Further Results on the Reverse-Order Law

Nobuo Shinozaki
Department of Mathematics
Keio University
832 Hiyoshi, Yokohama
Japan 223

and

Masaaki Sibuya
Tokyo Scientific Center
IBM Japan
3-2-12 Roppongi, Tokyo
Japan 106

Submitted by R. S. Varga

ABSTRACT

An explicit expression is obtained for a pair of generalized inverses (B^-,A^-) such that $B^-A^-=(AB)^+_{MN}$, and a class of pairs (B^-,A^-) of this property is shown. A necessary and sufficient condition for $(AB)^-$ to have the expression B^-A^- is also given.

1. INTRODUCTION

Let A and B be matrices of complex elements such that AB can be defined. T. N. E. Greville [7, 3] has given a necessary and sufficient condition for the reverse-order law of the Moore-Penrose inverse of matrices, $(AB)^+=B^+A^+$, to hold. His theorem has been extended in several directions. See, for example, [1, 2, 4, 5, 6, 8, 9]. The authors [9] have shown, among other things, that there always exists a pair of generalized inverses (B^-,A^-) such that $B^-A^-=(AB)^+$, but the pair was characterized indirectly. In Theorem 1 of this note we give an explicit expression for a pair (B^-,A^-) such that $B^-A^-=(AB)^+_{MN}$, the minimum-N-norm M-least-squares generalized inverse. [The Moore-Penrose inverse is its special case: $(AB)^+$

 0024-3795/79/050009+8$01.75

$=(AB)^+_{II}$.] The pair is not unique, and a class of pairs with this property is shown in Theorem 3, which is a generalization of a result by Barwick and Gilbert [1].

The inverse $(AB)^+_{MN}$ can always be expressed as B^-A^-, but it is not always possible to express a given generalized inverse $(AB)^-$ as B^-A^-. In Theorem 6 we give a necessary and sufficient condition on $(AB)^-$ to have the expression B^-A^-. The existence of $G\in\{A^-\}$ such that $B^-G\in\{(AB)^-\}$ for any B^- is also noticed.

We use the notation in Rao and Mitra's book [8]. A^- (A_r^-) belongs to $A\{1\}$ $(A\{1,2\})$ respectively as defined in Ben-Israel and Greville's book [3]. $A^-_{l(M)}$, where M is a square nonnegative definite matrix, is a generalized inverse of A such that $MAA^-_{l(M)}$ is hermitian. $A^-_{m(N)}$, where N is a square positive definite matrix, is a generalized inverse of A such that $NA^-_{m(N)}A$ is hermitian. A^+_{MN} is the unique matrix which is A_r^-, $A^-_{l(M)}$ and $A^-_{m(N)}$. $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the range space and the null space of A respectively.

2. FURTHER RESULTS ON THE REVERSE ORDER LAW

Let A,B be any matrices of complex elements with the product AB. Theorem 3.2 in [9] states that there exist $G_2\in\{B^-_{m(I)r}\}$ and $G_1\in\{A^-_{l(I)r}\}$ such that $G_2G_1=(AB)^+$, and they are constructed so that $G_2\in\{B^-_{l(A^*A)}\}$ and $G_1^*\in\{(A^*)^-_{l(BB^*)}\}$. In the following theorem we give an explicit expression for a pair (B^-,A^-) such that $B^-A^-=(AB)^+_{MN}$. We assume that M and N are positive definite matrices.

THEOREM 1.

(a) *Put*

$$G_2=B^-_{m(N)r}+(AB)^+_{MN}A-(AB)^+_{MN}ABB^-_{m(N)r} \tag{1}$$

and

$$G_1=A^-_{l(M)r}+B(AB)^+_{MN}-BG_2A^-_{l(M)r}. \tag{2}$$

Then $G_2\in\{B^-_{l(A^*MA)m(N)r}\}$, $G_1\in\{A^-_{l(M)r}\}$ *and* $G_2G_1=(AB)^+_{MN}$.

(b) *Put*

$$G_1=A^-_{l(M)r}+B(AB)^+_{MN}-A^-_{l(M)r}AB(AB)^+_{MN} \tag{3}$$

and

$$G_2 = B^-_{m(N)r} + (AB)^+_{MN}A - B^-_{m(N)r}G_1A. \tag{4}$$

Then $G_1 \in \{A^-_{l(M)r}\}$, $G_1^* \in \{(A^*)^-_{l(BN^{-1}B^*)}\}$, $G_2 \in \{B^-_{m(N)r}\}$ *and* $G_2G_1 = (AB)^+_{MN}$.

Proof. (a): Check $G_2BN^{-1}B^* = N^{-1}B^*$, $G_2BG_2 = G_2$ and $B^*A^*MABG_2 = B^*A^*MA$ to show that G_2 belongs to $\{B^-_{m(N)r}\}$ and $\{B^-_{l(A^*MA)}\}$. Check $A^*MAG_1 = A^*M$ and $G_1AG_1 = G_1$ to show $G_1 \in \{A^-_{l(M)r}\}$. Then $G_2G_1 = G_2B(AB)^+_{MN} = (AB)^+_{MN}$. (b) is a dual statement of (a). ∎

After submission of this paper the authors learned of a paper by Wibker, Howe and Gilbert [11] from the editor of this journal. The expressions in our Theorem 1 are essentially the same as that in their Theorem 2 and have simpler forms.

In the expressions (1) and (2) [(3) and (4)] the choice $B^-_{m(N)r}$ and $A^-_{l(M)r}$ is arbitrary, though two $B^-_{m(N)r}$'s and two $A^-_{l(M)r}$'s must be the same. So the pair (G_2, G_1) is not unique. Theorem 3 shows this fact more clearly. To prove the theorem, we note the following fact.

LEMMA 2.

(a) *Let* B^- *and* $(AB)^-_{l(M)}$ *be arbitrarily given. There exists* $G \in \{A^-\}$ *such that* $B^-G = B^-B(AB)^-_{l(M)}$, *which belongs to* $\{(AB)^-_{l(M)}\}$, *iff (if and only if)* $\mathfrak{R}(A^*) \cap \mathfrak{R}\{(B^-)^*\} \subset \mathfrak{R}(A^*MAB)$. *When the condition is satisfied, G can be taken from* $\{A^-_{l(M)}\}$.

(b) *Let* A^- *and* $(AB)^-_{m(N)}$ *be arbitrarily given. There exists* $G \in \{B^-\}$ *such that* $GA^- = (AB)^-_{m(N)}AA^-$, *which belongs to* $\{(AB)^-_{m(N)}\}$, *iff* $\mathfrak{R}(B) \cap \mathfrak{R}(A^-) \subset \mathfrak{R}(BN^{-1}B^*A^*)$. *When the condition is satisfied, G can be taken from* $\{B^-_{m(N)}\}$.

Proof. (a): The matrix G must satisfy the equations

$$AGA = A \quad \text{and} \quad B^-G = B^-B(AB)^-_{l(M)}, \tag{5}$$

which have particular solutions

$$G_1 = A^- + B(AB)^-_{l(M)} - A^-AB(AB)^-_{l(M)} \quad \text{and} \quad G_2 = B(AB)^-_{l(M)}, \tag{6}$$

respectively. Let C be any matrix such that $\mathcal{R}(C^*)=\mathcal{R}(A^*)\cap\mathcal{R}\{(B^-)^*\}$. Then the equations (5) have a common solution iff (see Shinozaki and Sibuya [10])

$$C[A^- + B(AB)^-_{l(M)} - A^-AB(AB)^-_{l(M)} - B(AB)^-_{l(M)}]A = 0,$$

or equivalently

$$C = CA^-AB(AB)^-_{l(M)}A \quad\Leftrightarrow\quad \mathcal{R}(C^*)\subset\mathcal{R}(A^*MAB). \tag{7}$$

If $G\in\{A^-_{l(M)}\}$ is wanted, replace the first equation of (5) by $A^*MAG=A^*M$, and A^- in (6) by $A^-_{l(M)}$. Then the first condition of (7) becomes $C(A^-_{l(M)} - A^-_{l(M)}AB(AB)^-_{l(M)})=0$, which is, however, equivalent to $\mathcal{R}(C^*)\subset\mathcal{R}(A^*MAB)$. (b) can be proved similarly. ■

THEOREM 3.

(a) *Let B_r^- be arbitrarily given. There exists $G\in\{A^-\}$ such that $B_r^-G=(AB)^+_{MN}$ iff $\mathcal{R}(B_r^-)\supset\mathcal{R}(N^{-1}B^*A^*)$ and $\mathcal{R}(A^*)\cap\mathcal{R}\{(B_r^-)^*\}\subset\mathcal{R}(A^*MAB)$. If such a matrix G exists, it can be taken from $\{A^-_{l(M)r}\}$.*

(b) *Let A_r^- be arbitrarily given. There exists $G\in\{B^-\}$ such that $GA_r^- = (AB)^+_{MN}$ iff $\mathcal{R}\{(A_r^-)^*\}\supset\mathcal{R}(MAB)$ and $\mathcal{R}(B)\cap\mathcal{R}(A_r^-)\subset\mathcal{R}(BN^{-1}B^*A^*)$. If such a G exists, it can be taken from $\{B^-_{m(N)r}\}$.*

Proof. (a): Since $\mathcal{R}\{(AB)^+_{MN}\}=\mathcal{R}(N^{-1}B^*A^*)$, the condition $\mathcal{R}(B_r^-)\supset\mathcal{R}(N^{-1}B^*A^*)$ is necessary. If the condition of the theorem is satisfied, $B_r^-B(AB)^+_{MN}=(AB)^+_{MN}$. If $G\notin\{A_r^-\}$, replace it by GAG_1, where $G_1\in\{A^-_{l(M)}\}$. (b) can be proved similarly. ■

REMARKS. (1) If $B_r^-\in\{B^-_{l(A^*MA)m(N)r}\}$, then $\mathcal{R}(B_r^-)=\mathcal{R}(N^{-1}B^*)\supset\mathcal{R}(N^{-1}B^*A^*)$ and $\mathcal{R}(A^*)\cap\mathcal{R}\{(B_r^-)^*\}\subset\mathcal{R}\{(B_r^-)^*B^*A^*\}=\mathcal{R}\{(B_r^-)^*B^*A^*MA\}=\mathcal{R}(A^*MAB)$. Therefore the conditions of (a) are satisfied. If $A_r^-\in\{A^-_{l(M)r}\}$ and $(A_r^-)^*\in\{(A^*)^-_{l(BN^{-1}B^*)}\}$, the conditions of (b) are satisfied.

(2) If $\operatorname{rank} AB = \operatorname{rank} A$, the condition $\mathcal{R}(A^*)\subset\mathcal{R}\{(B_r^-)^*\}\subset\mathcal{R}(A^*MAB)$ is satisfied for any B_r^-, and the remaining condition $\mathcal{R}(B_r^-)\supset\mathcal{R}(N^{-1}B^*A^*)$ is satisfied for any $B_r^-\in\{B^-_{m(N)r}\}$. A similar remark also holds for A_r^-.

(3) Barwick and Gilbert [1] have shown that $\mathcal{R}(A^*)\cap\mathcal{R}(B)\subset\mathcal{R}(A^*AB)$ iff there exists $G\in\{A^-_{l(I)r}\}$ such that $B^+G=(AB)^+$. This is a special case of Theorem 3(a).

COROLLARY 4. *Let* $(AB)_r^-$ *be arbitrarily given. There always exist* $G_2 \in \{B_r^-\}$ *and* $G_1 \in \{A_r^-\}$ *such that* $G_2G_1 = (AB)_r^-$.

Proof. This is clear because any reflexive generalized inverse X_r^- is X_{MN}^+ for the positive definite matrices $M = (XX_r^-)^* V_1 XX_r^- + (I - XX_r^-)^* V_2 (I - XX_r^-)$ and $N = (X_r^- X)^* V_3 X_r^- X + (I - X_r^- X)^* V_4 (I - X_r^- X)$, where the V_i's are any positive definite matrices. ■

Let, hereinafter, A and B be $l \times m$ and $m \times n$ matrices respectively, and r_{AB}, r_A and r_B be the ranks of AB, A and B respectively. When a generalized inverse $(AB)^-$ is given, it cannot always be expressed as B^-A^-. For example, $\text{rank}(AB)^-$ should not be greater than $\min(l, m, n)$. The following two theorems state the conditions for the decomposition. We define further $\mathcal{T} = \mathcal{R}\{(AB)^-\} \cap \mathcal{N}(B)$, $\dim \mathcal{T} = t$, $\mathcal{V} = \mathcal{R}[\{(AB)^-\}^*] \cap \mathcal{N}(A^*)$ and $\dim \mathcal{V} = v$.

THEOREM 5. *Let* $(AB)^-$ *be arbitrarily given. Then there exist* $G_2 \in \{B_r^-\}$ *and* $G_1 \in \{A_r^-\}$ *such that* $(AB)^- = G_2G_1$ *iff* $\mathcal{V} = \{0\}$ *and* $\mathcal{T} = \{0\}$.

Proof. We notice that $\mathcal{R}(B_r^-) \cap \mathcal{N}(B) = \{0\}$ for any B_r^-. Therefore, $\mathcal{T} = \{0\}$ if $(AB)^- = G_2G_1$, where $G_2 \in \{B_r^-\}$. Similarly we see that $\mathcal{V} = \{0\}$ if $(AB)^- = G_2G_1$, where $G_1 \in \{A_r^-\}$. Conversely, if $\mathcal{T} = \{0\}$, there exists a matrix $C_2 \in \{B_r^-\}$ such that $\mathcal{R}\{(AB)^-\} \subset \mathcal{R}(C_2)$ or $C_2B(AB)^- = (AB)^-$. There also exists a $C_1 \in \{A_r^-\}$ such that $(AB)^-AC_1 = (AB)^-$. Using these C_2 and C_1, we put

$$G_2 = C_2 + (AB)^- A - (AB)^- ABC_2$$

and

$$G_1 = C_1 + B(AB)^- - BG_2C_1.$$

Then we can check that $G_2 \in \{B_r^-\}$, $G_1 \in \{A_r^-\}$ and $G_2G_1 = (AB)^-$. ■

THEOREM 6. *Let* $(AB)^-$ *be arbitrarily given. Then there exist* $G_2 \in \{B^-\}$ *and* $G_1 \in \{A^-\}$ *such that* $G_2G_1 = (AB)^-$ *iff*

$$r_A + r_B + t + v \leqslant m + \text{rank}\{(AB)^-\}.$$

Proof. *Necessity:* Let $(AB)^-$ be expressed as G_2G_1, where $G_2 \in \{B^-\}$ and $G_1 \in \{A^-\}$. Then

$$\operatorname{rank}\{(AB)^-\} = \operatorname{rank}(G_2G_1)$$
$$= \operatorname{rank} G_1 - \dim\{\mathcal{R}(G_1) \cap \mathcal{N}(G_2)\}.$$

Since $\dim\{\mathcal{R}(G_1) \cap \mathcal{N}(G_2)\} \leqslant \dim\{\mathcal{N}(G_2)\} = m - \operatorname{rank} G_2$,

$$\operatorname{rank}\{(AB)^-\} \geqslant \operatorname{rank} G_1 + \operatorname{rank} G_2 - m$$
$$\geqslant r_A + r_B + t + v - m.$$

Sufficiency: Step 1. Let $\mathcal{S}$ be a subspace such that $\mathcal{R}\{(AB)^-\} = \mathcal{S} \oplus \mathcal{T}$, and let $\mathcal{U}$ be a subspace such that $\mathcal{R}[\{(AB)^-\}^*] = \mathcal{U} \oplus \mathcal{V}$. We note that $\mathcal{S} \cap \mathcal{N}(B) = \{0\}$ and $\mathcal{U} \cap \mathcal{N}(A^*) = \{0\}$. Then there exist $C_2 \in \{B_r^-\}$ and $C_1 \in \{A_r^-\}$ such that $\mathcal{R}(C_2) \supset \mathcal{S}$ and $\mathcal{R}(C_1^*) \supset \mathcal{U}$. Using these C_2 and C_1, we put

$$F_2 = C_2 + C_2B(AB)^- A - C_2B(AB)^- ABC_2$$

and

$$F_1 = C_1 + B(AB)^- AC_1 - BF_2C_1.$$

Then it can be verified that $F_2 \in \{B_r^-\}$, $\mathcal{R}(F_2) = \mathcal{R}(C_2)$, $F_1 \in \{A_r^-\}$, $\mathcal{R}(F_1^*) = \mathcal{R}(C_1^*)$ and $F_2F_1 = F_2B(AB)^-AF_1$. Further, if we put

$$G = F_1 + B(AB)^- (I - AF_1),$$

then $G \in \{A^-\}$ and $F_2G = F_2B(AB)^-$.

Step 2. Let $F_1 = PQ$ be a full-rank factorization, where P and Q are $m \times r_A$ and $r_A \times l$ matrices. Let $B(AB)^-(I - AF_1)$ be decomposed as UV, where V is a $v \times l$ matrix such that $\mathcal{R}(V^*) = \mathcal{V}$. Then

$$G = [P \ \vdots \ U]\begin{bmatrix} Q \\ \cdots \\ V \end{bmatrix}$$

and $\begin{bmatrix} Q \\ \cdots \\ V \end{bmatrix}$ is an $(r_A + v) \times l$ matrix which has full row rank. $[P \ \vdots \ U]$ is an $m \times (r_A + v)$ matrix whose rank is $s + q$, where $s = \dim \mathcal{S}$ and $q = \dim\{\mathcal{R}(G)$

$\cap \mathfrak{N}(F_2)\}$. Under the condition of the theorem there exists a matrix $\tilde{U}$ such that $[\, P \vdots U+\tilde{U}\,]$ has full column rank and $\mathfrak{R}(\tilde{U})\subset\mathfrak{N}(F_2)$. Actually $\tilde{U}$ can be constructed as follows: PP^+ is the orthogonal projector onto $\mathfrak{R}(P)$; let $U-PP^+U=KH$ be a full-rank factorization, where K has $s+q-r_A$ column vectors. Noting that there exists an $(m-r_B-q)$-dimensional subspace $\mathfrak{X}\subset\mathfrak{N}(F_2)$ which is virtually disjoint with $\mathfrak{R}(G)$, and that $m-r_B-q\geqslant r_A+v-s-q$ (condition of the theorem), we see that there exists an $l\times(r_A+v-s-q)$ matrix L such that $\mathfrak{R}(L)\subset\mathfrak{X}$ and the matrix $[\,K \vdots L\,]$ has full column rank. Let R be a matrix such that $\begin{bmatrix} H \\ \cdots \\ R \end{bmatrix}$ is nonsingular. Then we have the desired $\tilde{U}$ as $\tilde{U}=LR$. If we put $G_1=G+\tilde{U}V$, then it can be verified that $G_1\in\{A^-\}$, $F_2G_1=F_2B(AB)^-$ and $\mathfrak{R}(G_1^*)\supset\mathfrak{R}[\{(AB)^-\}^*]$.

Step 3. Since $\mathfrak{R}(G_1^*)\supset\mathfrak{R}[\{(AB)^-\}^*]$, there exists a matrix H which satisfies the equation $HG_1=(I-F_2B)(AB)^-$. Using this H, we put $G_2=F_2+(I-F_2B)H$. Then $G_2\in\{B^-\}$ and $G_2G_1=F_2G_1+(I-F_2B)HG_1=F_2B(AB)^-+(I-F_2B)(AB)^-=(AB)^-$, and the proof is completed. ■

COROLLARY 7. *Let $(AB)^-$ be arbitrarily given, and let $r_{AB}=r_A$ (or r_B). Then there exist $G_2\in\{B^-\}$ and $G_1\in\{A^-\}$ such that $G_2G_1=(AB)^-$ iff*

$$r_B+t\leqslant m \qquad (\text{or}\quad r_A+v\leqslant m).$$

Proof. This is clear, because $\operatorname{rank}\{(AB)^-\}=r_A+v$ (or r_B+t) if $r_{AB}=r_A$ (or r_B). ■

So far we have assumed that $(AB)^-$ is given. However, if we don't specify a generalized inverse $(AB)^-$, some interesting facts are found. For example, there exists $G\in\{A^-\}$ $[G\in\{B^-\}]$ such that $B^-G[GA^-]\in\{(AB)^-\}$ for any $B^-[A^-]$. In fact, put $G=A^-+B(AB)^--A^-AB(AB)^-$ $[G=B^-+(AB)^-A-(AB)^-ABB^-]$. Similarly we can show that there exists $G\in\{A^-\}$ $[G\in\{B^-\}]$ such that $B^-_{m(N)}G\in\{(AB)^-_{m(N)}\}$ $[GA^-_{l(M)}\in\{(AB)^-_{l(M)}\}]$ for any $B^-_{m(N)}$ $[A^-_{l(M)}]$.

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Received 27 September 1978; revised 8 November 1978