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Derivations of forms along a map: the framework for time-dependent second-order equations

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Abstract: A comprehensive theory is presented concerning derivations of scalar and vector-valued forms along the projection $\pi : \mathbb{R} \times TM \to \mathbb{R} \times M$. It is the continuation of previous work on derivations of forms along the tangent bundle projection and is prompted by the need for a scheme which is adapted to the study of time-dependent second-order equations. The overall structure of the theory closely follows the pattern of this preceding work, but there are many features which are certainly not trivial transcripts of the time-independent situation. As before, a crucial ingredient in the classification of derivations is a non-linear connection on the bundle π. In the presence of a given second-order system, such a connection is canonically defined and gives rise to two important operations: the dynamical covariant derivative, which is a derivation of degree 0, and the Jacobi endomorphism, which is a type $(1,1)$ tensor field along π. The theory is developed in such a way that all results readily apply to the more general situation of a bundle $\pi : J^1E \to E$, where E is fibred over $\mathbb{R}$, but need not be the trivial fibration $\mathbb{R} \times M \to \mathbb{R}$.

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1. Introduction

In [9] and [10] Martínez, Cariñena and Sarlet presented a comprehensive theory of derivations of forms along the tangent bundle projection $\tau : TM \to M$. One of the main motivations for this theory was to develop a calculus in which important concepts for the study of second-order differential equation fields (SODE) make their appearance in a most economical way, giving rise to formulas which stay close to analytical computations and yet give such computations a coordinate free backing. In the more traditional geometric approach to the study of a SODE, where tangent bundle geometry is the key issue, these concepts tend to be interpreted by tensorial objects of which half of the components are to some extent redundant. This is particularly evident

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in Sarlet's study of special forms and tensors associated to a SODE [14], which was the direct inducement for the work of Martínez et al. Another incentive for this work is the fact that important tensorial objects on TM are often scalar or vertical-vector-valued semi-basic forms (see e.g. [2,8]) and these can be put in direct correspondence with tensor fields along τ.

Two major applications of the calculus along τ have been developed so far. The first concerns a constructive characterization, mostly in terms of algebraic conditions, of systems of autonomous second-order equations which can be completely decoupled [11]. This application in itself calls for a generalization which can cover the situation of time-dependent systems. Obviously, one then wants to allow coordinate transformations, which realize the full separation, to depend on time as well. A second area where elements of the new calculus have proved to be highly efficient is the inverse problem of Lagrangian mechanics. It was already shown in [1] and [10] that the so-called Helmholtz conditions which characterize this inverse problem can be formulated in a very succinct way by means of properties of a metric tensor field along τ. More importantly, it has recently been shown [4] that the new approach also paves the way for solving the inverse problem, giving a geometrical content to the rather tricky analytical solution, presented by Douglas [6] for the case of two degrees of freedom. This recent development is in fact presented in a time-dependent setup and thus anticipates part of the results of the present paper.

The aim of this paper is to cover most of the results of [9] and [10] for a time-dependent framework. The type of space which is usually taken to carry a description of time-dependent second-order equations is the manifold $\mathbb{R} \times TM$ (see e.g. [3]) and it is then natural to let the projection $\pi : \mathbb{R} \times TM \to \mathbb{R} \times M$ take over the role of $\tau : TM \to M$ for the autonomous situation. That is what we will do indeed, but not without some precautions. The manifold $\mathbb{R} \times TM$ can be identified in a natural way with the jet space $J^1(\mathbb{R}, M)$ (see e.g. [16]). Such an identification, however, is not entirely harmless. Once the space $J^1(\mathbb{R}, M)$ has been endowed with the product structure coming from $\mathbb{R}\times TM$, objects are tensorially well defined when they behave the way they should under coordinate transformations which respect the product structure (i.e. do not mix time and position variables). Thinking of the jet bundle structure, however, one is tempted to allow for time-dependent coordinate transformations as well and, as already indicated above, this is a necessity for certain applications. Not all tensor fields, well defined on $\mathbb{R} \times TM$, transform covariantly under time-dependent coordinate transformations! Typical examples of dangerous objects in this respect are the vector field $\partial/\partial t$ and the dilation or Liouville vector field.

In view of what precedes, one of our principal guidelines will be to develop the calculus along π in such a way that time-dependent coordinate transformations cause no surprises. In other words, relying on the product structure of $\mathbb{R} \times TM$ and $\mathbb{R} \times M$ has to be avoided and as an interesting byproduct of this attitude all formulas will in fact remain perfectly valid in a more general setup, where $\mathbb{R} \times M$ is replaced by an arbitrary fibre bundle $E \to \mathbb{R}$ and π is the projection $\pi : J^1E \to E$. This guideline in itself does not preclude that there is no unique, natural way of extending the theory for the autonomous situation. At several stages of conceiving the basic ingredients for the classification of derivations, one has to make a choice and this may of course be a matter of personal preference. Generally speaking, the choices we make will be dictated by the wish to keep the structure of all formulas as closely as possible related to the autonomous case. We will briefly comment on alternative approaches along the way.

The scheme of the paper is as follows. Section 2 contains generalities about the structure of tensor fields along $\pi : \mathbb{R} \times TM \to \mathbb{R} \times M$ and the selection of a canonically defined 'vertical exterior derivative'. The classification of derivations of scalar forms along π is discussed in Section 3 and requires introducing a connection. The extension to vector-valued forms in Section 4 will lead us to the important vertical and horizontal covariant derivatives. Torsion and curvature of the connection are among the various concepts and properties that will come out of the study of commutators in Section 5. A digression on horizontal and vertical lifts in Section 6 will provide the necessary link with the traditional calculus on $\mathbb{R} \times TM$. Section 7 focusses on the case where the connection is coming from a given SODE. With regard to applications, this is the most important part of the paper. It highlights, in particular, the concepts of dynamical covariant derivative ∇ and Jacobi endomorphism Φ. The final section contains some immediate applications and comments on future developments.

2. Tensor fields along $\pi : \mathbb{R}\times TM \to \mathbb{R}\times M$ and the vertical exterior derivative

For general aspects of sections along a map and derivations we refer to [9]. Vector fields along π are sections of the pull back bundle $\pi^*(T(\mathbb{R} \times M))$ over $\mathbb{R} \times TM$. The set of vector fields along π, which is a module over $C^\infty(\mathbb{R} \times TM)$, is denoted by $\mathcal{X}(\pi)$. Similarly, $\bigwedge(\pi)$ will denote the graded algebra of scalar forms along π and $V(\pi)$ stands for the $\bigwedge(\pi)$-module of vector-valued forms along π. Obviously, we have $\bigwedge^0(\pi) \equiv C^\infty(\mathbb{R} \times TM)$ and $V^0(\pi) \equiv \mathcal{X}(\pi)$. Elements of $\mathcal{X}(\mathbb{R} \times M)$ or $\bigwedge(\mathbb{R} \times M)$, which through composition with π can be regarded as belonging to $\mathcal{X}(\pi)$ (respectively $\bigwedge(\pi)$), will be called *basic vector fields* (respectively *basic forms*). In coordinates, a tensor field along π is made up of tensor products of basic 1-forms and vector fields, with coefficients in $C^\infty(\mathbb{R} \times TM)$.

As is well known, $J^1(\mathbb{R}, M)$ can be identified with the submanifold of $T(\mathbb{R} \times M)$ consisting of tangent vectors with time-component 1. The corresponding natural injection defines the *canonical vector field along* π, denoted by $\mathbf{T}$. Its coordinate expression is given by

$$\mathbf{T} = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial q^i}. \tag{1}$$

One of the important features of $\mathbf{T}$ is that it preserves its form under *time-dependent coordinate transformations*, by which we mean transformations of the form $t' = t$, $q' = q'(q, t)$, with the induced affine transformation $v'^i = (\partial q'^i/\partial q^j)v^j + (\partial q'^i/\partial t)$ for the fibre coordinates of the fibration π. For this reason, for a local representation of elements of $\mathcal{X}(\pi)$, preference is given to the local basis $\{\mathbf{T}, \partial/\partial q^i\}$ over the coordinate basis $\{\partial/\partial t, \partial/\partial q^i\}$. Correspondingly, forms along π are best expressed in terms of the dual basis of $\bigwedge^1(\pi)$, consisting of dt and the contact forms $\theta^i = dq^i - v^i\, dt$.

We write $\overline{\mathcal{X}}(\pi)$ for the set of equivalence classes of vector fields along π, modulo $\mathbf{T}$ or, equivalently, the subset of $\mathcal{X}(\pi)$ consisting of elements $\overline{X}$ with the property $i_{\overline{X}}dt = 0$. Similarly, we set $\overline{V}(\pi) = \{\overline{L} \in V(\pi) \mid i_{\overline{L}}dt = 0\}$, where i_L, a derivation of $\bigwedge(\pi)$, is defined as in the standard calculus (see e.g. [7] or [9]). Every $L \in V(\pi)$ has a natural decomposition of the form

$$L = L^\circ \otimes \mathbf{T} + \overline{L}, \tag{2}$$

where $L^\circ = i_L\, dt$ and consequently $\overline{L} \in \overline{V}(\pi)$. In particular, the identity tensor field $I \in V^1(\pi)$ can be written as

$$I = dt \otimes \mathbf{T} + \overline{I}, \qquad \overline{I} = \theta^i \otimes \frac{\partial}{\partial q^i}. \tag{3}$$

We now come to the construction of a canonically defined *vertical exterior derivative* d^V on $\bigwedge(\pi)$. Recall that $\mathbb{R} \times TM$ carries a canonically defined type $(1,1)$ tensor field S (cf. [3]), which in coordinates has the form

$$S = \theta^i \otimes \frac{\partial}{\partial v^i}. \tag{4}$$

As discussed e.g. by Vondra [17], there are in fact 4 natural endomorphisms of vector fields on $\mathbb{R} \times TM$. Other constructions, however, make use of $\partial/\partial t$ or the dilation field $v^i\, \partial/\partial v^i$ and as such rely on the product structure of the manifold and its base. We here encounter a first element of choice for the development of our theory and the selection of S, of course, is in agreement with the motivations expressed in Section 1. There is a one-to-one correspondence between $\bigwedge(\pi)$ and the set of semi-basic forms on $\mathbb{R} \times TM$. The derivation $d_S = [i_S, d]$ of $\bigwedge(\mathbb{R} \times TM)$ maps semi-basic forms into semi-basic forms and thus carries over to a derivation of $\bigwedge(\pi)$, denoted by d^V. To see the meaning of d^V in a more direct way, observe first that there is a vertical lift construction on $\mathcal{X}(\pi)$, which provides a bijection between $\overline{\mathcal{X}}(\pi)$ and the set of vertical vector fields on $\mathbb{R} \times TM$. One way of defining this vertical lift goes as follows. First, for basic vector fields X, we set $X^V = S(X^{(1)})$, where $X^{(1)}$ denotes the prolongation of X. We then extend the definition to the whole of $\mathcal{X}(\pi)$ by linearity. If, for once, a general $X \in \mathcal{X}(\pi)$ is written in the coordinate basis as $X = X^0\, \partial/\partial t + X^i \partial/\partial q^i$, then

$$X^V = (X^i - v^i X^0)\frac{\partial}{\partial v^i}. \tag{5}$$

It is clear that $\mathbf{T}^V = 0$ and that conversely, therefore, every vertical vector field on $\mathbb{R} \times TM$ corresponds to a unique element of $\overline{\mathcal{X}}(\pi)$. As in [9], it is easy to argue that every derivation of $\bigwedge(\pi)$ is completely determined by its action on functions and basic 1-forms. This way, d^V can be defined directly by the rule: $\forall F \in C^\infty(\mathbb{R}\times TM), X \in \mathcal{X}(\pi)$,

$$d^V F(X) = X^V(F), \tag{6}$$

plus the requirement that d^V vanishes on basic 1-forms. For practical purposes, the result is that:

$$d^V F = (\partial F/\partial v^i)\theta^i, \qquad d^V(dt) = 0, \qquad d^V\theta^i = dt \wedge \theta^i. \tag{7}$$

It is easy to verify, using (7), that

$$d^V \circ d^V = dt \wedge d^V. \tag{8}$$

That $d^V \circ d^V \neq 0$ should not come as a surprise: it is a reflection of the fact that the Nijenhuis tensor of S is not zero. If, for $L \in V(\pi)$, d^V_L denotes the commutator $[i_L, d^V]$, we have that $d^V = d^V_I$ and learn more about the structure of d^V through the decomposition (3) of I. It is clear that $d^V_{dt\otimes\mathbf{T}}$ vanishes on functions, i.e. is a derivation of type i_* and accordingly (cf. [9]) must be representable in the form i_L for some $V \in V^2(\pi)$. From the action on dt and θ^i, it is easily seen that the L in question is $dt \wedge I$. Hence, we have

$$d^V = i_{dt\wedge I} + d^V_{\bar{I}}. \tag{9}$$

Comparison with (7) shows that

$$d^V_{\bar{I}} F = d^V F, \qquad d^V_{\bar{I}}(dt) = 0, \qquad d^V_{\bar{I}}\theta^i = 0, \tag{10}$$

from which it follows that $d^V_{\bar{I}} \circ d^V_{\bar{I}} = 0$. Again, a few comments are in order concerning alternative ways of selecting a type of exterior derivative which will afterall have an effect on the classification of all derivations. It is clear that $d^V_{\bar{I}}$ has nicer properties than d^V; it behaves very much like the vertical exterior derivative of the autonomous theory, with parametric dependence on the variable t, and as such also has trivial cohomology. We have nevertheless not chosen for $d^V_{\bar{I}}$ as fundamental derivative, because we preferred the d^V to be modeled (as in [9]) on some d_S on $\mathbb{R} \times TM$; the feeling is that somehow d^V comes first and $d^V_{\bar{I}}$ is derived from it subsequently. A result of our choice is that we will encounter some more inconveniences like (8), but it will also turn out that most of the interesting commutator relations in the end follow the same pattern as in the autonomous case. Needless to say, the selection of d^V as fundamental vertical exterior derivative does not preclude that $d^V_{\bar{I}}$ will play a prominent role in applications. The zero cohomology of $d^V_{\bar{I}}$ can be translated to an interesting property of d^V as well. To see this, one has to take into account the following decomposition of a general form $\omega \in \bigwedge(\pi)$: setting $\hat{\omega} = i_{\mathbf{T}}\omega$, we have

$$\omega = \tilde{\omega} + dt \wedge \hat{\omega}, \tag{11}$$

which defines $\tilde{\omega}$ in such a way that $i_{\mathbf{T}}\tilde{\omega} = 0$.

Proposition 2.1. *For $\omega \in \bigwedge^p(\pi)$, $d^V\omega = 0$ is equivalent to the existence of a $\beta \in \bigwedge^{p-1}(\pi)$, such that $\omega = d^V\beta + dt \wedge \beta$.*

Proof. If $\omega = d^V\beta + dt \wedge \beta$, the property $d^V\omega = 0$ trivially follows from (8). For the converse, using the decompositions (9) and (11), we first observe that

$$i_{dt\wedge I}\omega = dt \wedge i_I\omega = p\, dt \wedge \tilde{\omega}, \tag{12}$$

$$d_I^V\omega = d_I^V\tilde{\omega} - dt \wedge d_I^V\hat{\omega}. \tag{13}$$

It follows that $d^V\omega = 0$ is equivalent to $d_I^V\tilde{\omega} = 0$ and $d_I^V\hat{\omega} = p\tilde{\omega}$. The first of these implies that $\tilde{\omega} = d_I^V\tilde{\beta}$, for some $\tilde{\beta} \in \bigwedge^{p-1}(\pi)$ with $i_{\mathbf{T}}\tilde{\beta} = 0$, and the second condition subsequently implies: $\hat{\omega} = p\tilde{\beta} + d_I^V\tilde{\alpha}$, for some $\tilde{\alpha} \in \bigwedge^{p-2}(\pi)$ with $i_{\mathbf{T}}\tilde{\alpha} = 0$. Putting ω back together, we find that

$$\omega = d_I^V\tilde{\beta} + dt \wedge (p\tilde{\beta} + d_I^V\tilde{\alpha}).$$

Setting finally $\beta = \tilde{\beta} - dt \wedge \tilde{\alpha}$ and using the general rules (12) and (13), we obtain

$$d^V\beta = d_I^V\tilde{\beta} + (p-1)dt \wedge \tilde{\beta} + dt \wedge d_I^V\tilde{\alpha} = \omega - dt \wedge \tilde{\beta} = \omega - dt \wedge \beta,$$

which is the desired result. □

Going back to the decomposition (9) of d^V, we will now introduce the terminology of d_*^V-derivations in such a way that it applies to the main part d_I^V rather than to the full d^V.

Definition 2.2. A derivation of $\bigwedge(\pi)$ of type d_*^V is a derivation of the form d_L^V, with $i_L dt = 0$.

Computing the commutator $[d_L^V, d^V]$, with $L \in \bar{V}^r(\pi)$ say, one easily finds, using (8), that $D = d_L^V$ has the property

$$[D, d^V] + (-1)^r dt \wedge D = 0. \tag{14}$$

Further obvious properties of a derivation of type d_*^V are that it vanishes on basic functions and on dt. We want to show that these three properties completely characterize derivations of type d_*^V. To that end, observe first that for a basic 1-form α, we have

$$d^V\hat{\alpha} = \alpha - \hat{\alpha}\, dt, \qquad \alpha \in \textstyle\bigwedge^1(\mathbb{R} \times M). \tag{15}$$

This is, for example, easy to verify in coordinates. Using (15), it follows from (14) that for a basic 1-form α,

$$D\alpha = D\hat{\alpha} \wedge dt + \hat{\alpha}D(dt) + (-1)^r(d^V - dt\wedge)D\hat{\alpha} = \hat{\alpha}D(dt) + (-1)^r d^V D\hat{\alpha}. \tag{16}$$

In other words, a derivation with property (14) is completely determined by its action on functions and on dt.

Proposition 2.3. *A derivation D of $\bigwedge(\pi)$, of degree r, is of type d_*^V if and only if it vanishes on basic functions and on dt and has the property* (14).

Proof. It remains to be shown that a D with such properties is of the form d^V_L, with $L \in \bar{V}(\pi)$. To this end, given D we construct a derivation D' of degree $r-1$ by the requirements: $D'F = 0$ for all functions F, $D'\alpha = D\hat{\alpha}$ for basic 1-forms α. Since D' is of type i_*, it is of the form i_L and since $D'(dt) = 0$ we will actually have $L \in \bar{V}^r(\pi)$. The claim now is that the original D is d^V_L. Following the above remark, both derivations are completely determined by their action on functions. They trivially coincide on basic functions, so that it remains to compare their action on fibre linear functions of the form $\hat{\alpha}$, with $\alpha \in \bigwedge^1(\mathbb{R} \times M)$. We have

$$d^V_L\hat{\alpha} = i_L(\alpha - \hat{\alpha}\, dt) = i_L\alpha = D'\alpha = D\hat{\alpha},$$

which concludes the proof. □

Proposition 2.4. *Every derivation D of $\bigwedge(\pi)$, vanishing on basic functions, has a unique decomposition into the sum of a derivation of type i_* and a derivation of type d^V_*.*

Proof. From the given D, of degree r say, we construct a derivation D_2 by the following requirements: $D_2F = DF$ on functions, whereas on basic 1-forms α, inspired by (16), we impose:

$$D_2\alpha = (-1)^r d^V D\hat{\alpha}.$$

By construction, D_2 vanishes on basic functions and on dt. In view of the link between (14) and (16), D_2 further has the property (14) for its action on functions. We check that (14) also holds on basic 1-forms α:

$$\begin{aligned}[][D_2, d^V]\alpha + (-1)^r dt \wedge D_2\alpha &= -(-1)^r d^V D_2\alpha + (-1)^r dt \wedge D_2\alpha \\ &= -\, d^V d^V D\hat{\alpha} + dt \wedge d^V D\hat{\alpha} = 0.\end{aligned}$$

It follows that D_2 is of type d^V_*. The difference $D_1 = D - D_2$ vanishes on functions and therefore is of type i_*. It is easy to see that this decomposition is unique. □

As in [9] it appears that the full characterization and classification of derivations of $\bigwedge(\pi)$ requires some extra input, for the description of what happens with functions on the base manifold $\mathbb{R} \times M$. Before going into that in the next section, it is worthwhile pointing out again some pecularities about our notion of d^V_*-derivations. Note, for example, that d^V itself is not a derivation of type d^V_*, but d^V_I is! Also, as is seen from (14), derivations of type d^V_* do not commute with d^V, nor do they commute with d^V_I. These features may look unpleasant if one has the standard Frölicher-Nijenhuis calculus in mind, but are not too difficult to live with once one is aware of them.

3. Classification of derivations of $\Lambda(\pi)$

From now on, we assume to have a connection on the bundle π at our disposal, i.e., a splitting of the sequence

$$0 \to \mathrm{Vert}\,(\mathbb{R} \times TM) \xrightarrow{i} T(\mathbb{R} \times TM) \xrightarrow{j} \pi^*(T(\mathbb{R} \times M)) \to 0,$$

where Vert $(\mathbb{R}\times TM)$ denotes vertical tangent vectors (over $\mathbb{R}\times M$), i is the inclusion and the essential component of j is the projection $T\pi$. With the aid of such a connection we have a mechanism for lifting basic vector fields "horizontally" to corresponding vector fields on $\mathbb{R}\times TM$. We set

$$H_i = \left(\frac{\partial}{\partial q^i}\right)^H = \frac{\partial}{\partial q^i} - \Gamma_i^j \frac{\partial}{\partial v^j}, \qquad H_0 = \left(\frac{\partial}{\partial t}\right)^H = \frac{\partial}{\partial t} - \Gamma_0^i \frac{\partial}{\partial v^i}, \tag{17}$$

which identifies $n(n+1)$ so-called connection coefficients Γ_i^j, Γ_0^i. The horizontal lift construction trivially extends to $\mathcal{X}(\pi)$ by linearity. Of particular interest is $\mathbf{T}^H$, which defines a SODE on $\mathbb{R}\times TM$ with the following coordinate espression:

$$\mathbf{T}^H = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial q^i} - (\Gamma_0^i + v^j \Gamma_j^i)\frac{\partial}{\partial v^i}. \tag{18}$$

As in [9], the *horizontal exterior derivative* d^H on $\bigwedge(\pi)$ (associated to a given connection) is defined by the rule: $\forall F \in C^\infty(\mathbb{R}\times TM)$, $X \in \mathcal{X}(\pi)$,

$$d^H F(X) = X^H(F), \tag{19}$$

plus the requirement that d^H coincides with the ordinary exterior derivative on $\bigwedge(\mathbb{R}\times M)$ for basic forms. For computational purposes, using as before the local basis $\{dt, \theta^i\}$ to describe forms along π, d^H is completely determined by:

$$d^H F = H_i(F)\theta^i + \mathbf{T}^H(F)\,dt, \tag{20}$$

$$d^H(dt) = 0, \qquad d^H\theta^i = \Gamma_k^i \theta^k \wedge dt. \tag{21}$$

Definition 3.1. A derivation of $\bigwedge(\pi)$ is said to be of type d_*^H if it is of the form $d_L^H = [i_L, d^H]$ for some $L \in V(\pi)$.

A further digression on alternative selections is appropriate here. With the given connection comes a horizontal projector P_H on $\mathcal{X}(\mathbb{R}\times TM)$, which in coordinates is given by $P_H = dt\otimes H_0 + dq^i \otimes H_i$. This tensor field, however, has a natural decomposition in the form

$$P_H = dt \otimes \mathbf{T}^H + P_{\bar{H}}, \qquad P_{\bar{H}} = \theta^i \otimes H_i. \tag{22}$$

To $P_{\bar{H}}$ is associated, what is sometimes called a strong horizontal lift (cf. [5]). Whereas our definition of d^H is somehow governed by P_H, it is conceivable that somebody else would introduce two separate derivations at this point, one for each part of the decomposition (22). Yet another way of thinking of two separate derivations arises as follows. Similar to (9), we have

$$d^H = d_I^H = d_{dt\otimes\mathbf{T}}^H + d_{\bar{I}}^H, \tag{23}$$

whereby the two parts we encounter here are not exactly the two separate derivations referred to above. It must be said that there are grounds for paying attention to these two parts separately, coming from the jet bundle structure of $\mathbb{R}\times TM$. Indeed, we have the chain of inclusions of rings:

$$C^\infty(\mathbb{R}) \subset C^\infty(\mathbb{R}\times M) \subset C^\infty(\mathbb{R}\times TM) \subset \bigwedge(\pi),$$

which indicates that a possible approach towards the classification of derivations of $\bigwedge(\pi)$ would attribute a distinct role to properly chosen derivations with respect to (i.e. vanishing on) each of the subrings. The derivations i_L, d_I^V and d_I^H are suitable for such a purpose and at the end of such a process one then has to select another derivation which is an extension of the exterior derivative on the base $\mathbb{R}$. A possible candidate in that respect is $d^H_{dt\otimes\mathbf{T}}$, particularly since its square is zero. As in the discussions of the previous section, our prevailing feelings are that the full d^H is somehow born first, that other interesting derivations originate from it (e.g. by (23)) and that a classification of arbitrary derivations in terms of three components (instead of four) will more closely relate to the autonomous theory of [9] and [10].

Theorem 3.2. *Every derivation D of $\bigwedge(\pi)$, of degree r, has a unique decomposition in the form*

$$D = i_{L_1} + d^V_{L_2} + d^H_{L_3}, \tag{24}$$

with $L_1 \in V^{r+1}(\pi)$, $L_2 \in \bar{V}^r(\pi)$, $L_3 \in V^r(\pi)$.

Proof. For fixed $X_1,\ldots,X_r \in \mathcal{X}(\pi)$ and variable basic functions f, $Df(X_1,\ldots,X_r)$ maps basic functions into functions on $\mathbb{R}\times TM$ while satisfying a Leibnitz-type rule and therefore defines an element of $\mathcal{X}(\pi)$. Since the dependence on $X_1,\ldots,X_r$ is $C^\infty(\mathbb{R}\times TM)$-multilinear and skew-symmetric, we are actually looking at some $L_3 \in V^r(\pi)$ such that

$$L_3(X_1,\ldots,X_r)(f) = Df(X_1,\ldots,X_r).$$

The left-hand side is also $d^H_{L_3} f(X_1,\ldots,,X_r)$. It follows that $D - d^H_{L_3}$ vanishes on basic functions, so that Proposition 2.4 immediately yields the desired result. $\square$

For computational purposes, if we write a general $L \in V(\pi)$ in the form

$$L = L^0 \otimes \mathbf{T} + L^i \otimes \frac{\partial}{\partial q^i},$$

Dt and Dq^i determine consecutively L^0_3 and L^i_3. Subsequently, $(D - d^H_{L_3})(v^i)$ will provide us with the forms L^i_2. Finally, the action of $D - d^H_{L_3} - d^V_{L_2}$ on dt and θ^i will respectively yield L^0_1 and L^i_1.

4. Derivations of vector-valued forms and self-duality

For extending the action of a derivation D to the module of vector-valued forms $V(\pi)$ over the graded ring $\bigwedge(\pi)$, it suffices to specify the action on $\mathcal{X}(\pi)$ (or in fact on basic vector fields) in a way which is consistent with the already determined action on functions. Defining the extension of a commutator of two derivations to be the commutator of the extensions, the main issue is to define the action of i_L, d^V and d^H on $V(\pi)$. There are different ways of doing this, but what we regard as being the most natural procedure goes as follows. As in [9], we define i_L and d^V to vanish on basic vector fields, which is justified by the fact that they vanish on basic functions. We thus have,

$$d^V(\partial/\partial q^i) = 0, \qquad d^V\mathbf{T} = \bar{I}, \tag{25}$$

and exactly the same formulas for $d^V_{\bar{I}}$. Next, using the vertical projector $P_V = I_{\mathbb{R}\times TM} - P_H$, for an arbitrary $X \in \mathcal{X}(\pi)$, we define $d^H X \in V^1(\pi)$ by the following two prescriptions:

$$\forall Z \in \mathcal{X}(\pi), \qquad d^H X(Z)^V = P_V([Z^H, X^V]), \tag{26}$$

together with

$$i_{d^H X}\, dt = d^H(i_X\, dt). \tag{27}$$

It is easy to verify that this construction makes $d^H X$ tensorial indeed and satisfies the derivation requirement $d^H(FX) = d^H F \otimes X + F d^H X$, $\forall F \in C^\infty(\mathbb{R}\times TM)$. An important consequence of this extension of d^H, following from $\mathbf{T}^V = 0$, is that

$$d^H\mathbf{T} = 0, \qquad \text{and thus also} \qquad d^H_{\bar{I}}\mathbf{T} = 0. \tag{28}$$

For coordinate calculations, it is useful to know that

$$d^H \frac{\partial}{\partial q^i} = \left(\frac{\partial \Gamma^k_0}{\partial v^i} + v^j \frac{\partial \Gamma^k_j}{\partial v^i}\right) dt \otimes \frac{\partial}{\partial q^k} + \frac{\partial \Gamma^k_j}{\partial v^i}\theta^j \otimes \frac{\partial}{\partial q^k}, \tag{29}$$

where the two terms in the right-hand side correspond to the decomposition (23) of d^H.

Starting from a general derivation D of $V(\pi)$, one can consider its restriction to $\bigwedge(\pi)$ and regard this in turn as a derivation of $V(\pi)$ again, via the rules of extension which have just been adopted. The difference with the original derivation obviously vanishes on $\bigwedge(\pi)$: it is a derivation of type a_* and explicit formulas for the action of such derivations have been given in [9]. From Theorem 3.2, following exactly the line of proof of the autonomous theory in [9], we obtain the following classification result.

Theorem 4.1. *Every derivation D of $V(\pi)$, of degree r, can uniquely be written in the form*

$$D = i_{L_1} + d^V_{L_2} + d^H_{L_3} + a_Q, \tag{30}$$

with $L_1 \in V^{r+1}(\pi)$, $L_2 \in \bar{V}^r(\pi)$, $L_3 \in V^r(\pi)$, $Q \in \bigwedge^r(\pi)\otimes V^1(\pi)$.

An important class of derivations of degree 0 are the ones that have the property

$$D\langle X, \alpha\rangle = \langle DX, \alpha\rangle + \langle X, D\alpha\rangle, \qquad \alpha \in \textstyle\bigwedge^1(\pi), X \in \mathcal{X}(\pi) \tag{31}$$

and are said to be *self-dual* for this reason. In Section 3 of [10] one can find an extensive discussion of self-dual derivations in a way which is valid in a general setting and thus directly applies also to the present situation. The most important features for later use can be summarized as follows.

Let $d^{(1)}$ stand for either the vertical or horizontal exterior derivative on $V(\pi)$. Then, with $X \in \mathcal{X}(\pi)$, $d_X^{(1)}$ is not self-dual, so that two self-dual derivations can be constructed from it. On the one hand, the restriction of $d_X^{(1)}$ to $\bigwedge(\pi)$ can be extended to the whole of $V(\pi)$ by imposing the duality rule (31). This defines a derivation of Lie-derivative type, denoted by $\mathcal{L}_X^{(1)}$, giving rise to a bracket operation on $\mathcal{X}(\pi)$, say $[X,Y]_{(1)} = \mathcal{L}_X^{(1)}Y$. On the other hand, we can start from the restriction of $d_X^{(1)}$ to $\mathcal{X}(\pi)$ and use (31) again, this time to define a new action on $\bigwedge(\pi)$. This defines a derivation $\mathcal{D}_X^{(1)}$, depending linearly on the argument X and therefore said to be of covariant-derivative type. Clearly, by construction, the difference between $d_X^{(1)}$ and $\mathcal{L}_X^{(1)}$ is of type a_*, whereas the difference between $d_X^{(1)}$ and $\mathcal{D}_X^{(1)}$ is of type i_*. To find the element of $V^1(\pi)$ which will determine both difference terms, we can proceed as follows. First, it is easy to verify from the defining relations that $\mathcal{D}_X^{(1)}Y - \mathcal{D}_Y^{(1)}X - [X,Y]_{(1)}$ is $C^\infty(\mathbb{R}\times TM)$-linear in X and Y (and obviously skew-symmetric) and this way defines a 'torsion form' $T_{d^{(1)}} \in V^2(\pi)$. It then follows that

$$\mathcal{L}_X^{(1)} = d_X^{(1)} - a_{Q_X}, \quad \mathcal{D}_X^{(1)} = d_X^{(1)} - i_{Q_X}, \quad \text{with } Q_X = d^{(1)}X + i_X T_{d^{(1)}}. \tag{32}$$

We will learn from the analysis in the next section that the 'vertical torsion' is zero, while the 'horizontal torsion' corresponds exactly to the torsion of the non-linear connection we started from. To that end, it is useful to know that the characterizing property (31) of a self-dual derivation D, as has been proved in [10], is equivalent to

$$[D, i_L] = i_{DL}, \qquad \forall L \in V(\pi). \tag{33}$$

In view of their importance for applications, we list the following coordinate expressions for the action of a vertical and horizontal covariant derivative on the local basis of $\bigwedge^1(\pi)$ and $\mathcal{X}(\pi)$ and on functions $F \in C^\infty(\mathbb{R}\times TM)$. For

$$X = X^0\mathbf{T} + X^i\frac{\partial}{\partial q^i}, \tag{34}$$

using a notation like Γ^k_{ji} as shorthand for $\partial\Gamma^k_j/\partial v^i$, we have

$$\begin{gathered}\mathcal{D}^V_X F = X^V(F), \quad \mathcal{D}^V_X(dt) = 0, \quad \mathcal{D}^V_X\theta^i = -X^i\,dt,\\ \mathcal{D}^V_X\left(\frac{\partial}{\partial q^i}\right) = 0, \quad \mathcal{D}^V_X\mathbf{T} = X^i\frac{\partial}{\partial q^i} = \overline{X}\end{gathered} \tag{35}$$

and

$$\begin{gathered}\mathcal{D}^H_X F = X^H(F), \quad \mathcal{D}^H_X(dt) = 0, \quad \mathcal{D}^H_X\mathbf{T} = 0,\\ \mathcal{D}^H_X\theta^i = -[X^j\Gamma^i_{jk} + X^0(\Gamma^i_{jk}v^j + \Gamma^i_{0k})]\theta^k,\\ \mathcal{D}^H_X\left(\frac{\partial}{\partial q^i}\right) = [X^j\Gamma^k_{ji} + X^0(\Gamma^k_{ji}v^j + \Gamma^k_{0i})]\frac{\partial}{\partial q^k}.\end{gathered} \tag{36}$$

A special case of interest, certainly, concerns the covariant derivatives with respect to the canonical vector field $\mathbf{T}$. Since $d^V_{\mathbf{T}}$ is zero on functions and on vector fields, we have

$$\mathcal{D}^V_{\mathbf{T}} = 0 \tag{37}$$

by construction. This is, of course, also clear from (35). $\mathcal{D}^H_{\mathbf{T}}$ on the other hand is an important derivation and will manifest itself more distinctively when we consider the connection associated to a SODE.

As a final remark for this section, note that the action of the vertical and horizontal covariant derivative trivially extends to tensor fields of arbitrary type (as is true for all self-dual derivations). We can then define operators $\mathcal{D}^V$ and $\mathcal{D}^H$, which increase the covariant order of a tensor field U along π by 1 and are defined by

$$\mathcal{D}^V U(X,\ldots) = \mathcal{D}^V_X U(\ldots), \quad \mathcal{D}^H U(X,\ldots) = \mathcal{D}^H_X U(\ldots). \tag{38}$$

5. Commutators

The computation of commutators of interesting derivations is a rather boring story. We will limit ourselves to relations which are essential for introducing geometrical concepts such as torsion and curvature, and to identities which are frequently needed in applications.

Before starting, we collect a few simple properties of derivations which are often used in the subsequent analysis. For example, it is easy to verify that for arbitrary $L \in V(\pi)$

$$dt \wedge i_L = i_{dt\wedge L}, \quad dt \wedge d^V_L = d^V_{dt\wedge L}, \quad dt \wedge d^H_L = d^H_{dt\wedge L}. \tag{39}$$

Also frequently used are the properties

$$d^V_{L_1} i_{L_2} \alpha = i_{d^V_{L_1} L_2} \alpha, \qquad \text{for } \alpha \in \textstyle\bigwedge^1(\mathbb{R}\times M), \tag{40}$$

and

$$i_{L_1} i_{L_2} \sigma = i_{i_{L_1} L_2} \sigma, \qquad \text{for } \sigma \in \textstyle\bigwedge^1(\pi) \text{ or } \sigma \in V^1(\pi). \tag{41}$$

Thinking of a decomposition such as (2), one readily deduces from (15) that $d^V_{L^0\otimes\mathbf{T}}$ is a derivation of type i_*. It then follows from (40) that in fact, with $L^0 \in \bigwedge^\ell(\pi)$, we have

$$d^V_{L^0\otimes\mathbf{T}} = (-1)^\ell i_{d^V(L^0\otimes\mathbf{T})}. \tag{42}$$

One of the main tools in obtaining commutator relations is of course the graded Jacobi identity of derivations. The commutator of d^H and d^V trivially vanishes on basic functions and on dt. From the Jacobi identity involving another d^V, it follows from (8) that $[d^H, d^V]$ has the property (14). According to Proposition 2.3, it is therefore a derivation of type d^V_*, which means that there exists a $T \in \bar{V}^2(\pi)$, such that

$$[d^H, d^V] = d^V_T \qquad \text{on } \textstyle\bigwedge(\pi). \tag{43}$$

The tensor field T is called the *torsion* of the non-linear connection and is found to have the following coordinate expression

$$T = \tfrac{1}{2}(\Gamma^i_{kj} - \Gamma^i_{jk})\theta^k \wedge \theta^j \otimes \frac{\partial}{\partial q^i} + (v^j\Gamma^i_{jk} - \Gamma^i_k + \Gamma^i_{0k})\, dt \wedge \theta^k \otimes \frac{\partial}{\partial q^i}, \tag{44}$$

which is in good correspondence with the expression of the torsion in [17]. Comparing with the situation in the time-independent framework, we observe that not only do contact forms replace the coordinate 1-forms dq^k and do coefficient functions depend on the extra variable t, but there is also an additional term. Despite this fact, when one computes the derivation of type a_* which may come in when (43) is extended to vector-valued forms, one obtains formally the same result as in [10], namely:

$$[d^H, d^V] = d^V_T - a_{\mathcal{D}^V T} \qquad \text{on } V(\pi). \tag{45}$$

The commutator of d^H with a general d^V_*-derivation has a decomposition which will be useful below for arriving at further interesting relations. For $\bar{L} \in \bar{V}^\ell(\pi)$, one can verify, in coordinates for example, that

$$[d^H, d^V_{\bar{L}}] = (-1)^\ell i_{d^V d^H \bar{L}} + d^V_{d^H \bar{L}} - d^H_{d^V \bar{L}} \qquad \text{on } \textstyle\bigwedge(\pi). \tag{46}$$

Note in passing that $d^V(\bar{V}(\pi)) \subset \bar{V}(\pi)$ and also $d^H(\bar{V}(\pi)) \subset \bar{V}(\pi)$, which indicates that (46) truly represents a decomposition as guaranteed by Theorem 3.2. From (46) with $\bar{L} = \bar{I}$ and comparison with (43), one can learn that

$$d^V \bar{I} = dt \wedge \bar{I}, \qquad \text{or equivalently } d^V I = 0, \tag{47}$$

and more interestingly that

$$T = d^H \bar{I} = d^H d^V \mathbf{T}. \tag{48}$$

Turning next to the commutator of d^H with itself, which clearly vanishes on basic functions, we know that

$$\tfrac{1}{2}[d^H, d^H] = i_P + d^V_R \qquad \text{on } \textstyle\bigwedge(\pi),$$

for some $P \in V^3(\pi)$, $R \in \bar{V}^2(\pi)$. In fact, it is clear that also $i_P\, dt = 0$, i.e. $P \in \bar{V}^3(\pi)$. To specify P further, the trick is to compute from the above relation d^V_P, using the Jacobi identity and property (14) of d^V_R. It follows that

$$d^V_P = [d^H, d^V_T] + dt \wedge d^V_R,$$

which with the aid of (46) shows that

$$d^V T = 0, \qquad d^V d^H T = 0 \tag{49}$$

and $P = d^H T + dt \wedge R$. A similar calculation produces (always on $\bigwedge(\pi)$)

$$d^H_P = [d^H, d^V_R] = i_{d^V d^H R} + d^V_{d^H R} - d^H_{d^V R},$$

i.e., $P = -d^V R$ and

$$d^H R = 0. \tag{50}$$

From a combination of the two expressions for P, we also conclude that

$$d^H T + d^V R + dt \wedge R = 0. \tag{51}$$

The tensor field $R \in \bar{V}^2(\pi)$ represents the *curvature* of the non-linear connection and the properties (50) and (51) can be seen as *Bianchi identities*. Finally, extending the action to $V(\pi)$ again, we will have

$$\tfrac{1}{2}[d^H, d^H] = -i_{d^V R} + d^V_R + a_{\mathrm{Rie}}, \tag{52}$$

with $\mathrm{Rie} \in \bigwedge^2(\pi) \otimes V^1(\pi)$.

In coordinates, we have

$$R = \tfrac{1}{2} R^i_{kj}\, \theta^k \wedge \theta^j \otimes \frac{\partial}{\partial q^i} + R^i_{0j}\, dt \wedge \theta^j \otimes \frac{\partial}{\partial q^i}, \tag{53}$$

with

$$\begin{aligned} R^i_{kj} &= H_j(\Gamma^i_k) - H_k(\Gamma^i_j), \\ R^i_{0j} &= H_j(\Gamma^i_0) - H_0(\Gamma^i_j) + v^k(H_j(\Gamma^i_k) - H_k(\Gamma^i_j)), \end{aligned} \tag{54}$$

and one can verify that

$$\mathrm{Rie} = -\mathcal{D}^V R - R \otimes dt, \tag{55}$$

in other words, $\forall X, Y, Z \in \mathcal{X}(\pi)$:

$$\mathrm{Rie}(X,Y)Z = -(\mathcal{D}^V_Z R)(X,Y) - R(X,Y)\langle Z, dt\rangle.$$

For the commutator of two i_*-derivations we have (as a result of (41)) the usual formula:

$$[i_L, i_M] = i_{i_L M} - (-1)^{(\ell-1)(m-1)} i_{i_M L}. \tag{56}$$

An unusual feature is that the commutator of two d^V_*-derivations is not of type d^V_*. In fact, if D_1 and D_2 are two such derivations, of degree r_1 and r_2 respectively, it follows from the Jacobi identity that

$$[[D_1, D_2], d^V] = -2(-1)^{r_1 + r_2} dt \wedge [D_1, D_2], \tag{57}$$

which violates the property (14) of a d^V_*-derivation. In any event, since the commutator certainly vanishes on basic functions and on basic vector fields, we have a decomposition as in Proposition 2.4. The resulting relation reads as follows: for $\bar{L} \in \bar{V}^\ell(\pi)$, $\bar{M} \in \bar{V}^m(\pi)$,

$$[d^V_{\bar{L}}, d^V_{\bar{M}}] = -(-1)^{\ell+m} i_{dt \wedge [\bar{L}, \bar{M}]_V} + d^V_{[\bar{L}, \bar{M}]_V}. \tag{58}$$

The d^V_*-part in (58) must be seen as defining the bracket $[\bar{L}, \bar{M}]_V$. The i_*-part subsequently follows from using the same procedure as the one which identified the tensor field P above. To find the explicit meaning of the vertical bracket thus defined, it suffices to apply (58) to fibre linear functions $\hat{\alpha} = i_{\mathbf{T}}\alpha$ with $\alpha \in \bigwedge^1(\mathbb{R} \times M)$ and to make use of the property (40). One obtains:

$$[\bar{L}, \overline{M}]_V = d^V_{\bar{L}}\overline{M} - (-1)^{\ell m} d^V_{\overline{M}}\bar{L}. \tag{59}$$

It would seem natural to have a similar relation applying also to arbitrary vector-valued forms. There will be ground for imposing this as an extension of the vertical bracket after what follows.

Consider the commutator of an i_* and a d^V_*-derivation, which again decomposes as described by Proposition 2.4. The d^V_*-part is easy to identify from the action on functions $\hat{\alpha}$. We thus have, for $L \in V^\ell(\pi)$, $\overline{M} \in \bar{V}^m(\pi)$:

$$[i_L, d^V_{\overline{M}}] = i_A + d^V_{i_L\overline{M}},$$

for some A. The identification of A, which does not belong to $\bar{V}(\pi)$, is a rather tedious matter and will therefore not be described. One obtains that $A = (-1)^m[L, \overline{M}]_V$, provided we define this bracket by the right-hand side of (59), with L replacing $\bar{L}$. If we next enlarge the setting by considering an $M \in V(\pi)$, $M = \overline{M} + M^0 \otimes \mathbf{T}$ say, it is easy to compute the commutator $[i_L, d^V_{M^0\otimes\mathbf{T}}]$ with the aid of (42) and (56). Recollecting terms, one again can express the result in the form

$$[i_L, d^V_M] = (-1)^m i_{[L,M]_V} + d^V_{i_L M}, \tag{60}$$

provided the definition of the vertical bracket is further extended in the way indicated by (59). The nice feature about this end result is that it is formally identical to the situation in the calculus along $\tau : TM \to M$ (see [10]). Beware, however, that the right-hand side of (60) is not a decomposition in the strict sense of Theorems 3.2 or 4.1, because part of the second term will be of type i_*, when $i_L M$ does not belong to $\bar{V}(\pi)$.

The computation of the commutator $[d^V_L, d^V_M]$ for general $L, M \in V(\pi)$ is not a very thrilling story. We limit ourselves to the remark that the vector-valued form which determines its d^V_*-part is given by $d^V_L\overline{M} - (-1)^{\ell m} d^V_M\bar{L}$.

Concerning the commutator of i_L and d^H_M, one easily identifies, using (41), that the term of type d^H_* is $d^H_{i_L M}$ and subsequently can verify that the remaining part is of type i_*. By analogy with (60), it makes sense to use this i_*-part for defining the horizontal bracket $[L, M]_H$, i.e. we set

$$[i_L, d^H_M] = (-1)^m i_{[L,M]_H} + d^H_{i_L M}. \tag{61}$$

An explicit formula like (59) is not available for the horizontal bracket. The best approximation of such a formula is the following coordinate expression. Writing L and

M in their canonical decomposition (2) with for example $\bar{L} = L^i \otimes (\partial/\partial q^i)$, we have

$$[L, M]_H = \left(d_L^H M^i - (-1)^{\ell m} d_M^H L^i - \Gamma_k^i (L^k \wedge M^0 - (-1)^{\ell m} M^k \wedge L^0)\right) \otimes \frac{\partial}{\partial q^i} + (d_L^H M^0 - (-1)^{\ell m} d_M^H L^0) \otimes \mathbf{T}. \tag{62}$$

Since we have already introduced vertical and horizontal brackets, at least for vector fields, when discussing self-dual derivations of Lie-derivative type in the previous section, we have a question of consistency to verify. To that end, observe first that when M in (60) or (61) is taken to be a vector field X, the right-hand side reduces to the i_*-part. On the other hand, if d_X^V and d_X^H are viewed as derivations of scalar forms and then extended to corresponding Lie derivatives by duality, we have the property (33). It follows that

$$\mathcal{L}_X^V L = [X, L]_V, \qquad \mathcal{L}_X^H L = [X, L]_H, \tag{63}$$

and the subcase where L is also a vector field is indeed consistent with earlier considerations.

There is an interesting way now of reinterpreting the result $T = d^H I$ (cf. (48)). We have, using (61),

$$\begin{aligned} T(X, Y) &= i_Y i_X d^H I = i_Y d_X^H I - d_Y^H X \\ &= d_X^H Y - [X, Y]_H - d_Y^H X \\ &= \mathcal{D}_X^H Y - \mathcal{D}_Y^H X - [X, Y]_H. \end{aligned} \tag{64}$$

This confirms, as was announced in the previous section, that T relates to the 'horizontal torsion'.

That there is no 'vertical torsion' follows in the same way from $d^V I = 0$, or from the explicit formula (59) which for vector fields can be written in the form

$$[X, Y]_V = \mathcal{D}_X^V Y - \mathcal{D}_Y^V X. \tag{65}$$

To finish this section we want to arrive at the commutator relations of the important vertical and horizontal covariant derivatives. There are several ways of computing these; we choose to give first some information on commutators of d_X^V and d_Y^H-type derivations.

From the Jacobi identity for i_X, d^V and d^V one easily obtains that

$$[d_X^V, d^V] = \langle X, dt \rangle d^V - dt \wedge d_X^V.$$

The Jacobi identity for d_X^V, i_Y and d^V subsequently gives, with the aid of (60):

$$[d_X^V, d_Y^V] = d_{[X,Y]_V}^V + \langle X, dt \rangle\, d_Y^V - \langle Y, dt \rangle\, d_X^V - dt \wedge i_{[X,Y]_V}. \tag{66}$$

There are, however, several terms of type i_* hidden in the right-hand side of (66). In fact, following a remark made before, the d_*^V-part is determined by $d_X^V \bar{Y} - d_Y^V \bar{X}$, which is the same as $[\bar{X}, \bar{Y}]_V$ in view of (37). Recalling the definition of $\mathcal{D}_X^V$, which is an extension by duality of the action of d_X^V on vector fields, there are two observations which are important now. First of all, the extension by duality of a commutator is the commutator of the extensions. Secondly, for extensions in this direction, terms of type i_* clearly do not matter, since they vanish on vector fields. As a result, we conclude that

$$[\mathcal{D}^V_X, \mathcal{D}^V_Y] = \mathcal{D}^V_{[\overline{X},\overline{Y}]_V}. \tag{67}$$

For the computation of $[d^H_X, d^H_Y]$ and $[d^V_X, d^H_Y]$, we can fully rely on the corresponding calculations in [10]. Indeed, the results in [10] were essentially obtained from the Jacobi identity and relations like (59), (60) and (61), which are formally identical to the corresponding ones in [10]. The only difference which may and will occur is that the tensor field which determines the a_*-part in each result is expected to pick up extra terms in coordinates. Leaving this apart, the formulas for $[d^H_X, d^H_Y]$ and $[d^V_X, d^H_Y]$ are the same as (15) and (18) in [10]. In the process of extension by duality from vector fields to forms, derivations of type a_* create a corresponding derivation of type i_*. As in [10], we denote the resulting self-dual derivation by μ_A, where

$$\mu_A = a_A - i_A, \qquad A \in V^1(\pi). \tag{68}$$

This way we arrive at the following important commutator relations:

$$[\mathcal{D}^H_X, \mathcal{D}^H_Y] = \mathcal{D}^H_{[X,Y]_H} + \mathcal{D}^V_{R(X,Y)} + \mu_{\mathrm{Rie}(X,Y)}, \tag{69}$$

$$[\mathcal{D}^V_X, \mathcal{D}^H_Y] = \mathcal{D}^H_{\mathcal{D}^V_X Y} - \mathcal{D}^V_{\mathcal{D}^H_Y X} + \mu_{\theta(X,Y)}. \tag{70}$$

The last one can be interpreted as defining the tensor field θ, which is a type $(0,2)$ tensor field along π, taking values in $V^1(\pi)$. The coordinate expression of θ is found to be:

$$\theta = \left\{\Gamma^k_{ji\ell}\,\theta^i \otimes \theta^j + (\Gamma^k_{0i\ell} + v^j\Gamma^k_{ji\ell})\theta^i \otimes dt\right\} \otimes \left(\theta^\ell \otimes \frac{\partial}{\partial q^k}\right). \tag{71}$$

Although, in comparison with the autonomous case, there is indeed an extra term in the expression for θ, it has no effect on the following interesting property which follows from exactly the same calculation as in [10]:

$$\theta(X,Y) - \theta(Y,X) = -\mathcal{D}^V T(X,Y). \tag{72}$$

This shows that θ is symmetric for a torsionless connection, i.e. a connection generated by a SODE (see later).

6. Lifts and prolongations

The introduction of a connection has provided us with a horizontal lift operator from $\mathcal{X}(\pi)$ to $\mathcal{X}(\mathbb{R} \times TM)$, which together with the natural vertical lift gives rise to a decomposition of vector fields on $\mathbb{R} \times TM$ in two parts. Specifically, every $Z \in \mathcal{X}(\mathbb{R} \times TM)$ can be written in the form

$$Z = X^H + \overline{Y}^V, \qquad X \in \mathcal{X}(\pi),\ \overline{Y} \in \overline{\mathcal{X}}(\pi). \tag{73}$$

Indeed, if Z is given, X is pointwise defined by $X = \pi_* Z$ and the vertical vector field $Z - X^H$ then uniquely corresponds to a vector field $\overline{Y} \in \overline{\mathcal{X}}(\pi)$.

Vector fields on $\mathbb{R} \times TM$ of course have $2n+1$ components and, in the present context, are most appropriately expressed with respect to the local basis $\{\mathbf{T}^H, H_i, V_i = \partial/\partial v^i\}$. This may suggest that a decomposition of Z should contain three parts. It turns out, however, that the two-fold decomposition is most convenient and economical for discussing a number of general features. A corresponding three-fold decomposition can easily be obtained afterwards, if desired. For example, it suffices to consider the natural decomposition of X in the form (2), to obtain for Z itself a formula like

$$Z = \overline{X}^H + \overline{Y}^V + \langle X, dt\rangle \mathbf{T}^H. \tag{74}$$

This is of course related to the decomposition (22) of the horizontal projector.

Important for later calculations are the Lie brackets of horizontal and vertical lifts. Knowing that on functions F, we have $X^V(F) = \mathcal{D}^V_X F$ and $X^H(F) = \mathcal{D}^H_X F$, it immediately follows from (67) (69) and (70) that:

$$[X^V, Y^V] = ([\overline{X}, \overline{Y}]_V)^V, \tag{75}$$

$$[X^H, Y^V] = (\mathcal{D}^H_X Y)^V - (\mathcal{D}^V_Y X)^H, \tag{76}$$

$$[X^H, Y^H] = ([X, Y]_H)^H + (R(X, Y))^V. \tag{77}$$

A dual basis for expressing 1-forms on $\mathbb{R} \times TM$ is given by $\{dt, \theta^i, \eta^i\}$, where

$$\eta^i = dv^i + \Gamma^i_k\, dq^k + \Gamma^i_0\, dt.$$

There are corresponding lift operators for 1-forms along π. For $\alpha \in \bigwedge^1(\pi)$, we define $\alpha^H, \alpha^V \in \bigwedge^1(\mathbb{R} \times TM)$ as follows, taking into account that vector fields on $\mathbb{R} \times TM$ decompose as in (73):

$$\alpha^H(X^H) = \alpha(X), \qquad \alpha^H(\overline{X}^V) = 0, \tag{78}$$

$$\alpha^V(X^H) = 0, \qquad \alpha^V(\overline{X}^V) = \alpha(\overline{X}). \tag{79}$$

It is clear that α^H corresponds to a kind of pull back operation and that, referring to the decomposition (11) of $\bigwedge(\pi)$, we have $\alpha^V = \tilde{\alpha}^V$. Every $\rho \in \bigwedge^1(\mathbb{R} \times TM)$ can uniquely be written in the form

$$\rho = \alpha^H + \tilde{\beta}^V, \tag{80}$$

where $\tilde{\beta}$ is the semi-basic form $S(\rho)$, regarded as form along π, and α likewise is the semi-basic form $\rho - \tilde{\beta}^V$. As in [9], we have the property

$$dF = (d^H F)^H + (d^V F)^V. \tag{81}$$

The construction of various lifts of type $(1,1)$ tensor fields along π follows the same pattern. For any $U \in V^1(\pi)$, we define U^H and U^V by:

$$U^H(X^H) = U(X)^H, \qquad U^H(\overline{X}^V) = U(\overline{X})^V, \tag{82}$$

$$U^V(X^H) = U(X)^V, \qquad U^V(\overline{X}^V) = 0. \tag{83}$$

If U, in coordinates, is given by

$$U = u^i_j\,\theta^j \otimes \frac{\partial}{\partial q^i} + u^i\,dt \otimes \frac{\partial}{\partial q^i} + u_k\,\theta^k \otimes \mathbf{T} + u_0\,dt \otimes \mathbf{T},$$

we have

$$U^H = u^i_j(\theta^j \otimes H_i + \eta^j \otimes V_i) + u^i\,dt \otimes H_i + u_k\,\theta^k \otimes \mathbf{T}^H + u_0\,dt \otimes \mathbf{T}^H,$$
$$U^V = u^i_j\,\theta^j \otimes V_i + u^i\,dt \otimes V_i.$$

Having in mind the way the $(2n+1)\times(2n+1)$ coefficient matrix of a $(1,1)$ tensor on $\mathbb{R}\times TM$ can be separated into four blocks, it is further useful to consider the following lifts:

$$\begin{aligned} U^{H;H}(X^H) &= U(X)^H, & U^{H;H}(\overline{X}^V) &= 0,\\ U^{H;V}(X^H) &= U(X)^V, & U^{H;V}(\overline{X}^V) &= 0,\\ U^{V;H}(X^H) &= 0, & U^{V;H}(\overline{X}^V) &= U(\overline{X})^H,\\ U^{V;V}(X^H) &= 0, & U^{V;V}(\overline{X}^V) &= U(\overline{X})^V. \end{aligned}$$

We have $U^V = U^{H;V}$ and $U^H = U^{H;H} + U^{V;V}$. Note that the blocks involving a horizontal lift can be further separated into subblocks, in accordance with the earlier discussion of a three-fold splitting of vector fields on $\mathbb{R}\times TM$ (see (74)).

All interesting tensor fields on $\mathbb{R}\times TM$ come from simple tensor fields along π. For example, we have

$$S = I^V, \quad I_{\mathbb{R}\times TM} = I^H, \quad P_H = I^{H;H}, \quad P_V = I^{V;V}.$$

Also, putting

$$J = I^{H;V} - I^{V;H} = \theta^i \otimes V_i - \eta^i \otimes H_i, \tag{84}$$

we obtain a tensor field which is close to an almost complex structure, because

$$J^2 = -I_{\mathbb{R}\times TM} + dt \otimes \mathbf{T}^H. \tag{85}$$

Important constructions for the study of Lagrangian systems are the *Sasaki* and *Kähler lift* of a symmetric type $(0,2)$ tensor field g along π. It suffices to restrict the introduction of these concepts to symmetric g which have a coordinate expression of the form

$$g = g_{ij}\theta^i \otimes \theta^j.$$

In view of the symmetry, such tensor fields can be characterized by the requirement $\mathbf{T}\,\lrcorner\, g = g(\mathbf{T},\cdot) = 0$.

Definition 6.1. The Sasaki lift g^S of g is the symmetric type $(0,2)$ tensor field on $\mathbb{R}\times TM$, determined by

$$\begin{aligned} &g^S(X^H, Y^H) = g^S(X^V, Y^V) = g(X,Y),\\ &g^S(X^V, Y^H) = 0, \qquad\qquad \forall X,Y \in \mathcal{X}(\pi). \end{aligned}$$

Definition 6.2. The Kähler lift g^K of g is the 2-form on $\mathbb{R}\times TM$, determined by

$$g^K(X^H,Y^H)=g^K(X^V,Y^V)=0,$$
$$g^K(X^V,Y^H)=g(X,Y)=-g^K(X^H,Y^V).$$

In view of the form of g, it is clear that the arguments in both definitions can in fact be restricted to lifts of elements of $\overline{\mathcal{X}}(\pi)$. We have the properties

$$\begin{aligned} &g^S(JZ_1,JZ_2)=g^S(Z_1,Z_2),\\ &g^K(Z_1,Z_2)=g^S(Z_1,JZ_2), \qquad\qquad \forall Z_1,Z_2\in\mathcal{X}(\mathbb{R}\times TM),\\ &g^K(JZ_1,JZ_2)=g^K(Z_1,Z_2).\end{aligned}$$

In coordinates,

$$g^S=g_{ij}(\theta^i\otimes\theta^j+\eta^i\otimes\eta^j),\qquad g^K=g_{ij}\eta^j\wedge\theta^i.$$

We finally pay some attention to prolongations. The concept of prolongation of a vector field or "generalized vector field" is rather well known (see for example [16] or [12]). If X is an element of $\mathcal{X}(\pi)$, its prolongation X^1 is a vector field along the projection $\pi_{2,1}:J^2(\mathbb{R},M)\to J^1(\mathbb{R},M)$. We give a definition of X^1 here, which is perhaps not the most purely geometrical one, but turns out to be handy for obtaining certain properties and introducing prolongations of other objects.

Recall first the standard definition of prolongation of a basic function $f\in C^\infty(\mathbb{R}\times M)$. For an arbitrary section σ of $\mathbb{R}\times M\to\mathbb{R}$, whose first jet $j_t^1\sigma$ is a representation of a generic point in $\mathbb{R}\times TM$, one defines

$$f^1(j_t^1\sigma)=\frac{d}{ds}(f\circ\sigma)\bigg|_{s=t}.$$

Similarly, for $F\in C^\infty(\mathbb{R}\times TM)$, $F^1\in C^\infty(\mathbb{R}\times T^2M)$ is defined by

$$F^1(j_t^2\sigma)=\frac{d}{ds}(F\circ j^1\sigma)\bigg|_{s=t}.$$

In coordinates (t,q,v) on $\mathbb{R}\times TM$ and (t,q,v,a) on $\mathbb{R}\times T^2M$, we have

$$f^1=\frac{\partial f}{\partial t}+v^i\frac{\partial f}{\partial q^i},\qquad F^1=\frac{\partial F}{\partial t}+v^i\frac{\partial F}{\partial q^i}+a^i\frac{\partial F}{\partial v^i}.$$

We have the property: $\forall F,G\in C^\infty(\mathbb{R}\times TM)$,

$$(FG)^1=(\pi_{2,1}^*F)G^1+F^1(\pi_{2,1}^*G).\tag{86}$$

For $X\in\mathcal{X}(\pi)$, in a way similar to the construction in [9], $X^1\in\mathcal{X}(\pi_{2,1})$ can be determined by the requirement: $\forall f\in C^\infty(\mathbb{R}\times M)$,

$$X^1(f^1)=X(f)^1-\langle X,dt\rangle^1(\pi_{2,1}^*f^1).\tag{87}$$

Writing X for example in the form $X = X^0\mathbf{T} + \overline{X}^i(\partial/\partial q^i)$, we have

$$X^1 = X^0\mathbf{T}^1 + \overline{X}^i\frac{\partial}{\partial q^i} + \dot{\overline{X}}^i\frac{\partial}{\partial v^i},$$

where $\dot{\overline{X}}^i = \mathbf{T}^1(\overline{X}^i)$.

The action of the tensor field S (see (4)) extends to $\mathcal{X}(\pi_{2,1})$ by the pointwise construction: $S(Y)(a) = S_{\pi_{2,1}(a)}(Y(a))$. One can then prove the following interesting property (using (87)):

$$(FX)^1 = (\pi_{2,1}^* F)X^1 + F^1 S(X^1). \tag{88}$$

The prolongation of differential forms amounts essentially to a kind of "total time derivative" operation. We limit ourselves to 1-forms $\alpha \in \bigwedge^1(\pi)$, for which the prolongation α^1 can be introduced, thanks to the definition (87), by duality as follows: $\forall X \in \mathcal{X}(\pi)$,

$$\alpha^1(X^1) = \alpha(X)^1 - \langle X, dt\rangle^1(\pi_{2,1}^*\hat{\alpha}). \tag{89}$$

If α is written in the form (cf. (11)) $\alpha = \hat{\alpha}\, dt + \alpha_i\theta^i$, we have

$$\alpha^1 = \dot{\hat{\alpha}}\, dt + \dot{\alpha}_i\theta^i + \alpha_i\dot{\theta}^i, \qquad \dot{\theta}^i = dv^i - a^i\, dt.$$

Similar definitions can be given for general scalar or vector-valued forms along π. In the latter situation, one needs in addition the notion of vertical lift: for $L \in V^\ell(\pi)$, $L^V \in V^\ell(\mathbb{R} \times TM)$ is defined by,

$$L^V(Z_1, \ldots, Z_\ell) = L(\pi_* Z_1, \ldots, \pi_* Z_\ell)^V.$$

To fix the idea, the prolongation of a type $(1,1)$ tensor field U now can be defined as follows:

$$U^1(X^1) = U(X)^1 - \langle X, dt\rangle^1\, U(\mathbf{T})^V \circ \pi_{2,1}, \tag{90}$$

and for a U of the form $\alpha \otimes X$, we have the property

$$(\alpha \otimes X)^1 = \pi_{2,1}^*\alpha \otimes X^1 + \alpha^1 \otimes S(X^1). \tag{91}$$

Our interest in prolongations in this context comes from the fact that they give rise to another lift operation from objects along π to objects on $\mathbb{R} \times TM$, whenever we have a second-order system Γ at our disposal. Indeed, regarding a SODE as a section γ of the bundle $\pi_{2,1}$, we can define maps $I_\Gamma : \bigwedge(\pi) \to \bigwedge(\mathbb{R}\times TM)$ and $J_\Gamma : V(\pi) \to V(\mathbb{R}\times TM)$ by:

$$I_\Gamma : \omega \mapsto \omega^1 \circ \gamma, \qquad J_\Gamma : L \mapsto L^1 \circ \gamma,$$

except for the functions $F \in C^\infty(\mathbb{R}\times TM)$, for which I_Γ is taken to be the identity map. From the defining relations (87),(89),(90) and property (88), we immediately obtain the following properties:

$$J_\Gamma(FX) = FJ_\Gamma X + \Gamma(F)X^V, \tag{92}$$

$$\langle J_\Gamma X, I_\Gamma\alpha\rangle = \mathcal{L}_\Gamma\langle X, \alpha\rangle - \hat{\alpha}\mathcal{L}_\Gamma\langle X, dt\rangle, \tag{93}$$

$$J_\Gamma U(J_\Gamma X) = J_\Gamma(U(X)) - \mathcal{L}_\Gamma\langle X, dt\rangle U(\mathbf{T})^V. \tag{94}$$

It can be proved as in [9], or from a coordinate calculation, that the image sets $\mathcal{X}_\Gamma = J_\Gamma(\mathcal{X}(\pi))$ and $\mathcal{X}_\Gamma^* = I_\Gamma(\bigwedge^1(\pi))$ are exactly the sets introduced in [15] for describing symmetries and adjoint symmetries of a SODE. In other words, they are characterized by,

$$\mathcal{X}_\Gamma = \{Z \in \mathcal{X}(\mathbb{R} \times TM) \mid S([\Gamma, Z]) = 0\},$$

$$\mathcal{X}_\Gamma^* = \{\alpha \in \textstyle\bigwedge^1(\mathbb{R} \times TM) \mid \mathcal{L}_\Gamma(S(\alpha)) = \alpha - \langle \Gamma, \alpha \rangle\, dt\}.$$

Having at last brought a SODE Γ into the picture, we are now ready to discuss the important extra features of the case where the connection is coming from a SODE.

7. The case of a connection associated to second-order dynamics

Let us present in some detail how the connection associated to a SODE Γ can be conceived. In the spirit of the general concept of connection, referred to at the beginning of Section 3, we are looking for a map $\xi : \pi^*(T(\mathbb{R} \times M)) \to T(\mathbb{R} \times TM)$, linear in its vector argument, such that $j \circ \xi$ is the identity map. Taking an arbitrary $z \in \pi^*(T(\mathbb{R} \times M))$, formally represented in the form $(t, q, v, w_{(t,q)})$ with $w_{(t,q)} \in T_{(t,q)}(\mathbb{R} \times M)$, we can choose any basic vector field $X \in \mathcal{X}(\mathbb{R} \times M)$ with the property $X(t,q) = w_{(t,q)}$, and define $\xi(z)$ to be the value at (t, q, v) of the following vector field:

$$X^H = \tfrac{1}{2}\left(X^1 + [X^V, \Gamma] + \langle X, dt \rangle \Gamma\right). \tag{95}$$

It is straightforward to verify, for example by a coordinate calculation, that this construction matches all requirements. The horizontal lift is subsequently extended to vector fields along π by imposing linearity over $C^\infty(\mathbb{R} \times TM)$. We can in fact arrive at an explicit formula for this extended definition as follows. If X is a basic vector field and F a function on $\mathbb{R} \times TM$, FX^1 is a vector field on $\mathbb{R} \times TM$ while $(FX)^1$ is a vector field along $\pi_{2,1}$. It follows from the property (88) that we can write

$$FX^1 = (FX)^1 \circ \gamma - (F^1 \circ \gamma) X^V,$$

where γ is any section of $\pi_{2,1}$. Choosing, in particular, γ to be the section associated to Γ, we can use this in the right-hand side of (95) to write down a formula for $(FX)^H$, which then automatically applies to all $Y \in \mathcal{X}(\pi)$ and reads as follows

$$Y^H = \tfrac{1}{2}\left(J_\Gamma Y + [Y^V, \Gamma] + \langle Y, dt \rangle \Gamma\right). \tag{96}$$

It is clear from the definition of J_Γ that $J_\Gamma \mathbf{T} = \Gamma$. Since $\mathbf{T}^V = 0$, we obtain the interesting conclusion that Γ is horizontal:

$$\mathbf{T}^H = \Gamma. \tag{97}$$

Recall that the tensor field $\mathcal{L}_\Gamma S$ on $\mathbb{R} \times TM$ has the following properties (see e.g. [3]): $\mathcal{L}_\Gamma S(Y^V) = Y^V$, $\mathcal{L}_\Gamma S(H_i) = -H_i$, $\mathcal{L}_\Gamma S(\Gamma) = 0$. Using the by now familiar representation of a general $Y \in \mathcal{X}(\pi)$ in the form $Y = Y^0 \mathbf{T} + \bar{Y}^i (\partial/\partial q^i)$, it is then trivial to verify that the projections P_H and P_V are given by

$$P_H = \tfrac{1}{2}(I_{\mathbb{R} \times TM} - \mathcal{L}_\Gamma S + dt \otimes \Gamma), \tag{98}$$

$$P_V = \tfrac{1}{2}(I_{\mathbb{R} \times TM} + \mathcal{L}_\Gamma S - dt \otimes \Gamma). \tag{99}$$

Denoting the right-hand sides of the given second-order equations by $f^i(t,q,v)$, the connection coefficients are found to be

$$\Gamma_i^j = -\frac{1}{2}\frac{\partial f^i}{\partial v^j}, \qquad \Gamma_0^i = -f^i - v^k\Gamma_k^i. \tag{100}$$

Before proceeding, it is worth repeating the difference between "weak" and "strong" horizontal lifts referred to before, because the distinction between the two, in the present context, is merely a matter of a different sign in one of the terms. Indeed, the projection $P_{\overline{H}}$ in (22) is now defined by

$$P_{\overline{H}} = \tfrac{1}{2}(I_{\mathbb{R}\times TM} - \mathcal{L}_\Gamma S - dt \otimes \Gamma)$$

and correspondingly, the strong horizontal lift of a vector field along π could be defined as:

$$X^{\overline{H}} = \tfrac{1}{2}\left(J_\Gamma X + [X^V, \Gamma] - \langle X, dt\rangle \Gamma\right) = \overline{X}^H.$$

Again, whenever there is need to, it is a simple matter to pass to a three-fold decomposition of vector fields as in (74), but most of the subsequent results in this section bear great ressemblance to the time-independent framework when we stick to the two-fold decomposition, with the horizontal lift (96).

Proposition 7.1. *The connection associated to* Γ *is torsion-free. Conversely, every connection with zero torsion comes from a* SODE.

Proof. With connection coefficients of the form (100), it is a matter of direct verification that the components of the torsion T (see (44)) are identically zero. Conversely, assuming $T = 0$, we have $\Gamma^i_{kj} - \Gamma^i_{jk} = 0$, which implies that $\Gamma^i_k = -\frac{1}{2}(\partial g^i/\partial v^k)$ for some functions g^i. The second term in (44) subsequently tells us that $v^j\Gamma^i_j + g^i + \Gamma^i_0 = -h^i(t,q)$ for some basic functions h^i. Setting $f^i = g^i + h^i$, we get expressions of the form (100) for the connection coefficients, which means that the connection is indeed generated by the SODE corresponding to the functions f^i. □

With the aid of Γ, we have at least two operations at our disposal for constructing tensor fields on $\mathbb{R}\times TM$: the first one is the lift J_Γ (or I_Γ), the second one is the process of Lie derivation with respect to Γ of previously obtained objects. As in [10], we will discover new important concepts for the calculus along π by looking at the decomposition into horizontal and vertical parts of the results of these operations.

From the construction of $J_\Gamma X$, it is obvious that $\pi_*(J_\Gamma X) = X$, which means that X^H is the horizontal part of $J_\Gamma X$. Its vertical part relates to an element of $\overline{\mathcal{X}}(\pi)$ or can uniquely be associated to an element of $\mathcal{X}(\pi)$, if we add a prescription for fixing the component along $\mathbf{T}$. The resulting new element of $\mathcal{X}(\pi)$ originates in any event from some operation on the original X; we denote it by ∇X.

Definition 7.2. For $X \in \mathcal{X}(\pi)$, $\nabla X \in \mathcal{X}(\pi)$ is uniquely defined by

$$J_\Gamma X = X^H + (\nabla X)^V, \qquad i_{\nabla X}dt = \Gamma(i_X dt). \tag{101}$$

Note the similarity here with the way we fixed $d^H X$ by (26),(27).

For the particular case of $\mathbf{T}$, it follows from (97) and $\Gamma(i_{\mathbf{T}}dt) = 0$ that

$$\nabla \mathbf{T} = 0. \tag{102}$$

Consider similarly the decomposition of a 1-form $I_\Gamma \alpha$ according to (80). Looking at the coordinate expression of α^1, it is clear that the vertical part of $I_\Gamma \alpha$ can be written as α^V. The horizontal part then uniquely determines another 1-form along π.

Definition 7.3. For $\alpha \in \bigwedge^1(\pi)$, $\nabla\alpha \in \bigwedge^1(\pi)$ is defined by

$$I_\Gamma \alpha = (\nabla\alpha)^H + \alpha^V. \tag{103}$$

Proposition 7.4. *Defining the action of ∇ on functions to be $\nabla F = \Gamma(F)$, ∇ is a self-dual derivation of degree 0 of $V(\pi)$, which we call the dynamical covariant derivative associated to Γ.*

Proof. From the defining relations (101), it is easy to show with the aid of (92) that $\nabla(FX) = F(\nabla X) + \Gamma(F)X$. Hence, putting $\nabla F = \Gamma(F)$, the operator ∇ becomes a degree zero derivation of $\mathcal{X}(\pi)$. Next, using the property (93) on the one hand and the decompositions of $J_\Gamma X$, $I_\Gamma \alpha$ in conjunction with the definitions (78), (79) on the other hand, we obtain the following two expressions,

$$\langle J_\Gamma X, I_\Gamma \alpha \rangle = \mathcal{L}_\Gamma \langle X, \alpha \rangle - \hat{\alpha}\, \mathcal{L}_\Gamma \langle X, dt \rangle = \langle X, \nabla\alpha \rangle + \langle \nabla X - \langle \nabla X, dt \rangle \mathbf{T}, \alpha \rangle,$$

which show that

$$\langle X, \nabla\alpha \rangle + \langle \nabla X, \alpha \rangle = \nabla(\langle X, \alpha \rangle).$$

In agreement with (31), we conclude that ∇ is self-dual. □

For practical calculations, we have to know, apart from (102) that

$$\nabla\theta^i = -\Gamma^i_j \theta^j, \qquad \nabla\, dt = 0, \qquad \nabla \frac{\partial}{\partial q^i} = \Gamma^k_i \frac{\partial}{\partial q^k}. \tag{104}$$

It is further important to remember that the action of ∇, as is true for every self-dual derivation, extends to tensor fields along π of any type. For example, if $U \in V^1(\pi)$ is of the form $U = u^i_j\, \theta^j \otimes (\partial/\partial q^i)$, we have

$$\nabla U = (\Gamma(u^i_j) + \Gamma^i_k u^k_j - u^i_k \Gamma^k_j)\, \theta^j \otimes \frac{\partial}{\partial q^i}.$$

It is of interest to have a look also at the decomposition of $J_\Gamma U$ for a general $U \in V^1(\pi)$. Since a local basis of vector fields on $\mathbb{R} \times TM$ can be constructed (away from the zero section over $\mathbb{R} \times M$) out of elements of the set $\mathcal{X}_\Gamma$, it is sufficient to evaluate $J_\Gamma U$ on

$J_\Gamma X$. Using first (94) and (101) and subsequently (82) and (83), we obtain

$$\begin{aligned} J_\Gamma U(J_\Gamma X) &= U(X)^H + (\nabla U(X) + U(\nabla X))^V - \mathcal{L}_\Gamma\langle X, dt\rangle\, U(\mathbf{T})^V \\ &= U^H(X^H) + (\nabla U)^V(X^H) + U(\overline{\nabla X})^V \\ &= U^H(X^H + (\nabla X)^V) + (\nabla U)^V(X^H) \\ &= (U^H + (\nabla U)^V)(J_\Gamma X), \end{aligned}$$

so that

$$J_\Gamma U = U^H + (\nabla U)^V. \tag{105}$$

We now turn to the decomposition of $\mathcal{L}_\Gamma$-derivatives.

Proposition 7.5. *For all $X \in \mathcal{X}(\pi)$, we have:*

1. $$\mathcal{L}_\Gamma X^V = -\overline{X}^H + (\nabla X)^V \tag{106}$$

2. *There exists a tensor field $\Phi \in V^1(\pi)$, which is determined by*

$$\mathcal{L}_\Gamma X^H = (\nabla X)^H + \Phi(X)^V, \qquad i_\Phi\, dt = 0. \tag{107}$$

Proof. We know that $J_\Gamma X$, as an element of $\mathcal{X}_\Gamma$, is characterized by the condition $S(\mathcal{L}_\Gamma(J_\Gamma X)) = 0$, or equivalently $\mathcal{L}_\Gamma(S(J_\Gamma X)) = \mathcal{L}_\Gamma S(J_\Gamma X)$. Applying $\mathcal{L}_\Gamma S$ to this relation and using the well-known property

$$(\mathcal{L}_\Gamma S)^2 = I_{\mathbb{R}\times TM} - dt \otimes \Gamma \tag{108}$$

together with the decomposition of $J_\Gamma X$, we obtain

$$\mathcal{L}_\Gamma S(\mathcal{L}_\Gamma X^V) = X^H + (\nabla X)^V - \langle X, dt\rangle\Gamma = \overline{X}^H + (\nabla X)^V.$$

Applying $\mathcal{L}_\Gamma S$ again, knowing that $\mathcal{L}_\Gamma S(H_i) = -H_i$, $\mathcal{L}_\Gamma S(Y^V) = Y^V$ for any Y and $\langle \mathcal{L}_\Gamma X^V, dt\rangle = 0$, the first property follows. Observe next that $\mathcal{L}_\Gamma X^V = \mathcal{L}_\Gamma(S(X^H)) = \mathcal{L}_\Gamma S(X^H) + S(\mathcal{L}_\Gamma X^H) = -\overline{X}^H + S(\mathcal{L}_\Gamma X^H)$. Comparing with (106), it follows that $S(\mathcal{L}_\Gamma X^H) = S(\nabla X^H)$. Moreover, $\langle \mathcal{L}_\Gamma X^H, dt\rangle = \Gamma\langle X, dt\rangle = \langle \nabla X^H, dt\rangle$ from the second part of (101). We thus see that $(\nabla X)^H$ is the horizontal part of $\mathcal{L}_\Gamma X^H$. Its vertical part defines an element of $\overline{\mathcal{X}}(\pi)$, which must come from some operation Φ on X. Computing $\mathcal{L}_\Gamma(FX^H)$, it is easily seen that Φ is a $C^\infty(\mathbb{R}\times TM)$-linear map and we can ensure that it takes values in $\overline{\mathcal{X}}(\pi)$ by imposing the additional prescription $\langle \Phi(X), dt\rangle = 0$ for all X. The second statement now follows. □

Definition 7.6. The tensor field $\Phi \in V^1(\pi)$, defined by (107), is called the *Jacobi endomorphism* associated to Γ.

Remark. Exactly as in [10], one can verify by duality that for any $\alpha \in \bigwedge^1(\pi)$:

$$\mathcal{L}_\Gamma \alpha^H = \alpha^V + (\nabla\alpha)^H, \qquad \mathcal{L}_\Gamma \alpha^V = (\nabla\alpha)^V - (\Phi(\alpha))^H. \tag{109}$$

In the same way, using the results of Proposition 7.5, we can obtain decompositions of the $\mathcal{L}_\Gamma$-derivative of the various lifts of a type $(1,1)$ tensor field U along π. We limit ourselves to listing the following two results:

$$\mathcal{L}_\Gamma U^H = (\nabla U)^H + [\Phi, U]^V - (U(dt) \otimes \mathbf{T})^{H;H}, \tag{110}$$

$$\mathcal{L}_\Gamma U^V = (\nabla U)^{H;V} - U^{H;H} + U^{V;V} + (U(dt) \otimes \mathbf{T})^{H;H}. \tag{111}$$

Applying (107) to the case $X = \mathbf{T}$ and using (97) and (102), one sees that $\Phi(\mathbf{T}) = 0$. To obtain a coordinate expression for the important tensor Φ, it then suffices to compute $\mathcal{L}_\Gamma(\partial/\partial q^i)^H$. The result reads,

$$\Phi = -\left(\frac{\partial f^i}{\partial q^j} + \Gamma(\Gamma^i_j) + \Gamma^i_k\Gamma^k_j\right)\theta^j \otimes \frac{\partial}{\partial q^i}. \tag{112}$$

Compared to the autonomous case (see Proposition 7.5 in [10]), the dynamical covariant derivative has a very simple decomposition now and the Jacobi endomorphism is directly a curvature component. As a matter of fact, we have

$$\nabla = \mathcal{D}^H_{\mathbf{T}}, \tag{113}$$

$$\Phi = i_{\mathbf{T}} R. \tag{114}$$

Both properties are simultaneously obtained from (107) if we recall that $\Gamma = \mathbf{T}^H$ and use (77) for computing $[\mathbf{T}^H, X^H]$. We find

$$(\nabla X)^H + \Phi(X)^V = ([\mathbf{T}, X]_H)^H + (R(\mathbf{T}, X))^V.$$

The conclusion (114) follows from the fact that both Φ and $i_{\mathbf{T}}R$ take values in $\overline{\mathcal{X}}(\pi)$. The first property is a consequence of the relation (64), taking into account that the torsion is zero and that $\mathcal{D}^H_X\mathbf{T} = 0$ (see (36)). Note in passing that as a result of the vanishing torsion and the property $d^H\mathbf{T} = 0$, (32) shows that

$$\nabla = \mathcal{D}^H_{\mathbf{T}} = \mathcal{L}^H_{\mathbf{T}} = d^H_{\mathbf{T}}. \tag{115}$$

Proposition 7.7. *The exterior derivatives of the Jacobi endomorphism are related to the curvature of the connection in the following way*

$$d^V\Phi + 2\,dt \wedge \Phi = 3R, \qquad d^H\Phi = \nabla R. \tag{116}$$

Proof. As an auxiliary property, note first that from (32) applied to the vertical exterior derivative, and knowing that $\mathcal{D}^V_{\mathbf{T}} = 0$, we find

$$d^V_{\mathbf{T}} = i_{d^V\mathbf{T}} = i_{\bar{I}} = i_I - dt \wedge i_{\mathbf{T}}. \tag{117}$$

Since the torsion is zero, the Bianchi identities (50) and (51) simplify to $d^H R = 0$ and $d^V R + dt \wedge R = 0$. We thus get, making use also of (114) and (115):

$$\begin{aligned} d^V\Phi &= d^V i_{\mathbf{T}} R = d^V_{\mathbf{T}} R - i_{\mathbf{T}} d^V R \\ &= i_I R - dt \wedge \Phi + i_{\mathbf{T}}(dt \wedge R) \\ &= 2R - dt \wedge \Phi + R - dt \wedge \Phi, \end{aligned}$$

$$d^H\Phi = d^H i_{\mathbf{T}} R = d^H_{\mathbf{T}} R = \nabla R,$$

from which the statements (116) follow. □

A decomposition such as (11) for scalar forms of course also applies to vector-valued forms. In the case of the curvature tensor we know by (114) that $\hat{R} = \Phi$, i.e. we can write R in the form

$$R = \tilde{R} + dt \wedge \Phi. \tag{118}$$

It is of interest for applications to see what the properties (116) imply for the essential part $\tilde{R}$ of the curvature. From (9) we have $d^V\Phi = d^V_{\bar{I}}\Phi + dt \wedge i_I\Phi = d^V_{\bar{I}}\Phi + dt \wedge \Phi$. Similarly, from (23) and (115) we find: $d^H\Phi = d^H_{\bar{I}}\Phi + dt \wedge \nabla\Phi = d^H_{\bar{I}}\Phi + \nabla(dt \wedge \Phi)$.

Comparison with (116) reveals that

$$d^V_{\bar{I}}\Phi = 3\tilde{R}, \qquad d^H_{\bar{I}}\Phi = \nabla\tilde{R}, \tag{119}$$

which are direct analogues of the results for the autonomous theory.

We close this section with some interesting commutators involving the dynamical covariant derivative. From the Jacobi identity applied to $i_{\mathbf{T}}$, d^H and d^V, knowing that d^H and d^V commute because we have zero torsion, it follows that

$$[d^H_{\mathbf{T}}, d^V] = -[d^V_{\mathbf{T}}, d^H] = -[i_{\bar{I}}, d^H] = -d^H_{\bar{I}}.$$

Moreover, with the decomposition (9) of d^V in mind, it is clear that $[dt \wedge i_I, \nabla] = dt \wedge [i_I, \nabla] = 0$. We thus conclude that

$$[\nabla, d^V] = [\nabla, d^V_{\bar{I}}] = -d^H_{\bar{I}}. \tag{120}$$

The commutator of ∇ and d^H is more involved and so we omit it. Of more importance for applications are the commutators of general vertical and horizontal covariant derivatives with the dynamical covariant derivative. They are easy to calculate from the general commutators (69) and (70) because ∇ is itself a horizontal covariant derivative. To simplify the resulting expressions, it will be useful to express exterior derivatives of a general $U \in V^1(\pi)$ in terms of covariant derivatives. We have,

$$\begin{aligned} d^V U(X,Y) &= i_X d^V U(Y) = ((d^V_X - d^V i_X)U)(Y) \\ &= \mathcal{D}^V_X U(Y) + (i_{d^V X} U)(Y) - \mathcal{D}^V_Y(U(X)), \end{aligned}$$

from which it follows that

$$d^V U(X,Y) = (\mathcal{D}^V_X U)(Y) - (\mathcal{D}^V_Y U)(X). \tag{121}$$

In deriving this result we have made use of (32) and the fact that there is no "vertical torsion". In the present case of a connection associated to a SODE, there is also no "horizontal torsion", so that in a similar way:

$$d^H U(X,Y) = (\mathcal{D}^H_X U)(Y) - (\mathcal{D}^H_Y U)(X). \tag{122}$$

Consider now the commutator (70) for the case that $Y = \mathbf{T}$. As a preliminary remark, it is clear from (70) that $\theta(\mathbf{T},\mathbf{T}) = 0$, which means that the tensor θ can never have terms involving $dt \otimes dt$. In the case of zero torsion, we know from (72) that θ is symmetric, which then implies that it cannot have terms involving a single dt either and thus that also $\theta(X, \mathbf{T}) = 0$. In fact it is easy to verify that the second term in the coordinate expression (71) vanishes identically when the connection coefficients are of the form (100). Making further use of the property $\mathcal{D}^V_X \mathbf{T} = \overline{X}$ (see (35)), we conclude that

$$[\mathcal{D}^V_X, \nabla] = \mathcal{D}^H_{\overline{X}} - \mathcal{D}^V_{\nabla X} = \mathcal{D}^H_X - \mathcal{D}^V_{\nabla X} - \langle X, dt\rangle \nabla. \tag{123}$$

Consider next the commutator (69) for $Y = \mathbf{T}$. For the first term, note that $[X, \mathbf{T}]_H = \mathcal{D}^H_X \mathbf{T} - \mathcal{D}^H_{\mathbf{T}} X = -\nabla X$. In the second term, we recognize that $R(X, \mathbf{T}) = -\Phi(X)$ because of (114). Concerning the third term, we make the following computation, in which we take advantage of the relations (55), (114), (121), (116) and (118): for arbitrary $Z \in \mathcal{X}(\pi)$,

$$\begin{aligned}
\mathrm{Rie}(X, \mathbf{T})Z &= -(\mathcal{D}^V_Z R)(X, \mathbf{T}) + \langle Z, dt\rangle \Phi(X) \\
&= -\mathcal{D}^V_Z(R(X, \mathbf{T})) + R(\mathcal{D}^V_Z X, \mathbf{T}) + R(X, \overline{Z}) + \langle Z, dt\rangle \Phi(X) \\
&= \mathcal{D}^V_Z(\Phi(X)) - \Phi(\mathcal{D}^V_Z X) + R(X, Z) + 2\langle Z, dt\rangle \Phi(X) \\
&= (\mathcal{D}^V_X \Phi)(Z) - d^V\Phi(X, Z) + R(X, Z) + 2\langle Z, dt\rangle \Phi(X) \\
&= (\mathcal{D}^V_X \Phi)(Z) - 2R(X, Z) + 2(dt \wedge \Phi)(X, Z) + 2\langle Z, dt\rangle \Phi(X) \\
&= (\mathcal{D}^V_X \Phi)(Z) - 2\tilde{R}(X, Z) + 2\langle Z, dt\rangle \Phi(X).
\end{aligned}$$

Collecting results, we conclude that

$$[\mathcal{D}^H_X, \nabla] = -\mathcal{D}^H_{\nabla X} - \mathcal{D}^V_{\Phi(X)} + \mu_{(\mathcal{D}^V_X \Phi - 2 i_X \tilde{R} + 2 dt \otimes \Phi(X))}. \tag{124}$$

8. Applications and comments

At this stage, the reader will not dispute that developing the "calculus along π" has led to ... a large number of formulas. So what may be the purpose of this game? The applications we will discuss in this section are in a way merely reformulations of known results in the present language. Yet, they will be sufficient to underscore the main general advantage of this new approach: it provides the most economical formulation of properties and problems, staying as closely as possible to the analytical equations which in the end will have to be tackled and yet giving them a coordinate free, geometrical meaning. Needless to say, we are convinced that this new formulation will lead to a better understanding and truly new results. As a matter of fact, a couple of quite non-trivial applications have already been worked out, but they require the space of a full-scale paper and so cannot be discussed within the scope of the present general theory.

As is well known, a dynamical symmetry of a second-order system Γ is a vector field Z on $\mathbb{R} \times TM$, satisfying $\mathcal{L}_\Gamma Z = h\,\Gamma$ for some function h. If we write Z locally in the form $Z = \tau\Gamma + \mu^i(\partial/\partial q^i) + \nu^i(\partial/\partial v^i)$, the symmetry requirement in principle gives rise to three conditions, but from a computational point of view, if one has to set up determining equations for finding dynamical symmetries, there is only one system of second-order partial differential equations that matters. To be precise, one of the three conditions will merely fix h in terms of τ, another one will tell us that ν^i must be $\Gamma(\mu^i)$, which is just the analytical content of saying that $Z = J_\Gamma X$ for some $X \in \mathcal{X}(\pi)$, and with this information the last condition will give rise to the equations

$$\Gamma^2(\mu^i) - \frac{\partial f^i}{\partial v^j}\Gamma(\mu^j) - \frac{\partial f^i}{\partial q^j}\mu^j = 0. \tag{125}$$

It is exactly this set of equations which appears in a coordinate free way in the description of symmetries in our new calculus.

Proposition 8.1. *For $X \in \mathcal{X}(\pi)$, $J_\Gamma X$ is a dynamical symmetry of Γ if and only if*

$$\nabla^2 \overline{X} + \Phi(\overline{X}) = 0. \tag{126}$$

Proof. Using the decomposition (101) and the results of Proposition 7.5, we have

$$\begin{aligned}\mathcal{L}_\Gamma(J_\Gamma X) &= (\nabla X)^H + (\Phi(X))^V - (\nabla \overline{X})^H + (\nabla^2 X)^V \\ &= (\nabla^2 \overline{X} + \Phi(\overline{X}))^V + (\Gamma\langle X, dt\rangle)\Gamma,\end{aligned}$$

from which the result immediately follows. □

Not surprisingly, one can make similar observations for the dual notion of adjoint symmetries. As discussed in [15], adjoint symmetries of Γ essentially are 1-forms on $\mathbb{R} \times TM$ of type $I_\Gamma\alpha$, which under the action of the tensor $\mathcal{L}_\Gamma S$ become invariant. In coordinates, the determining equations for adjoint symmetries are second-order partial differential equations for the leading coefficients, which are exactly the adjoints of the linear equations (125). Their coordinate free representation is given by the following result.

Proposition 8.2. *For $\alpha \in \bigwedge^1(\pi)$, $I_\Gamma\alpha$ is an adjoint symmetry of Γ if and only if (with $\alpha = \tilde{\alpha} + \hat{\alpha}\, dt$),*

$$\nabla^2\tilde{\alpha} + \Phi(\tilde{\alpha}) = 0. \tag{127}$$

Proof. We have made use before of the following properties of the tensor field $\mathcal{L}_\Gamma S$ on $\mathbb{R} \times TM$: $\mathcal{L}_\Gamma S(X^V) = X^V$, $\mathcal{L}_\Gamma S(X^H) = -\overline{X}^H = -X^H + \langle X, dt\rangle\Gamma$. Using the definitions (78) and (79) of the horizontal and vertical lifts of a 1-form, it is an easy matter to obtain the following dual properties:

$$\mathcal{L}_\Gamma S(\alpha^V) = \alpha^V = \tilde{\alpha}^V, \qquad \mathcal{L}_\Gamma S(\alpha^H) = -\tilde{\alpha}^H.$$

We now want to express that $\mathcal{L}_\Gamma S(I_\Gamma\alpha)$ must be invariant. Using the decomposition (103) of $I_\Gamma\alpha$ and the decompositions (109) of $\mathcal{L}_\Gamma$-derivatives, one arrives at:

$$\mathcal{L}_\Gamma(\mathcal{L}_\Gamma S(I_\Gamma\alpha)) = -(\nabla^2\tilde{\alpha} + \Phi(\tilde{\alpha}))^H,$$

from which the desired characterization directly follows. □

An interesting subclass of adjoint symmetries, as discussed in [15], consist of those which identify a potential Lagrangian for Γ. We rederive this result to illustrate that it can be detected directly within the present framework. Note also that despite the presence of extra time-components in many formulas, the following results look formally identical to the corresponding ones of the autonomous calculus in [10].

Proposition 8.3. *A regular $L \in C^\infty(\mathbb{R} \times TM)$ is a Lagrangian for Γ if and only if*

$$\nabla\theta_L = d^H L, \quad \text{where} \quad \theta_L = d^V L + L\,dt. \tag{128}$$

Proof. From [15], we know that Lagrangians correspond to exact 1-forms in the set $\mathcal{X}^*_\Gamma$, i.e. L is a Lagrangian, provided we have

$$\mathcal{L}_\Gamma(S(dL)) = dL - \Gamma(L)\,dt.$$

Observe first that for a general $\alpha \in \bigwedge^1(\pi)$, we have $S(\alpha^H) = 0$ and $S(\alpha^V) = \tilde{\alpha}^H$. Applied to the case of $dL = (d^V L)^V + (d^H L)^H$, knowing that $\langle \mathbf{T}, d^V L\rangle = 0$, we get $S(dL) = (d^V L)^H$. Using (109), the above criterion then easily translates to the desired result. □

The 1-form θ_L, regarded as semi-basic form on $\mathbb{R} \times TM$ is of course the familiar Poincaré–Cartan 1-form. In the next result, we transfer the denomination "adjoint symmetry" to elements of $\bigwedge^1(\pi)$ which satisfy the condition (127) (a similar convention can be adopted for symmetries).

Proposition 8.4. *If α is an adjoint symmetry of Γ, which can be written as $d^V F$ for some function F, the function $L = \Gamma(F)$ (provided it is regular) is a Lagrangian for Γ. Conversely, every Lagrangian of the form $\Gamma(F)$ determines an adjoint symmetry.*

Proof. Making use of the commutator (120), we have

$$\begin{aligned}(\nabla^2 + \Phi)(d^V F) &= \nabla(d^V \nabla F - d^H F + d^H_{dt\otimes\mathbf{T}} F) + d^V_\Phi F \\ &= \nabla(d^V \nabla F + (\nabla F)dt) - \nabla d^H F + d^V_\Phi F.\end{aligned}$$

From the Jacobi identity applied to $i_\mathbf{T}$, d^H and d^H, we have $[\nabla, d^H] = -\frac{1}{2}[[d^H, d^H], i_\mathbf{T}]$. Restricting this property to the action on functions F, it follows from (52) and (56) that

$$[\nabla, d^H](F) = -[d^V_R, i_\mathbf{T}](F) = i_\mathbf{T} i_R d^V F = i_{i_\mathbf{T}R} d^V F = d^V_\Phi F.$$

As a result, we get

$$(\nabla^2 + \Phi)(d^V F) = \nabla(d^V \nabla F + (\nabla F)dt) - d^H \nabla F,$$

and the conclusion follows from the two previous propositions. □

We conclude with the most economical formulation of the so-called Helmholtz conditions concerning the inverse problem of Lagrangian mechanics. A geometrical version of these conditions, for time-dependent systems, was given e.g. in [3]. It involves conditions on a 2-form on the $(2n+1)$-dimensional space $\mathbb{R} \times TM$, which in the end is going to be the Cartan 2-form $d\theta_L$. When expressed in terms of the local basis $\{dt, \theta^i, \eta^i\}$ of 1-forms, however, $d\theta_L$ is actually fully determined by a symmetric $(n \times n)$-matrix g_{ij} (the Hessian of the Lagrangian). From an analytical point of view, the inverse problem concerns the search for a multiplier matrix g_{ij} which will turn the given equations $\ddot{q}^j = f^j$ into Euler-Lagrange equations. A concise analytical formulation of the Helmholtz conditions in this n-dimensional setting can be found e.g. in [13]. The theorem below gives precisely a coordinate free version of these conditions and the key to it is the fact that the Cartan 2-form is really the Kähler lift of a symmetric type $(0,2)$ tensor field along π.

Theorem 8.5. *The* SODE Γ *is (locally) Lagrangian if and only if there exists a non-degenerate symmetric type* $(0,2)$ *tensor field* g *along* π, *with the property* $\mathbf{T} \lrcorner g = 0$, *such that:* $\nabla g = 0$, $\Phi \lrcorner g$ *is symmetric,* $\mathcal{D}^V g|_{\overline{\mathcal{X}}(\pi)}$ *is symmetric.*

Proof. From the defining relations of the Kähler lift of g (see Definition 6.2) and the results of Proposition 7.5, one easily obtains the following relations:

$$\begin{aligned}
&\mathcal{L}_\Gamma g^K(X^H, Y^H) = g(X, \Phi(Y)) - g(\Phi(X), Y), \\
&\mathcal{L}_\Gamma g^K(X^V, Y^V) = 0, \\
&\mathcal{L}_\Gamma g^K(X^H, Y^V) = -\nabla g(X, Y).
\end{aligned}$$

In fact, these relations can be taken over from the autonomous case (see [10]), since they rely on formulas which look almost identical. The only difference is that we have $\overline{X}^H$ in the right-hand side of (106) and not X^H. But this difference does not matter, because $\mathbf{T} \lrcorner g = 0$. We conclude that the conditions $\nabla g = 0$ and $\Phi \lrcorner g$ symmetric are equivalent to $\mathcal{L}_\Gamma g^K = 0$. We further have $g^K(X^V, Y^V) = 0$ by definition and $i_\Gamma g^K = 0$ from $\mathbf{T} \lrcorner g = 0$. The above cited result of Crampin et al [3] thus says that there is only one more requirement to be satisfied, which here translates to:

$$i_{\overline{X}^H} dg^K(Y^V, Z^V) = 0, \qquad \forall X, Y, Z \in \mathcal{X}(\pi).$$

Since for any vector field Y, we have $Y^V = \overline{Y}^V$, all arguments in this condition come from elements of $\overline{\mathcal{X}}(\pi)$. This entails that further manipulations of it follow exactly the same pattern as in the autonomous case. We may therefore conclude that this last requirement is equivalent to $\mathcal{D}^V g|_{\overline{\mathcal{X}}(\pi)}$ being symmetric. $\square$

Since at present it is not known what future applications of this new calculus might bring, we have tried in this paper to bring together all essential ingredients of the theory and have elaborated only on those formulas which are thought to be sufficiently relevant. Very likely, a number of applications will only require a limited number of these formulas. Those involving the different kinds of covariant derivatives and the Jacobi endomorphism would seem to be the most important ones. This is at least what two extensive applications in progress are indicating. In a recent preprint [4], a true break-through has been achieved in understanding in geometrical terms how Douglas has solved the inverse problem for the case $n = 2$ [6]. This should lead also to new results for higher dimensions in the future. Incidentally, the analysis in [4] was anticipating on the results for time-dependent systems of the present paper and, in particular, starts from the above characterization of the inverse problem. This was possible because most formulas formally look identical to their counterparts for time-independent systems, when the action is restricted to vector fields in $\overline{\mathcal{X}}(\pi)$ (and forms of the type $\tilde{\omega}$ in (11)).

A second highly non-trivial application concerns the generalization to time-dependent systems of the study of second-order systems which are totally separable into individual equations. Experience has shown, in that problem, that it is better to do the analysis in the more general setup of the first jet extension J^1E of an arbitrary fibre bundle $E \to \mathbb{R}$. In such a framework, we still have contact forms θ^i and a canonical vector field $\mathbf{T}$ along $\pi : J^1E \to E$ at our disposal. Also, the complementary part for decomposing vector fields along π is well defined. Indeed, the set $\overline{\mathcal{X}}(\pi)$ then is simply defined as consisting of those vector fields along π, whose value is everywhere vertical over $\mathbb{R}$. Since none of the constructions in the present paper was relying on the product structure of $\mathbb{R} \times M$ or $\mathbb{R} \times TM$, we can claim that our results are by no means resticted to the case $E = \mathbb{R} \times M$. The separability analysis is in preparation.

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References

[1] J.F. Cariñena and E. Martínez, Generalized Jacobi equation and inverse problem in classical mechanics, in: V.V. Dodonov and V.I. Manko, eds., *Group Theoretical Methods in Physics*, Proc. 18th Int. Colloquium 1990, Moscow, USSR, Vol. II (Nova Science Publishers, New York 1991) 59–64.

[2] M. Crampin, Generalized Bianchi identities for horizontal distributions, *Math. Proc. Camb. Phil. Soc.* **94** (1983) 125–132.

[3] M. Crampin, G.E. Prince and G. Thompson, A geometrical version of the Helmholtz conditions in time-dependent Lagrangian dynamics, *J. Phys. A: Math. Gen.* **17** (1984) 1437–1447.

[4] M. Crampin, W. Sarlet, E. Martínez, G.B. Byrnes and G.E. Prince, Towards a geometrical understanding of Douglas's solution of the inverse problem of the calculus of variations, *Inverse Problems* **10** (1994) 245–260.

[5] M. de León and P.R. Rodrigues, *Methods of Differential Geometry in Analytical Mechanics*, Mathematics Studies **158** (North-Holland, Amsterdam 1989).

[6] J. Douglas, Solution of the inverse problem of the calculus of variations, *Trans. Amer. Math. Soc.* **50** (1941) 71–128.

[7] A. Frölicher and A. Nijenhuis, Theory of vector-valued differential forms, *Proc. Ned. Acad. Wetensch. Sér. A* **59** (1956) 338–359.

[8] J. Grifone, Structure presque tangente et connexions I, *Ann. Inst. Fourier* **22** (1) (1972) 287–334.

[9] E. Martínez, J.F. Cariñena and W. Sarlet, Derivations of differential forms along the tangent bundle projection, *Diff. Geom. Appl.* **2** (1992) 17–43.

[10] E. Martínez, J.F. Cariñena and W. Sarlet, Derivations of differential forms along the tangent bundle projection II, *Diff. Geom. Appl.* **3** (1993) 1–29.

[11] E. Martínez, J.F. Cariñena and W. Sarlet, Geometric characterization of separable second-order equations, *Math. Proc. Camb. Phil. Soc.* **113** (1993) 205–224.

[12] P.J. Olver, *Application of Lie Groups to Differential Equations*, Graduate Texts in Mathematics **107** (Springer-Verlag, New York, 1986).

[13] W. Sarlet, The Helmholtz conditions revisited. A new approach to the inverse problem of Lagrangian dynamics, *J. Phys. A: Math. Gen.* **15** (1982) 1503–1517.

[14] W. Sarlet, Geometrical structures related to second-order equations, in: D. Krupka and A. Švec. eds., *Differential Geometry and Its Applications*, Proc. Conf. 1986, Brno, Czechoslovakia (J.E. Purkyně University, Brno, 1987) 279–288.

[15] W. Sarlet, G.E. Prince and M. Crampin, Adjoint symmetries for time-dependent second-order equations, *J. Phys. A: Math. Gen.* **23** (1990) 1335–1347.

[16] D.J. Saunders, *The Geometry of Jet Bundles*, London Math. Soc. Lect. Notes Series **142** (Cambridge University Press, 1989).

[17] A. Vondra, Connections in the geometry of non-autonomous regular higher-order dynamics, Thesis, Masaryk University, Brno, 1991.