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An Application of the Continuous Selection Theorem to the Study of the Fixed Points of Multivalued Mappings

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Applying the continuous selection theorem given by K. Przesławski and L. Rybiński (Michael selection theorem under weak lower semicontinuity assumption, (*Proc. Amer. Math. Soc.*, in press), we state the result on continuous dependence of the fixed points of set-valued contractions and the Krasnoselski type fixed-point theorem for multivalued mappings.

1. INTRODUCTION

Let Y be a nonempty closed subset of a Banach space $(Z, \|\cdot\|)$. Let $B = \{y \in Z : \|y\| \leqslant 1\}$ denote the closed unit ball and $B^0 = \{y \in Z : \|y\| < 1\}$ denote the open unit ball in Z. Denote

$$\mathcal{N}(Y) = \{C \subset Y : C \neq \varnothing\},$$

$$\mathscr{Cl}(Y) = \{C \in \mathcal{N}(Y) : C \text{ is closed}\},$$

$$\mathscr{CCl}(Y) = \{C \in \mathscr{Cl}(Y) : C \text{ is convex}\}.$$

For $y \in Z$, $C, D \in \mathcal{N}(Y)$ set $\operatorname{dist}(y, D) = \inf\{\|y - v\| : v \in D\}$, $\delta(C, D) = \sup\{\operatorname{dist}(y, D) : y \in C\}$, $\mathscr{D}(C, D) = \max\{\delta(C, D), \delta(D, C)\}$, $y + C = \{y + v : v + C\}$, $rC = \{rv : v \in C\}$ for $r \in \mathbb{R}$. Assume that X is a topological space. Let us consider a multivalued contraction mapping with the parameter $x \in X$ and closed convex values, i.e., the mapping $H: X \times Y \to \mathscr{CCl}(Y)$ satisfying

$$\mathscr{D}(H(x, y_1), H(x, y_2)) \leqslant K \|y_1 - y_2\| \quad \text{for} \quad x \in X, \quad y_1, y_2 \in Y, \tag{C}$$

where $K < 1$. From [1] it follows that for every $x \in X$ *the fixed-point set* $P_H(x) = \{y \in Y : y \in H(x, y)\}$ is nonempty and closed. The properties of

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related *fixed-point set-valued mappings* $P_H: X \to \mathscr{C}l(Y)$ have been studied recently, in various settings, in [2, 5–8]. In [8] we have shown that if X is paracompact and perfectly normal and the functions $x \to \operatorname{dist}(v, H(x, y))$, $v, y \in Y$, are continuous, then P_H has a *retractive representation*; i.e., there exists a continuous mapping $h: X \times Y \to Y$ such that $h(x, y) \in P_H(x)$, and $h(x, y) = y$ iff $y \in P_H(x)$ for every (x, y). The basic tool in proving that relation was the Michael Selection Theorem [3]. Now, using the continuous selection theorem from [9] we give a similar result under considerably weaker continuity assumptions for the maps $H(\cdot, y)$, $y \in Y$.

Recall that a multivalued mapping $F: X \to \mathscr{N}(Y)$ is called *lower semicontinuous* (l.s.c.) at $x_0 \in X$ iff for every $\varepsilon > 0$ and $z \in F(x_0)$ there exists a neighborhood U_z of x_0 such that

$$z \in \bigcap \{F(x) + \varepsilon B^0 : x \in U_z\}.$$

We say that F is *weakly lower semicontinuous at* $x_0 \in X$ (w.l.s.c.) iff for every $\varepsilon > 0$ and every neighborhood V of x_0 there is a point $x_1 \in V$ such that for every $z \in F(x_1)$ there is a neighborhood U_z of x_0 such that

$$z \in \bigcap \{F(x) + \varepsilon B^0 : x \in U_z\}.$$

We say that F is l.s.c. (w.l.s.c.) iff F is l.s.c. (w.l.s.c.) at every $x \in X$. It is known that F is l.s.c. iff the functions $x \to \operatorname{dist}(v, F(x))$, $v \in Y$, are upper semicontinuous in the usual sense. Straightforward checking shows that if F is l.s.c., then F is w.l.s.c., but not conversely (see [9]).

2. Fixed-Point Results

Our main result reads as follows.

THEOREM 1. *Let X be a paracompact and perfectly normal topological space and Y be a closed subset of a Banach space $(Z, \|\cdot\|)$. Assume that $H: X \times Y \to \mathscr{C}\mathscr{C}l(Y)$ satisfies* (C) *and is such that for every $y \in Y$ the multivalued mapping $H(\cdot, y)$ is w.l.s.c. Then there exists a continuous mapping $h: X \times Y \to Y$ such that $h(x, y) \in P_H(x)$ for every $(x, y) \in X \times Y$.*

First we will prove two auxiliary lemmas.

LEMMA 1. *For every continuous mapping $f: X \times Y \to Y$ the mapping $(x, y) \to H(x, f(x, y))$ is w.l.s.c.*

Proof. Denote $w = (x, y) \in X \times Y$ and

$$\mathscr{H}(w, v) = H(pr_X(w), v) = H(x, v)$$

for $w \in X \times Y$, $v \in Y$. We need to show that the mapping $w \to \mathscr{H}(w, f(w))$ is w.l.s.c. Note that $\mathscr{H}(\cdot, v)$ is w.l.s.c. for every $v \in Y$, by virtue of [9, Lemma 3]. Clearly by (C), we have also

$$\mathscr{D}(\mathscr{H}(w, v_1), \mathscr{H}(w, v_2)) \leqslant K \|v_1 - v_2\|. \tag{1}$$

Fix $\varepsilon > 0$, a point $w_0 \in X \times Y$, and a neighborhood W of w_0. Choose a neighborhood $V \subset W$ of w_0 such that $\|f(w) - f(w_0)\| < 3^{-1}\varepsilon$ for each $w \in V$. In V choose a point w_1 such that for every $z \in \mathscr{H}(w_1, f(w_0))$ there exists a neighborhood U_z of w_0 such that

$$z \in \bigcap \{\mathscr{H}(w, f(w_0)) + 3^{-1}\varepsilon B^0 : w \in U_z\}. \tag{2}$$

Now let z_1 be an arbitrary point in $\mathscr{H}(w_1, f(w_1))$. By (1), there exists $z \in \mathscr{H}(w_1, f(w_0))$ such that

$$\|z - z_1\| \leqslant \sqrt{K}\, \|f(w_1) - f(w_0)\| < 3^{-1}\varepsilon.$$

For such z choose a neighborhood U_z such that (2) holds. Since

$$\mathscr{H}(w, f(w_0)) + 3^{-1}\varepsilon B^0 \subset \mathscr{H}(w, f(w)) + 3^{-1}\varepsilon B^0 + 3^{-1}\varepsilon B^0,$$

for every $w \in V$, we have

$$z \in \bigcap \{\mathscr{H}(w, f(w)) + 2 \cdot 3^{-1}\varepsilon B^0 : w \in U_z \cap V\}.$$

Then, consequently

$$z_1 \in \bigcap \{\mathscr{H}(w, f(w)) + \varepsilon B^0 : w \in U_z \cap V\}. \qquad \text{Q.E.D.}$$

From the above lemma it follows, in particular, that H is w.l.s.c. with respect to both variables jointly (set $f(x, y) = y$).

LEMMA 2. *For every $L > 1$, $M > K$, and continuous mappings $f_1, f_2 : X \times Y \to Y$ such that f_2 is a selection of the multivalued mapping $(x, y) \to H(x, f_1(x, y))$, there exists a continuous selection f_3 of the multivalued mapping $(x, y) \to H(x, f_2(x, y))$ such that*

$$\|f_3(x, y) - f_2(x, y)\| \leqslant LM \|f_2(x, y) - f_1(x, y)\|$$

and

$$\operatorname{dist}(f_3(x, y), H(x, f_3(x, y))) \leqslant M \|f_3(x, y) - f_2(x, y)\|$$

for every $(x, y) \in X \times Y$.

Proof. Since $f_2(x, y) \in H(x, f_1(x, y))$, then

$$\operatorname{dist}(f_2(x, y), H(x, f_2(x, y))) \leqslant \mathscr{D}(H(x, f_1(x, y)), H(x, f_2(x, y))) \leqslant K \|f_2(x, y) - f_1(x, y)\|$$

for $(x, y) \in X \times Y$.

Thus

$$H(x, f_2(x, y)) \cap (f_2(x, y) + M \|f_2(x, y) - f_1(x, y)\|) \neq \varnothing$$

for every (x, y). By Lemma 1 and [9, Lemma 2], the multivalued mapping $G: (x, y) \to H(x, f_2(x, y)) \cap (f_2(x, y) + LM \|f_2(x, y) - f_1(x, y)\| B)$ is w.l.s.c. Since the product $X \times Y$ is paracompact (see [4]), G has a continuous selection, say f_3, by virtue of [9, Theorem 1]. Obviously f_3 satisfies the first inequality, the second inequality follows as for f_2. Q.E.D.

Proof of Theorem 1. We will construct a sequence of continuous mappings $h_n: X \times Y \to Y$ such that

(1°) $\|h_n(x, y) - h_{n-1}(x, y)\| \leqslant LM \|h_{n-1}(x, y) - h_{n-2}(x, y)\|$, for $n = 2, 3, \ldots$,

(2°) $\operatorname{dist}(h_n(x, y), H(x, h_n(x, y))) \leqslant M \|h_n(x, y) - h_{n-1}(x, y)\|$, for $n = 1, 2, \ldots$,

for every (x, y), where $M \in (K, 1)$ and $L \in (1, M^{-1})$. If such a sequence is defined, then the rest of the proof is routine. Indeed, we have

$$\|h_n(x, y) - h_{n-1}(x, y)\| \leqslant (LM)^{n-1} \|h_1(x, y) - h_0(x, y)\|.$$

Therefore $(h_n(x, y))$ is a Cauchy sequence for every (x, y), hence convergent. Set $h(x, y) = \lim h_n(x, y)$. It is easily seen that

$$\|h(x, y) - h_n(x, y)\| \leqslant \sum_{m=n}^{\infty} (LM)^m \|h_1(x, y) - h_0(x, y)\|.$$

Since the sets $H(x, y)$ are closed and for arbitrary large n we have

$$\begin{aligned} &\operatorname{dist}(h(x, y), H(x, h(x, y))) \\ &\quad \leqslant \|h(x, y) - h_n(x, y)\| + \operatorname{dist}(h_n(x, y), H(x, h(x, y))) \\ &\quad \leqslant (1 + K) \|h(x, y) - h_n(x, y)\| + M \|h_n(x, y) - h_{n-1}(x, y)\|, \end{aligned}$$

then $h(x, y) \in H(x, h(x, y))$, i.e., $h(x, y) \in P_H(x)$, for every $(x, y) \in X \times Y$. Since for arbitrary large n the mapping h_n is continuous and the function

$(x, y) \to \|h_1(x, y) - h_0(x, y)\|$ is continuous, then taking into account the inequality

$$\begin{aligned}\|h(x, y) - h(x_0, y_0)\| &\leqslant \|h(x, y) - h_n(x, y)\| + \|h_n(x, y) - h_n(x_0, y_0)\| \\ &\quad + \|h_n(x_0, y_0) - h(x_0, y_0)\| \\ &\leqslant \sum_{m=n}^{\infty} (LM)^m \|h_1(x, y) - h_0(x, y)\| \\ &\quad + \|h_n(x, y) - h_n(x_0, y_0)\| \\ &\quad + \sum_{m=n}^{\infty} (LM)^m \|h_1(x_0, y_0) - h_0(x_0, y_0)\|,\end{aligned}$$

we conclude that h is continuous at every $(x_0, y_0) \in X \times Y$. The sequence (h_n) is constructed by induction. Set $h_0(x, y) = y$, and applying [9, Theorem 1], choose a continuous selection h_1 of the multivalued mapping H. Assume that the mappings $h_1, ..., h_m$ satisfying (1°), (2°) are defined. Applying Lemma 2 with $f_1 = h_{m-1}$, $f_2 = h_m$, we get the mapping $f_3 = h_{m+1}$ satisfying (1°) and (2°). Q.E.D.

Remark. If we assume additionally that there exists a continuous mapping $m: X \times Y \to [0, \infty)$ such that $\text{dist}(y, H(x, y)) \leqslant m(x, y)$ and $\text{dist}(y, H(x, y)) = 0$ implies $m(x, y) = 0$ (this implies actually that $(x, y) \to \text{dist}(y, H(x, y))$ is continuous at every $(x, y) \in \text{Graph } P_H$), then we can take in the above proof as h_1 a continuous selection of the multivalued mapping $(x, y) \to H(x, y) \cap (y + Lm(x, y) B)$. Then for $y \in P_H(x)$ we have $h_n(x, y) = y$ for $n = 1, 2, ...$, and consequently $h(x, y) = y$. In this case h is a retractive representation for P_H.

Using Theorem 1 we get the following improvement of the Krasnoselskii type fixed-point theorem given in [7, Theorem 2].

THEOREM 2. *Let the assumptions of Theorem 1 be satisfied. Assume additionally that the set Y is convex bounded and $\Gamma: Y \to X$ is a continuous mapping such that $\Gamma(Y)$ is a relatively compact subset of X. Then there exists a point $w \in Y$ such that $w \in H(\Gamma(w), w)$.*

Proof. Choose a mapping h satisfying the assertion of Theorem 1. Fix $y \in Y$ and define the continuous mapping $g: Y \to Y$ by $g(v) = h(\Gamma(v), y)$. Since g maps the closed convex bounded subset of a Banach space into its relatively compact subset, then g has a fixed point by virtue of the Schauder Theorem. Thus there exists a point $w \in Y$ such that

$$w = g(w) = h(\Gamma(w), y) \in H(\Gamma(w), h(\Gamma(w), y)) = H(\Gamma(w), w). \quad \text{Q.E.D.}$$

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