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On a Minimal Counterexample to Dade's Projective Conjecture

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INTRODUCTION

Dade's projective conjecture (in the form we prefer to state it; see also [1]) predicts that whenever Z is a central p-subgroup of a finite group H, λ is a linear character of Z, and B is a p-block of H whose defect group strictly contains Z, then we should have

$$\sum_{\sigma \in \mathcal{N}(H,Z)/H} (-1)^{|\sigma|} k_d(B_\sigma, \lambda) = 0.$$

Here, $\mathcal{N}(H, Z)$ consists of the chains (strictly increasing under usual inclusion) of p-subgroups of H of the form $\sigma = (Q_0(=Z) < Q_1 \cdots < Q_n)$, where each $Q_i \lhd Q_n$. There is an obvious conjugation action of H on $\mathcal{N}(H, Z)$, and $\mathcal{N}(H, Z)/H$ indicates that we are taking a set of representatives for the H-orbits under this action. The stabilizer of the chain σ is denoted by H_σ, and B_σ denotes the sum of the Brauer correspondent blocks of B for the subgroup H_σ. The notation $k_d(B_\sigma, \lambda)$ means that we consider the irreducible (complex) characters χ associated to the blocks B_σ whose restriction to Z is a multiple of λ and which satisfy $p^d \chi(1)_p = |H_\sigma|_p$. This is the number of irreducible characters of defect d in B_σ which lie over λ.

It is no loss of generality to assume that $Z = O_p(H)$, for otherwise the alternating sums in question all vanish for trivial reasons.

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In this paper, we show that the cancellation theorem proved by the second author in [4] can be used in conjunction with some of the Clifford theory for blocks developed by E. C. Dade in [1] to further restrict the structure of a minimal counterexample to Dade's projective conjecture (henceforth referred to as DPC). One of the points of the approach we adopt here is that we wish to pursue as far as possible the reduction of the (relatively easily stated) basic form of DPC to a group as close to simple as possible. In the announcements made so far in Dade's own programme of reductions, it appears that the conjectures involved have become increasingly complex. We feel therefore that the direct approach mentioned above has some value. This is not just for aesthetic reasons, but also for the convenience of those who wish to contribute to the programme of verifying the conjectures, so that:

(a) They will know precisely what would be required to be checked to complete an inductive proof of the conjecture.

(b) The simplest possible form of the conjecture is left to be verified.

We prove here that a (putative) minimal counterexample G to DPC has a unique conjugacy class of components (a component being a quasi-simple subnormal subgroup). It already follows from Theorem 1 of [3] that such a group G has its Fitting subgroup $F(G) \leq Z(G)$. The ultimate objective of any programme of Clifford-theoretic reductions for this conjecture is to reduce to the case where a purported counterexample G has $F^*(G)/Z(G)$, a non-Abelian simple group, where, as usual, $F^*(G)(= E(G)F(G))$ denotes the generalized Fitting subgroup of G, $E(G)$ being the central product of the components of G.

From the point of view of the logical structure of performing such a reduction, it seems that our main result should afford a simplification, since, even if it were to prove necessary to make stronger assumptions for inductive reasons to complete the final step of the reduction, it would not be necessary to prove the stronger statements for *all* finite groups: at worst, it would only be necessary to prove such stronger statements for finite groups H with $F(H) \leq Z(H)$ and $E(H)$ a central product of a single conjugacy class of components.

As in [3], the ring R denotes here a complete discrete valuation ring of characteristic 0 whose residue field $F = R/J(R)$ is algebraically closed of characteristic p, and which also contains sufficiently many p-power roots of unity. The field of fractions of R is denoted by $\mathbb{K}$.

Our main theorem here is:

THEOREM 1. *Suppose that the formula appearing in DPC fails to hold* (*for some defect d and some linear character*, λ, *of the central subgroup* $Z = O_p(G)$) *for the block B of RG, and that first* $[G : Z(G)]$ *and then* $|G|$ *have* *been minimized subject to such a failure occurring. Then whenever* M *is a non-central normal subgroup of* G *containing* $Z(G)$, *we have* $C_G(M) = Z(G)$ *and* $F^*(M) > F(M)$.

Remarks. The conclusion of Theorem 1 is equivalent to asserting that G has central Fitting subgroup and a unique conjugacy class of components. For if G has a unique conjugacy class of components and a central Fitting subgroup, then any non-central normal subgroup M of G which contains $Z(G)$ must contain $F^*(G)$, in which case we certainly have $F^*(M) > F(M)$ and $C_G(M) \leq Z(F(G)) = Z(G)$. On the other hand, if G satisfies the conclusion of the theorem, then G has components, as $F^*(G) > F(G)$. Furthermore, $F(G) \leq C_G(E(G)) = Z(G)$ (the last equality by hypothesis). Finally, G must have a single conjugacy class of components, for if M_1 is the central product of a conjugacy class of components of G, then any component of G not contained in M_1 would be contained in $C_G(M_1) = Z(G)$, which is absurd.

We note also that the choice of G forces λ to be a faithful linear character of Z (otherwise, DPC would still fail for a block of $RG/ker\,\lambda$).

1. DADE CORRESPONDENCE

Let us consider a finite group X with a normal subgroup Y and an irreducible character ζ of Y. Let I be the inertial subgroup of ζ in X. Let B be a block of RX. Let $Irr(B, \zeta)$ denote the set of irreducible characters in B which lie over (X-conjugates of) ζ, and assume that $Irr(B, \zeta)$ is non-empty. Then each element of $Irr(B, \zeta)$ is induced from a (unique) irreducible character of I which lies over ζ (we let $Irr(I, \zeta)$ denote the set of such characters of I, and we retain similar notation throughout).

Let $\beta \in Irr(I, \zeta)$, and suppose that $\gamma = Ind_I^X(\beta)$ lies in B. Let B' be the block of RI containing β. Then (as is well known) all irreducible characters in B' which lie over ζ induce to irreducible characters in $Irr(B, \zeta)$. For a routine (and instructive) computation shows that for each $a \in X$, we have $\gamma(a^X)/\gamma(1) = \beta(a^X \cap I)/\beta(1)$ (where, by a slight abuse, we let a^X denote the sum in RX of the elements of the conjugacy class of a in X, an abuse we will continually indulge in). More generally, it follows for the same reason that characters in $Irr(I, \zeta)$ which lie in the same p-block induce to irreducible characters in the same p-block of X. However, it need not be the case that irreducible characters in $Irr(I, \zeta)$ which lie in different blocks of I induce to irreducible characters in different blocks of G.

Now let us (now, and for the remainder of this section) consider the case that $X = I$; that is to say, ζ is X-stable. By standard Clifford theory, there is a finite central extension $\widetilde{X}$ of X/Y having a cyclic central subgroup $\widetilde{W}$, which in turn has a linear character $\tilde{\mu}$ such that there is a bijection between $Irr(X, \zeta)$ and $Irr(\widetilde{X}, \tilde{\mu})$. Furthermore, this bijection may be explicitly realised as follows: there is a finite central extension

$$1 \to \widehat{W} \to \widehat{X} \to X \to 1$$

with $\widehat{W}$ cyclic such that $\widehat{X}$ has a normal subgroup $\widehat{Y}$ (canonically isomorphic to Y) such that $\widehat{Y} \cap \widehat{W} = 1_{\widehat{X}}$. The irreducible character of $\widehat{Y}$ canonically identified with ζ may be extended to an irreducible character of $\widehat{X}$, denoted by $\hat{\zeta}$. Then $\hat{\zeta}$ lies over a unique (faithful) linear character of $\widehat{W}$, say $\bar{\hat{\mu}}$ (we use the complex conjugate here for notational convenience elsewhere).

The group $\widetilde{X}$ is $\widehat{X}/\widehat{Y}$, which has a cyclic central subgroup $\widetilde{W}$, canonically isomorphic to $\widehat{W}$ (in fact the image of $\widehat{W}$ under the natural homomorphism). The linear character $\tilde{\mu}$ is the linear character of $\widetilde{W}$ canonically identified with $\hat{\mu}$. Given an irreducible character $\tilde{\alpha}$ of $\widetilde{X}$ which lies over $\tilde{\mu}$, we may regard it as an irreducible character $\hat{\alpha}$ of $\widehat{X}$ which lies over $\bar{\hat{\mu}}$, and then $\alpha = \hat{\alpha}\hat{\zeta}$ may be viewed as an irreducible character of X which lies over ζ. All elements of $Irr(X, \zeta)$ arise uniquely in this fashion.

We now describe briefly how *Dade correspondent* blocks of $\widetilde{X}$ may be constructed. Strictly speaking, we are usually interested not in the blocks themselves, but in those irreducible characters in them which lie over the linear character $Res^{\widetilde{W}}_{O_p(\widetilde{W})}(\tilde{\mu})$ of $O_p(\widetilde{W})$. This reconciles our usage with the use of blocks of twisted group algebras in Dade [1]. Once this fact is noted, what follows below is just a more elementary description of the constructions in Dade [1, especially Sects. 13 and 14].

Let $\hat{\alpha}\hat{\zeta}$ be as above. Choose an element $\hat{a} \in \widehat{X}$ whose image in $\widetilde{X}$ is denoted by $\tilde{a}$. Let $\widehat{H}(\hat{a})$ be the full pre-image in $\widehat{X}$ of $C_{\widetilde{X}}(\tilde{a})$. Notice that $\widehat{H}(\hat{a})$ contains $\widehat{Y}\widehat{W}$, so that $\hat{\zeta}$ restricts to an irreducible character of $\widehat{H}(\hat{a})$. In particular, $\left[\widehat{H}(\hat{a}) : C_{\widehat{X}}(\hat{a})\right]\hat{\zeta}(\hat{a})/\hat{\zeta}(1)$ is an element of R. We note that $\hat{a}^{\widehat{Y}}$, the $\widehat{Y}$-conjugacy class sum of a, is already represented by a scalar matrix in the representation affording $\hat{\zeta}$ (even if $\hat{a} \notin \widehat{Y}$. The point to notice is that this element commutes with all of $\widehat{Y}$ and $Res^{\widehat{X}}_{\widehat{Y}}(\hat{\zeta})$ is irreducible).

It follows that $\left[\widehat{H}(\hat{a}) : C_{\widehat{X}}(\hat{a})\right]\hat{\zeta}(\hat{a})/\hat{\zeta}(1)$ is an element of $J(R)$ unless $\widehat{Y}C_{\widehat{X}}(\hat{a})$ contains a Sylow p-subgroup of $\widehat{H}(\hat{a})$.

Notice that we now have

$$\frac{\left[\widehat{X} : C_{\widehat{X}}(\hat{a})\right]\hat{\alpha}(\hat{a})}{\hat{\alpha}(1)} = \frac{\left[\widetilde{X} : C_{\widetilde{X}}(\tilde{a})\right]\tilde{\alpha}(\tilde{a})}{\tilde{\alpha}(1)} \frac{\left[\widehat{H}(\hat{a}) : C_{\widehat{X}}(\hat{a})\right]\hat{\zeta}(\hat{a})}{\hat{\zeta}(1)}.$$

It follows from this equation that if $\tilde{\alpha}$ and $\tilde{\beta}$ are irreducible characters of $Irr(\widetilde{X}, \tilde{\mu})$ which belong to the same p-block, then the corresponding irreducible characters α and β both lie in the same p-block of X and lie over ζ. (The observant reader might be concerned by our passage from $\widehat{X}$ to X. If the element $\hat{a}$ is conjugate to $\hat{a}\hat{z}$ for a non-identity element $\hat{z}$ of $\widehat{W}$, then $\hat{\zeta}(\hat{a}) = 0$, so that $\hat{\alpha}(\hat{a}) = 0$. Otherwise, $\hat{a}$ has the same number of conjugates in $\widehat{X}$ as its image does in X. In either case, the left-hand side of the equation above tells us how to evaluate the corresponding central character of RX.)

Given the block B of RX such that $Irr(B, \zeta)$ is non-empty, consider the set of all irreducible characters $\tilde{\alpha} \in Irr(\widetilde{X}, \tilde{\mu})$ such that $\alpha \in Irr(B, \zeta)$. This forms the set of irreducible characters lying over $Res^{\widetilde{W}}_{O_p(\widetilde{W})}(\tilde{\mu})$ in a sum of blocks of $R\widetilde{X}$ lying over $Res^{\widetilde{W}}_{O_{p'}(\widetilde{W})}(\tilde{\mu})$. We call the blocks which occur in this sum the set of *Dade correspondents* of B associated to ζ, and in the opposite direction, we say that B is a Dade correspondent of any of these blocks. The elements of $Irr(B, \zeta)$ are in bijection with the irreducible characters lying over $Res^{\widetilde{W}}_{O_p(\widetilde{W})}(\tilde{\mu})$ in the Dade correspondents of B associated to ζ. The elements of defect d in the former set are in bijection with the elements of defect $\tilde{d}$ in the latter set, where $\tilde{d} = d + \log_p(\zeta(1)_p|\widetilde{W}|_p/|Y|_p)$. Each block of $R\widetilde{X}$ which lies over $Res^{\widetilde{W}}_{O_{p'}(\widetilde{W})}(\tilde{\mu})$ occurs as the Dade correspondent of a unique block of RX.

We give an elementary proof that Dade correspondence commutes with Brauer correspondence in the situation that we consider. This may also be deduced from 14.3 of [1]. As usual, if χ is an irreducible character of a finite group M, we let ω_χ denote the central character of RM defined by $\omega_\chi(c) = \frac{\chi(c)}{\chi(1)}$ for all $c \in Z(RM)$. We let ω_B denote the central character of FM associated to the block B of RM, which may be computed by calculating the residue (mod $J(R)$) of ω_χ for any $\chi \in Irr(B)$. For ease of notation, we retain the abuse introduced earlier that a^M denotes the class sum of $a \in M$ in RM. When L is a subgroup of M, we will let π_L denote the projection from RM onto RL with kernel $R[M\backslash L]$.

Let T be a subgroup of X which contains Y. Then, still using ζ, we may repeat the earlier arguments with T in place of X, defining $\widehat{T}$ and $\widetilde{T}$ in the obvious way. Let $\tilde{b}$ be a block of $R\widetilde{T}$ with $Irr(\tilde{b}, \tilde{\lambda})$ non-empty, and suppose that the Brauer correspondent $\tilde{b}^{\widetilde{X}} = \widetilde{B}$ is defined. Let b and B be the Dade correspondents of $\tilde{b}$ and $\widetilde{B}$, respectively. To show that Dade correspondence commutes with Brauer correspondence in this situation, it suffices to demonstrate that the Brauer correspondent of b for X is defined and is B.

Let $\tilde{\gamma} \in Irr(\tilde{b}, \tilde{\mu})$ and $\tilde{\alpha} \in Irr(\widetilde{B}, \tilde{\mu})$. Choose $\hat{a} \in \widehat{X}$. We must show that $\omega_{\hat{b}} \circ \pi_{\widehat{T}}(\hat{a}^{\widehat{X}}) \equiv \omega_{\widehat{B}}(\hat{a}^{\widehat{X}})$ (mod $J(R)$).

An easy computation shows that (with congruences taken (mod $J(R)$))

$$\omega_{\hat{b}} \circ \pi_{\widehat{T}}(\hat{a}^{\widehat{X}}) \equiv \omega_{\hat{\gamma}} \circ \pi_{\widehat{T}}(\hat{a}^{\widehat{X}}) = \frac{\mathrm{Ind}_{\widehat{T}}^{\widehat{X}}(\hat{\gamma})[\hat{a}^{\widehat{X}}]}{\mathrm{Ind}_{\widehat{T}}^{\widehat{X}}(\hat{\gamma})[1]}$$

$$= \frac{\mathrm{Ind}_{\widehat{T}}^{\widehat{X}}(\tilde{\gamma}\hat{\zeta})[\hat{a}^{\widehat{X}}]}{\mathrm{Ind}_{\widehat{T}}^{\widehat{X}}(\tilde{\gamma}\hat{\zeta})[1]} = [\widehat{X} : C_{\widehat{X}}(\hat{a})] \frac{\mathrm{Ind}_{\widehat{T}}^{\widehat{X}}(\tilde{\gamma}\hat{\zeta})[\hat{a}]}{\mathrm{Ind}_{\widehat{T}}^{\widehat{X}}(\tilde{\gamma}\hat{\zeta})[1]}.$$

This is equal to

$$\frac{1}{[\widetilde{X} : C_{\widetilde{X}}(\tilde{a})]} \cdot \frac{[\widehat{X} : C_{\widehat{X}}(\hat{a})]\hat{\zeta}(\hat{a})}{\hat{\zeta}(1)} \cdot \frac{\mathrm{Ind}_{\widetilde{T}}^{\widetilde{X}}(\tilde{\gamma})[\tilde{a}^{\widetilde{X}}]}{\mathrm{Ind}_{\widetilde{T}}^{\widetilde{X}}(\tilde{\gamma})[1]} (*).$$

We recall that $\widehat{H}(\hat{a})$ is the full inverse image in $\widehat{X}$ of $C_{\widetilde{X}}(\tilde{a})$, so that we have

$$[\widehat{H}(\hat{a}) : C_{\widehat{X}}(\hat{a})] = \frac{[\widehat{X} : C_{\widehat{X}}(\hat{a})]}{[\widetilde{X} : C_{\widetilde{X}}(\tilde{a})]}.$$

Furthermore, we recall that $[\widehat{H}(\hat{a}) : C_{\widehat{X}}(\hat{a})]\hat{\zeta}(\hat{a})/\hat{\zeta}(1) \in R$.

Since $\gamma \in \tilde{b}$, we now see that $(*)$ is congruent (mod $J(R)$) to

$$\frac{[\widehat{H}(\hat{a}) : C_{\widehat{X}}(\hat{a})]\hat{\zeta}(\hat{a})}{\hat{\zeta}(1)} \cdot \omega_{\tilde{b}} \circ \pi_{\widetilde{T}}(\tilde{a}^{\widetilde{X}})(**).$$

Since $\tilde{b}^{\widetilde{X}} = \widetilde{B}$, we see that $(**)$ is congruent (mod $J(R)$) to

$$\frac{[\widehat{H}(\hat{a}) : C_{\widehat{X}}(\hat{a})]\hat{\zeta}(\hat{a})}{\hat{\zeta}(1)} \omega_{\widetilde{B}}(\tilde{a}^{\widetilde{X}}) \equiv \frac{[\widehat{H}(\hat{a}) : C_{\widehat{X}}(\hat{a})]\hat{\zeta}(\hat{a})}{\hat{\zeta}(1)} \omega\alpha(\tilde{a}^{\widetilde{X}}).$$

This last expression is equal to $\omega_{\hat{\alpha}}(\hat{a}^{\widehat{X}})$, as we have already seen.

We now have $\omega_{\hat{b}} \circ \pi_{\widehat{T}}(\hat{a}^{\widehat{X}}) \equiv \omega_{\widehat{B}}(\hat{a}^{\widehat{X}})$ (mod $J(R)$), which suffices to show that $b^X = B$.

2. THE CENTRALIZER OF A NORMAL SUBGROUP

Let G be our purported minimal counterexample to DPC. We know from [3] that $F(G) \leq Z(G)$. Let M be the full pre-image in G of a minimal normal subgroup of $G/Z(G)$. Then $M = Z(G)E$, where E is a central product of a single G-conjugacy class of components of G (and $E = M'$). Let $N = C_G(E)$. Then N is a normal subgroup of G such that $M \cap N = Z(G)$.

Let B be a block of RG for which the equality predicted by DPC fails to hold (with respect to the linear character λ of the central p-subgroup $Z = O_p(G)$, for some defect d). We suppose for the moment that $N > Z(G)$. We will show presently that this assumption leads to a contradiction. By Theorem 1 of [4] (applied with our M in the role of N of that theorem), we know that B covers a G-stable block of RM, say b, whose defect group strictly contains Z.

Also from Theorem 1 of [4] (again applied to our M), we know that

$$\sum_{\sigma \in \mathcal{N}(M, Z)/G} (-1)^{|\sigma|} k_d(B_\sigma, \lambda) \neq 0$$

(using our assumption of failure of DPC). Notice that $N \leq G_\sigma$ for each $\sigma \in \mathcal{N}(M, Z)$. For $\zeta \in Irr(N, \lambda)$, we count the contribution to the above alternating sum from irreducible characters of defect d which lie over G-conjugates of ζ. Let ζ^G denote the G-orbit of ζ.

Then we may choose ζ so that

$$\sum_{\sigma \in \mathcal{N}(M, Z)/G} (-1)^{|\sigma|} \sum_{\gamma \in \zeta^G/G_\sigma} k_d(B_\sigma, \gamma) \neq 0.$$

Notice that in the double sum, we are really summing over G-orbits of ordered pairs (σ, γ), where $\sigma \in \mathcal{N}(M, Z)$ and γ is a G-conjugate of ζ. We may change the order of summation to obtain

$$\sum_{\sigma \in \mathcal{N}(M, Z)/I_G(\zeta)} (-1)^{|\sigma|} k_d(B_\sigma, \zeta) \neq 0.$$

Let B' denote the sum of blocks of $RI_G(\zeta)$ whose irreducible characters lying over ζ induce to characters in B. Then the central character, ω_B, of FG associated to B satisfies $\omega_B(x^G) = \omega_{B'}(\pi_I(x^G))$, where x^G denotes the class sum of x and π_I denotes vector space projection onto $FI_G(\zeta)$ with kernel $F[G \backslash I_G(\zeta)]$.

On the other hand, we also have $\omega_B(x^G) = \omega_{B_\sigma}(\pi_\sigma(x^G))$, where π_σ denotes vector space projection onto FG_σ with kernel $F[G \backslash G_\sigma]$, and ω_{B_σ} denotes any of the central characters associated to the sum of blocks B_σ.

A similar remark applies within $I_G(\zeta)$, so we see that

$$\sum_{\sigma \in \mathcal{N}(M, Z)/I_G(\zeta)} (-1)^{|\sigma|} k_d(B_\sigma, \zeta) = \sum_{\sigma \in \mathcal{N}(M, Z)/I_G(\zeta)} (-1)^{|\sigma|} k_d(B'_\sigma, \zeta) \neq 0$$

(it may be helpful to recall that $1_{B_\sigma} = Br_{V^\sigma}(1_B)$, and that induction of irreducible characters is defect preserving).

There must be a block B^* which is an indecomposable summand of B' such that

$$\sum_{\sigma \in \mathcal{N}(M, Z)/I_G(\zeta)} (-1)^{|\sigma|} k_d(B^\star_\sigma, \zeta) \neq 0.$$

It remains to show that we may replace $B^\star$ by the sum of the Dade correspondent blocks associated to ζ of an appropriate central extension of $I_G(\zeta)/N$ and obtain a similar non-vanishing alternating sum. Once this is done, we will easily obtain a contradiction.

Let I denote $I_G(\zeta)(\geq MN)$. As before, we construct an appropriate central extension of I, say $\widehat{I}$, by a cyclic central subgroup $\widehat{W}$ so that ζ extends to an irreducible character, $\hat{\zeta}$ say, of $\widehat{I}$. We let $\tilde{I} = \widehat{I}/\widehat{N}$, where $\widehat{N}$ is the natural isomorphic copy of N inside $\widehat{I}$, satisfying $\widehat{N} \cap \widehat{W} = 1_{\widehat{I}}$. Notice that $[\widetilde{I} : Z(\widetilde{I})] \leq [I_G(\zeta) : N] < [G : Z(G)]$, so that DPC holds in $\widetilde{I}$ (and all its sections) by the choice of G.

Now $N = C_G(M)$ by definition, and $Z(I)$ centralizes $NM \geq F^*(G)$, so that $Z(I) \leq Z(F(G)) = Z(G)$. Also $[O_p(I), M] = [O_p(I), F^*(N)] = 1$, so that $[O_p(I), F^*(G)] = 1$ and $O_p(I) \leq Z(G)$. Hence $Z = O_p(I)$. We note also that $O_p(\widetilde{M}) = O_p(\widetilde{W})$, as MN/N is a direct product of non-Abelian simple groups (where $\widetilde{M}$ is the full preimage in $\widetilde{I}$ of MN/N). Now if L is a subgroup of G such that $[L, M] \leq N$, then we have $[L, M] \leq M \cap N = Z(G)$, so that $[L, M'] = 1$ by the three subgroups lemma. Hence $[L, M] = 1$, as $M = M'Z(G)$. It follows that MN/N has trivial centralizer in G/N. Consequently, we conclude that $F(\widetilde{I}) = \widetilde{W}$.

Since $M \cap N = Z(G)$, it is elementary to check that there is a natural bijection between $\mathcal{N}(M, Z)$ and $\mathcal{N}(MN/N, 1)$, induced by the correspondence of p-subgroups which lets $Q \geq Z$ correspond to QN/N. Consequently, there is a natural bijection between $\mathcal{N}(M, Z)$ and $\mathcal{N}(\widetilde{M}, O_p(\widetilde{W}))$, induced by letting the p-subgroup $Q(\geq Z)$ of M correspond to the unique Sylow p-subgroup of the (nilpotent) pre-image of QN/N in $\widetilde{I}$.

Furthermore, these bijections respect chain stabilizers and conjugacy in the obvious fashion. (For example, the full pre-image of a p-subgroup of MN/N in MN has the form QN for some p-subgroup Q of M (containing Z). If P and Q are p-subgroups (containing Z) of M such that PN/N and QN/N are conjugate via an element xN of I/N, then $P^x \leq (QN \cap M) = Q(M \cap N) = QZ(G)$, so that (as P is a p-group) $P^x \leq QZ = Q$. Similarly, we deduce that $Q^{x^{-1}} \leq P$, so that $P^x = Q$.) It is also worth noting that if Q is a p-subgroup of M containing Z, and $xN \in C_{I/N}(QN/N)$, then for each $u \in Q$, we have $u^x \in uN \cap M = uZ(G)$, so that $u^x \in uZ$ as u is a p-element. If x is p-regular, this forces $u^x = u$ by a standard and well-known argument. Hence the p-regular elements in $C_{I/N}(QN/N)$ are precisely the images of the p-regular elements of $C_I(Q)$.

For a chain $\sigma \in \mathcal{N}(M, Z)$, let $\tilde{\sigma}$ be the corresponding chain in $\mathcal{N}(\widetilde{M}, O_p(\widetilde{W}))$. We recall our equation

$$\sum_{\sigma \in \mathcal{N}(M, Z)/I_G(\zeta)} (-1)^{|\sigma|} k_d(B^\star_\sigma, \zeta) \neq 0.$$

Let $\widetilde{B}$ denote the set of Dade correspondents of $B^{\star}$ associated to ζ. Then the fact that Dade correspondence and Brauer correspondence commute allows us to conclude that $\widetilde{B}_{\tilde{\sigma}}$ is the sum of the Dade correspondents of $(B^{\star})_{\sigma}$. Hence we obtain

$$\sum_{\sigma \in \mathcal{N}(\widetilde{M},\, O_p(\widetilde{W}))/\widetilde{I}} (-1)^{|\tilde{\sigma}|} k_{\tilde{d}}\Big(\widetilde{B}_{\tilde{\sigma}},\, Res^{\widetilde{W}}_{O_p(\widetilde{W})}(\tilde{\mu})\Big) \neq 0,$$

where $\tilde{d} = d + \log_p\big(\zeta(1)\big|\widetilde{W}\big|/|N|\big)$.

Since DPC holds in $\widetilde{I}$ and all its sections, the proof of Theorem 1 of [4] may be applied within $\widetilde{I}$ to each block summand of $\widetilde{B}$. This tells us that

$$\sum_{\sigma \in \mathcal{N}(\widetilde{M},\, O_p(\widetilde{W}))/\widetilde{I}} (-1)^{|\tilde{\sigma}|} k_{\tilde{d}}\Big(\widetilde{B}_{\tilde{\sigma}},\, Res^{\widetilde{W}}_{O_p(\widetilde{W})}(\tilde{\mu})\Big) = \sum_{\sigma \in \mathcal{N}(\widetilde{I},\, O_p(\widetilde{W}))/\widetilde{I}} (-1)^{|\tilde{\sigma}|} k_{\tilde{d}}\Big(\widetilde{B}_{\tilde{\sigma}},\, Res^{\widetilde{W}}_{O_p(\widetilde{W})}(\tilde{\mu})\Big),$$

unless some block summand of $\widetilde{B}$ covers a block of $\widetilde{M}$ with central defect group.

Since this alternating sum is non-zero, while DPC holds for $\widetilde{I}$, we must conclude that one of the block summands of $\widetilde{B}$ covers a block of $\widetilde{M}$ which has defect group $O_p(\widetilde{W})$.

By the results of Külshammer and Robinson [2, especially Corollary 2.4 and Theorem 2.5], this means that some block in $B^{\star}$ contains an irreducible character, γ say, lying over an N-projective irreducible character of MN. Let θ be an irreducible constituent of $Res^{I}_{MN}(\gamma)$. Then θ has the form $\hat{\zeta}\tilde{\beta}$ for some irreducible character $\tilde{\beta}$ of $\widetilde{M}$, lying in a block with defect group $O_p(\widetilde{W})$.

But, as a character of MN, θ factorizes as a product $\zeta\beta$ for some irreducible character β of N. As projective representations (in Schur's sense), the projective representations of M affording β and $\tilde{\beta}$ may be afforded by the same underlying RM-module (the representing matrices for a given group element differ only by a root of unity scalar multiple in the two representations), and the M-algebra structure of the associated endomorphism ring is exactly the same in both cases. Since the module affording $\tilde{\beta}$ is $O_p(\widetilde{W})$-projective, we see that the RM-module affording β is Z-projective.

This means in turn that some block in $B^{\star}$ covers a block of RM with defect group Z. Hence B covers a block of M with central defect group, contrary to hypothesis.

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