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On the leading eigenvalue of neutron transport models

Mustapha Mokhtar-Kharroubi

Département de Mathématiques, Université de Franche-Comté, 16 Route de Gray, 25030 Besançon, France

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Abstract

We give variational characterizations of the leading eigenvalue of neutron transport-like operators. The proofs rely on sub- and super-eigenvalues. Various bounds of the leading eigenvalue are derived.

Keywords: Positivity; Leading eigenvalue; Sub-eigenvalue; Super-eigenvalue; Transport equations

1. Introduction

This paper provides a new approach of the leading eigenvalue for neutron transport-like equations. The so-called time eigenvalue of the fundamental mode (i.e. the leading eigenvalue) of neutron transport operators plays a basic role in nuclear reactor theory, e.g., in pulsed experiments [6, Chapter 5] or in the stochastic description of neutron chain fissions [3]. This eigenvalue or, more generally, the peripheral spectrum of such operators is strongly related to their positivity properties (in the lattice sense); see [17] and references therein. In the same spirit, positivity plays an essential role in reactor criticality; see [14] and references therein. We refer to [10, Chapter 5] and references therein for the known results on the leading eigenvalue of neutron transport operators. Motivated by transport theory, the present paper is devoted to *variational characterizations* of the lead-

E-mail address: mokhtar@math.univ-fcomte.fr.

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ing eigenvalue for a class of perturbed operators of the form $A = T + K$ where T is an unbounded operator with a *positive resolvent* and K is a bounded *positive* operator. If we denote respectively by $s(T)$ and $s(A)$ the spectral bound of T and A and if some power of $(\lambda - T)^{-1}K$ is compact ($\lambda > s(T)$), then it is known that $s(A)$ is the leading eigenvalue of A once $s(T) < s(A)$ [16]. Here, this leading eigenvalue is handled by means of sub-eigenvalues or super-eigenvalues. Roughly speaking, we prove that $\lambda \in]s(T), s(A)[$ if and only if λ is a *sub-eigenvalue*, i.e. there exists a nonnegative (non-trivial) φ such that $A\varphi \geqslant \lambda\varphi$. We show also that $\lambda \in]s(A), \infty[$ if and only if λ is a *super-eigenvalue*, i.e. there exists a nonnegative (non-trivial) φ such that $A\phi \leqslant \lambda\phi$. It follows that $s(A)$ can be characterized as the supremum of sub-eigenvalues or the infimum of super-eigenvalues. This provides us with *max–inf* and *min–sup* principles for the leading eigenvalue. This first part of our work, of more functional analytic character, is in the spirit of I. Marek [9] who deals, in particular, with variational characterizations of spectral radius of certain positive operators. In the second part, devoted specifically to neutron transport, we show how to derive in a systematic manner, from the above (abstract) variational principles, upper and lower bounds of the leading eigenvalue in terms of various physical parameters. This paper resumes some results from a longer preliminary version [12] containing additional results and references. We present now our general framework. Let $\Omega \subset R^N$ be a smooth and bounded open set and let μ be a positive Radon measure on R^N with support V. We refer to V as the velocity space. We assume in this paper that V is *bounded away from zero*, i.e. $0 \notin V$. We refer to [12] for the case $0 \in V$. Let T be the advection operator in $L^p(\Omega \times V) := L^p(\Omega \times V; dx\, d\mu(v))$ $(1 \leqslant p < \infty)$

$$T\varphi = -v \cdot \frac{\partial \varphi}{\partial x} - \sigma(x, v)\varphi(x, v), \quad \varphi \in D(T)$$

with domain

$$W^p_{0-} = \left\{\varphi \in L^p(\Omega \times V);\ v \cdot \frac{\partial \varphi}{\partial x} \in L^p(\Omega \times V),\ \varphi = 0 \text{ on } \Gamma_-\right\}$$

where $\Gamma_- := \{(x, v) \in \partial\Omega \times V;\ \ v \cdot n(x) < 0\}$ and $n(x)$ is the outward unit vector at $x \in \partial\Omega$. The real and bounded measurable function $\sigma(\cdot,\cdot)$ is the collision frequency while the scattering (or collision) operator is

$$K : \varphi \in L^p(\Omega \times V) \to \int_V k(x, v, v')\varphi(x, v')\, d\mu(v') \in L^p(\Omega \times V).$$

Finally, the neutron transport operator is given by

$$A : \varphi \in W^p_{0-} \to -v \cdot \frac{\partial \varphi}{\partial x} - \sigma(x, v)\varphi(x, v) + \int_V k(x, v, v')\varphi(x, v')\, d\mu(v')$$

with the same domain as the advection operator T. The cross sections $\sigma(\cdot,\cdot)$ and $k(\cdot, \cdot, \cdot)$ are *nonnegative* in accordance with the physical theory. The spectral bound of T, $s(T) = \sup\{\operatorname{Re}\lambda;\ \lambda \in \sigma(T)\}$, is characterized in full generality in [18]: $s(T) = -\lambda^*$ where

$$\lambda^* = \lim_{t \to \infty} \inf_{\{(x,v) \in \Omega \times V;\ t < \tau(x,-v)\}} t^{-1} \int_0^t \sigma(x + sv, v)\, ds$$

and $\tau(x,v) := \inf\{s > 0;\ x - sv \notin \Omega\}$ is the "*exit time*" function. In particular $s(T) = -\infty$ under our assumption $0 \notin V$. Let $X_+ := L^p_+(\Omega \times V)$ be the positive cone of the space $L^p(\Omega \times V)$. We assume for all the sequel there exists an integer n such that

$$\big[(\lambda - T)^{-1}K\big]^n \text{ is compact in } L^p(\Omega \times V) \tag{1}$$

for $\lambda > s(T)$. There exists a vast literature on such compactness properties which goes back to the sixtees; see [10, Chapter 4] and references therein. More recent results are given in [11,13]. In particular, if $1 < p < \infty$ and if, for *fixed* $x \in \Omega$, the collision operator is compact from $L^p(V)$ into itself then (1) is satisfied with $n = 1$ provided the linear hyperplanes have zero μ-measure. On the other hand, for $p = 1$, if for fixed $x \in \Omega$ the collision operator is weakly compact from $L^1(V)$ into itself then, under suitable assumptions on μ, (1) is satisfied for *some* $n > 1$. Those assumptions are satisfied by the usual continuous or multigroup models used in transport theory. According to Gohberg–Schmulyan's alternative (see [16]), (1) implies that $\sigma(A) \cap \{\lambda; \operatorname{Re}\lambda > s(T)\}$ consists (at most) of isolated eigenvalues with finite algebraic multiplicities. Thus $\sigma(A) \cap \{\lambda; \operatorname{Re}\lambda > s(T)\} \neq \emptyset$ if and only if $s(T) < s(A)$ where $s(A)$ is the spectral bound of A and $s(A)$ is an eigenvalue of $T + K$ (actually the leading one) associated with a *nonnegative* eigenfunction [16]. Besides the compactness hypothesis (1) we assume that

$$r_\sigma\big[(\lambda - T)^{-1}K\big] > 0 \quad \text{for all } \lambda > s(T). \tag{2}$$

We do not discuss in details this assumption here. We note, however, that, according to B. De Pagter's theorem [5], the last assumption is certainly satisfied if, besides (1),

$$(\lambda - T)^{-1}K \text{ is irreducible} \tag{3}$$

i.e. for each $\varphi \in L^p_+(\Omega \times V)$, $\varphi \neq 0$, and $\psi \in L^{p'}_+(\Omega \times V)$, $\psi \neq 0$ (p' is the conjugate exponent of p), there exists an integer m (depending a priori on φ and ψ) such that $\langle [(\lambda - T)^{-1}K]^m \varphi, \psi\rangle > 0$ where $\langle \cdot,\cdot \rangle$ denotes the duality pairing between $L^p(\Omega \times V)$ and $L^{p'}(\Omega \times V)$. This is satisfied, for instance, when μ is the Lebesgue measure on R^N or on spheres (multigroup models), Ω is convex and $k(x,v,v') > 0$ a.e. We refer to [10, Chapter 5] and references therein for more general irreducibility results. We point out that such strict positivity assumptions are, to some extent, also "necessary." Indeed, if $k(x,v,v') = 0$ for $|v| \geqslant |v'|$ (which arises in the slowing down theory of "superthermal" fission neutrons in a moderator [6]) then the point spectrum is *empty* regardless of the size of Ω [7]. The mathematical reason behind this emptiness is the quasinilpotence of the collision operator (see [10, Chapter 5]). We recall, however, that under (1), (3) and the assumption that the velocity space is bounded away from zero, the point spectrum is never empty [10, Chapter 5, Theorem 5.12]. More precisely, $r_\sigma[(\lambda - T)^{-1}K] > 0$ ($\lambda > s(T)$) is an algebraically simple eigenvalue of $(\lambda - T)^{-1}K$. By Gohberg–Schmulyan's alternative, the nonincreasing function $\lambda \in]s(T), \infty[\to r_\sigma[(\lambda - T)^{-1}K]$ is actually (strictly) *decreasing*. Moreover, it is continuous because $r_\sigma[(\lambda - T)^{-1}K]$ is an algebraically simple eigenvalue [8]. It follows, under (1), (3), that A has a leading eigenvalue if and only if

$$\lim_{\lambda \to s(T)} r_\sigma\big[(\lambda - T)^{-1}K\big] > 1.$$

In such a case, $s(A)$ is the unique $\bar\lambda > s(T)$ such that

$$r_\sigma\big[(\bar\lambda - T)^{-1}K\big] = 1. \tag{4}$$

In particular, when the velocity space is bounded away from zero, then $s(T) = -\infty$ and then convexity arguments [10, Chapter 5, Theorem 5.12] show that

$$\lim_{\lambda \to -\infty} r_\sigma\big[(\lambda - T)^{-1}K\big] = \infty$$

so a leading eigenvalue does exist. To our knowledge, until now, the analysis of the fundamental eigenvalue relied essentially on (4). In particular, this makes an approximation theory of the leading eigenvalue based on (4) quite involved [2,4]. Our purpose here is to show that we can avoid this auxiliary "Peierls operator" and give (variational) characterizations of $s(A)$ in terms of A itself and suitable test functions. Moreover, those characterizations will provide us with computable upper and lower bounds of the leading eigenvalue in terms of various physical parameters, in particular in terms of the "exit time" $\tau(\cdot,\cdot)$. We point out that our mathematical analysis does not rely directly on (3), i.e. on the irreducibility of $(\lambda - T)^{-1}K$. Actually, besides (1), (2), we merely need either

$$\text{There exists a } \textit{positive} \text{ eigenfunction of } (\lambda - T)^{-1}K \tag{5}$$

corresponding to its spectral radius, or

$$\text{There exists a } \textit{positive} \text{ eigenfunction of } \big[(\lambda - T)^{-1}K\big]' \tag{6}$$

corresponding to its spectral radius, where $[(\lambda - T)^{-1}K]'$ is the dual operator to $(\lambda - T)^{-1}K$. Note that the existence of *nonnegative* eigenfunctions (corresponding to the spectral radius) is already contained in (1), (2). We note that both (5), (6) are consequences of the irreducibility of $(\lambda - T)^{-1}K$. However, in practice, it is much easier to verify the irreducibility of $(\lambda - T)^{-1}K$ than (5), (6). Unless otherwise stated, the basic assumptions (1), (2) are made for *all* the sequel and will not be repeated in the different statements.

2. Variational characterizations of the spectral bound

This section is devoted to several (variational) characterizations of the spectral bound of $A = T + K$,

$$s(A) := \sup\big\{\operatorname{Re}\lambda;\ \lambda \in \sigma(A)\big\},$$

where T is an unbounded operator with a *positive resolvent* and K is a bounded *positive* operator. Those characterizations are obtained from preliminary results based on (1), (2) and (5) *or* (6). To this end, it is useful to recall a general (abstract) characterization of $s(A)$ [19] which does not rely on such assumptions:

$$\lambda > s(A) \quad \text{iff} \quad \lambda > s(T) \text{ and } r_\sigma\big[(\lambda - T)^{-1}K\big] < 1, \tag{7}$$

where $s(T)$ is the spectral bound T. Note that a priori $s(A) \geqslant s(T)$.

Lemma 1. *Let* $E := \{\lambda > s(T);\ \exists\varphi \in D(T) \cap X_+ - \{0\},\ A\varphi - \lambda\varphi \in X_+\}$. *Then* $E = \{\lambda > s(T);\ r_\sigma[(\lambda - T)^{-1}K] \geqslant 1\}$.

Proof. Let $\hat{\lambda} \in E$. Then $\hat{\lambda} > s(T)$ and there exists $\varphi \in D(T) \cap X_+$, $\varphi \neq 0$, such that $A\varphi - \hat{\lambda}\varphi \in X_+$, i.e. $T\varphi + K\varphi \geqslant \hat{\lambda}\varphi$. Since $(\hat{\lambda} - T)^{-1}$ is positive then $\varphi \leqslant (\hat{\lambda} - T)^{-1}K\varphi$ and $\varphi \leqslant [(\hat{\lambda} - T)^{-1}K]^k\varphi$ $(\forall k \in N)$ so

$$\big\|\big[(\hat{\lambda} - T)^{-1}K\big]^k\big\| \geqslant 1 \quad (\forall k \in N).$$

Hence $r_\sigma[(\hat{\lambda} - T)^{-1}K] \geqslant 1$ and $E \subset \{\lambda > s(T);\, r_\sigma[(\lambda - T)^{-1}K] \geqslant 1\}$. Now, let $\lambda > s(T)$ and $r_\sigma[(\lambda - T)^{-1}K] \geqslant 1$. Then $\alpha := r_\sigma[(\lambda - T)^{-1}K]$ is an *eigenvalue* of $(\lambda - T)^{-1}K$ associated with a nonnegative eigenvector φ, i.e. $(\lambda - T)^{-1}K\varphi = \alpha\varphi$. This is possible because $(\lambda - T)^{-1}K$ is a positive operator (so $r_\sigma[(\lambda - T)^{-1}K] \in \sigma((\lambda - T)^{-1}K)$, [15]) and power compact. Thus $\frac{1}{\alpha}K\varphi = (\lambda - T)\varphi$ and $\lambda\varphi - T\varphi \leqslant K\varphi$ i.e. $\lambda \in E$ which shows the reverse inclusion. $\square$

Corollary 1. *We have $s(T) < s(A)$ if and only if E is not empty. In such a case $s(A) = \sup\{\lambda;\, \lambda \in E\}$.*

$$\{\lambda > s(T);\ \exists\varphi \in D(T) \cap X_+ - \{0\},\, A\varphi - \lambda\varphi \in X_+\} \neq \emptyset.$$

Proof. Let $\bar{\lambda} = \sup\{\lambda \in E\}$. Then $\bar{\lambda} = \sup\{\lambda > s(T);\ r_\sigma[(\lambda - T)^{-1}K] \geqslant 1\}$ by Lemma 1. It follows that $r_\sigma[(\lambda - T)^{-1}K] < 1\ \forall\lambda > \bar{\lambda}$ and, by (7), $s(A) \leqslant \bar{\lambda}$. On the other hand, there exists a sequence $\{\hat{\lambda_k}\} \subset E$ such that $\hat{\lambda}_k \to \bar{\lambda}$. By Lemma 1, $r_\sigma[(\hat{\lambda}_k - T)^{-1}K] \geqslant 1\ \forall k$. The set $\{\lambda > s(T);\, r_\sigma[(\lambda - T)^{-1}K] \geqslant 1\}$ is *closed* in $]s(T), +\infty[$ because the mapping

$$\lambda \in \big]s(T), +\infty\big[\to r_\sigma\big[(\lambda - T)^{-1}K\big]$$

is *upper-semicontinuous* ([8, Theorem 3.1 and Remark 3.3, p. 208]). Hence $r_\sigma[(\bar{\lambda} - T)^{-1}K] \geqslant 1$ and, according to (7), $\bar{\lambda} \leqslant s(A)$. $\square$

Lemma 2 [12, Lemma 2]. *If* (6) *is satisfied then $\lambda > s(T) \to r_\sigma[(\lambda - T)^{-1}K]$ is continuous.*

Theorem 1. *For each $\varphi \in D(T) \cap X_+ - \{0\}$, set*

$$\tau(\varphi) := \sup\big\{\lambda > s(T);\ A\varphi - \lambda\varphi \in X_+\big\}$$

with the convention $\tau(\varphi) = s(T)$ if $\{\lambda > s(T);\, A\varphi - \lambda\varphi \in X_+\} = \emptyset$. Then

$$s(T) < s(A) \quad \textit{if and only if} \quad \sup_{\varphi \in D(T) \cap X_+} \tau(\varphi) > s(T).$$

In such a case

$$\sup_{\varphi \in D(T) \cap X_+} \tau(\varphi) = s(A).$$

Moreover, if (6) *is satisfied then $\tau(\varphi) = s(A)$ if and only if $A\varphi = s(A)\varphi$.*

Proof. Let $s(T) < s(A)$ and $\psi \in D(T) \cap X_+ - \{0\}$ be a corresponding eigenvector, then $A\psi - s(A)\psi = 0$ and $\tau(\psi) \geqslant s(A) > s(T)$ so

$$\sup_{\varphi \in D(T) \cap X_+} \tau(\varphi) \geqslant s(A) > s(T). \tag{8}$$

Conversely, if

$$\sup_{\varphi \in D(T)\cap X_+} \tau(\varphi) > s(T),$$

then for *all* $\varphi \in D(T) \cap X_+ - \{0\}$ such that $\tau(\varphi) > s(T)$ we have $A\varphi - \tau(\varphi)\varphi \in X_+$. According to Corollary 1, $\tau(\varphi) \leqslant s(A)$ and consequently

$$\sup_{\varphi \in D(T)\cap X_+} \tau(\varphi) \leqslant s(A)$$

which ends the first part of the statement. Assume (6), then, by Lemma 2, the leading eigenvalue $s(A)$ is characterized by $r_\sigma[(s(A) - T)^{-1}K] = 1$. Now suppose that $\tau(\varphi) = s(A)$, i.e.

$$T\varphi + K\varphi - s(A)\varphi \geqslant 0. \tag{9}$$

We claim that (9) is an *equality*. Otherwise $\varphi \leqslant (s(A) - T)^{-1}K\varphi$ would *not* be an equality so, using a *positive* eigenfunction φ' of $[(s(A) - T)^{-1}K]'$ corresponding to its spectral radius,

$$\begin{aligned}\langle \varphi, \varphi' \rangle &< \big\langle \big(s(A) - T\big)^{-1} K\varphi, \varphi' \big\rangle = \big\langle \varphi, \big[\big(s(A) - T\big)^{-1} K\big]'\varphi' \big\rangle \\ &= r_\sigma\big[\big(s(A) - T\big)^{-1} K\big] \langle \varphi, \varphi' \rangle = \langle \varphi, \varphi' \rangle\end{aligned}$$

which is a contradiction. □

Theorem 2. *Let* $X_+^* = \{\varphi \in L^p(\Omega \times V);\ \varphi(x, v) > 0$ *a.e.*$\}$. *We assume that* (5) *is satisfied. Then* $\sup_{\varphi \in D(T)\cap X_+^*} \tau(\varphi) = s(A)$.

Proof. A priori $\sup_{\varphi \in D(T)\cap X_+^*} \tau(\varphi) \leqslant s(A)$. When $s(T) < s(A)$, there exists a *positive* eigenfunction ψ associated with $s(A)$. Indeed, $A\psi = T\psi + K\psi = s(A)\psi$ amounts to $(s(A) - T)^{-1}K\psi = \psi$ and by (5), we can choose $\psi \in X_+^*$. Hence $\sup_{\varphi \in D(T)\cap X_+^*} \tau(\varphi) \geqslant s(A)$ and this ends the proof. □

Theorem 3. *We assume that* (5) *is satisfied. Then the spectral bound* $s(A)$ *is given by*

$$\max_{\varphi \in D(T)\cap X_+^*} \max\left\{\inf \frac{A\varphi}{\varphi},\ s(T)\right\}$$

where $\inf \frac{A\varphi}{\varphi}$ *denotes the essential infimum of* $\frac{A\varphi}{\varphi}$.

Proof. According to Theorem 2, $s(A) = \sup_{\varphi \in D(T)\cap X_+^*} \tau(\varphi)$. On the other hand, for each $\varphi \in D(T) \cap X_+^*$, $\tau(\varphi)$ is equal to

$$\begin{aligned}&\begin{cases} \sup\big\{\lambda > s(T);\ A\varphi - \lambda\varphi \in X_+\big\} & \text{if } \{\lambda > s(T);\ A\varphi - \lambda\varphi \in X_+\} \neq \emptyset, \\ s(T), & \text{otherwise}\end{cases} \\ &\quad = \begin{cases} \sup\big\{\lambda > s(T);\ \frac{A\varphi}{\varphi} \geqslant \lambda \text{ a.e.}\big\} & \text{if } \inf \frac{A\varphi}{\varphi} > s(T), \\ s(T) & \text{otherwise}\end{cases} \\ &\quad = \max\left\{\inf \frac{A\varphi}{\varphi}, s(T)\right\}\end{aligned}$$

and this ends the proof. $\square$

We give now some "dual" results.

Lemma 3. *Let* (6) *be satisfied. Let*

$$\hat{E} := \{\lambda > s(T);\ \exists \varphi \in D(T) \cap X_+ \text{ and } \lambda\varphi - A\varphi \in X_+ - \{0\}\}.$$

Then $\hat{E} = \{\lambda > s(T);\ r_\sigma[(\lambda - T)^{-1}K] < 1\}$.

Proof. Let $\lambda > s(T)$ be such that $r_\sigma[(\lambda - T)^{-1}K] < 1$. Then $\lambda > s(A)$ and

$$(\lambda - A)^{-1} = \sum_{k=0}^{\infty} [(\lambda - T)^{-1}K]^k (\lambda - T)^{-1} \geqslant 0.$$

Let $\psi \in X_+ - \{0\}$ and $\varphi := (\lambda - A)^{-1}\psi$. Then $\varphi \in D(T) \cap X_+$ and $\lambda\varphi - A\varphi \in X_+ - \{0\}$. Conversely, let $\lambda \in \hat{E}$. Then $\lambda\varphi - T\varphi - K\varphi \in X_+ - \{0\}$ for some $\varphi \in D(T) \cap X_+$. This implies that

$$\varphi - (\lambda - T)^{-1}K\varphi \in X_+ - \{0\}. \tag{10}$$

By (6), there exists $\psi' \in X'_+$, $\psi' > 0$ a.e., an eigenvector associated to the spectral radius of the *dual* operator $[(\lambda - T)^{-1}K]'$ (which is equal to that of $(\lambda - T)^{-1}K$). Then (10) implies $\langle \varphi - (\lambda - T)^{-1}K\varphi, \psi' \rangle > 0$, i.e.

$$\langle \varphi, \psi' \rangle > \langle (\lambda - T)^{-1}K\varphi, \psi' \rangle = \langle \varphi, [(\lambda - T)^{-1}K]'\psi' \rangle = r_\sigma[(\lambda - T)^{-1}K]\langle \varphi, \psi' \rangle$$

whence

$$r_\sigma[(\lambda - T)^{-1}K] < 1 \quad \text{and} \quad \hat{E} \subset \{\lambda > s(T);\ r_\sigma[(\lambda - T)^{-1}K] < 1\}$$

since $\langle \varphi, \psi' \rangle > 0$. $\square$

Theorem 4. *Let* (6) *be satisfied. For each* $\varphi \in D(T) \cap X_+ - \{0\}$ *define*

$$\hat{\tau}(\varphi) := \inf\{\lambda > s(T);\ \lambda\varphi - A\varphi \in X_+ - \{0\}\} \tag{11}$$

with the convention that $\hat{\tau}(\varphi) = +\infty$ *if* $\{\lambda > s(T);\ \lambda\varphi - A\varphi \in X_+ - \{0\}\}$ *is empty. Then* $s(A) = \inf_{\varphi \in D(T) \cap X_+ - \{0\}} \hat{\tau}(\varphi)$.

Proof. It follows from Lemma 3 and (7) that $\hat{E} \neq \emptyset$ and $s(A) = \inf\{\lambda; \lambda \in \hat{E}\}$. Let $\varphi \in D(T) \cap X_+ - \{0\}$ be such that $\hat{\tau}(\varphi) < +\infty$. Let $\lambda > \hat{\tau}(\varphi)$ be arbitrary. By assumption $\lambda\varphi - A\varphi \in X_+ - \{0\}$ and it follows that $s(A) \leqslant \lambda$ whence $s(A) \leqslant \hat{\tau}(\varphi)$. Finally $s(A) \leqslant \inf_{\varphi \in D(T) \cap X_+ - \{0\}} \hat{\tau}(\varphi)$ because φ is arbitrary. Conversely, let $\lambda > s(A)$ be arbitrary. According to Lemma 3, there exists $\varphi \in D(T) \cap X_+$ such that $\lambda\varphi - A\varphi \in X_+ - \{0\}$ so that $\hat{\tau}(\varphi) \leqslant \lambda$. Hence $\inf_{\varphi \in D(T) \cap X_+ - \{0\}} \hat{\tau}(\varphi) \leqslant s(A)$ and we are done. $\square$

Theorem 5. *We assume that* (5) *and* (6) *are satisfied. Then the spectral bound* $s(A)$ *is given by*

$$\min_{\varphi \in D(T) \cap X_+^*} \max\left\{\sup \frac{A\varphi}{\varphi},\ s(T)\right\},$$

where $\sup \frac{A\varphi}{\varphi}$ *denotes the essential supremum of* $\frac{A\varphi}{\varphi}$.

Proof. We note that if $\{\lambda > s(T);\ \lambda\varphi - A\varphi \in X_+ - \{0\}\} = \emptyset$ for all $\varphi \in X_+ - \{0\}$ then $\sup \frac{A\varphi}{\varphi} \leqslant s(T)$ for all $\varphi \in X_+^*$ and

$$\min_{\varphi\in D(T)\cap X_+^*} \max\left\{\sup \frac{A\varphi}{\varphi},\ s(T)\right\} = s(T).$$

Otherwise,

$$\left\{\lambda > s(T);\ \lambda\varphi - A\varphi \in X_+ - \{0\}\right\} \neq \emptyset$$

for some $\varphi \in X_+ - \{0\}$ and $s(A) > s(T)$. A priori $s(A) \leqslant \inf_{\varphi\in D(T)\cap X_+^*} \hat{\tau}(\varphi)$. It follows from (5) there exists $\psi \in D(T) \cap X_+^*$ such that $A\psi = s(A)\psi$ and consequently, for all $\varepsilon > 0$, $(s(A) + \varepsilon)\psi - A\psi \in X_+ - \{0\}$ so $\hat{\tau}(\psi) \leqslant s(A) + \varepsilon$ for all $\varepsilon > 0$ or $\hat{\tau}(\psi) \leqslant s(A)$. Hence $\inf_{\varphi\in D(T)\cap X_+^*} \hat{\tau}(\varphi) \leqslant s(A)$ and finally $s(A) = \inf_{\varphi\in D(T)\cap X_+^*} \hat{\tau}(\varphi)$. On the other hand, for $\varphi \in D(T) \cap X_+^*$, $\hat{\tau}(\varphi)$ is equal to

$$\begin{aligned}\inf\left\{\lambda > s(T);\ \lambda\varphi - A\varphi \in X_+ - \{0\}\right\} &= \inf\left\{\lambda > s(T);\ \lambda > \frac{A\varphi}{\varphi} \text{ a.e.}\right\}\\ &= \max\left\{\sup \frac{A\varphi}{\varphi}, s(T)\right\}\end{aligned}$$

and this ends the proof. □

Remark 1. Note that $A\varphi \geqslant \tau(\varphi)\varphi$, i.e. $\tau(\varphi)$ is the best "sub-eigenvalue" of A corresponding to $\varphi \in D(T)\cap X_+$ and $\sup_{\varphi\in D(T)\cap X_+} \tau(\varphi) = s(A)$. In particular, the leading eigenvalue $s(A)$ can be approximated by "sub-eigenvalues." The corresponding "sub-eigenfunctions" also approximate the leading eigenfunction. Indeed, we show [12, Theorems 6, 7] that if $\{\varphi_k\} \subset W^p_{0-} \cap X_+$ (appropriately normalized) is such that $A\varphi_k \geqslant \lambda_k\varphi_k$ with $\lambda_k \to s(A)$, then $\|\varphi_k - \varphi\| \to 0$ where φ is an (appropriately normalized) eigenfunction of A associated with $s(A)$.

3. Applications to neutron transport

This section is concerned with the derivation of upper and lower bounds for the leading eigenvalue of neutron transport operators. According to Theorem 1, for each $\varphi \in W^p_{0-} \cap X_+,\ \varphi \neq 0$, $s(A)$ is greater or equal to each real λ satisfying the inequality

$$-v\cdot\frac{\partial\varphi}{\partial x} - \sigma(x,v)\varphi(x,v) + \int_V k(x,v,v')\varphi(x,v')\,d\mu(v') \geqslant \lambda\varphi \quad \text{a.e.} \tag{12}$$

On the other hand, according to Theorem 4, for each $\varphi \in W^p_{0-} \cap X_+,\ \varphi \neq 0,\ \ s(A)$ is less than or equal to each real λ satisfying the inequality

$$-v\cdot\frac{\partial\varphi}{\partial x} - \sigma(x,v)\varphi(x,v) + \int_V k(x,v,v')\varphi(x,v')\,d\mu(v') \leqslant \lambda\varphi$$

with a strict inequality on a set of positive measure. Consider the operator

$$K_\tau^\sigma : \varphi \in L^p(V) \to \int \frac{k(x,v,v')\tau(x,v')}{1+\sigma(x,v)\tau(x,v)} \varphi(v')\, d\mu(v') \in L^p(V)$$

indexed by the spatial parameter $x \in \Omega$ even if, for the simplicity of notations, we do not make explicit its dependence on x. Define the parameter

$$\hat{\beta} := \sup\bigl\{\beta \geqslant 0;\ \exists \varphi \in L_+^p(V), \varphi \neq 0,\ K_\tau^\sigma \varphi \geqslant \beta\varphi\ (\forall x \in \Omega)\bigr\}. \tag{13}$$

Lemma 4. *We assume that for each $x \in \Omega$*

$$\varphi \in L^p(V) \to \int k(x,v,v')\varphi(v')\, d\mu(v') \in L^p(V)$$

is compact if $p > 1$ or weakly compact if $p = 1$. Then the set

$$I := \bigl\{\beta \geqslant 0;\ \exists \varphi \in L_+^p(V), \varphi \neq 0,\ K_\tau^\sigma \varphi \geqslant \beta\varphi\ \forall x \in \Omega\bigr\}$$

is closed and consequently there exists $\hat{\varphi} \in L_+^p(V), \hat{\varphi} \neq 0$, such that $K_\tau^\sigma \hat{\varphi} \geqslant \hat{\beta}\hat{\varphi}$ $(\forall x \in \Omega)$.

Proof. We first note that by a *domination* argument [1] K_τ^σ is compact if $p > 1$ or weakly compact if $p = 1$. The set $I \subset R^+$ is clearly a bounded interval containing zero. Let $I \neq \{0\}$ and $\{\beta_j\} \subset I$, $\beta_j \to \hat{\beta}$. Then $K_\tau^\sigma \varphi_j \geqslant \beta_j \varphi_j$ $(\varphi_j \in L_+^p(V), \varphi_j \neq 0)$. We argue with fixed $x \in \Omega$. For $p > 1$, we choose the normalization

$$\bigl\| K_\tau^\sigma \varphi_j \bigr\|_{L^p(V)} = 1.$$

A subsequence $\{\varphi_{j_k}\}_k$ converges weakly to some $\hat{\varphi} \geqslant 0$ satisfying $K_\tau^\sigma \hat{\varphi} \geqslant \hat{\beta}\hat{\varphi}$. Now the compactness of K_τ^σ (for each $x \in \Omega$) shows that

$$\bigl\| K_\tau^\sigma \varphi_j - K_\tau^\sigma \hat{\varphi} \bigr\|_{L^p(V)} \to 0$$

and consequently

$$\bigl\| K_\tau^\sigma \hat{\varphi} \bigr\|_{L^p(V)} = 1 \quad \text{and} \quad \hat{\varphi} \neq 0.$$

For $p = 1$, we use $[K_\tau^\sigma]^2 \varphi_j \geqslant \beta_j^2 \varphi_j$ with the normalization

$$\bigl\| \bigl[K_\tau^\sigma\bigr]^2 \varphi_j \bigr\|_{L^1(V)} = 1.$$

Note that $[K_\tau^\sigma]^2$ is compact as a square of a weakly compact operator. The domination above shows that $\{\varphi_j\}$ is weakly compact in $L^1(V)$ and we argue as previously. □

Theorem 6. *We assume that $\hat{\beta} \geqslant 1$. Then*

$$s(A) \geqslant (\hat{\beta} - 1)\left(\frac{1}{\tau_{\max}} + \sigma_{\min}\right)$$

where

$$\tau_{\max} := \sup_{(x,v) \in \Omega \times V} \tau(x,v) \quad \textit{and} \quad \sigma_{\min} := \inf_{(x,v) \in \Omega \times V} \sigma(x,v).$$

Proof. Let $\psi \in L^p_+(V)$, $\varphi \neq 0$. Define the test function $\varphi(x,v) := \tau(x,v)\psi(v)$. Note that $\tau(\cdot,\cdot)$ is bounded since $\tau(x,v) \leqslant \frac{d}{v_{\min}}$ where $v_{\min}$ is the minimum speed. Thus $\varphi \in L^p_+(\Omega \times V)$. Moreover, one sees that $\varphi \in W^p_{0-}$ and $-v \cdot \frac{\partial \varphi}{\partial x} = -\psi(v)$. According to Theorem 1, for all $\psi \in L^p_+(V)$, $\psi \neq 0$, $s(A)$ is greater or equal to every λ satisfying

$$-\big(1+\sigma(x,v)\tau(x,v)\big)\psi(v) + \int_V k(x,v,v')\tau(x,v')\psi(v')\,d\mu(v') \geqslant \lambda\tau(x,v)\psi(v). \tag{14}$$

We look for $\psi \in L^p_+(V)$, $\varphi \neq 0$ and $\lambda > -\infty$ satisfying (14). This is equivalent to

$$\int_V k(x,v,v')\tau(x,v')\psi(v')\,d\mu(v') \geqslant \big[1+\sigma(x,v)\tau(x,v)+\lambda\tau(x,v)\big]\psi(v)$$

or to

$$\int \frac{k(x,v,v')\tau(x,v')}{1+\sigma(x,v)\tau(x,v)}\psi(v')\,d\mu(v') \geqslant \left[1+\frac{\lambda\tau(x,v)}{1+\sigma(x,v)\tau(x,v)}\right]\psi(v)$$

i.e.

$$K^\sigma_\tau\psi \geqslant \left(1+\frac{\lambda\tau(x,v)}{1+\sigma(x,v)\tau(x,v)}\right)\psi(v).$$

By Lemma 4, $K^\sigma_\tau\hat{\varphi} \geqslant \hat{\beta}\hat{\varphi}$ so it suffices that

$$1+\frac{\lambda\tau(x,v)}{1+\sigma(x,v)\tau(x,v)} \leqslant \hat{\beta} \quad \forall (x,v) \in \Omega \times V$$

i.e.

$$\lambda \leqslant (\hat{\beta}-1)\left(\frac{1}{\tau(x,v)}+\sigma(x,v)\right) \quad \forall (x,v) \in \Omega \times V$$

or equivalently

$$\begin{aligned}\lambda &\leqslant \inf_{(x,v)}(\hat{\beta}-1)\left(\frac{1}{\tau(x,v)}+\sigma(x,v)\right) = (\hat{\beta}-1)\inf_{(x,v)}\left(\frac{1}{\tau(x,v)}+\sigma(x,v)\right)\\ &= (\hat{\beta}-1)\left(\frac{1}{\tau_{\max}}+\sigma_{\min}\right)\end{aligned}$$

since $\hat{\beta}-1 \geqslant 0$. $\square$

Corollary 2. *Let V be bounded. We assume that*

$$\bar{\beta} := \inf_{(x,v)} \int \frac{k(x,v,v')\tau(x,v')}{1+\sigma(x,v)\tau(x,v)}\,d\mu(v') \geqslant 1. \tag{15}$$

Then

$$s(A) \geqslant (\bar{\beta}-1)\left(\frac{1}{\tau_{\max}}+\sigma_{\min}\right).$$

Proof. We note that (15) expresses that

$$K_\tau^\sigma \varphi \geqslant \bar{\beta}\varphi \quad \text{with the choice } \varphi = 1$$

so that $\bar{\beta} \leqslant \hat{\beta}$. $\square$

Remark 2. Theorem 6 depends heavily on the assumption $\hat{\beta} \geqslant 1$. If $\hat{\beta} < 1$ and if

$$\inf_{(x,v)\in\Gamma_-} \int_V k(x,v,v')\tau(x,v')\,d\mu(v') > 1,$$

then we can also derive a lower bound of $s(A)$; see [12, Theorem 7 and Corollary 3].

To derive *upper bounds* of $s(A)$, define the "dual" parameter to $\hat{\beta}$ (see (13))

$$\hat{\alpha} := \inf_{\psi\in L^p_{*+}(V)} \sup_{(x,v)} \frac{K_\tau^\sigma \psi}{\psi}. \tag{16}$$

Theorem 7. *If* $\hat{\alpha} < 1$ *then*

$$s(A) \leqslant (\hat{\alpha} - 1)\left(\frac{1}{\tau_{\max}} + \sigma_{\min}\right).$$

Proof. Let $\psi \in L^p_{*+}(V)$ be such that

$$\sup_{(x,v)} \frac{K_\tau^\sigma \psi}{\psi} < 1. \tag{17}$$

Use a test function of the form $\varphi(x,v) := \tau(x,v)\psi(v)$. By Theorem 4, $s(A)$ is less than or equal to every λ satisfying

$$-\big(1 + \sigma(x,v)\tau(x,v)\big)\psi(v) + \int_V k(x,v,v')\tau(x,v')\psi(v')\,d\mu(v')$$
$$< \lambda\tau(x,v)\psi(v) \quad \text{a.e.},$$

or equivalently

$$\frac{K_\tau^\sigma \psi}{\psi} < \left(1 + \frac{\lambda\tau(x,v)}{1 + \sigma(x,v)\tau(x,v)}\right) \quad \text{a.e.}$$

This is satisfied if

$$\lambda > \sup_{(x,v)} \left(\frac{K_\tau^\sigma \psi}{\psi} - 1\right)\left(\frac{1}{\tau(x,v)} + \sigma(x,v)\right). \tag{18}$$

Note that

$$\left(\frac{K_\tau^\sigma \psi}{\psi} - 1\right) \leqslant 0 \quad \text{and} \quad \frac{1}{\tau(x,v)} + \sigma(x,v) \geqslant \frac{1}{\tau_{\max}} + \sigma_{\min}$$

so

$$\left(\frac{K_\tau^\sigma \psi}{\psi} - 1\right)\left(\frac{1}{\tau(x,v)} + \sigma(x,v)\right) \leqslant \left(\frac{K_\tau^\sigma \psi}{\psi} - 1\right)\left(\frac{1}{\tau_{\max}} + \sigma_{\min}\right)$$

and

$$\sup_{(x,v)}\left(\frac{K_\tau^\sigma\psi}{\psi}-1\right)\left(\frac{1}{\tau(x,v)}+\sigma(x,v)\right)\leqslant\sup_{(x,v)}\left(\frac{K_\tau^\sigma\psi}{\psi}-1\right)\left(\frac{1}{\tau_{\max}}+\sigma_{\min}\right).$$

Thus (18) is satisfied if

$$\lambda>\sup_{(x,v)}\left(\frac{K_\tau^\sigma\psi}{\psi}-1\right)\left(\frac{1}{\tau_{\max}}+\sigma_{\min}\right)=\left(\sup_{(x,v)}\frac{K_\tau^\sigma\psi}{\psi}-1\right)\left(\frac{1}{\tau_{\max}}+\sigma_{\min}\right)$$

and

$$s(A)\leqslant\left(\sup_{(x,v)}\frac{K_\tau^\sigma\psi}{\psi}-1\right)\left(\frac{1}{\tau_{\max}}+\sigma_{\min}\right)$$

for all $\psi\in L^p_{*+}(V)$ satisfying (17). Hence

$$s(A)\leqslant\left(\inf_{\psi\in L^p_{*+}(V)}\sup_{(x,v)}\frac{K_\tau^\sigma\psi}{\psi}-1\right)\left(\frac{1}{\tau_{\max}}+\sigma_{\min}\right)$$

which ends the proof. □

Corollary 3. *Let V be bounded. If*

$$\bar{\alpha}:=\sup_{(x,v)}\int\frac{k(x,v,v')\tau(x,v')}{1+\sigma(x,v)\tau(x,v)}\,d\mu(v')<1$$

then

$$s(A)\leqslant(\bar{\alpha}-1)\left(\frac{1}{\tau_{\max}}+\sigma_{\min}\right).$$

Proof. We note that $\bar{\alpha}=\sup_{(x,v)}\frac{K_\tau^\sigma\psi}{\psi}$ with $\psi=1$ so $\bar{\alpha}\geqslant\hat{\alpha}$. □

Remark 3. When $\hat{\alpha}\geqslant 1$ we can also derive an upper bound of $s(A)$ provided that

$$\sup_{(x,v)\in\Gamma_-}\int k(x,v,v')\tau(x,v')\,d\mu(v')<1;$$

see [12, Theorem 9 and Corollary 5].

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