and $-\boldsymbol{\varepsilon}$ define the same hyperplane of attack. □

Since each attack hyperplane can be represented by two different $\boldsymbol{\varepsilon}$, we will only deal with the vector in which the first non-zero ε_i is 1.

Definition 3. A *complete linear solution* of the n queens problem in d dimensions is a set of n non-attacking queens, where the ith queen is at the location $(c_1 i + e_1, \ldots, c_d i + e_d)$ where $i = 1, 2, \ldots, n$.

Without loss of generality, we can assume all of the e's are zero by the translational invariance of the modular board.

So with these definitions, we can now consider the problem of maximizing the number of non-attacking queens that can be put on a d-dimensional chessboard modulo n. We begin by noting that,

$$M(n,d) \leqslant n, \tag{8}$$

due to the pigeonhole principle. In addition,

$$M(n,d+1) \leqslant M(n,d), \tag{9}$$

since any set of non-attacking queens in $(d+1)$-dimensions is also a set of non-attacking queens in d-dimensions simply by ignoring the last coordinate.

3. Constructing a complete solution

Theorem 4. *If* $\gcd(n,(2^d - 1)!) = 1$, *then* $M(n,d) = n$.

Proof. Consider the set of n queens located at $(i, 2i, \ldots, 2^{d-1} i)$, where $i = 1, 2, \ldots, n$. We claim that no two of these queens attack each other. Suppose instead that there exists distinct $j, k \in \{1, 2, \ldots, n\}$ and some $(\varepsilon_1, \ldots, \varepsilon_d)$ such that:

$$\left(\sum_{i=1}^{d} \varepsilon_i 2^{i-1}\right) j \equiv \left(\sum_{i=1}^{d} \varepsilon_i 2^{i-1}\right) k \pmod{n}. \tag{10}$$

Given the trivial upper bound,

$$\left|\sum_{i=1}^{d} \varepsilon_i 2^{i-1}\right| \leqslant 2^d - 1, \tag{11}$$

we see the above sum is relatively prime to n, so we can cancel it from Eq. (10) to get

$$j \equiv k \pmod{n}, \tag{12}$$

a contradiction. □

4. Conditions on existence of a complete solution

In the preceding section we have seen that a complete linear solution exists for certain values of n. In this section we find restrictions on the existence of complete solutions in all cases not previously constructed. To do so, we will need the following lemma.

Lemma 5. *Let S be any k element subset of $\{1,2,\dots,m\}$. If $2^k > m$, then two disjoint subsets of S have the same sum modulo m.*

Proof. There are 2^k subsets of S. Since $2^k > m$, by the pigeonhole principle two subsets of S will have the same sum modulo m. If we remove all common elements from these two subsets, they will be disjoint and still be congruent modulo m. □

Lemma 5 leads us directly to two results. The first shows that if n is fairly small, then two non-attacking queens cannot be placed on the board. This will have some deeper implications to be discussed later in the paper.

Theorem 6. *If $n \leqslant 2^d - 1$, then $M(n,d) = 1$.*

Proof. Without loss of generality, we place the first queen on the origin. Assume we can put a second non-attacking queen on the board. Let the location of the second queen be $\boldsymbol{x} = (x_1, x_2, \dots, x_d)$. Since $n \leqslant 2^d - 1$, we have from Lemma 5 that there are two disjoint subsets of the x_i's whose sums are congruent modulo n. Denote these two subsets by A and B. Consider the hyperplane generated by $\boldsymbol{\varepsilon}$, in which ε_i is 1 if $x_i \in A$, -1 if $x_i \in B$, and 0 otherwise. Then the two queens attack each other in this hyperplane, since $\boldsymbol{\varepsilon} \cdot \boldsymbol{x} \equiv 0 \pmod n$. Thus, $M(n,d) = 1$. □

A second consequence of Lemma 5 provides us with a weaker converse of Theorem 4. The cases constructed in Theorem 4 are the only cases constructible by that method. Moreover, if a complete solution exists for any other values of n, the solution will be non-linear.

Theorem 7. *If $\gcd(n,(2^d - 1)!) > 1$, then there does not exist a complete linear solution on the modulo n chessboard in d-dimensions.*

Proof. Let p be a prime that divides $\gcd(n,(2^d - 1)!)$. Then $p|n$ and $p \leqslant 2^d - 1$. Assume there is a complete linear solution, $(c_1 i, c_2 i, \dots, c_d i)$ where $i = 1,2,\dots,n$. We know all the c's are distinct. If not, there would exist j and k such that $c_j = c_k$, and all queens would be in the hyperplane defined by

$$x_j - x_k \equiv 0 \pmod n. \tag{13}$$

So consider the set $C = \{c_1, \dots, c_d\}$. Since $p \leqslant 2^d - 1$, two disjoint subsets of C have the same sum modulo p. Denote them by A and B. Now define the hyperplane such that ε_j is 1 if $c_j \in A$, -1 if $c_j \in B$, 0 otherwise. In this hyperplane, for some m:

$$\sum_{j=1}^{d} \varepsilon_j c_j \equiv mp \pmod n. \tag{14}$$

Consider any queen in the linear solution where $i = an/p$, $a = 1, 2, \ldots, p$. Then for any such queen we have

$$\sum_{j=1}^{d} \varepsilon_j c_j i \equiv 0 \pmod{n}. \tag{15}$$

Thus all p of these queens attack each other. Therefore, there can be no complete linear solution in this case. □

Now we will turn our attention to a stronger theorem which tells us when no complete solution can exist. But first, we need the following lemmas.

Lemma 8. *If p is an odd prime, then for all α, $\sum_{i=1}^{p^\alpha} i^{p-1} \not\equiv 0 \pmod{p^\alpha}$.*

Proof. We proceed by induction on α. For $\alpha = 1$, by Fermat's Little Theorem, we have that if p does not divide i, then

$$i^{p-1} \equiv 1 \pmod{p}. \tag{16}$$

Thus

$$\sum_{i=1}^{p} i^{p-1} \equiv \sum_{i=1}^{p-1} i^{p-1} \equiv \sum_{i=1}^{p-1} 1 \equiv p - 1 \not\equiv 0 \pmod{p}. \tag{17}$$

Now we will assume the assertion is true for $\alpha - 1$. Then

$$\sum_{i=1}^{p^\alpha} i^{p-1} \equiv \sum_{j=0}^{p-1} \sum_{i=1}^{p^{\alpha-1}} (i + jp^{\alpha-1})^{p-1} \pmod{p^\alpha} \tag{18}$$

$$\equiv \sum_{j=0}^{p-1} \sum_{i=1}^{p^{\alpha-1}} \sum_{k=0}^{p-1} \binom{p-1}{k} j^k p^{k(\alpha-1)} i^{p-1-k} \pmod{p^\alpha}, \tag{19}$$

and if $\alpha \geqslant 2$ and $k \geqslant 2$ then

$$p^{k(\alpha-1)} \equiv 0 \pmod{p^\alpha}, \tag{20}$$

and the sum simplifies to

$$\sum_{i=1}^{p^\alpha} i^{p-1} \equiv \sum_{j=0}^{p-1} \sum_{i=1}^{p^{\alpha-1}} (i^{p-1} + j(p-1)p^{\alpha-1} i^{p-2}) \pmod{p^\alpha}. \tag{21}$$

Summing over the index j yields

$$\sum_{i=1}^{p^\alpha} i^{p-1} \equiv p \sum_{i=1}^{p^{\alpha-1}} i^{p-1} + \frac{1}{2}(p-1)^2 p^\alpha \sum_{i=1}^{p^{\alpha-1}} i^{p-2} \tag{22}$$

$$\equiv p \sum_{i=1}^{p^{\alpha-1}} i^{p-1} \pmod{p^\alpha}. \tag{23}$$

Since,

$$\sum_{i=1}^{p^{\alpha-1}} i^{p-1} \not\equiv 0 \pmod{p^{\alpha-1}}, \tag{24}$$

we know that

$$\sum_{i=1}^{p^{\alpha}} i^{p-1} \not\equiv 0 \pmod{p^{\alpha}}. \qquad \square \tag{25}$$

Lemma 9. *If p is an odd prime and* $\sum_{i=1}^{n} i^{p-1} \equiv 0 \pmod{n}$, *then p does not divide n.*

Proof. Assume p divides n, and write $n = p^{\alpha}m$ where p does not divide m. Then,

$$0 \equiv \sum_{i=1}^{n} i^{p-1} \equiv \sum_{j=0}^{m-1} \sum_{i=1}^{p^{\alpha}} (i + jp^{\alpha})^{p-1} \equiv m \sum_{i=1}^{p^{\alpha}} i^{p-1} \pmod{p^{\alpha}}. \tag{26}$$

Since m is relatively prime to p^{α}, we can cancel the m to obtain

$$\sum_{i=1}^{p^{\alpha}} i^{p-1} \equiv 0 \pmod{p^{\alpha}}. \tag{27}$$

However, this contradicts the previous lemma. $\square$

Lemma 9 is an essential ingredient in the proof of our final theorem. Now we find it necessary to break the set of attack hyperplanes into different families, based on the number of non-zero ε_i's, in their representation. Let τ_s denote the set of all $\boldsymbol{\varepsilon}$ such that exactly s of $\boldsymbol{\varepsilon}$'s components are non-zero. The last step in preparation for the proof is the following algebraic identity.

Lemma 10. *Let* $\boldsymbol{x} = (x_1, \ldots, x_d)$. *Then*

$$\sum_{i=1}^{d} (-2)^{d-i} \sum_{\forall \boldsymbol{\varepsilon} \in \tau_i} (\boldsymbol{\varepsilon} \cdot \boldsymbol{x})^{2d-2} = 0. \tag{28}$$

Proof. Consider the polynomial given by the inner sum:

$$\sum_{\forall \boldsymbol{\varepsilon} \in \tau_i} (\varepsilon_1 x_1 + \cdots + \varepsilon_d x_d)^{2d-2}. \tag{29}$$

It will only contain terms of the form $x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}$, where all the exponents are even. Any term with odd exponents occurs exactly as many times with a positive sign as it does with a negative sign, and sums to zero. But a term with all even exponents always occurs with a positive sign in this sum. Consider the term $x_{j_1}^{2\alpha_1} \cdots x_{j_k}^{2\alpha_k}$, where $\alpha_i > 0$, and $\alpha_1 + \cdots + \alpha_k = d - 1$. It will always occur with the same coefficient in the expansion of $\binom{d-k}{i-k} 2^{i-1}$ terms of the sum. And each of these terms is multiplied by

$(-2)^{d-i}$ from the outer sum. So the number of times the term occurs is:

$$\sum_{i=k}^{d}(-2)^{d-i}\binom{d-k}{i-k}2^{i-1}=(-1)^{d}2^{d-1}\sum_{i=k}^{d}(-1)^{i}\binom{d-k}{i-k}=0. \tag{30}$$

Thus the total sum is zero. □

We mention one final point before we prove the next theorem. Consider any hyperplane of attack $\varepsilon=(\varepsilon_1,\ldots,\varepsilon_d)$, and a set of n non-attacking queens, where the ith queen is at the location $x^{(i)}=(x_1^{(i)},\ldots,x_d^{(i)})$. Then for $i\neq j$, $\varepsilon\cdot x^{(i)}\neq\varepsilon\cdot x^{(j)}$, since no two queens attack each other. Thus the set $\{\varepsilon\cdot x^{(i)} \pmod n,\ i=1,\ldots,n\}$ is merely a permutation of the numbers $1,\ldots,n$.

Theorem 11. *If* $M(n,d)=n$, *then* $\gcd(n,(2d-1)!)=1$.

Proof. We proceed by induction on d. The base case $d=2$ is the well-known result of the original modular n-queens problem.

Now assume that the statement is true for $d-1$. We consider two cases: (i) $2d-1$ is composite, and (ii) $2d-1$ is prime.

Case (i): By Eq. (9), we know that since $M(n,d)=n$, then $M(n,d-1)=n$. Then by the inductive hypothesis $\gcd(n,(2d-3)!)=1$. But since $2d-1$ is not prime, all prime factors of $(2d-1)!$ have already occured in $(2d-3)!$, so we get the desired condition $\gcd(n,(2d-1)!)=1$.

Case (ii): In a manner almost identical to case (i), we obtain the result that $\gcd(n,(2d-2)!)=1$. So now all we must show is that if $2d-1$ is prime and $M(n,d)=n$, then $2d-1$ does not divide n. Consider the following equation. From Lemma 10, we know that the sum over the index i is zero. Thus,

$$\sum_{j=1}^{n}\sum_{i=1}^{d}(-2)^{d-i}\sum_{\forall\varepsilon\in\tau_i}(\varepsilon\cdot x^{(j)})^{2d-2}=0. \tag{31}$$

Switching order of the summation, we see that

$$\sum_{i=1}^{d}(-2)^{d-i}\sum_{\forall\varepsilon\in\tau_i}\sum_{j=1}^{n}(\varepsilon\cdot x^{(j)})^{2d-2}\equiv 0 \pmod n. \tag{32}$$

Since the numbers $\varepsilon\cdot x^{(j)}$, for $j=1,\ldots,n$ represent a permutation of $1,\ldots,n$, the inner most sum simplifies to:

$$\sum_{j=1}^{n}(\varepsilon\cdot x^{(j)})^{2d-2}\equiv\sum_{j=1}^{n}j^{2d-2} \pmod n. \tag{33}$$

Combining the above result with the fact that there are $\binom{d}{i}2^{i-1}$ elements in τ_i, we obtain

$$\sum_{i=1}^{d}(-2)^{d-i}\binom{d}{i}2^{i-1}\sum_{j=1}^{n}j^{2d-2}\equiv(-2)^{d-1}\sum_{j=1}^{n}j^{2d-2}\equiv 0 \pmod n. \tag{34}$$

But we know by the base case that n is odd, thus

$$\sum_{j=1}^{n} j^{2d-2} \equiv 0 \pmod{n}. \tag{35}$$

So, by Lemma 9, $2d-1$ does not divide n. Therefore, $\gcd(n,(2d-1)!) = 1$. □

5. Conclusions

The previous theorems serve to give us insight into when a complete solution can exist. If a complete solution exists for the d dimensional modular n chessboard problem, no prime less than $2d$ can divide n, but we can only generate solutions when no prime less than 2^d divides n. In two dimensions, $2 \cdot 2 = 2^2$, so we have a full characterization of when a complete solution exists. In higher dimensions, these two cases diverge, and there exists a class of integers n where a solution may exist, but we were unable to generate one. However, we were able to show that no linear solutions could exist for this class of n.

In two dimensions, we have seen that any time a solution exists, then a linear solution exists. So in higher dimensions, we are led to wonder if for some n a complete solution can exist without the existence of a linear solution. This leads us to the following conjecture.

Conjecture 1. For all positive integers n and d, $M(n,d) = n$ if and only if $\gcd(n,(2^{d-1})!) = 1$.

In the two dimensional case complete solutions exist if and only if both 2 and 3 do not divide n. If these kinds of divisibility requirements were to generalize to higher dimensions, Theorem 6 would imply the above conjecture.

Several other questions also remain open. Determining $M(n,d)$ in the case of incomplete solutions is still wide open. For some integer m_d depending on d, can $M(n,d)$ be determined based on the value of n modulo some m_d, in a manner similar to that of the two dimensional case [2–5]. It would also be interesting to determine a lower bound for $M(n,d)$, independent of any modular value of n. For instance, in two dimensions we know $M(n,2) \geqslant n-2$ [5]. Does there exist some δ_d such that $M(n,d) \geqslant n - \delta_d$? If there is, Theorem 6 implies that $\delta_d \geqslant 2^d - 2$.

Another interesting line of investigation would be to generalize the original n-queens problem to higher dimensions, using analagous hyperplanes of attack. Does there exist some integer n_d, such that if $n \geqslant n_d$, then n non-attacking queens can be placed on the non-modular d-dimensional chessboard? In two dimensions, Ahrens has shown $n_2 = 4$. We know a complete solution to the modular problem will also serve as a solution to the non-modular problem as well.

Acknowledgements

This work was done under the supervision of Joseph A. Gallian, at the University of Minnesota, Duluth, with financial support from the National Science Foundation (grant number DMS-9225045) and the National Security Agency (grant number MDA 904-91-H-0036). The author wishes to thank Joseph Gallian and Mike Reid for their encouragement and suggestions. The author is also grateful for the referee's comments and reference to the article by Monsky.

References

[1] W. Ahrens, Mathematische Unterhaltungen und Spiele (Berlin, 1910).

[2] O. Heden, On the modular n-queen problem, Discrete Math. 102 (1992) 155–161.

[3] T. Klöve, The modular n-queen problem, Discrete Math. 19 (1977) 289–291.

[4] T. Klöve, The modular n-queen problem II, Discrete Math. 36 (1981) 33–48.

[5] P. Monsky, Solution of problem E3162, Amer. Math. Monthly 96 (1989) 258–259.