ELSEVIER
Discrete Applied Mathematics 51 (1994) 269–275
DISCRETE APPLIED MATHEMATICS

On the computation of pfaffians[†]

G. Galbiati[a] and F. Maffioli[b,*]

[a] *Università di Pavia, Italy*
[b] *Politecnico di Milano, Milano, Italy*

(Received 21 October 1991)

Abstract

We present an efficient algorithm for computing the pfaffian of a matrix whose elements belong to an integral domain. Relevant applications are exact value problems in matching and matroid theory.

1. Introduction

This work deals with efficient ways of computing the pfaffian of a skew-symmetric matrix whose elements belong to an integral domain.

Pfaffians play an important role in matching problems [9], and in their natural generalization into matroid parity problems [3, 8].

Since for every skew-symmetric matrix of even order it is well known [7] that its determinant is equal to the square of the pfaffian of the matrix, in the solution of existence versions of the above mentioned matching and matroid parity problems, the computation of the pfaffian can be substituted by the computation of the determinant. However, in the solution of the corresponding exact value problems, computing pfaffians becomes essential [2, 3]. The possibility of directly computing pfaffians was already pointed out in [1].

This paper presents an efficient algorithm for computing the pfaffian of a matrix whose elements belong to an integral domain. When applied to an integral matrix, similarly to Edmonds' algorithm for computing the determinant [4], this algorithm works with elements of bounded magnitude, namely bounded above by the magnitude of any minor of the given matrix.

*Corresponding author.
[†]Sponsored in part by the Italian Ministry of University and Scientific and Technological Research (MURST) under contract MPI 40%.

SSDI 0166-218X(92)00034-V

Section 2 defines the pfaffian of a matrix and recalls some of its basic properties. Section 3 restates for the reader's convenience the algorithm of [4] for computing determinants, giving a proof of its correctness. Section 4 shows in a similar spirit how to compute pfaffians. Section 5 discusses some issues related to applications and to the parallel implementation of the algorithms.

2. The pfaffian of a matrix

Let $\Gamma = (\gamma_{ij})$ be an $n \times m$ matrix. For $1 \leqslant i_1 < \cdots < i_r \leqslant n$, $1 \leqslant j_1 < \cdots < j_r \leqslant m$, $r \in \{1, \ldots, \min\{n, m\}\}$, we denote by

$$\Gamma\begin{pmatrix} i_1, \ldots, i_r \\ j_1, \ldots, j_r \end{pmatrix}$$

the *minor* of Γ of *order r*, made with the indicated rows and columns, respectively, that is the determinant of the matrix $G = (g_{hk})$ with $g_{hk} = \gamma_{i_h j_k}$ for $h, k \in \{1, \ldots, r\}$.

Let now Γ be a $2n \times 2n$ skew-symmetric matrix and let P be the set of all partitions of the set $\{1, 2, \ldots, 2n\}$ into pairs. If $\pi = \{\{i_1, j_1\}, \ldots, \{i_n, j_n\}\}$ is one of these partitions, let $\sigma(\pi)$ be the sign of the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & 2n-1 & 2n \\ i_1 & j_1 & \cdots & i_n & j_n \end{pmatrix}$$

and let

$$\alpha(\pi) = \sigma(\pi) \sum_{h=1}^{n} \gamma_{i_h j_h}.$$

Then the *pfaffian* of matrix Γ is defined as

$$pf(\Gamma) = \sum_{\pi \in P} \alpha(\pi). \tag{1}$$

Similarly as for determinants, we shall consider pfaffians obtained from some rows and columns of Γ: for $1 \leqslant i_1 < \cdots < i_{2r} \leqslant 2n$, $r \in \{1, \ldots, n\}$, we denote by $\Gamma(i_1, \ldots, i_{2r})$ the pfaffian of the $2r \times 2r$ skew-symmetric matrix $G = (g_{hk})$ with $g_{hk} = \gamma_{i_h i_k}$ for $h, k \in \{1, \ldots, 2r\}$. Such a pfaffian is called a *p-minor* of Γ of *order r*.

A well-known result from linear algebra is the following theorem.

Theorem 2.1. (Lang [7], Lovász and Plummer [9]). *For every skew-symmetric matrix Γ of even order $2n$:*

(a) $\mathrm{pf}^2(\Gamma) = \det(\Gamma)$;

(b) $\mathrm{pf}(A\ \Gamma\ A^{\mathrm{T}}) = \det(A)\ \mathrm{pf}(\Gamma)$, *where A is any square matrix of order $2n$.*[1]

Using suitable matrices A (see for instance [5, p. 130]) it is immediate to see that Theorem 2.1(b) implies three properties, similar to the three well-known elementary

[1]Note that the pfaffian defined in (1) is $(-1)^{n(n-1)/2}$ times the pfaffian defined in [7].

properties of determinants, concerning multiplication of rows by a constant, sum of rows, and interchanging of rows (or columns).

Corollary 2.2. *For every skew-symmetric matrix Γ of even order and for any constant λ:*

(a) *for every i, multiplying by λ both row i and column i has the effect of multiplying the pfaffian by λ;*

(b) *for every $i \neq j$, adding both to row i and to column i, λ times row j and λ times column j, respectively, does not change the value of the pfaffian;*

(c) *for every $i \neq j$, interchanging both rows i, j and columns i, j has the effect of changing the sign of the pfaffian.*

3. Computing determinants

For the reader's convenience in this section we restate the polynomial algorithm given in [4] for computing the determinant of a square integral matrix. The following recursive procedure $\mathbf{DET}(c, n, \Gamma)$ receives as input an integer $c \neq 0$ and an $n \times n$ integer matrix $\Gamma = (\gamma_{ij})$, whose minors of any order $r \in \{1, \ldots, n\}$ are all known to be divisible by c^{r-1}, and returns $\det(\Gamma)/c^{n-1}$. As a particular case, $\mathbf{DET}(1, n, \Gamma)$ returns the determinant of any $n \times n$ integer matrix Γ.

procedure $\mathbf{DET}(c, n, \Gamma)$:
1. **if** $n = 1$ **then return** γ_{11} **else**
2. **if** $\gamma_{nk} \neq 0$ for some $k \in \{1, \ldots, n\}$ **then**
   begin
3. **if** $k \neq n$ **then** interchange in Γ columns k, n;
4. **for each** $i, j \in \{1, \ldots, n-1\}$ **do**
   $\delta_{ij} \leftarrow \frac{1}{c}(\gamma_{nn}\gamma_{ij} - \gamma_{in}\gamma_{nj})$;
5. $\Delta \leftarrow (\delta_{ij})$;
   $d \leftarrow \gamma_{nn}$;
6. **comment** $d \neq 0$ and since $\delta_{ij} = \frac{1}{c}\Gamma\binom{i,\ n}{j,\ n}$,
   Δ is an integer $(n-1) \times (n-1)$ matrix;
7. **return if** $k \neq n$ **then** $-\mathbf{DET}(d, n-1, \Delta)$ **else** $\mathbf{DET}(d, n-1, \Delta)$
   end
8. **else return** 0.

The correctness of this procedure relies upon the following property.

Property 3.1. *In the recursive call at line 7—with arguments $d, n-1$ and Δ—the following holds true:*

$$\frac{1}{d^{r-1}}\Delta\begin{pmatrix} i_1, \ldots, i_r \\ j_1, \ldots, j_r \end{pmatrix} = \frac{1}{c^r}\Gamma\begin{pmatrix} i_1, \ldots, i_r, i_{r+1} = n \\ j_1, \ldots, j_r, j_{r+1} = n \end{pmatrix} \tag{2}$$

for all minors of Δ of order $r \in \{1, \ldots, n-1\}$.

Proof. It is immediate to see that by bordering Δ with a column with elements $\delta_{in} = 0$ $(i = 1, \ldots, n-1)$ and a row with elements $\delta_{nj} = \gamma_{nj} (j = 1, \ldots, n)$, we get an $n \times n$ matrix Δ such that

$$\Delta\begin{pmatrix} i_1, \ldots, i_r, i_{r+1} = n \\ j_1, \ldots, j_r, j_{r+1} = n \end{pmatrix} = \gamma_{nn}\, \Delta\begin{pmatrix} i_1, \ldots, i_r \\ j_1, \ldots, j_r \end{pmatrix}. \tag{3}$$

Moreover, the equalities

$$\delta_{ij} = \frac{1}{c}(\gamma_{ij}\,\gamma_{nn} - \gamma_{in}\,\gamma_{nj})$$

for $i = 1, \ldots, n-1$ and $j = 1, \ldots, n$ can be given the following interpretation. For $i = 1, \ldots, n-1$, the ith row of Δ is obtained by subtracting γ_{in} times the last row of Γ from γ_{nn} times the ith row of Γ, and dividing the result by c. The well-known properties of determinants imply therefore that the left-hand side of (3) is also equal to

$$\frac{\gamma_{nn}^r}{c^r}\,\Gamma\begin{pmatrix} i_1, \ldots, i_r, i_{r+1} = n \\ j_1, \ldots, j_r, j_{r+1} = n \end{pmatrix}.$$

Recalling that $\gamma_{nn} = d \neq 0$, (2) follows immediately. □

In order to conclude that procedure **DET** is correct, we observe that at the time of the recursive call, Γ may differ from the matrix given in input only because of the columns interchange in line 3, and therefore property 3.1 not only implies that all elements of Δ are integers, but also that all minors of Δ of order $r \in \{1, \ldots, n-1\}$ are divisible by d^{r-1}. As a consequence, **DET**$(d, n-1, \Delta)$ can indeed be called. Moreover, (2) with $r = n-1$ implies that

$$\frac{1}{d^{n-2}} \det(\Delta) = \frac{1}{c^{n-1}} \det(\Gamma),$$

i.e. the value returned in line 7 is correct. When (eventually) **DET**(c, n, Γ) is called with $n = 1$, then $(1/c^{n-1}) \det(\Gamma) = \gamma_{11}$ is the correct value to be returned.

Note that Property 3.1 ensures that if the first call is **DET**$(1, n, \Gamma)$, then all elements of the matrices which are computed in all subsequent recursive executions, are equal (in absolute value) to certain minors of the original matrix Γ. It follows that if μ is the maximum absolute value of an element in the original matrix, the magnitude of each element of the subsequent matrices is bounded by $(n!\mu^n)$. The overall number of operations (additions, multiplications and divisions) on these elements is $O(n^3)$.

4. Computing pfaffians

In a similar spirit, we propose an algorithm for computing pfaffians, using Corollary 2.2. The following recursive procedure **PF**(c, n, Γ) receives as input an integer $c \neq 0$ and a $2n \times 2n$ integer skew-symmetric matrix $\Gamma = (\gamma_{ij})$, whose p-minors of any order $r \neq \{1, \ldots, n\}$ are all known to be divisible by c^{r-1}, and returns $\mathrm{pf}(\Gamma)/c^{n-1}$.

As a particular case, **PF**$(1, n, \Gamma)$ returns the pfaffian of any $2n \times 2n$ integer skew-symmetric matrix.

procedure PF(c, n, Γ):
1. **if** $n = 1$ **then return** γ_{12} **else**
2. **if** $\gamma_{2n-1,k} \neq 0$ for some $k \in \{1, \ldots, 2n\}$ **then**
   begin
3. **if** $k \neq 2n$ **then** interchange in Γ rows k, $2n$ and columns k, $2n$;
4. **for each** $i, j \in \{1, \ldots 2n-2\}$ **do**
   $\delta_{ij} \leftarrow \frac{1}{c}(\gamma_{2n-1,2n}\gamma_{ij} - \gamma_{i,2n-1}\gamma_{j,2n} + \gamma_{j,2n-1}\gamma_{i,2n})$;
5. $\Delta \leftarrow (\delta_{ij})$;
   $d \leftarrow \gamma_{2n-1,2n}$;
6. **comment** $d \neq 0$, and since for $i < j$, $\delta_{ij} = \frac{1}{c}\Gamma(i, j, 2n-1, 2n) = -\delta_{ji}$ and $\delta_{ii} = 0$, then Δ is an integer $(2n-2) \times (2n-2)$ skew-symmetric matrix;
7. **return if** $k \neq 2n$ **then** $-$ **PF**$(d, n-1, \Delta)$ **else PF**$(d, n-1, \Delta)$
   end
8. **else return** 0

The correctness of this procedure is a consequence of the following property.

Property 4.1. *In the recursive call at line 7—with arguments d, $n-1$ and Δ, respectively—the following holds true;*

$$\frac{1}{d^{r-1}}\Delta(i_1, \ldots, i_{2r}) = \frac{1}{c^r}\Gamma(i_1, \ldots, i_{2r}, 2n-1, 2n) \tag{4}$$

for all p-minors of Δ of order $r \in \{1, \ldots, n-1\}$.

Proof. Border Δ with two columns and two rows, whose (not necessarily integer) elements are specified as follows:

$$\delta_{i,2n-1} = -\delta_{2n-1,i} = \frac{1}{c}\gamma_{i,2n-1}, \quad i = 1, \ldots, 2n-2;$$

$$\delta_{i,2n} = \delta_{2n,i} = 0, \quad i = 1, \ldots, 2n-2;$$

$$\delta_{2n-1,2n-1} = \delta_{2n,2n} = 0; \qquad \delta_{2n-1,2n} = -\delta_{2n,2n-1} = \frac{1}{c}.$$

It is then immediate to see that

$$\Delta(i_1, \ldots, i_{2r}, 2n-1, 2n) = \frac{1}{c}\Delta(i_1, \ldots, i_{2r}). \tag{5}$$

Moreover, Δ can be seen as the $2n \times 2n$ matrix obtained from Γ through the following operations:

(i) for $i = 1, \ldots, 2n-2$, multiply row i by $\gamma_{2n-1,2n}$, and subtract from it $\gamma_{i,2n}$ times row $2n-1$;

(ii) for $i = 1, \ldots, 2n-2$, multiply column i by $\gamma_{2n-1,2n}$, and subtract from it $\gamma_{i,2n}$ times columns $2n-1$;

(iii) divide elements of the resulting matrix by $c\gamma_{2n-1,2n}$.

Applying Corollary 2.2(a) and (b) to the submatrix of Γ with rows and columns $i_1, \ldots, i_{2r}, 2n-1, 2n$, operations (i) and (ii) above have the effect of multiplying the pfaffian by $\gamma_{2n-1,2n}^{2r}$, whereas (by definition of pfaffian) operation (iii) causes the pfaffian to be divided by $(c\gamma_{2n-1,2n})^{r+1}$. We have therefore

$$\Delta(i_1, \ldots, i_{2r}, 2n-1, 2n) = \frac{\gamma_{2n-1,2n}^{r-1}}{c^{r+1}} \Gamma(i_1, \ldots, i_{2r}, 2n-1, 2n). \tag{6}$$

Recalling that $\gamma_{2n-1,2n} = d \neq 0$, Property 4.1 follows from (5) and (6). □

Similarly as for procedure **DET**(c, n, Γ), Property 4.1 and Corollary 2.2(c) imply that all elements of Δ are integers and all p-minors of order r are divisible by d^{r-1}. Moreover, (4) with $r = n-1$ implies that

$$\frac{1}{d^{n-2}} \operatorname{pf}(\Delta) = \frac{1}{c^{n-1}} \operatorname{pf}(\Gamma)$$

and line 7 is therefore correct.

Even in this case if the first call is **PF**$(1, n, \Gamma)$, then (4), and the interpretation of δ_{ij} in the comment of line 6, imply that all δ_{ij}'s computed in line 4 are equal (in absolute value) to certain p-minors of the original matrix Γ. It follows that the time complexity of **PF**$(1, n, \Gamma)$ is the same as that of **DET**$(1, n, \Gamma)$.

We conclude this section by observing that both algorithms **DET** and **PF** also work if the elements of the original matrices are not drawn from the ring of integers but from any integral domain.

5. On some applications and parallel complexity issues

It is straightforward to obtain parallel versions of procedures **DET** and **PF** that use $O(n)$ operations and $O(n^2)$ processors. It is well known that the computation of determinants in an arbitrary ring can be done by algorithms that use only $O(\log^2 n)$ operations and a polynomial number of processors, thus implying that such computation lies in the complexity class Arithmetic NC^2 [6, p. 913]. However, these algorithms are based on properties which are peculiar to determinants, such as those expressed by the Cayley–Hamilton theorem, and do not appear to generalize to pfaffians. To our knowledge no ad hoc parallel algorithms for computing pfaffians have appeared in the literature, which use $O(\log^k n)$ arithmetic operations, for some constant k, and a polynomial number of processors.

Many classical applications of an algorithm for computing the pfaffian of a matrix are surveyed in [9]. More recently in [2, 3] it has been shown that the notion of pfaffian plays a key role in the solution of the problems of existence and construction of a perfect matching having a given value in an integrally weighted graph, or, more generally, of a base of a matroid having a given value and respecting parity constraints.

Acknowledgement

We are grateful to Francisco Barahona for encouraging us to write this note.

References

[1] F. Barahona, Balancing signed toroidal graphs in polynomial time, Int. Rept., Depto de Matematicas, Universidad de Chile (1981).
[2] P.M. Camerini, G. Galbiati and F. Maffioli, Modular arithmetic and randomization for exact matroid problems, in: Proceedings of the 1st Conference on Integer Programming and Combinatorial Optimization, Waterloo (1990) 101–114.
[3] P.M. Camerini, G. Galbiati and F. Maffioli Random pseudo-polynomial algorithms for exact matroid problems, J. Algorithms 13 (1992) 258–273.
[4] J. Edmonds, System of distinct representatives and linear algebra, J. Res. Nat Bur. Standards (B) 71 (1967) 241–245.
[5] F.R. Gantmacher, The Theory of Matrices, Vol. 1 (Chelsea, New York, 1959).
[6] R.M. Karp and V. Ramachandran, Parallel algorithms for shared-memory machines, in: J. van Leeuwen, ed., Handbook of Computer Science, Vol. A: Algorithms and Complexity (Elsevier, Amsterdam and MIT Press, Cambridge, MA, 1990) 869–941.
[7] S. Lang, Algebra (Addison-Wesley, Reading, MA, 1965).
[8] L. Lovász, On determinants, matching and random algorithms, Fund. Comput. Theory 79 (Akademie Verlag, Berlin, 1979) 565–574.
[9] L. Lovász and M.D. Plummer, Matching Theory (North-Holland, Amsterdam, 1986).