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Constants of Weitzenböck derivations and invariants of unipotent transformations acting on relatively free algebras

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Abstract

In commutative algebra, a Weitzenböck derivation is a nonzero triangular linear derivation of the polynomial algebra $K[x_1,\dots,x_m]$ in several variables over a field K of characteristic 0. The classical theorem of Weitzenböck states that the algebra of constants is finitely generated. (This algebra coincides with the algebra of invariants of a single unipotent transformation.) In this paper we study the problem of finite generation of the algebras of constants of triangular linear derivations of finitely generated (not necessarily commutative or associative) algebras over K assuming that the algebras are free in some sense (in most of the cases relatively free algebras in varieties of associative or Lie algebras). In this case the algebra of constants also coincides with the algebra of invariants of some unipotent transformation.

The main results are the following: (1) We show that the subalgebra of constants of a factor algebra can be lifted to the subalgebra of constants. (2) For all varieties of associative algebras which are not nilpotent in Lie sense the subalgebras of constants of the relatively free algebras of rank $\geqslant 2$ are not finitely generated. (3) We describe the generators of the subalgebra of constants for all factor

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algebras $K\langle x, y\rangle/I$ modulo a $GL_2(K)$-invariant ideal I. (4) Applying known results from commutative algebra, we construct classes of automorphisms of the algebra generated by two generic 2×2 matrices. We obtain also some partial results on relatively free Lie algebras.

1. Introduction

We fix a base field K of characteristic 0, an integer $m \geqslant 2$ and a set of symbols $X = \{x_1, \ldots, x_m\}$. We call the elements of X variables. Sometimes we shall use other symbols, e.g., y, z, y_i, etc., for the elements of X. We denote by V_m the vector space with basis X.

Let $K\langle X\rangle = K\langle x_1, \ldots, x_m\rangle$ be the free unitary associative algebra freely generated by X over K. The elements of $K\langle X\rangle$ are linear combinations of words $x_{j_1} \cdots x_{j_n}$ in the noncommuting variables X. The general linear group $GL_m = GL_m(K)$ acts naturally on the vector space V_m and this action is extended diagonally on $K\langle X\rangle$ by the rule

$$g(x_{j_1} \cdots x_{j_n}) = g(x_{j_1}) \cdots g(x_{j_n}), \quad g \in GL_m,\ x_{j_1}, \ldots, x_{j_n} \in X.$$

All associative algebras which we consider in this paper are homomorphic images of $K\langle X\rangle$ modulo ideals I which are invariant under this GL_m-action. We shall use the same symbols x_j and X for the generators and the whole generating set of $K\langle X\rangle/I$. Most of the algebras in our considerations will be relatively free algebras in varieties of unitary associative algebras. Examples of relatively free algebras are the polynomial algebra $K[X]$ and the free algebra $K\langle X\rangle$ which are free, respectively, in the varieties of all commutative algebras and all associative algebras. We also shall consider Lie algebras which are homomorphic images of the free Lie algebra with X as a free generating set modulo ideals which are also GL_m-invariant.

Let A be any (not necessarily associative or Lie) algebra over K. Recall that the K-linear operator δ acting on A is called a derivation of A if

$$\delta(uv) = \delta(u)v + u\delta(v) \quad \text{for all } u, v \in A.$$

The elements $u \in A$ which belong to the kernel of δ are called constants of δ and form a subalgebra of A which we shall denote by A^δ. The derivation δ is locally nilpotent if for any $u \in A$ there exists a positive integer n such that $\delta^n(u) = 0$. If δ is a locally nilpotent derivation of A, then the linear operator of A

$$\exp\delta = 1 + \frac{\delta}{1!} + \frac{\delta^2}{2!} + \cdots$$

is well defined and is an automorphism of the K-algebra A. It is easy to see that A^δ coincides with the subalgebra of fixed points (or invariants) of $\exp\delta$ which we shall denote by $A^{\exp\delta}$. The mapping $\alpha \to \exp(\alpha\delta)$, $\alpha \in K$, defines an additive action of K on A. It is well known that for polynomial algebras every additive action of K is of this kind, see for more details Snow [68]. See also Drensky and Yu [30] for relations between exponents of locally nilpotent derivations and automorphisms φ with the property that the orbit $\{\varphi^n(a) \mid n \in \mathbb{Z}\}$ of each $a \in A$ spans a finite dimensional vector space in the noncommutative case.

If $A = K\langle X\rangle/I$ for some GL_m-invariant ideal I, then the derivation δ of A is called triangular, if $\delta(x_j)$, $j = 1, \dots, m$, belongs to the subalgebra of A generated by $x_1, \dots, x_{j-1}$. Clearly, the triangular derivations are locally nilpotent. If δ acts linearly on the vector space $V_m = \sum_{j=1}^m Kx_j \subset A$, then it is called linear.

If δ is a triangular derivation, then $\exp\delta$ is a triangular automorphism of A, with the property

$$\exp\delta(x_j) = x_j + f_j(x_1, \dots, x_{j-1}), \quad j = 1, \dots, m,$$

where $f_j(x_1, \dots, x_{j-1})$ depends on $x_1, \dots, x_{j-1}$ only. Every triangular automorphism φ of this form can be presented in the form $\varphi = \exp\delta$ for some triangular derivation

$$\delta = \log(\varphi) = \frac{\varphi - 1}{1} - \frac{(\varphi - 1)^2}{2} + \frac{(\varphi - 1)^3}{3} - \frac{(\varphi - 1)^4}{4} + \cdots.$$

(The K-linear operator δ of A is well defined because the linear operators $(\varphi - 1)^k$ map every $f(x_1, \dots, x_j) \in A$ to a polynomial depending on $x_1, \dots, x_j$ only, $\deg_{x_j}(\varphi - 1)^k f \leqslant \deg_{x_j} f - k$ and $(\varphi - 1)K = 0$.)

Every locally nilpotent linear derivation δ is triangular with respect to a suitable basis of V_m and the automorphism $\exp\delta$ is a unipotent linear transformation (i.e., an automorphism of the algebra A which acts as a unipotent linear operator on V_m).

In commutative algebra, the triangular linear derivations of the polynomial algebra $K[X] = K[x_1, \dots, x_m]$ are called Weitzenböck derivations. The classical theorem of Weitzenböck [75] states that the algebra of constants of such a derivation is finitely generated. This algebra coincides with the algebra of invariants of a single unipotent transformation.

In this paper we study the problem of finite generation of the algebras of constants of triangular linear derivations of (usually noncommutative) algebras $K\langle X\rangle/I$ where the ideal I is GL_m-invariant. As in the commutative case, the algebra of constants coincides with the algebra of invariants of some unipotent transformation. The paper is organized as follows. Below we assume that δ is a nonzero triangular linear derivation of $K\langle X\rangle$ which induces a derivation (which we shall also denote by δ) on the factor algebras of $K\langle X\rangle$ modulo GL_m-invariant ideals.

In Section 2 we present a short survey on constants of locally nilpotent derivations and invariant theory both in the commutative and noncommutative case, giving some motivation for our investigations. We believe that some of the results exposed there can serve as a motivation and inspiration for further investigations on noncommutative algebras.

Section 3 presents a summary of the results on the Weitzenböck derivations of polynomial algebras which we need in the next sections.

In Section 4 we are interested in the problem of lifting the constants: If $I \subset J$ are two GL_m-invariant ideals of $K\langle X\rangle$, then we show that the subalgebra of constants $(K\langle X\rangle/J)^\delta$ can be lifted to the subalgebra of constants $(K\langle X\rangle/I)^\delta$. In the special case of algebras with two generators x, y we may assume that $\delta(x) = 0$, $\delta(y) = x$. Then the subalgebra of constants is spanned by elements which have a very special behaviour under the action of the general linear group GL_2, the so called highest weight vectors. This allows to involve classical combinatorial techniques as theory of generating functions and representation theory of general linear groups.

In Section 5 we present various examples of subalgebras of constants of relatively free associative algebras. In particular, he handle the case of the free algebra $K\langle x, y\rangle$ and show that the algebra of constants is generated by x and a set of $SL_2(K)$-invariants which we describe explicitly. As a consequence, we obtain a similar generating set for all factor algebras $K\langle x, y\rangle/I$.

Section 6 considers relatively free algebras $F_m(\mathfrak{W})$ in varieties $\mathfrak{W}$ of associative algebras. It is known that every variety $\mathfrak{W}$ is either nilpotent in Lie sense or contains the algebra of 2×2 upper triangular matrices. We show that for all $\mathfrak{W}$ which are not nilpotent in Lie sense the subalgebras of constants $F_m(\mathfrak{W})^\delta$ are not finitely generated.

In Section 7 we apply results from commutative algebra and construct classes of automorphisms of the relatively free algebra $F_2(\operatorname{var} M_2(K))$. This algebra is isomorphic to the algebra generated by two generic 2×2 matrices x and y. The centre of the associated generic trace algebra (which coincides with the algebra of invariants of two 2×2 matrices under simultaneous conjugation by GL_2) is generated by the traces of x, y and xy and the determinants of x and y and is isomorphic to the polynomial algebra in five variables. We want to mention that up till now most of the investigations have been performed in the opposite direction. The automorphisms of $F_2(\operatorname{var} M_2(K))$ and of the trace algebra have been used to produce automorphisms of the polynomial algebra in five variables, see, e.g., Bergman [7], Alev and Le Bruyn [1], Drensky and Gupta [26].

Finally, we obtain also some partial results on relatively free Lie algebras.

2. Survey

2.1. Motivation from commutative algebra

Locally nilpotent derivations of the polynomial algebra $K[X] = K[x_1, \ldots, x_m]$ have been studied for many decades and have had significant impact on different branches of algebra and invariant theory, see, e.g., the books by Nowicki [61] and van den Essen [33].

Let G be a subgroup of GL_m and let $K[X]^G = K[x_1, \ldots, x_m]^G$ be the algebra of G-invariants. The problem for finite generation of $K[X]^G$ was the main motivation for the famous Hilbert Fourteenth Problem [44]. The theorem of Emmy Noether [60] gives the finite generation of $K[X]^G$ for finite groups G. More generally, the Hilbert–Nagata theorem states the finite generation of $K[X]^G$ for reductive groups G, see, e.g., [16].

The first counterexample of Nagata [58] to the Hilbert Fourteenth Problem was the nonfinitely generated algebra of invariants $K[x_1, \ldots, x_{32}]^G$ of a specially constructed triangular linear group G. Today, most of the known counterexamples have been obtained (or can be obtained) as algebras of constants of some derivations. This includes the original counterexample of Nagata, see Derksen [14] who was the first to recognize the connection between the Hilbert Fourteenth Problem and constants of derivations (but his derivations were not always locally nilpotent) and the counterexample of Daigle and Freudenburg [12] of a triangular (but not linear) derivation of $K[x_1,\dots,x_5]$ with not finitely generated algebra of constants. For more counterexamples to the Hilbert Fourteenth Problem we refer to the recent survey by Freudenburg [39].

The theorem of Weitzenböck gives the finite generation of the algebra of constants for a triangular linear derivation or, equivalently, for the algebra of invariants of a single unipotent transformation. (This contrasts to the counterexample of Nagata described above.) The original proof of Weitzenböck from 1932 was for $K=\mathbb{C}$. Later Seshadri [63] found a proof for any field K of characteristic 0. A simple proof for $K=\mathbb{C}$ using ideas from [63] has been recently given by Tyc [72]. To the best of our knowledge, no constructive proof, with effective estimates of the degree of the generators of the algebra of constants has been given up till now.

For each dimension m there are only finite number of essentially different Weitzenböck derivations to study: Up to a linear change of the coordinates, the Weitzenböck derivations δ are in one-to-one correspondence with the partition $(p_1+1,p_2+1,\dots,p_s+1)$ of m, where $p_1\geqslant p_2\geqslant\cdots\geqslant p_s\geqslant 0$, $(p_1+1)+(p_2+1)+\cdots+(p_s+1)=m$, and the correspondence is given in terms of the Jordan normal form $J(\delta)$ of the matrix of the derivation

$$J(\delta)=\begin{pmatrix} J_1 & 0 & \cdots & 0\\ 0 & J_2 & \cdots & 0\\ \vdots & \vdots & \dots & \vdots\\ 0 & 0 & \cdots & J_s\end{pmatrix},\quad \text{where}\quad J_i=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & 0 & \cdots & 0 & 0\end{pmatrix}$$

is the $(p_i+1)\times(p_i+1)$ Jordan cell with zero diagonal.

Another important application of locally nilpotent derivations is the construction of candidates for wild automorphisms of polynomial algebras, see, e.g., the survey of Drensky and Yu [31]. A typical example is the following. If δ is a Weitzenböck derivation of $K[x_1,\dots,x_m]$ and $0\neq w\in K[x_1,\dots,x_m]^\delta$, then $\Delta=w\delta$ is also a locally nilpotent derivation of $K[x_1,\dots,x_m]$ with the same algebra of constants as δ and $\exp\Delta$ is an automorphism of $K[x_1,\dots,x_m]$. By the theorem of Martha Smith [67], all such automorphisms are stably tame and become tame if extended to $K[x_1,\dots,x_m,x_{m+1}]$ by $(\exp\Delta)(x_{m+1})=x_{m+1}$. The famous Nagata automorphism of $K[x,y,z]$, see [59], also can be obtained in this way: We define the derivation δ by

$$\delta(x)=-2y,\qquad \delta(y)=z,\qquad \delta(z)=0,\qquad w=xz+y^2\in K[x,y,z]^\delta,$$

and for $\Delta=w\delta$ the Nagata automorphism is $\nu=\exp\Delta$:

$$\begin{aligned}
\nu(x)&=x+(-2y)\frac{w}{1!}+(-2z)\frac{w^2}{2!}=x-2\big(xz+y^2\big)y-\big(xz+y^2\big)^2z,\\
\nu(y)&=y+z\frac{w}{1!}=y+\big(xz+y^2\big)z,\\
\nu(z)&=z.
\end{aligned}$$

Recently Shestakov and Umirbaev [64] proved that the Nagata automorphism is wild. It is interesting to mention that their approach is based on Poisson algebras and methods of noncommutative, and even nonassociative, algebras.

There are few exceptions of locally nilpotent derivations and their exponents which do not arise immediately from triangular derivations: the derivations of Freudenburg (obtained with his local slice construction [38]) and the automorphisms of Drensky and Gupta (obtained by methods of noncommutative algebra, [26]). Later, Drensky, van den Essen and Stefanov [24] have shown that the automorphisms from [26] also can be obtained in terms of locally nilpotent derivations and are stably tame.

2.2. *Noncommutative invariant theory*

An important part of noncommutative invariant theory is devoted to the study of the algebra of invariants of a linear group $G \subset GL_m$ acting on the free associative algebra $K\langle X\rangle = K\langle x_1, \dots, x_m\rangle$, relatively free algebras $F_m(\mathfrak{W})$ in varieties of associative algebras $\mathfrak{W}$, the free Lie algebra $L_m = L(X)$ and relatively free algebras $L_m(\mathfrak{V})$ in varieties of Lie algebras $\mathfrak{V}$. For more detailed exposition we refer to the surveys on noncommutative invariant theory by Formanek [35], Drensky [22] and the survey on algorithmic methods for relatively free semigroups, groups and algebras by Kharlampovich and Sapir [47].

2.2.1. *Free associative algebras*

By a theorem of Lane [53] and Kharchenko [45], the algebra of invariants $K\langle X\rangle^G$ is always a free algebra (independently of the properties of $G \subset GL_m$). By the theorem of Dicks and Formanek [15] and Kharchenko [45], if G is finite, then $K\langle X\rangle^G$ is finitely generated if and only if G is cyclic and acts on $V_m = \sum_{j=1}^m Kx_j$ as a group of scalar multiplications. This result was generalized for a much larger class of groups by Koryukin [48] who also established a finite generation of $K\langle X\rangle^G$ if we equip it with a proper action of the symmetric group.

Recall that if V is a multigraded vector space which is a direct sum of its multihomogeneous components $V^{(n_1,\dots,n_m)}$, then the Hilbert series of V is defined as the formal power series

$$H(V, t_1, \dots, t_m) = \sum \dim\bigl(V^{(n_1,\dots,n_m)}\bigr) t_1^{n_1} \cdots t_m^{n_m}.$$

If V is "only" graded with homogeneous components $V^{(n)}$, then its Hilbert series is

$$H(V, t) = \sum_{n \geqslant 0} \dim\bigl(V^{(n)}\bigr) t^n.$$

Dicks and Formanek [15] proved also an analogue of the Molien formula for the Hilbert series of $K\langle X\rangle^G$, $|G| < \infty$, which was generalized for compact groups G by Almkvist, Dicks and Formanek [4] (an analogue of the Molien–Weyl formula in classical invariant theory). In particular, Almkvist, Dicks and Formanek showed that the Hilbert series of the algebra of invariants $K\langle X\rangle^g$ is an algebraic function if g is a unipotent matrix. (Hence the same holds for the algebra of constants $K\langle X\rangle^\delta$ for a Weitzenböck derivation δ.)

2.2.2. Relatively free associative algebras

Let $f(x_1,\dots,x_m)\in K\langle x_1,x_2,\dots\rangle$ be an element of the free algebra of countable rank. Recall that $f(x_1,\dots,x_m)=0$ is a polynomial identity for the algebra A if $f(a_1,\dots,a_m)=0$ for all $a_1,\dots,a_m\in A$. The algebra is called PI, if it satisfies some nontrivial polynomial identity. The class of all algebras satisfying a given set $U\subset K\langle x_1,x_2,\dots\rangle$ of polynomial identities is called the variety of associative algebras defined by the system U. We shall denote the varieties by German letters. If $\mathfrak{W}$ is a variety, then $T(\mathfrak{W})$ is the ideal of $K\langle x_1,x_2,\dots\rangle$ consisting of all polynomial identities of $\mathfrak{W}$ and the algebra

$$F_m(\mathfrak{W})=K\langle x_1,\dots,x_m\rangle/\big(K\langle x_1,\dots,x_m\rangle\cap T(\mathfrak{W})\big)$$

is the relatively free algebra of rank m in $\mathfrak{W}$. The ideals $K\langle x_1,\dots,x_m\rangle\cap T(\mathfrak{W})$ of $K\langle x_1,\dots,x_m\rangle$ are invariant under all endomorphisms of $K\langle x_1,\dots,x_m\rangle$ and, in particular, are GL_m-invariant.

Most of the work on invariant theory of relatively free algebras is devoted to the description of the varieties $\mathfrak{W}$ such that $F_m(\mathfrak{W})^G$ is finitely generated for all $m=2,3,\dots,$ and all groups $G\subset GL_m$ from a given class $\mathfrak{G}$. The description of such varieties for the class of all finite groups is given in different terms by several authors, starting with Kharchenko [46], see the surveys by Formanek [35], Drensky [22], Kharlampovich and Sapir [47]. In particular, the finite generation of $F_m(\mathfrak{W})^G$ for all finite groups holds if and only if all finitely generated algebras of $\mathfrak{W}$ are weakly noetherian (i.e., noetherian with respect to two-sided ideals) which is equivalent to the fact that $\mathfrak{W}$ satisfies a polynomial identity of a very special form. One of the simplest descriptions is the following (see [20]): $F_m(\mathfrak{W})^G$ is finitely generated for all $m\geqslant 2$ and all finite groups $G\subset GL_m$ if and only if $F_2(\mathfrak{W})^g$ is finitely generated for the linear transformation g defined by $g(x_1)=-x_1$, $g(x_2)=x_2$.

If we consider the finite generation of $F_m(\mathfrak{W})^G$ for the class all reductive groups G, then the results of Vonessen [74], Domokos and Drensky [17] give that $F_m(\mathfrak{W})^G$ is finitely generated for all reductive G if and only if the finitely generated algebras in $\mathfrak{W}$ are one-side noetherian. For unitary algebras this means that $\mathfrak{W}$ satisfies the Engel identity $[x_2,x_1,\dots,x_1]=0$.

Concerning the Hilbert series of subalgebras of invariants of relatively free algebras, Formanek [35] generalized the Molien–Weyl formula for the Hilbert series of $K[x_1,\dots,x_m]^G$ for G finite or compact to the case of any relatively free algebra, expressing the Hilbert series of $F_m(\mathfrak{W})^G$ in terms of the Hilbert series $H(F_m(\mathfrak{W}),t_1,\dots,t_m)$. If G is finite, then $H(F_m(\mathfrak{W})^G,t)$ involves the eigenvalues of all $g\in G$. By a theorem of Belov [5], the Hilbert series of $F_m(\mathfrak{W})$ is always a rational function and this implies that $H(F_m(\mathfrak{W})^G,t)$ is also rational for G finite. For reductive G the rationality of $H(F_m(\mathfrak{W})^G,t)$ is known only for varieties $\mathfrak{W}$ satisfying a nonmatrix polynomial identity, see Domokos and Drensky [17].

2.2.3. Lie algebras

We shall mention few results only. By a theorem of Bryant [9], if G is a nontrivial finite linear group, then the algebra of fixed points of the free Lie algebra L_m^G is never finitely generated. This result was extended by Drensky [21] to the fixed points of all relatively free algebras $L_m(\mathfrak{V})$ (and also for all finite $G\neq 1$) for nonnilpotent varieties $\mathfrak{V}$ of Lie algebras. We refer also to the work done by several authors and mainly by Bryant, Kovacz and Stöhr about fixed points of free Lie algebras in the modular case, see, e.g., [10] and the references therein.

2.3. Derivations of free algebras

The algebra of constants of the formal partial derivatives $\partial/\partial x_j$, $j = 1, \ldots, m$, for $K\langle X\rangle = K\langle x_1, \ldots, x_m\rangle$ was described by Falk [34]. It is generated by all Lie commutators $[[\ldots[x_{j_1}, x_{j_2}], \ldots], x_{j_n}]$, $n \geqslant 2$. Specht [69] applied products of such commutators in the study of algebras with polynomial identities, see also Drensky [19] or the book [23] for further application to the theory of PI-algebras. It is known, see Gerritzen [40], that in this case the algebra of constants is free, see also Drensky and Kasparian [28] for an explicit basis. (The freedom of the algebra of constants of the partial derivatives of $K[X]$ does not follow immediately from the result of Lane [53] and Kharchenko [45]. The derivations $\partial/\partial x_j$ are locally nilpotent and their exponents $\exp(\partial/\partial x_j)$ generate a group of automorphisms of $K\langle X\rangle$ which consists of all translations of the form $x_i \to x_i + a_i$, $a_i \in \mathbb{Z}$. Although this group is a subgroup of the affine group, we cannot apply directly [53] and [45] because the group is not linear.)

Similar study of the algebra of constants in a very large class of (not only associative) algebras was performed by Gerritzen and Holtkamp [41] and Drensky and Holtkamp [27]. We shall finish the survey section with the following, probably folklore known lemma.

Lemma 2.1. *Let $\mathfrak{W}$ be any variety of algebras and let $F(\mathfrak{W})$ be the relatively free algebra of any rank. Every mapping from the free generating set to $F(\mathfrak{W})$ can be extended to a derivation.*

Proof. We shall prove the lemma for relatively free associative algebras of finite or countable rank only. The same considerations work in the case of any infinite rank. Let $\delta_0 : \{x_1, x_2, \ldots\} \to F_\infty(\mathfrak{W})$ be any mapping and let $T(\mathfrak{W})$ be the T-ideal of $K\langle x_1, x_2, \ldots\rangle$ of all polynomial identities of $\mathfrak{W}$. We fix $f_1, \ldots, f_m \in K\langle X\rangle$ such that $\delta_0(x_j) = f_j + T(\mathfrak{W}) \in F_\infty(\mathfrak{W})$, $j = 1, 2, \ldots$. Since every mapping $\{x_1, x_2, \ldots\} \to K\langle x_1, x_2, \ldots\rangle$ can be extended to a derivation of $K\langle x_1, x_2, \ldots\rangle$, it is sufficient to show that the derivation Δ of $K\langle x_1, x_2, \ldots\rangle$ defined by $\Delta(x_j) = f_j$, $j = 1, 2, \ldots$, has the property $\Delta(T(\mathfrak{W})) \subset T(\mathfrak{W})$. Since the field K is of characteristic 0, if $u(x_1, \ldots, x_m)$ belongs to $T(\mathfrak{W})$, then the multihomogeneous components of u also are in $T(\mathfrak{W})$ and we may assume that $u(x_1, \ldots, x_m) \in T(\mathfrak{W})$ is multihomogeneous. The partial linearization $u_j(x_1, \ldots, x_m, x_{m+1})$ in x_j of $u(x_1, \ldots, x_m)$, i.e., the linear component in x_{m+1} of $u(x_1, \ldots, x_{j-1}, x_j + x_{m+1}, x_{j+1}, \ldots, x_m)$ also belongs to $T(\mathfrak{W})$. It is easy to see that Δ acts on $u(x_1, \ldots, x_m)$ by

$$\Delta\big(u(x_1, \ldots, x_m)\big) = \sum_{j=1}^{m} u_j\big(x_1, \ldots, x_m, \Delta(x_j)\big).$$

Since $u_j(x_1, \ldots, x_m, \Delta(x_j)) \in T(\mathfrak{W})$ we obtain that $\Delta(u) \in T(\mathfrak{W})$ and this means that Δ induces a derivation δ on $F_\infty(\mathfrak{W}) = K\langle x_1, x_2, \ldots\rangle / T(\mathfrak{W})$ with the additional property $\delta(x_j) = f_j$, and δ extends δ_0. This implies also the case of $F_m(\mathfrak{W})$: If $f_1, \dots, f_m \in F_m(\mathfrak{W})$, then we extend the mapping to a derivation of $F_\infty(\mathfrak{W})$ (e.g., by $\delta_0(x_j) = 0$ for $j > m$). Then the restriction to $F_m(\mathfrak{W})$ of the derivation of $F_\infty(\mathfrak{W})$ is a derivation of $F_m(\mathfrak{W})$. $\square$

3. Weitzenböck derivations of polynomial algebras

Since we consider nonzero Weitzenböck derivations only, without loss of generality we may assume that the derivation δ is in its Jordan normal form, $\delta(x_1) = 0$, $\delta(x_2) = x_1$ and the set of variables $X = \{x_1, \dots, x_m\}$ is a Jordan basis of $V_m = \sum_{j=1}^m Kx_j$. If the rank of δ is equal to $m-1$ (i.e., $\delta(x_j) = x_{j-1}$, $j = 2, \dots, m$), then δ is called the basic Weitzenböck derivation of $K[X]$. The following proposition, see [61], gives the description of the algebras of constants of any Weitzenböck derivation. (It is a very special case of the more general situation of an arbitrary locally nilpotent derivation.) For our purposes we work in the localization of the polynomial algebra $K[X][x_1^{-1}] = K[x_1, x_2, \dots, x_m][x_1^{-1}]$ consisting of all polynomials in $x_1, \dots, x_m$ allowing negative degrees of x_1. Since x_1 is a constant (i.e., $\delta(x_1) = 0$), we may extend δ to a derivation of $K[X][x_1^{-1}]$.

Proposition 3.1. *Let $\delta^{p_j+1}(x_j) = 0$, $j = 1, \dots, m$, and let*

$$z_j = \sum_{k=0}^{p_j} \frac{\delta^k(x_j)}{k!}(-x_2)^k x_1^{p_j-k}, \quad j = 3, 4, \dots, m.$$

(i) $(K[X][x_1^{-1}])^\delta = K[x_1, z_3, z_4, \dots, z_m][x_1^{-1}]$;
(ii) $K[X]^\delta = K[X] \cap (K[X][x_1^{-1}])^\delta$.

Example 3.2. If δ is a basic Weitzenböck derivation, then

$$z_3 = x_3x_1^2 - \frac{x_2^2x_1}{2} = \frac{x_1}{2}\left(2x_3x_1 - x_2^2\right), \quad z_4 = x_1\left(x_4x_1^2 - x_3x_2x_1 + \frac{x_2^3}{3}\right), \quad \dots,$$

$$z_j = (-1)^j \frac{j}{(j+1)!} x_1 \left(x_2^{j+1} + \frac{(j+1)!}{j} \sum_{k=0}^{j-1} (-1)^{j-k} \frac{1}{(j+1)!} x_1^{j-k} x_2^k x_{j+1-k}\right).$$

Corollary 3.3. *For any Weitzenböck derivation δ, the transcendence degree* (*i.e., the maximal number of algebraically independent elements*) *of $K[x_1, \dots, x_m]^\delta$ is equal to $m-1$.*

The explicit form of the generators of $K[x_1, \dots, x_m]^\delta$ is known for small m only. Tan [71] presented an algorithm for computing the generators of the algebra of constants of a basic derivation. It was generalized by van den Essen [32] for any locally nilpotent derivation assuming that the finite generation of the algebra of constants is known. The algorithm involves Gröbner bases techniques.

Examples 3.4. We have selected few examples of the generating sets of the algebra of constants, all of them taken from [61]. For δ being a basic Weitzenböck derivation (with $\delta(x_1)=0$ and $\delta(x_j)=x_{j-1}$, $j=2,\dots,m$):

$$K[x_1,x_2]^\delta = K[x_1], \qquad K[x_1,x_2,x_3]^\delta = K\bigl[x_1, x_2^2-2x_1x_3\bigr],$$

$$K[x_1,x_2,x_3,x_4]^\delta = K\Bigl[x_1, x_2^2-2x_1x_3, x_2^3-3x_1x_2x_3+3x_1^2x_4,$$
$$x_2^2x_3^2-2x_2^3x_4+6x_1x_2x_3x_4-\frac{8}{3}x_1x_3^3-3x_1^2x_4^2\Bigr]$$

(see [61, Example 6.8.2]),

$$K[x_1,x_2,x_3,x_4,x_5]^\delta = K\bigl[x_1, x_2^2-2x_1x_3, 2x_2x_4-x_3^2-2x_1x_5, x_2^3-3x_1x_2x_3+3x_1^2x_4,$$
$$6x_2^2x_5-6x_2x_3x_4+2x_3^3-12x_1x_3x_5+9x_1x_4^2\bigr]$$

(see [61, Example 6.8.4]).

For δ nonbasic, $\delta(x_2)=x_1$, $\delta(x_4)=x_3$, $\delta(x_1)=\delta(x_3)=0$ (see [61, Proposition 6.9.5]):

$$K[x_1,x_2,x_3,x_4]^\delta = K[x_1,x_3,x_1x_4-x_2x_3],$$

for δ defined by $\delta(x_3)=x_2$, $\delta(x_2)=x_1$, $\delta(x_5)=x_4$, $\delta(x_1)=\delta(x_4)=0$ (see [61, Example 6.8.5]):

$$K[x_1,x_2,x_3,x_4,x_5]^\delta = K\bigl[x_1,x_4,x_1x_5-x_2x_4, x_2^2-2x_1x_3, 2x_3x_4^2-2x_2x_4x_5+x_1x_5^2\bigr].$$

Remark 3.5. Springer [70] found a formula for the Hilbert series of the algebra of invariants of $SL_2(K)$ acting on the forms of degree d. This is equivalent to the description of the Hilbert series of the algebra of constants of the basic Weitzenböck derivation of $K[x_1,\dots,x_{d+1}]$. Almkvist [2,3] related these invariants with invariants of the modular action of a cyclic group of order p.

4. Lifting and description of the constants

We need the following easy lemma.

Lemma 4.1. *Let $G\subset H$ be groups and let the H-module M be completely reducible. If $N\subset M$ is an H-submodule and $\bar m\in M/N$ is a G-invariant, then $\bar m$ can be lifted to a G-invariant $m\in M$.*

Proof. Let P be an H-complement of N in M, i.e., $M=N\oplus P$. Since $M/N\cong P$, there exists an element $m\in P$ which maps on $\bar m$ under the natural homomorphism $M\to M/N$. Since $\bar m$ is G-invariant, we obtain that $\overline{G(m)}=G(\bar m)=\bar m$. Taking into account that

$m_1, m_2 \in P$, $m_1 \neq m_2$, implies that $\bar{m}_1 \neq \bar{m}_2$ in M/N, and $G(P) = P$, we deduce that $G(m) = m$ in M, i.e., m is G-invariant. □

Proposition 4.2. *Let $K\langle X\rangle = K\langle x_1, \dots, x_m\rangle$ be the free associative algebra with the canonical GL_m-action, and let $I \subset J$ be GL_m-invariant two-sided ideals of $K\langle X\rangle$. Then for every subgroup G of GL_m, the G-invariants of $K\langle X\rangle/J$ can be lifted to G-invariants of $K\langle X\rangle/I$.*

Proof. The statement follows immediately from Lemma 4.1 because, as a GL_m-module, $K\langle X\rangle$ is completely reducible. □

Corollary 4.3. *If $I \subset J$ are GL_m-invariant two-sided ideals of $K\langle X\rangle$ and δ is a Weitzenböck derivation on $K\langle X\rangle$, then the algebra of constants $(K\langle X\rangle/J)^\delta$ can be lifted to the algebra of constants $(K\langle X\rangle/I)^\delta$.*

Proof. The corollary is a straightforward consequence of Proposition 4.2 because the algebras of constants $(K\langle X\rangle/J)^\delta$ and $(K\langle X\rangle/I)^\delta$ coincide, respectively, with the algebras of g-invariants $(K\langle X\rangle/J)^g$ and $(K\langle X\rangle/I)^g$, where $g = \exp\delta$ is the linear transformation corresponding to δ. □

Corollary 4.4. *Let $I \subset J$ be GL_m-invariant two-sided ideals of $K\langle X\rangle$ and let δ be a Weitzenböck derivation on $K\langle X\rangle$. If the algebra of constants $(K\langle X\rangle/J)^\delta$ is not finitely generated, then $(K\langle X\rangle/I)^\delta$ is also not finitely generated.*

Remark 4.5. Corollary 4.4 holds also for Lie algebras and other free algebras including free (special or not) Jordan algebras and the absolutely free algebra $K\{x_1, \dots, x_m\}$.

Now we shall describe the algebras of constants in the case of two variables, assuming that $K\langle x_1, x_2\rangle = K\langle x, y\rangle$ and $\delta(x) = 0$, $\delta(y) = x$.

Recall that any irreducible polynomial GL_2-module $W(\lambda_1, \lambda_2)$ has a unique (up to a multiplicative constant) element $w(x, y)$ which is bihomogeneous of degree (λ_1, λ_2) and is called the highest weight vector of $W(\lambda_1, \lambda_2)$. For any GL_2-invariant homomorphic image $K\langle x, y\rangle/I$ of $K\langle x, y\rangle$ the algebra of constants $(K\langle x, y\rangle/I)^\delta$ coincides with the algebra of g-invariants $(K\langle x, y\rangle/I)^g$ where $g = \exp\delta$. Since $g(x) = x$, $g(y) = x + y$ and $\operatorname{char} K = 0$, the algebra of g-invariants coincides with the algebra of invariants of the unitriangular group $UT_2(K)$. Hence, as in Almkvist, Dicks and Formanek [4], we may use Theorem 3.3(i) of De Concini, Eisenbud and Procesi [13] and obtain:

Theorem 4.6. *For any GL_2-invariant ideal I of $K\langle x, y\rangle$ the algebra of constants $(K\langle x, y\rangle/I)^\delta$ is spanned by the highest weight vectors of the GL_2-irreducible components of $K\langle x, y\rangle/I$.*

Remarks 4.7. (1) A direct proof of Theorem 4.6 can be obtained using the criterion of Koshlukov [49] which states: A multihomogeneous of degree $\lambda = (\lambda_1, \dots, \lambda_m)$ polynomial $w(x_1, \dots, x_m) \in K\langle x_1, \dots, x_m\rangle$ is a highest weight vector of an irreducible

GL_m-submodule $W(\lambda)$ of $K\langle x_1,\dots,x_m\rangle$ if and only if for all partial linearizations $w_j(x_1,\dots,x_m,x_{m+1})$ of $w(x_1,\dots,x_m)$ one has $w_j(x_1,\dots,x_m,x_i)=0$ for all $i<j$.

(2) By Almkvist, Dicks and Formanek [4] the algebra $(K\langle x_1,\dots,x_m\rangle)^{UT_m(K)}$ of all $UT_m(K)$-invariants coincides with the vector space spanned by all highest weight vectors $w(x_1,\dots,x_m)\in W(\lambda)\subset K\langle x_1,\dots,x_m\rangle$, when $\lambda=(\lambda_1,\dots,\lambda_m)$ runs on the set of all partitions in not more than m parts.

(3) Following Almkvist, Dicks and Formanek [4], for any unipotent transformation g of $K\langle x_1,\dots,x_m\rangle$ (and hence for any Weitzenböck derivation δ) one can define a GL_2-action on $K\langle x_1,\dots,x_m\rangle$ and on the factor algebras $K\langle x_1,\dots,x_m\rangle/I$ modulo GL_m-invariant ideals, such that $(K\langle x_1,\dots,x_m\rangle)^g$ and $(K\langle x_1,\dots,x_m\rangle/I)^g$ are spanned by the highest weight vectors with respect to the GL_2-action.

The necessary background on symmetric functions which we need can be found, e.g., in the book by Macdonald [57]. Any symmetric function in m variables $f(t_1,\dots,t_m)$ which can be expressed as a formal power series has the presentation

$$f(t_1,\dots,t_m)=\sum_{\lambda} m(\lambda)S_\lambda(t_1,\dots,t_m),$$

where $S_\lambda(t_1,\dots,t_m)$ is the Schur function corresponding to the partition $\lambda=(\lambda_1,\dots,\lambda_m)$ and $m(\lambda)$ is the multiplicity of $S_\lambda(t_1,\dots,t_m)$ in $f(t_1,\dots,t_m)$. This presentation agrees with the theory of polynomial representations of GL_m because the Schur functions play the role of characters of the irreducible polynomial GL_m-representations. In our case this relation gives the following: If $K\langle X\rangle/I$ for some GL_m-invariant ideal I, then the Hilbert series of $K\langle X\rangle/I$ has the presentation

$$H\bigl(K\langle X\rangle/I,t_1,\dots,t_m\bigr)=\sum_{\lambda} m(\lambda)S_\lambda(t_1,\dots,t_m),$$

if and only if $K\langle X\rangle/I$ is decomposed as a GL_m-module as

$$K\langle X\rangle/I\cong\sum_{\lambda} m(\lambda)W(\lambda).$$

In the case of two variables the Schur functions have the following simple expression

$$S_{(\lambda_1,\lambda_2)}(t_1,t_2)=(t_1t_2)^{\lambda_2}\frac{t_1^{\lambda_1-\lambda_2+1}-t_2^{\lambda_1-\lambda_2+1}}{t_1-t_2}.$$

Drensky and Genov [25] defined the multiplicity series of

$$f(t_1,t_2)=\sum_{\lambda} m(\lambda)S_\lambda(t_1,t_2)$$

as the formal power series

$$M(f)(t,u)=\sum_{\lambda} m(\lambda)t^{\lambda_1}u^{\lambda_2},$$

or, if one introduces a new variable $v=tu$, as

$$M'(f)(t,v)=\sum_{\lambda} m(\lambda)t^{\lambda_1-\lambda_2}v^{\lambda_2}.$$

The relation between the symmetric function and its multiplicity series is

$$f(t_1,t_2)=\frac{t_1M'(f)(t_1,t_1t_2)-t_2M'(f)(t_2,t_1t_2)}{t_1-t_2}.$$

Theorem 4.6 gives that the Hilbert series of the algebra of constants $(K\langle x,y\rangle/I)^{\delta}$ (with respect to the bigrading) is equal to the multiplicity series of the Hilbert series of $K\langle x,y\rangle/I$:

Corollary 4.8. *For any GL_2-invariant ideal I of $K\langle x,y\rangle$ and for the basic Weitzenböck derivation δ*

$$H\big(\big(K\langle x,y\rangle/I\big)^{\delta},t,u\big)=M\big(H\big(K\langle x,y\rangle/I,t_1,t_2\big)\big)(t,u).$$

If we consider the usual grading, Corollary 4.8 has the form

$$H\big(\big(K\langle x,y\rangle/I\big)^{\delta},t\big)=M\big(H\big(K\langle x,y\rangle/I,t_1,t_2\big)\big)(t,t)=M'\big(H\big(K\langle x,y\rangle/I,t_1,t_2\big)\big)\big(t,t^2\big).$$

We shall apply Corollary 4.8 in the next section in the concrete description of the generators of the constants in $K\langle x,y\rangle$ and, more generally, in any relatively free associative algebra.

5. Examples and concrete generators of algebras of constants

We start this section with several examples when we determine completely the algebras of constants and their generators. We shall consider algebras of rank 2 and 3 only and shall denote the free generators by x,y and x,y,z, respectively. We shall handle the case of basic Weitzenböck derivations δ only, assuming that $\delta(x)=0$, $\delta(y)=x$ (and $\delta(z)=y$ if the rank of the algebra is equal to 3).

Example 5.1. Let $\mathfrak{L}_2$ be the variety of associative algebras defined by the identity $[[x,y],z]=0$. By the theorem of Krakowski and Regev [51], $\mathfrak{L}_2$ coincides with the variety generated by the infinite dimensional Grassmann algebra. The S_n-cocharacter sequence of $\mathfrak{L}_2$ is equal to

$$\chi_n(\mathfrak{L}_2)=\sum_{k=1}^{n}\chi_{(k,1^{n-k})},\quad n\geqslant 1,$$

see [51]. In virtue of the correspondence between cocharacters and Hilbert series, see [6] and [18] (or the book [23]) the Hilbert series of the relatively free algebra $F_m(\mathfrak{L}_2)$ is equal to

$$H\big(F_m(\mathfrak{L}_2), t_1, \dots, t_m\big) = 1 + \sum_{k \geqslant 1} \sum_{l=0}^{m-1} S_{(k,1^l)}(t_1, \dots, t_m).$$

It is well known that $F_m(\mathfrak{L}_2)$ has a basis

$$x_1^{a_1} \cdots x_m^{a_m} [x_{i_1}, x_{i_2}] \cdots [x_{i_{2p-1}}, x_{i_{2p}}], \quad 1 \leqslant i_1 < i_2 < \cdots < i_{2p-1} < i_{2p} \leqslant m,$$

see, for example, Bokut and Makar-Limanov [8] or the book [23]. The commutators $[x_i, x_j]$ are in the centre of $F_m(\mathfrak{L}_2)$ and satisfy the relations

$$[x_{\sigma(1)}, x_{\sigma(2)}] \cdots [x_{\sigma(2p-1)}, x_{\sigma(2p)}] = (\operatorname{sign} \sigma)[x_1, x_2] \cdots [x_{2p-1}, x_{2p}], \quad \sigma \in S_{2p}.$$

Let $m = 2$. Then $F_2(\mathfrak{L}_2)$ has a basis

$$\big\{x^a y^b, x^a y^b [x, y] \;\big|\; a, b \geqslant 0\big\}.$$

Its Hilbert series and the related multiplicity series are, respectively,

$$H\big(F_2(\mathfrak{L}_2), t_1, t_2\big) = \frac{1 + t_1 t_2}{(1 - t_1)(1 - t_2)} = \sum_{n \geqslant 0} S_{(n)}(t_1, t_2) + \sum_{n \geqslant 2} S_{(n-1,1)}(t_1, t_2),$$

$$M\big(H\big(F_2(\mathfrak{L}_2), t_1, t_2\big)\big)(t, u) = \sum_{n \geqslant 0} t^n + \sum_{n \geqslant 2} t^{n-1} u = \frac{1 + tu}{1 - t}.$$

By Corollary 4.8,

$$H\big(\big(F_2(\mathfrak{L}_2)\big)^{\delta}, t, u\big) = \frac{1 + tu}{1 - t}.$$

Since the vector subspace of $F_2(\mathfrak{L}_2)$ spanned by x^n, $n \geqslant 0$, and $x^{n-2}[x, y]$, $n \geqslant 2$, consists of δ-constants and has the same Hilbert series as $(F_2(\mathfrak{L}_2))^{\delta}$, we obtain that it coincides with the algebra of constants. This immediately implies that the algebra $(F_2(\mathfrak{L}_2))^{\delta}$ is generated by x and $[x, y]$.

Let $m = 3$. Then $F_3(\mathfrak{L}_2)$ has a basis

$$\big\{x^a y^b z^c, x^a y^b z^c [x, y], x^a y^b z^c [x, z], x^a y^b z^c [y, z] \;\big|\; a, b, c \geqslant 0\big\}$$

and the commutator ideal C of $F_3(\mathfrak{L}_2)$ is a free $K[x, y, z]$-module with free generators $[x, y], [x, z], [y, z]$. By Examples 3.4, $K[x, y, z]^{\delta} = K[x, y^2 - 2xz]$. We may choose

$y^2-xz-zx$ as a lifting in $(F_3(\mathfrak{L}_2))^\delta$ of $y^2-2xz\in (K[x,y,z])^\delta$. Hence $(F_3(\mathfrak{L}_2))^\delta$ is generated by x, $y^2-xz-zx$ and some elements in the commutator ideal C. Every element of $K[x,y,z]$ can be written in a unique way as

$$f_0\bigl(x,y^2-2xz\bigr)+\sum_{n\geqslant 1} f_n\bigl(x,y^2-2xz\bigr)z^n+\sum_{n\geqslant 1} g_n\bigl(x,y^2-2xz\bigr)yz^{n-1}.$$

The elements in C have the form

$$f=\alpha(x,y,z)[x,y]+\beta(x,y,z)[x,z]+\gamma(x,y,z)[y,z],\quad \alpha,\beta,\gamma\in K[x,y,z].$$

If f is a δ-constant, then

$$0=\delta(f)=\bigl(\delta(\alpha)+\beta\bigr)[x,y]+\bigl(\delta(\beta)+\gamma\bigr)[x,z]+\delta(\gamma)[y,z].$$

In this way, $f\in (F_3(\mathfrak{L}_2))^\delta$ if and only if

$$\delta(\gamma)=0,\qquad \delta(\beta)=-\gamma,\qquad \delta(\alpha)=-\beta.$$

We present $\beta(x,y,z)$ in the form

$$\beta=f_0+\sum_{n\geqslant 1}\bigl(f_nz^n+g_nz^{n-1}y\bigr),\quad f_0,f_n,g_n\in\bigl(K[x,y,z]\bigr)^\delta,$$

and calculate, bearing in mind that $y^2=(y^2-2xz)+2xz$,

$$\begin{aligned}-\gamma=\delta(\beta)&=\sum_{n\geqslant 1}\bigl(nf_nz^{n-1}y+(n-1)g_nz^{n-2}y^2+xg_nz^{n-1}\bigr)\\&=\sum_{n\geqslant 1}\bigl((n-1)g_n\bigl(y^2-2xz\bigr)z^{n-2}+(2n-1)xg_nz^{n-1}+nf_nz^{n-1}y\bigr).\end{aligned}$$

This easily implies that $f_n=0$, $n\geqslant 1$, $g_n=0$, $n\geqslant 2$, and $\beta=f_0+g_1y$, $f_0,g_1\in K[x,y^2-2xz]=(K[x,y,z])^\delta$. Hence $\gamma=-g_1x$. Continuing in this way, we obtain the final form of α,β,γ:

$$\alpha=\alpha_0z+\alpha_1y+\alpha_2,\qquad \beta=-\alpha_0y-\alpha_1x,\qquad \gamma=\alpha_0x.$$

Hence the part of the algebra of constants of $F_3(\mathfrak{L}_2)$ which belongs to the commutator ideal C is spanned as a $(K[x,y,z])^\delta$-module by

$$[x,y],\quad y[x,y]-x[x,z],\quad z[x,y]-y[x,z]+x[y,z],$$

and $(F_3(\mathfrak{L}_2))^\delta$ is generated by

$$x,\quad y^2-xz-zx,\quad [x,y],\quad y[x,y]-x[x,z],\quad z[x,y]-y[x,z]+x[y,z].$$

Example 5.2. Let us consider the variety $\mathfrak{M}$ of all "metabelian" associative algebras defined by the identity $[x_1, x_2][x_3, x_4] = 0$. It is well known that $F_2(\mathfrak{M})$ has a basis

$$\left\{x^a y^b,\ x^a y^b[x, y]x^c y^d \;\middle|\; a, b, c, d \geqslant 0\right\}.$$

We shall write the element $x^a y^b[x, y]x^c y^d$ as $[x, y]x_1^a y_1^b x_2^c y_2^d$. In this way, the commutator ideal C of $F_2(\mathfrak{M})$ is a free cyclic $K[x, y]$-bimodule (or a free cyclic $K[x_1, y_1, x_2, y_2]$-module) with the $K[x, y]$-action defined by

$$x[x, y] = [x, y]x_1, \quad y[x, y] = [x, y]y_1, \quad [x, y]x = [x, y]x_2, \quad [x, y]y = [x, y]y_2.$$

The Hilbert series of $F_2(\mathfrak{M})$ is

$$H\big(F_2(\mathfrak{M}), t_1, t_2\big) = \frac{1}{(1-t_1)(1-t_2)} + \frac{t_1t_2}{(1-t_1)^2(1-t_2)^2}.$$

One can calculate directly the S_n-cocharacter of $\mathfrak{M}$ using the Young rule as in [23] or to apply techniques of [25] to see that the multiplicity series of the Hilbert series of $F_2(\mathfrak{M})$ is

$$M'\big(H\big(F_2(\mathfrak{M}), t_1, t_2\big)\big)(t, v) = \frac{1}{1-t} + \frac{v}{(1-t)^2(1-v)}.$$

By Corollary 4.8 this is also the Hilbert series of the algebra of constants $(F_2(\mathfrak{M}))^\delta$. We consider the linearly independent highest weight vectors

$$x^n, \quad n \geqslant 0, \qquad [x, y]x_1^p x_2^q (x_1y_2 - y_1x_2)^r, \quad p, q, r \geqslant 0.$$

They span a graded vector subspace of $(F_2(\mathfrak{M}))^\delta$ and its Hilbert series coincides with the Hilbert series of $(F_2(\mathfrak{M}))^\delta$. Hence the above highest weight vectors span $(F_2(\mathfrak{M}))^\delta$. Since the square of the commutator ideal C is equal to 0, the element x together with all $[x, y](x_1y_2 - y_1x_2)^r$, $r \geqslant 0$, is a minimal generating set of $(F_2(\mathfrak{M}))^\delta$ and the algebra of constants is not finitely generated.

Now we start with the description of the constants of the free algebra $K\langle x, y\rangle$ which will gives also the description of the constants in any two-generated associative algebra.

Proposition 5.3. *The Hilbert series of the algebra of constants $(K\langle x, y\rangle)^\delta$ are*

$$H\big(\big(K\langle x, y\rangle\big)^\delta, t, u\big) = \sum_{(\lambda_1,\lambda_2)} \left(\binom{\lambda_1+\lambda_2}{\lambda_2} - \binom{\lambda_1+\lambda_2}{\lambda_2-1}\right) t^{\lambda_1} u^{\lambda_2}$$

$$= \frac{1-\sqrt{1-4v}}{2v} \cdot \frac{1}{1 - \frac{1-\sqrt{1-4v}}{2v}t},$$

where $v = tu$ and, in one variable,

$$H\big((K\langle x,y\rangle)^\delta, t\big) = \sum_{p\geqslant 0}\left(\binom{2p}{p}t^{2p} + \binom{2p+1}{p}t^{2p+1}\right).$$

Proof. By Corollary 4.8 the Hilbert series of the algebra of constants $(K\langle x,y\rangle)^\delta$ is equal to the multiplicity series of the Hilbert series of $K\langle x,y\rangle$. By representation theory of general linear groups, the multiplicity m_λ of the irreducible GL_m-module $W(\lambda)$ in $K\langle x_1,\dots,x_m\rangle$ for the partition λ of n is equal to the degree d_λ of the irreducible S_n-character χ_λ. By the hook formula, for $\lambda = (\lambda_1,\lambda_2)$

$$d_\lambda = \frac{(\lambda_1+\lambda_2)!\,(\lambda_1-\lambda_2+1)}{(\lambda_1+1)!\,\lambda_2!} = \binom{\lambda_1+\lambda_2}{\lambda_2} - \binom{\lambda_1+\lambda_2}{\lambda_2-1}.$$

This gives the expression for $H((K\langle x,y\rangle)^\delta, t, u)$. If we set there $u = t$ we obtain that the coefficient of t^{2p} is equal to

$$\sum_{i=0}^{p}\left(\binom{2p}{i} - \binom{2p}{i-1}\right) = \binom{2p}{p},$$

and similarly for the coefficient of t^{2p+1}. In order to obtain the formula in terms of t and v we can either use the known formulas for the summation of formal power series with binomial coefficients or proceed in the following way using ideas from [25]. The Hilbert series of $K\langle x,y\rangle$ is equal to

$$f(t_1,t_2) = H\big(K\langle x,y\rangle, t_1, t_2\big) = \frac{1}{1-(t_1+t_2)}.$$

It is sufficient to show that the multiplicity series of $f(t_1,t_2)$ is

$$M'(f)(t,v) = \frac{1-\sqrt{1-4v}}{2v}\cdot\frac{1}{1-\frac{1-\sqrt{1-4v}}{2v}t}.$$

Since the multiplicity series of any symmetric function $f(t_1,t_2)\in K[[t_1,t_2]]$ is a uniquely determined formal power series in $K[[t,v]]$, it is sufficient to show that the expansion of

$$\frac{1-\sqrt{1-4v}}{2v}\cdot\frac{1}{1-\frac{1-\sqrt{1-4v}}{2v}t}$$

is in $K[[t,v]]$ (which is obvious because $1-\sqrt{1-4v} = \sum_{n\geqslant 1} a_n v^n$ for some $a_n\in K$ and $(1-\sqrt{1-4v})/(2v)\in K[[v]]$) and to use the formula

$$\frac{t_1M'(f)(t_1,t_1t_2) - t_2M'(f)(t_2,t_1t_2)}{t_1-t_2} = f(t_1,t_2).$$

Direct verification shows that for

$$g(t,v)=\frac{1-\sqrt{1-4v}}{2v}\cdot\frac{1}{1-\frac{1-\sqrt{1-4v}}{2v}t},$$

$$\frac{t_1g(t_1,t_1t_2)-t_2g(t_2,t_1t_2)}{t_1-t_2}=\frac{1}{1-(t_1+t_2)}$$

which gives that $g(t,v)=M'(f)(t,v)$. $\square$

By the theorem of Lane [53] and Kharchenko [45], the algebra of constants $(K\langle X\rangle)^{\delta}$ is a graded free algebra and hence has a homogeneous system of free generators. The following theorem describes the generating function of the set of free generators.

Theorem 5.4. *The generating function of any bihomogeneous system of free generators of $(K\langle X\rangle)^{\delta}$ with respect to the variables t and $v=tu$ is*

$$a(t,v)=t+\frac{1-\sqrt{1-4v}}{2}.$$

Proof. If $a(t,v)$ is the generating function of the set of free generators of $(K\langle X\rangle)^{\delta}$, then the Hilbert series of $(K\langle X\rangle)^{\delta}$ is

$$H\big(\big(K\langle X\rangle\big)^{\delta},t,v\big)=\frac{1}{1-a(t,v)}.$$

Applying Proposition 5.3 we obtain that

$$\frac{1}{1-a(t,v)}=\frac{1-\sqrt{1-4v}}{2v}\cdot\frac{1}{1-\frac{1-\sqrt{1-4v}}{2v}t}$$

and the expression of $a(t,v)$ is a result of easy calculations. $\square$

Corollary 5.5. *The algebra of constants $(K\langle x,y\rangle)^{\delta}$, where $\delta(x)=0$, $\delta(y)=x$, is generated by x and by $SL_2(K)$-invariants.*

Proof. An element $f(x,y)\in K\langle x,y\rangle$ is an SL_2-invariant if and only if it is a linear combination of highest weight vectors of a GL_2-submodules $W(\lambda_1,\lambda_1)$. By Theorem 4.6, the δ-constants are linear combinations of highest weight vectors $w_{(\lambda_1,\lambda_2)}$, and $w_{(\lambda_1,\lambda_2)}$ is bihomogeneous of degree (λ_1,λ_2). Hence we obtain that the set of SL_2-invariants coincides with the linear combinations of bihomogeneous elements of degree (p,p). The only nonzero coefficients of the Hilbert series $H((K\langle x,y\rangle)^{SL_2},t,u)$ are for $v^n=(tu)^n$ and $H((K\langle x,y\rangle)^{SL_2},t,u)$ is obtained from $H((K\langle x,y\rangle)^{\delta},t,v)$ by replacing t with 0. Hence Theorem 5.4 gives that the set of homogeneous generators of the algebra of δ-constants is spanned by x and SL_2-invariants. $\square$

Corollary 4.3 gives immediately:

Corollary 5.6. *For any GL_2-invariant ideal I of $K\langle x, y\rangle$ the algebra of constants $(K\langle x, y\rangle/I)^{\delta}$, where $\delta(x) = 0$, $\delta(y) = x$, is generated by x and by $SL_2(K)$-invariants.*

Remark 5.7. By Almkvist, Dicks and Formanek [4, Example 5.10], the Hilbert series of the algebra of SL_2-invariants of $K\langle x, y\rangle$ is

$$H\big(\big(K\langle x, y\rangle\big)^{SL_2}, v\big) = \frac{1-\sqrt{1-4v}}{2v} = \sum_{n\geqslant 0} \frac{1}{n+1}\binom{2n}{n} v^n,$$

and the coefficient of v^n is the $(n+1)$st Catalan number c_{n+1}. (By definition c_n is the number of possibilities to distribute parentheses in the sum $1+1+\cdots+1$ of n units, see, e.g., [42].) This agrees with Proposition 5.3 because $H((K\langle x, y\rangle)^{SL_2}, v)$ is obtained from

$$H\big(\big(K\langle x, y\rangle\big)^{\delta}, t, v\big) = \frac{1-\sqrt{1-4v}}{2v} \cdot \frac{1}{1-\frac{1-\sqrt{1-4v}}{2v}t}$$

by replacing t with 0.

Theorem 5.4 gives that the generating function of a homogeneous system of free generators of $(K\langle X\rangle)^{SL_2}$ is

$$b(v) = \frac{1-\sqrt{1-4v}}{2} = vH\big(\big(K\langle x, y\rangle\big)^{SL_2}, v\big).$$

Since $v = tu$ is of second degree, the number of generators of $(K\langle x, y\rangle)^{SL_2}$ of degree $2n$ is equal to the nth Catalan number.

Below we give an inductive procedure to construct an infinite set of free generators of the algebra $(K\langle x, y\rangle)^{SL_2}$.

Algorithm 5.8. The following infinite procedure gives a complete set $\{w_1, w_2, \ldots\}$ of free generators of the algebra $(K\langle x, y\rangle)^{SL_2}$. We set $w_1 = [x, y]$. If we have already constructed all free generators $w_1, w_2, \ldots, w_k$ of degree $\leqslant 2n$, then we form all c_{n+1} products $w_{i_1}\cdots w_{i_s}$ of degree $2n$, which we number as ω_j, $j = 1, \ldots, c_{n+1}$, and add to the system of generators the c_{n+1} elements

$$w_{k+j} = x\omega_j y - y\omega_j x = xw_{i_1}\cdots w_{i_s}y - yw_{i_1}\cdots w_{i_s}x, \quad j = 1, \ldots, c_{n+1}.$$

The first several elements of the generating set are:

$$w_1 = [x, y], \quad w_2 = x[x, y]y - y[x, y]x,$$
$$w_3 = xw_1^2y - yw_1^2x = x[x, y]^2y - y[x, y]^2x,$$
$$w_4 = xw_2y - yw_2x = x\big(x[x, y]y - y[x, y]x\big)y - y\big(x[x, y]y - y[x, y]x\big)x.$$

Proof. By Remark 5.7 and by inductive arguments, we may assume that the number of products $\omega_j = w_{i_1} \cdots w_{i_s}$ of degree $2n$ is equal to the Catalan number c_{n+1}. Hence the number of words $x\omega_j y - y\omega_j x$, all of degree $2(n+1)$ is also equal to c_{n+1} which agrees with the number of free generators of degree $2(n+1)$. Clearly, if ω_j is an SL_2-invariant, the element $x\omega_j y - y\omega_j x$ is also an SL_2-invariant. Hence it is sufficient to show that all products $w_{j_1} \cdots w_{j_p}$ of degree $2(n+1)$ and all $x\omega_j y - y\omega_j x$ are linearly independent.

We introduce the lexicographic ordering on $K\langle x, y\rangle$ assuming that $x < y$. Then by induction we prove that the minimal monomials $z_{k_1} \cdots z_{k_{2n+2}}$, $z_k \in \{x, y\}$, of $w_{j_1} \cdots w_{j_p}$ and $x\omega_j y - y\omega_j x$ have the property that the number of x's in every beginning $z_{k_1} \cdots z_{k_q}$ of $z_{k_1} \cdots z_{k_{2n+2}}$ is bigger or equal to the number of y's. For example, the minimal monomial of $w_2 = x[x, y]y - y[x, y]x$ is $xxyy$, all its beginnings are $x, xx, xxy, xxyy$ and the number of entries of x and y are $(1, 0), (2, 0), (2, 1), (2, 2)$, respectively. Similarly, the minimal monomial of

$$w_1 w_2^2 = [x, y]\big(x[x, y]y - y[x, y]x\big)\big(x[x, y]y - y[x, y]x\big)$$

is $xyxxyyxxyy$ and the entries of x and y in the beginnings are

$$(1, 0), (1, 1), (2, 1), (3, 1), (3, 2), (3, 3), (4, 3), (5, 3), (5, 4), (5, 5).$$

Pay attention that the first place where the number of x's is equal to the number of y's, namely the beginning xy, corresponds to the beginning $w_1 = [x, y]$ in $w_1 w_2^2$ and the rest of the minimal monomial $xxyyxxyy$ has the same property.

We shall show that the products $w_{j_1} \cdots w_{j_p}$ (including the case $p = 1$ of a product of one free generator $x\omega_j y - y\omega_j x$) are in a one-to-one correspondence with the words $z_{k_1} \cdots z_{k_{2n+2}}$ in x and y with the property that the number of x's in every beginning $z_{k_1} \cdots z_{k_q}$ is bigger or equal to the number of y's. Let $\omega = w_{j_1} \cdots w_{j_p}$ be a product of elements of the constructed set. If $p = 1$, i.e., $\omega = w_j$ is in the set, then $w_j = x\omega' y - y\omega' x$ and the minimal monomial $z_1 \cdots z_{2n}$ of ω' has the property that the number of x's in every beginning of $z_1 \cdots z_{2n}$ is bigger or equal to the corresponding number of y's. Since the minimal monomial of w_j is $xz_1 \cdots z_{2n} y$, we obtain that in every of its proper beginnings the number of occurrences of x is strictly bigger than the number of entries of y. If $p > 1$, then, reading the minimal word from left to right, the first place where the numbers of the x's and the y's is the same, is the end of w_{j_1}. This arguments combined with induction easily imply that the different products $w_{j_1} \cdots w_{j_p}$ have different minimal monomials and each word corresponds to some product $w_{j_1} \cdots w_{j_p}$. Hence the products $w_{j_1} \cdots w_{j_p}$ are linearly independent and this completes the proof. □

Corollary 5.9. *For any variety $\mathfrak{W}$ of associative algebras which does not contain the metabelian variety $\mathfrak{M}$, the algebra of constants $F_2(\mathfrak{W})^\delta$ is finitely generated.*

Proof. It is well known that any variety $\mathfrak{W}$ which does not contain $\mathfrak{M}$ satisfies some Engel identity $[x_2, x_1, \ldots, x_1] = 0$. By a theorem of Latyshev [54] any finitely generated PI-algebra satisfying a nonmatrix polynomial identity, satisfies also some identity of the form $[x_1, x_2] \cdots [x_{2k-1}, x_{2k}] = 0$. Applying this result to $F_2(\mathfrak{W})$ we obtain that $F_2(\mathfrak{W})$

is solvable as a Lie algebra, and, by a theorem of Higgins [43] $F_2(\mathfrak{W})$ is Lie nilpotent. (Actually Zelmanov [76] proved the stronger result that any Lie algebra over a field of characteristic zero satisfying the Engel identity is nilpotent.)

By Drensky [19], for any nilpotent variety $\mathfrak{W}$, and for a fixed positive integer m, the vector space $B_m(\mathfrak{W})$ of so called proper polynomials in $F_m(\mathfrak{W})$ is finite dimensional. Using the relation

$$F_m(\mathfrak{W}) \cong K[x_1, \ldots, x_m] \otimes_K B_m(\mathfrak{W})$$

between the GL_m-modules $F_m(\mathfrak{W})$ and $B_m(\mathfrak{W})$ and the Young rule, we can derive the following. There exists a positive constant p such that the nonzero irreducible components $W(\lambda_1, \ldots, \lambda_m)$ of the GL_m-module $F_m(\mathfrak{W})$ satisfy the restriction $\lambda_2 \leqslant p$. Hence the subalgebra $F_2(\mathfrak{W})^{SL_2}$ of SL_2-invariants of $F_2(\mathfrak{W})$ (which is spanned on the highest weight vectors of $W(\lambda_1, \lambda_1)$ with $\lambda_1 \leqslant p$) is finite dimensional. Now the statement follows from Corollary 5.6 because $F_2(\mathfrak{W})^\delta$ is generated by x and the finite dimensional vector space $F_2(\mathfrak{W})^{SL_2}$. $\square$

Corollary 5.9 inspires the following:

Question 5.10. Is it true that, for $m \geqslant 2$ and for a fixed nonzero Weitzenböck derivation δ, the algebra of constants $F_m(\mathfrak{W})^\delta$ is finitely generated if and only if the variety of associative algebras $\mathfrak{W}$ does not contain the metabelian variety $\mathfrak{M}$?

Corollary 4.3, Example 5.2 and Corollary 5.9 show that the answer to this question is affirmative for $m = 2$. In the next section we shall show that the algebra of constants $F_m(\mathfrak{W})^\delta$ is not finitely generated if $\mathfrak{W}$ contains $\mathfrak{M}$.

6. Constants of relatively free associative algebras

First we shall work in the free metabelian associative algebra $F_m(\mathfrak{M})$ where the metabelian variety is defined by the polynomial identity $[x_1, x_2][x_3, x_4] = 0$. We need an embedding of $F_m(\mathfrak{M})$ into a wreath product. For this purpose, let $Y = \{y_1, \ldots, y_m\}$, $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_m\}$ be three sets of commuting variables and let

$$M = \sum_{i=1}^{m} a_i K[U, V]$$

be the free $K[U, V]$-module of rank m generated by $\{a_1, \ldots, a_m\}$. Clearly, M has also a structure of a free $K[Y]$-bimodule with the action of $K[Y]$ defined by

$$y_j a_i = a_i u_j, \quad a_i y_j = a_i v_j, \quad i, j = 1, \ldots, m.$$

Define the trivial multiplication $M \cdot M = 0$ on M and consider the algebra

$$W = K[Y] \leftthreetimes M,$$

which is similar to the abelian wreath product of Lie algebras, see [66] (M is an ideal of W with multiplication by $K[Y]$ induced by the bimodule action of $K[Y]$ on M). Obviously W satisfies the metabelian identity and hence belongs to $\mathfrak{M}$. The following proposition is a partial case of the main result of Lewin [56], see also Umirbaev [73] for further applications of this construction to automorphisms of relatively free associative algebras.

Proposition 6.1. *The mapping $\iota : x_j \to y_j + a_j$, $j = 1, \ldots, m$, defines an embedding ι of $F_m(\mathfrak{M})$ into $W = K[Y] \leftthreetimes M$.*

Proposition 6.2. *For any nontrivial Weitzenböck derivation δ of the free metabelian associative algebra $F_m(\mathfrak{M})$ of rank $m \geqslant 2$, the algebra of constants $F_m(\mathfrak{M})^\delta$ is not finitely generated.*

Proof. The derivation δ acts as a linear operator on the vector space with basis $\{x_1, \ldots, x_m\}$ and we define in a similar way the action of δ on the vector spaces with bases $\{y_1, \ldots, y_m\}$ and $\{a_1, \ldots, a_m\}$: If $\delta(x_j) = \sum_{i=1}^m \alpha_{ij} x_j$, $\alpha_{ij} \in K$, then $\delta(y_j) = \sum_{i=1}^m \alpha_{ij} y_j$ and $\delta(a_j) = \sum_{i=1}^m \alpha_{ij} a_j$, $j = 1, \ldots, m$. As in the proof of Lemma 2.1 we can show that this action δ defines a derivation on W and on the polynomial algebra $K[U, V]$ (which we denote also by δ). Additionally, we consider the embedding ι of $F_m(\mathfrak{M})^\delta$ as a subalgebra in W, as stated in Proposition 6.1. By definition $\delta(\iota(x_j)) = \delta(y_j + a_j) = \iota(\delta(x_j))$ and hence if $\delta(f(X)) = 0$ in $F_m(\mathfrak{M})$, then the same holds for the image $\iota(f)$ of f in W. In this way, ι embeds the algebra of constants $F_m(\mathfrak{M})^\delta$ into the algebra of constants W^δ.

As till now, we assume that $\delta(x_1) = 0$ and $\delta(x_2) = x_1$. If the algebra of constants $F_m(\mathfrak{M})^\delta$ is generated by a finite set $\{f_1, \ldots, f_n\}$, then, as elements of W,

$$\iota(f_k) = g_k(Y) + \sum_{i=1}^m a_i h_{ik}(U, V), \quad g_k(Y) \in K[Y],\ h_{ik}(U, V) \in K[U, V],$$

$i = 1, \ldots, m$, $k = 1, \ldots, n$, and

$$g_k(Y), b_k = \sum_{i=1}^m a_i h_{ik}(U, V), \quad k = 1, \ldots, n,$$

are also constants. Hence $\iota(F_m(\mathfrak{M})^\delta)$ is a subalgebra of the subalgebra of W^δ generated by the union of the finite sets

$$\{g_1, \ldots, g_n\} \subset K[Y]^\delta, \quad \{b_1, \ldots, b_n\} \subset M^\delta.$$

This implies that $F_m(\mathfrak{M})^\delta$ is a subalgebra of

$$W_0 = K[Y]^\delta \leftthreetimes \sum_{k=1}^n b_k K[U]^\delta K[V]^\delta.$$

By Corollary 3.3 the transcendence degree of $K[Y]^\delta$ is equal to $m-1$ and hence the transcendence degree of $K[U]^\delta K[V]^\delta$ is equal to $2(m-1)$. Since, see, e.g., the book by Krause and Lenagan [52], the Gelfand–Kirillov dimension of a commutative algebra is equal to the transcendence degree of the algebra, we easily derive that the Gelfand–Kirillov dimension of the algebra W_0 is bounded from above by $2(m-1)$. On the other hand, the vector space $\iota([x_1,x_2])K[U,V]$ is contained in $\iota(F_m(\mathfrak{M}))$ and is a free $K[U,V]$-module generated by $a_1(v_2-u_2)+a_2(u_1-v_1)$. Since $\iota([x_1,x_2])\in M^\delta$, we obtain that $\iota([x_1,x_2])K[U,V]^\delta$ is a free $K[U,V]^\delta$-module. By Corollary 3.3 the transcendence degree of $K[U,V]^\delta$ is equal to $2m-1$, and hence the Gelfand–Kirillov dimension of the $K[U,V]^\delta$-module is equal to $2m-1$. This is also a lower bound for the Gelfand–Kirillov dimension of $F_m(\mathfrak{M})^\delta$ which contradicts with the inequality $\mathrm{GKdim}(F_m(\mathfrak{M})^\delta)\leqslant \mathrm{GKdim}(W_0)\leqslant 2(m-1)$. □

Remark 6.3. In the notation of Proposition 6.2, if $b_1,\dots,b_k$ is a finite number of elements in M^δ, then the subalgebra of $K[Y]^\delta \leftthreetimes M^\delta$ generated by $K[Y]^\delta$ and $b_1,\dots,b_k$, contains only a finite number of elements $\iota([x_1,x_2])(u_1v_2-u_2v_1)^n$. This can be seen in the following way. We consider the localization of the polynomial algebra $K[Y][y_1^{-1}]=K[y_1,y_2,\dots,y_m][y_1^{-1}]$, and similarly $K[U][u_1^{-1}]$, $K[V][v_1^{-1}]$. Then we define

$$W'=K[Y]\big[y_1^{-1}\big]\leftthreetimes MK\big[u_1^{-1},v_1^{-1}\big].$$

Since y_1,u_1,v_1 are δ-constants, we can extend the action of δ as a derivation on W to a derivation on W'. Let $\delta^{p_j+1}(y_j)=0$, $j=1,\dots,m$, and let us define

$$\tilde{y}_j=\sum_{k=0}^{p_j}\frac{\delta^k(y_j)}{k!}(-y_2)^k y_1^{p_j-k},\quad j=3,4,\dots,m,$$

and similarly $\tilde{y}_j,\tilde{u}_j,\tilde{v}_j$. Let also $\tilde{w}_2=u_1v_2-u_2v_1$. By Proposition 3.1

$$\big(K[Y]\big[y_1^{-1}\big]\big)^\delta=K\big[y_1,y_1^{-1}\big][\tilde{y}_3,\tilde{y}_4,\dots,\tilde{y}_m],$$
$$\big(K[U,V]\big[u_1^{-1},v_1^{-1}\big]\big)^\delta=K\big[u_1,v_1,u_1^{-1},v_1^{-1}\big][\tilde{u}_3,\dots,\tilde{u}_m,\tilde{v}_3,\dots,\tilde{u}_m,\tilde{w}_2].$$

The algebra generated by $K[Y]^\delta$ and $b_1,\dots,b_k$ is a subalgebra of

$$\big(K[Y]\big[y_1^{-1}\big]\big)^\delta\leftthreetimes\sum_{j=1}^k b_j\big(K[U]\big[u_1^{-1}\big]\big)^\delta\big(K[V]\big[v_1^{-1}\big]\big)^\delta$$

and hence its elements have the form

$$f(\tilde{y}_3,\dots,\tilde{y}_m)+\sum_{j=1}^m b_jf_j(\tilde{u}_3,\dots,\tilde{u}_m,\tilde{v}_3,\dots,\tilde{v}_m),$$

where f and f_j are polynomials with coefficients depending respectively on y_1,y_1^{-1} and $u_1,v_1,u_1^{-1},v_1^{-1}$. Since $\tilde{u}_3,\dots,\tilde{u}_m,\tilde{v}_3,\dots,\tilde{u}_m,\tilde{w}_2$ are algebraically independent on

$K[u_1, v_1, u_1^{-1}, v_1^{-1}]$, and the finite number of elements $b_1, \dots, b_k$ contains only a finite number of summands, we cannot present all elements $\iota([x_1, x_2])(u_1v_2 - u_2v_1)^n = (a_1(v_2 - u_2) + a_2(u_1 - v_1))\tilde{w}_2^n$ in the form

$$\big(a_1(v_2 - u_2) + a_2(u_1 - v_1)\big)\tilde{w}_2^n = \sum_{j=1}^{m} b_j f_{jn}(\tilde{u}_3, \dots, \tilde{u}_m, \tilde{v}_3, \dots, \tilde{v}_m).$$

Theorem 6.4. *Let $\mathfrak{W}$ be a variety of associative algebras containing the metabelian variety $\mathfrak{M}$. Then for any $m \geqslant 2$ and for any fixed nonzero Weitzenböck derivation δ, the algebra of constants $F_m(\mathfrak{W})^\delta$ is not finitely generated.*

Proof. By Corollary 4.3 the algebra $F_m(\mathfrak{M})^\delta$ is a homomorphic image of $F_m(\mathfrak{W})^\delta$. Now the proof follows immediately because $F_m(\mathfrak{M})^\delta$ is not finitely generated by Proposition 6.2. □

Remark 6.5. Using the elements $\iota([x_1, x_2])(u_1v_2 - u_2v_1)^n$, $n \geqslant 0$, from Remark 6.3 for any variety $\mathfrak{W}$ containing the metabelian variety $\mathfrak{M}$ and any nontrivial Weitzenböck derivation δ we can construct an infinite set of constants which is not contained in any finitely generated subalgebra of $F_m(\mathfrak{W})^\delta$. Again, we assume that $\delta(x_1) = 0$, $\delta(x_2) = x_1$. Let l_u and r_u be, respectively, the operators of left and right multiplication by $u \in F_m(\mathfrak{W})$. Consider the elements

$$(l_{x_1} r_{x_2} - l_{x_2} r_{x_1})^n [x_1, x_2], \quad n \geqslant 0.$$

All these elements are constants which are liftings of the constants from Remark 6.3 and hence any finitely generated subalgebra of $F_m(\mathfrak{W})^\delta$ does not contain $(l_{x_1} r_{x_2} - l_{x_2} r_{x_1})^n [x_1, x_2]$ for sufficiently large n.

Corollary 6.6. *Let $\mathfrak{W}$ be a variety of associative algebras containing the metabelian variety $\mathfrak{M}$. Then for any $m \geqslant 2$ the algebra $F_m(\mathfrak{W})^{UT_m}$ of $UT_m(K)$-invariants is not finitely generated.*

Proof. Let the algebra $F_m(\mathfrak{W})^{UT_m}$ be finitely generated. By Remarks 4.7, the algebra $(K\langle x_1, \dots, x_m\rangle)^{UT_m}$, and hence also $F_m(\mathfrak{W})^{UT_m}$ is spanned by all highest weight vectors. Hence $F_m(\mathfrak{W})^{UT_m}$ is generated by a finite system of highest weight vectors $w(x_1, \dots, x_m) \in W(\lambda) \subset F_m(\mathfrak{W})^{UT_m}$. Hence $F_m(\mathfrak{W})^{UT_m}$ is multigraded and has a finite multihomogeneous set of generators. The generators which depend on x_1 and x_2 only, generate the subalgebra spanned by all highest weight vectors $w(x_1, \dots, x_m) \in W(\lambda_1, \lambda_2, 0, \dots, 0)$. This subalgebra coincides with the algebra of UT_2-invariants of $F_2(\mathfrak{W})$ and hence with the algebra of constants of the Weitzenböck derivation δ of $F_2(\mathfrak{W})$ defined by $\delta(x_1) = 0$, $\delta(x_2) = x_1$. By Theorem 6.4 for $m = 2$ (or by Corollary 4.3 and Example 5.2) $F_2(\mathfrak{W})^\delta$ is not finitely generated. Hence the algebra $F_m(\mathfrak{W})^{UT_m}$ cannot be finitely generated. □

Remark 6.7. Let $\mathfrak{W}$ be a Lie nilpotent variety of associative algebras and let m be a fixed positive integer. Using the approach of [19] (as in the proof of Corollary 5.9), and the fact that $F_m(\mathfrak{W})$ is a direct sum of GL_m-modules of the form $W(\lambda_1, \ldots, \lambda_m)$ with $\lambda_2 \leqslant p$ for some p, one can show that there exists a finite system of highest weight vectors $w_i(x_1, \ldots, x_k) \in F_m(\mathfrak{W})$, $i = 1, \ldots, k$, such that all highest weight vectors of $F_m(\mathfrak{W})$ are linear combinations of $x^n w_i(x_1, \ldots, x_k)$. Hence the algebra $F_m(\mathfrak{W})^{UT_m}$ of UT_m-invariants is generated by x and $w_i(x_1, \ldots, x_k)$, $i = 1, \ldots, k$. Hence $F_m(\mathfrak{W})^{UT_m}$ is finitely generated.

7. Generic 2 × 2 matrices

In this section we construct classes of automorphisms of the relatively free algebra $F_2(\operatorname{var} M_2(K))$. This algebra is isomorphic to the algebra generated by two generic 2×2 matrices x and y. So, the results are stated in the natural setup of the trace algebra. We start with the necessary background, see Formanek [36], Alev and Le Bruyn [1], or Drensky and Gupta [26].

We consider the polynomial algebra in 8 variables $\Omega = K[x_{ij}, y_{ij} \mid i, j = 1, 2]$. The algebra R of two generic 2×2 matrices

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$$

is the subalgebra of $M_2(\Omega)$ generated by x and y. We denote by C the centre of R and by $\bar{C}$ the algebra generated by all the traces of elements from R. Identifying the elements of $\bar{C}$ with 2×2 scalar matrices we denote by T the generic trace algebra generated by R and $\bar{C}$. It is well known that $\bar{C}$ is generated by

$$\operatorname{tr}(x), \quad \operatorname{tr}(y), \quad \det(x), \quad \det(y), \quad \operatorname{tr}(xy)$$

and is isomorphic to the polynomial algebra in five variables.

Proposition 7.1 (Formanek, Halpin, and Li [37]). *The vector subspace of C consisting of all polynomials without constant term is a free $\bar{C}$-module generated by $[x, y]^2$.*

For our purposes it is more convenient to replace in T (as in [1]) the generic matrices x and y by the generic traceless matrices

$$x_0 = x - \frac{1}{2}\operatorname{tr}(x), \qquad y_0 = y - \frac{1}{2}\operatorname{tr}(y)$$

and assume that T is generated by x_0, y_0, $\operatorname{tr}(x)$, $\operatorname{tr}(y)$, $\det(x_0)$, $\det(y_0)$, $\operatorname{tr}(x_0 y_0)$. A further reduction is to use the formulas

$$\det(x_0) = -\frac{1}{2}\operatorname{tr}\bigl(x_0^2\bigr), \qquad \det(y_0) = -\frac{1}{2}\operatorname{tr}\bigl(y_0^2\bigr),$$

and to replace the determinants by $\mathrm{tr}(x_0^2)$ and $\mathrm{tr}(y_0^2)$. In this way, we may assume that $\bar{C}$ is generated by

$$p = \mathrm{tr}(x), \qquad q = \mathrm{tr}(y), \qquad u = \mathrm{tr}\big(x_0^2\big), \qquad v = \mathrm{tr}\big(y_0^2\big), \qquad t = \mathrm{tr}(x_0 y_0).$$

Then $[x, y]^2 = t^2 - uv$ and

$$T = \bar{C} + \bar{C}x_0 + \bar{C}y_0 + \bar{C}[x_0, y_0]$$

is a free $\bar{C}$-module generated by $1, x_0, y_0, [x_0, y_0]$.

The defining relations of the algebra generated by the 2×2 traceless matrices x_0 and y_0 are $[x_0^2, y_0] = [y_0^2, x_0] = 0$, see, e.g., [55] or [29] for the case of characteristic 0 and [50] for the case of an arbitrary infinite base field. More generally, the defining relations of the algebra generated by m generic 2×2 traceless matrices $y_1, \ldots, y_m$ are $[v_1^2, v_2] = 0$, where v_1, v_2 run on the set of all Lie elements in $K\langle y_1, \ldots, y_m\rangle$ which is a restatement of the theorem of Razmyslov [62] for the weak polynomial identities of $M_2(K)$. An explicitly written system of defining relations consists of $[y_i^2, y_j] = 0$, $[y_i y_j + y_j y_i, y_k] = 0$, $i, j, k = 1, \ldots, m$, and the standard polynomials $s_4(y_{i_1}, y_{i_2}, y_{i_3}, y_{i_4}) = 0$, $1 \leqslant i_1 < i_2 < i_3 < i_4 \leqslant m$, see [29].

Lemma 7.2. *Every mapping $\delta : \{p, q, x_0, y_0\} \to T$ such that*

$$\delta(p), \delta(q) \in \bar{C}, \qquad \delta(x_0), \delta(y_0) \in \bar{C}x_0 + \bar{C}y_0 + \bar{C}[x_0, y_0]$$

can be extended to a derivation of T.

Proof. The defining relations of T are

$$[p, q] = [p, x_0] = [p, y_0] = [q, x_0] = [q, y_0] = 0,$$

together with the defining relations of the subalgebra generated by x_0, y_0. It is sufficient to see that the extension of δ (inductively, by the rule $\delta(fg) = \delta(f)g + f\delta(g)$) to a derivation on T is well defined, i.e., sends the defining relations to 0. For the relations involving p and q this can be checked directly:

$$\delta\big([p, q]\big) = \big[\delta(p), q\big] + \big[p, \delta(q)\big] = 0,$$

analogously for $\delta([p, x_0]), \delta([p, y_0]), \delta([q, x_0]), \delta([q, y_0])$, because $p, q, \delta(p), \delta(q)$ are in the centre of T. The condition for the defining relations of the algebra generated by x_0, y_0 can be proved using the universal properties of this algebra or directly: Since $x_0^2, y_0^2, x_0y_0 + y_0x_0, [x_0, y_0]^2$ are in the centre of T, and $x_0[x_0, y_0] + [x_0, y_0]x_0 = y_0[x_0, y_0] + [x_0, y_0]y_0 = 0$, if $\delta(x_0) = ax_0 + by_0 + c[x_0, y_0]$, $a, b, c \in \bar{C}$, then

$$\big(\delta(x_0)\big)^2 = a^2x_0^2 + b^2y_0^2 + c^2[x_0, y_0]^2 + ab(x_0y_0 + y_0x_0),$$
$$\delta(x_0)x_0 + x_0\delta(x_0) = ax_0^2 + b(x_0y_0 + y_0x_0)$$

are in the centre of T and $\delta([x_0^2, y_0]) = 0$. In the same way $\delta([y_0^2, x_0]) = 0$. $\square$

Example 7.3. Let us consider the basic Weitzenböck derivation δ defined on the relatively free algebra $F_2(\operatorname{var} M_2(K))$ in its realization as the generic trace algebra generated by generic 2×2 matrices x and y by $\delta(x)=0$, $\delta(y)=x$. We extend δ to the trace algebra T by

$$\begin{gathered}\delta(p)=\delta\bigl(\operatorname{tr}(x)\bigr)=\operatorname{tr}\bigl(\delta(x)\bigr),\\ \delta(q)=\delta\bigl(\operatorname{tr}(y)\bigr)=\operatorname{tr}\bigl(\delta(y)\bigr),\\ \delta(x_0)=0,\qquad \delta(y_0)=x_0,\\ \delta(u)=\delta\bigl(\operatorname{tr}\bigl(x_0^2\bigr)\bigr)=\operatorname{tr}\bigl(\delta\bigl(x_0^2\bigr)\bigr),\\ \delta(v)=\delta\bigl(\operatorname{tr}\bigl(y_0^2\bigr)\bigr)=\operatorname{tr}\bigl(\delta\bigl(y_0^2\bigr)\bigr),\\ \delta(t)=\delta\bigl(\operatorname{tr}(x_0y_0)\bigr)=\operatorname{tr}\bigl(\delta(x_0y_0)\bigr).\end{gathered}$$

By Lemma 7.2 this is possible. Direct calculations give that

$$\delta(p)=0,\quad \delta(q)=p,\quad \delta(u)=0,\quad \delta(t)=u,\quad \delta(v)=2t.$$

Replacing v with $2v_1$, we obtain that the action of δ on $\bar{C}=K[p,q,u,t,v_1]$ is as in Examples 3.4. Hence

$$\begin{aligned}(\bar{C})^{\delta}&=K\bigl[p,u,pt-qu,t^2-2uv_1,2p^2v_1-2pqt+q^2u\bigr]\\ &=K\bigl[p,u,pt-qu,t^2-uv,q^2u-2pqt+p^2v\bigr].\end{aligned}$$

The generators of $(\bar{C})^{\delta}$ satisfy the relation

$$u\bigl(q^2u-2pqt+p^2v\bigr)+p^2\bigl(t^2-uv\bigr)=(pt-qu)^2.$$

If $w\in(\bar{C})^{\delta}$, then $\exp(w\delta)$ is an automorphism of T. If t^2-uv divides w, then $\exp(w\delta)$ is an automorphism also of R. This automorphism acts on R as

$$\exp(w\delta):x\to x,\qquad \exp(w\delta):y\to y+wx,$$

where $w=(t^2-uv)w_1(p,u,pt-qu,t^2-uv,q^2u-2pqt+p^2v)$ for some polynomial w_1. Such automorphisms (fixing x) were studied in the PhD thesis of Chang [11].

Example 7.4. Now we shall modify Example 7.3 in the following way. We use Lemma 7.2 and define the derivation δ of T by

$$\delta(p)=\alpha_1u+\beta_1t+\gamma_1v,\qquad \delta(q)=p+\alpha_2u+\beta_2t+\gamma_2v,$$

$\alpha_i,\beta_i,\gamma_i\in\bar{C}$, $i=1,2$,

$$\delta(u)=0,\qquad \delta(t)=u,\qquad \delta(v)=2t.$$

This derivation is locally nilpotent and acts on the generic matrices $x = \frac{1}{2}\mathrm{tr}(x) + x_0$ and $y = \frac{1}{2}\mathrm{tr}(y) + y_0$ by

$$\delta(x) = \frac{1}{2}(\alpha_1 u + \beta_1 t + \gamma_1 v), \qquad \delta(y) = x + \frac{1}{2}(\alpha_2 u + \beta_2 t + \gamma_2 v).$$

The matrix of the linear operator δ acting on the vector space $Kp + Kq + Ku + Kt + Kv$ (with respect to the basis $\{p, q, u, t, v\}$) is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & \alpha_2 & 0 & 1 & 0 \\ \beta_1 & \beta_2 & 0 & 0 & 2 \\ \gamma_1 & \gamma_2 & 0 & 0 & 0 \end{pmatrix}$$

and has rank 3 or 4 depending on whether $\gamma_1 = 0$ or $\gamma_1 \neq 0$. Hence its Jordan normal form is one of the following matrices:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & \\ & & & & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \\ & & & 0 & 1 \\ & & & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Examples 3.4 give concrete systems of generators of the algebras of constants of $(\bar{C})^\delta$ and hence automorphisms of the algebras T and R.

For example, if we fix $\delta(p) = v$, $\delta(q) = p$, then δ is a basic derivation with

$$\delta(q) = p, \qquad \delta(p) = v, \qquad \delta(v) = 2t, \qquad \delta(t) = u, \qquad \delta(u) = 0.$$

Considering $\bar{C} = K[q/2, p/2, v/2, t, u]$, we obtain after some easy calculations that the algebra of constants is generated by

$$u, \quad t^2 - uv, \quad tp - qu - \frac{v^2}{4},$$
$$t^3 - \frac{3}{2}utv + \frac{3}{2}u^2 p, \quad 3t^2 q - \frac{3}{2}tvp + \frac{v^3}{4} - 3uvq + \frac{9}{4}up^2.$$

In this case δ acts on x and y by

$$\delta(x) = \frac{1}{2}\mathrm{tr}\big(y_0^2\big) = \frac{1}{2}v, \qquad \delta(y) = x.$$

If w is in $(\bar{C})^\delta$, then

$$\exp(w\delta): x \to x + \frac{wv}{2.1!} + \frac{w^2t}{2!} + \frac{w^3u}{3!},$$
$$\exp(w\delta): y \to y + \frac{wx}{1!} + \frac{w^2v}{2.2!} + \frac{w^3t}{3!} + \frac{w^4u}{4!}.$$

If w is divisible by $t^2 - uv$, then $\exp(w\delta)$ is also an automorphism of R. Since all these automorphisms $\exp(w\delta)$ are obtained by the construction of Martha Smith [67], they induce stably tame automorphisms of $\bar{C} = K[p, q, u, t, v]$.

8. Relatively free Lie algebras

We start with few examples for the algebras of constants of relatively free algebras. By the well-known dichotomy a variety of Lie algebras either satisfies the Engel condition (and by the theorem of Zelmanov [76] is nilpotent) or contains the metabelian variety $\mathfrak{A}^2$ (which consists of all solvable of class 2 Lie algebras and is defined by the identity $[[x_1, x_2], [x_3, x_4]] = 0$). Since the finitely generated nilpotent Lie algebras are finite dimensional, the problem for the finite generation of the algebras of constants of relatively free nilpotent Lie algebras is solved trivially.

The bases of the free polynilpotent Lie algebras were described by Shmelkin [65]. Considering relatively free algebras of rank 2, we assume that the algebra is generated by x and y and the basic Weitzenböck derivation δ is defined by $\delta(x) = 0$, $\delta(y) = x$.

Example 8.1. Let $L_2(\mathfrak{A}^2) = L_2/L_2''$ be the free metabelian Lie algebra of rank 2. It has a basis

$$\bigl\{x, y, [y, x, \underbrace{x, \dots, x}_{a \text{ times}}, \underbrace{y, \dots, y}_{b \text{ times}}] \bigm| a, b \geqslant 0\bigr\}.$$

It is well known (and can be also obtained by simple arguments from the Hilbert series of $L_m(\mathfrak{A}^2)$) that the nth cocharacter of the variety $\mathfrak{A}^2$ is

$$\chi_1\bigl(\mathfrak{A}^2\bigr) = \chi_{(1)}, \qquad \chi_n\bigl(\mathfrak{A}^2\bigr) = \chi_{(n-1,1)}, \quad n \geqslant 2.$$

The corresponding highest weight vectors are

$$w_{(1)} = x, \qquad w_{(n-1,1)} = [y, x, \underbrace{x, \dots, x}_{n-2 \text{ times}}], \quad n \geqslant 2.$$

Hence the algebra of constants $L_2(\mathfrak{A}^2)^\delta$ is generated by x and $[x, y]$.

Example 8.2. The free abelian-by-{nilpotent of class 2} Lie algebra $L_2(\mathfrak{A}\mathfrak{N}_2) = L_2/[L_2, L_2, L_2]'$ satisfies the identity

$$\bigl[[x_1, x_2, x_3], [x_4, x_5, x_6]\bigr] = 0$$

and has a basis

$$\bigl\{x, y, [x, y], \bigl[y, x, \underbrace{x, \ldots, x}_{a \text{ times}}, \underbrace{y, \ldots, y}_{b \text{ times}}, \underbrace{[x, y], \ldots, [x, y]}_{c \text{ times}}\bigr] \bigm| a + b > 0,\ c \geqslant 0\bigr\}.$$

Its Hilbert series is

$$\begin{aligned} H\bigl(L_2(\mathfrak{AN}_2), t_1, t_2\bigr) &= t_1 + t_2 + t_1 t_2 + \frac{t_1 t_2 (t_1 + t_2)}{(1 - t_1)(1 - t_2)(1 - t_1 t_2)} \\ &= S_{(1)}(t_1, t_2) + S_{(1^2)}(t_1, t_2) + \sum_{\lambda_1 > \lambda_2 \geqslant 1} S_{(\lambda_1, \lambda_2)}(t_1, t_2) \end{aligned}$$

and the highest weight vectors of $L_2(\mathfrak{AN}_2)$ are

$$x, \qquad [x, y], \qquad \bigl[y, x, \underbrace{x, \ldots, x}_{a \text{ times}}, \underbrace{[x, y], \ldots, [x, y]}_{c \text{ times}}\bigr], \quad a > 0,\ c \geqslant 0.$$

Hence the algebra $L_2(\mathfrak{AN}_2)^\delta$ is generated by x and $[x, y]$.

Example 8.3. We consider the relatively free algebra $L_2(\operatorname{var} sl_2(K))$ of the variety of Lie algebras generated by the algebra of 2×2 traceless matrices. This algebra is isomorphic to the Lie algebra generated by the generic 2×2 traceless matrices x_0, y_0 considered in Section 7. By Drensky [18], as a GL_2-module $L_2(\operatorname{var} sl_2(K))$ has the decomposition

$$L_2\bigl(\operatorname{var} sl_2(K)\bigr) \cong W(1) \oplus \sum W(\lambda_1, \lambda_2),$$

where the summation runs on all $\lambda = (\lambda_1, \lambda_2)$ such that $\lambda_2 > 0$ and at least one of the integers λ_1, λ_2 is odd. The highest weight vectors of $W(\lambda_1, \lambda_2)$ are given in [18] but we do not need their concrete form for our purposes. The algebra of constants $L_2(\operatorname{var} sl_2(K))^\delta$ is bigraded. Assuming that the degree of x corresponds to t and the degree of y is $u = v/t$, the Hilbert series of $L_2(\operatorname{var} sl_2(K))^\delta$ is

$$\begin{aligned} H\bigl(L_2\bigl(\operatorname{var} sl_2(K)\bigr)^\delta, t, v\bigr) &= t + v\Bigl(\sum_{p,q \geqslant 0} t^p v^q - \sum_{p,q \geqslant 0} t^{2p} v^{2q+1}\Bigr) \\ &= t + \frac{v}{(1 - t)(1 - v)} - \frac{v^2}{(1 - t)^2 (1 - v)^2}. \end{aligned}$$

If $L_2(\operatorname{var} sl_2(K))^\delta$ is finitely generated, we may fix a finite system of bigraded generators. For every homogeneous $f \in L_2(\operatorname{var} sl_2(K))^\delta$ we have $\deg_x f \geqslant \deg_y f$. Hence the subalgebra spanned on the homogeneous components of bidegree (n, n), n odd, is also finitely generated. This subalgebra is infinite dimensional and its Hilbert series is obtained from the Hilbert series $H(L_2(\operatorname{var} sl_2(K))^\delta, t, v)$ by the substitution $t = 0$, i.e.,

$$H\bigl(L_2\bigl(\operatorname{var} sl_2(K)\bigr)^\delta, 0, v\bigr) = \frac{v}{1 - v} - \frac{v^2}{(1 - v)^2}.$$

Besides, the subalgebra is abelian because the commutator of any two highest weight vectors $w_{(2p+1,2p+1)}$ and $w_{(2q+1,2q+1)}$ is a highest weight vector $w_{(2(p+q+1),2(p+q+1))}$ which does not participate in the decomposition of $L_2(\operatorname{var} sl_2(K))^\delta$. Since the finitely generated abelian Lie algebras are finite dimensional, we obtain a contradiction which gives that $L_2(\operatorname{var} sl_2(K))^\delta$ cannot be finitely generated.

Example 8.4. The free abelian-by-{nilpotent of class 3} Lie algebra $L_2(\mathfrak{A}\mathfrak{N}_3) = L_2/[L_2, L_2, L_2, L_2]'$ has a basis consisting of x, y and commutators of the form

$$\big[y, x, \underbrace{x, \ldots, x}_{a \text{ times}}, \underbrace{y, \ldots, y}_{b \text{ times}}, \underbrace{[x, y], \ldots, [x, y]}_{c \text{ times}}, \underbrace{[y, x, x], \ldots, [y, x, x]}_{d \text{ times}}, \underbrace{[y, x, y], \ldots, [y, x, y]}_{e \text{ times}}\big],$$

with some natural restrictions of $a, b, c, d, e \geqslant 0$ which guarantee that these commutators are different from zero and, up to a sign, pairwise different. If the algebra of constants $L_2(\mathfrak{A}\mathfrak{N}_3)^\delta$ is finitely generated, then it has a generating set consisting of a finite number of bihomogeneous elements $w_1, \ldots, w_k$ of degree $\geqslant 4$ (and bidegree (n_1, n_2), where $n_1 \geqslant n_2$) and constants of degree $\leqslant 3$ (i.e., $x, [x, y], [y, x, x]$). Since the commutators of length $\geqslant 4$ commute, we derive that $L_2(\mathfrak{A}\mathfrak{N}_3)^\delta$ is a sum of the Lie subalgebra N generated by $x, [x, y], [y, x, x]$ and the N-module generated by $w_1, \ldots, w_k$. The following elements are constants:

$$u_n = \sum_{\rho,\sigma,\ldots,\tau \in S_2} \operatorname{sign}(\rho\sigma \cdots \tau)\big[y, x, x, z_{\rho(1)}, z_{\sigma(1)}, \ldots, z_{\tau(1)}, [x, y, z_{\rho(1)}],$$
$$[x, y, z_{\sigma(1)}], \ldots, [x, y, z_{\tau(1)}]\big],$$

where $\{z_1, z_2\} = \{x, y\}$ and, in the summation, $\rho, \sigma, \ldots, \tau$ run on n copies of the symmetric group S_2. They are homogeneous of bidegree $(2n + 2, 2n + 1)$ and hence can be written as linear combinations of commutators involving a w_i, several $[x, y]$ and not more than one x or $[y, x, x]$. But this is impossible because for sufficiently large n one cannot obtain the summands of u_n

$$\big[y, x, x, \underbrace{x, \ldots, x}_{n \text{ times}}, \underbrace{[x, y, y], \ldots, [x, y, y]}_{n \text{ times}}\big].$$

Hence the algebra $L_2(\mathfrak{A}\mathfrak{N}_3)^\delta$ is not finitely generated.

Example 8.5. Let $m > 2$ and let δ be the Weitzenböck derivation of the free metabelian Lie algebra $L_m(\mathfrak{A}^2)$ defined by $\delta(x_2) = x_1$, $\delta(x_j) = 0$ for $j \neq 2$. Then, since $L_m(\mathfrak{A}^2)$ has a basis consisting of x_j and all commutators $[x_{i_1}, x_{i_2}, \ldots, x_{i_n}]$ with $i_1 > i_2 \leqslant i_3 \leqslant \cdots \leqslant i_n$, then the free generators x_j, $j \neq 2$, and the commutators which do not include x_2 are constants. It is easy to see that the commutators with x_2 are of the form

$$u' = [x_2, \underbrace{x_1, \ldots, x_1}_{a \text{ times}}, \underbrace{x_2, \ldots, x_2}_{b \text{ times}}, x_{i_k}, \ldots, x_{i_n}], \quad a > 0,\ b \geqslant 0,\ i_k > 2,$$

$$u'' = [x_i, \underbrace{x_1, \ldots, x_1}_{a \text{ times}}, \underbrace{x_2, \ldots, x_2}_{b \text{ times}}, x_{i_k}, \ldots, x_{i_n}], \quad a \geqslant 0,\ b > 0,\ i, i_k > 2.$$

It is easy to see that a linear combination of u' and u'' is a constant if and only if it contains as summands only u' with $b = 0$ and does not contain any u''. Hence the algebra of constants $L_m(\mathfrak{A}^2)^\delta$ is generated by $x_1, x_j,\ j > 2$, and $[x_1, x_2]$.

Example 8.6. Let $m > 2$ and let δ be the Weitzenböck derivation of the free abelian-by-{nilpotent of class 2} Lie algebra $L_m(\mathfrak{A}\mathfrak{N}_2)$ defined, as in the previous example, by $\delta(x_2) = x_1,\ \delta(x_j) = 0$ for $j \neq 2$. We define a GL_2-action on $L_m(\mathfrak{A}\mathfrak{N}_2)$ assuming that GL_2 fixes $x_3, \ldots, x_m$ and acts canonically on the linear combinations of x_1, x_2. Then the subspaces of $L_m(\mathfrak{A}\mathfrak{N}_2)$ which are homogeneous in each variable $x_3, \ldots, x_m$ are GL_2-invariant. This easily implies that the algebra of constants $L_m(\mathfrak{A}\mathfrak{N}_2)^\delta$ is multigraded and $\deg_{x_1} f \geqslant \deg_{x_2} f$ for each multihomogeneous constant f. If the algebra $L_m(\mathfrak{A}\mathfrak{N}_2)^\delta$ is finitely generated, then as in Example 8.4, it is generated by $x_1, [x_1, x_2], x_3, x_4, \ldots, x_m$ and a finite system $w_1, \ldots, w_k$ of homogeneous elements of degree $\geqslant 3$. Then $L_m(\mathfrak{A}\mathfrak{N}_2)^\delta$ is a sum of the subalgebra N generated by $x_1, [x_1, x_2], x_3, x_4, \ldots, x_m$ and the N-module generated by $w_1, \ldots, w_k$. The constants

$$\sum_{\rho,\sigma,\ldots,\tau \in S_2} \operatorname{sign}(\rho\sigma\cdots\tau) u_n = \big[x_1, x_2, x_{\rho(1)}, x_{\sigma(1)}, \ldots, x_{\tau(1)}, [x_3, x_{\rho(1)}], \\ [x_3, x_{\sigma(1)}], \ldots, [x_3, x_{\tau(1)}]\big],$$

where in the summation $\rho, \sigma, \ldots, \tau$ run on n copies of the symmetric group S_2, are homogeneous of degree $(n+1, n+1, n, 0, \ldots, 0)$ and arguments as in Example 8.4 show that this is impossible. Hence the algebra $L_m(\mathfrak{A}\mathfrak{N}_2)^\delta$ cannot be finitely generated.

In the above examples, the matrix of the Weitzenböck derivation δ (as a linear operator acting on the vector space with basis $\{x_1, \ldots, x_m\}$) is of rank 1. This gives rise to the following natural problem.

Problem 8.7. *If the matrix of the Weitzenböck derivation δ is of rank 1, find the exact frontier where the algebra of constants $L_m(\mathfrak{W})^\delta$ becomes finitely generated, i.e., describe all varieties of Lie algebras $\mathfrak{W}$ and all integers $m > 1$ such that the algebra $L_m(\mathfrak{W})^\delta$ is finitely generated.*

Finally, we shall give the solution of this problem in the case of rank $\geqslant 2$.

Theorem 8.8. *Let $\mathfrak{W}$ be a nonnilpotent variety of Lie algebras and let δ be a Weitzenböck derivation of the relatively free algebra $L_m(\mathfrak{W})$, $m \geqslant 3$. If the rank of the matrix of δ is $\geqslant 2$, then the algebra of constants $L_m(\mathfrak{W})^\delta$ is not finitely generated.*

Proof. As in the associative case, it is sufficient to establish the theorem for the metabelian variety of Lie algebras only. We consider the abelian wreath product of Lie algebras

$$W_m = (Ky_1 \oplus \cdots \oplus Ky_m) \leftthreetimes \sum_{j=1}^{m} a_j K[y_1, \ldots, y_m],$$

where $[y_i, y_j] = [a_i f_i, a_j f_j] = 0$ and $[a_i f_i, y_j] = a_i f_i y_j$ $(f_i, f_j \in K[y_1, \ldots, y_m])$. Then by the theorem of Shmelkin [66] the mapping $\iota : x_j \to a_j + y_j$, $j = 1, \ldots, m$, defines an embedding of the free metabelian Lie algebra $L_m(\mathfrak{A}^2)$ into W_m. We assume that δ is in its normal Jordan form (and $\delta(x_2) = x_1$, $\delta(x_1) = 0$). Hence the fixed part of $Ky_1 \oplus \cdots \oplus Ky_m$ is of dimension $m - \operatorname{rank}(\delta) \leqslant m - 2$ and is spanned on some free generators $x_{j_1} = x_1, x_{j_2}, \ldots, x_{j_p}$, $p \leqslant m - 2$. If the algebra $L_m(\mathfrak{A}^2)^\delta$ is finitely generated, then it is a sum of $Kx_1 \oplus Kx_{j_2} \oplus \cdots \oplus Kx_{j_p}$ and a finitely generated $K[x_1, x_{j_2}, \ldots, x_{j_p}]$-submodule of the commutator ideal $L_m(\mathfrak{A}^2)'$. But, as in the associative case, this is impossible because the image of this module under ι should contain, for example, $\iota([x_2, x_1])K[y_1, \ldots, y_m]^\delta$ and the transcendence degree of $K[y_1, \ldots, y_m]^\delta$ is equal to $m - 1$.

One can see directly, that if $\delta(x_3) = x_2$, then a finitely generated subalgebra of $\iota(L_m(\mathfrak{A}^2)^\delta)$ cannot contain all constants

$$\iota\bigl([x_2, x_1]\bigr)\bigl(x_2^2 - 2x_1x_3\bigr)^n, \quad n \geqslant 0.$$

Similarly, if $\delta(x_4) = x_3, \delta(x_3) = 0$, then $\iota(L_m(\mathfrak{A}^2)^\delta)$ cannot contain all

$$\iota\bigl([x_2, x_1]\bigr)(x_1x_4 - x_2x_3)^n, \quad n \geqslant 0. \qquad \square$$

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