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A class of multi-parameter eigenvalue problems for perturbed p-Laplacians

F. Güngör, M. Hasanov *

Dogus University, Department of Mathematics, 34722 Kadikoy, Istanbul, Turkey

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ABSTRACT

This paper is devoted to multi-parameter eigenvalue problems for one-dimensional perturbed p-Laplacians, modelling travelling waves for a class of nonlinear evolution PDE. Dispersion relations between the eigen-parameters, the existence of eigenfunctions and positive eigenfunctions, variational principles for eigenvalues and constructing solutions in the analytical and implicit forms are the main subject of this paper. We use both variational and analytical methods.

1. Introduction

In this paper we study a class of multi-parameter eigenvalue problems, which naturally arise when we search travelling waves for evolution p-Laplacian equations (see E. Di Benedetto and M.A. Herrero [8]):

$$\begin{aligned} &iv_t - \operatorname{div}\big(|\nabla v|^{p-2}\nabla v\big) = \lambda|v|^{p-2}v, \quad p > 1, \\ &v|_{\partial Q} = 0, \end{aligned} \tag{1.1}$$

where $v := v(t,x,y)$, $t > 0$, λ is a parameter and $Q = \{(x,y) \mid x \in (0,1),\ \ y \in \mathbb{R}\}$ is the infinite rectangle in $\mathbb{R}^2$. Now, we look for wave solutions to Eq. (1.1) in the form $v(t,x,y) = e^{i(wt-ky)}u(x)$, where $u(x)$ is a real-valued function and $(w,k) \in \mathbb{R} \times \mathbb{R}$. We have $\nabla v = (v_x, v_y) = e^{i(wt-ky)}(u', -iku)$ and $|\nabla v| = (k^2u^2 + u'^2)^{1/2}$. For convenience we introduce the notation: $\nabla_k u := (ku, u')$. Then, $|\nabla v| = |\nabla_k u| = (k^2u^2 + u'^2)^{1/2}$. By using this notation and putting $v(t,x,y) = e^{i(wt-ky)}u(x)$ into (1.1) we obtain that $v(t,x,y)$ is a solution to Eq. (1.1) if and only if $u(x)$ is an eigenfunction of the following multi-parameter perturbed eigenvalue problem which is the main concern of this paper (for similar problems in the case of $k = 0$, we refer to J.P.G. Azorero and I.P. Alonso [5]):

$$\begin{aligned} &-wu + k^2|\nabla_k u|^{p-2}u - \big(|\nabla_k u|^{p-2}u'\big)' = \lambda|u|^{p-2}u, \quad p > 1, \\ &u(0) = u(1) = 0, \end{aligned} \tag{1.2}$$

where $u := u(x)$ and $|\nabla_k u| = (k^2u^2 + u'^2)^{1/2}$. We note that in the case of $k = 0$ we obtain a standard two-parameter eigenvalue problem for one-dimensional p-Laplacians. For this reason we refer to problem (1.2) as a multi-parameter eigenvalue

* Corresponding author.
E-mail addresses: fgungor@dogus.edu.tr (F. Güngör), hasanov61@yahoo.com (M. Hasanov).

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problem for perturbed p-Laplacians, which is the central topic of this paper. We suggest new analytical techniques to analyze problem (1.2). We mention that the paper [15] by G. Morosanu, D. Souroujon, S. Tersian devoted to a problem in a different context, also includes some similar techniques to those we use in Section 4.

We now give the definition of a solution to Eq. (1.2).

Definition 1.1. $0 \neq u \in W_0^{1,p}(0,1)$ is a solution to Eq. (1.2) if and only if

$$-w\int_0^1 uv\,dx + k^2\int_0^1 |\nabla_k u|^{p-2}uv\,dx + \int_0^1 |\nabla_k u|^{p-2}u'v'\,dx = \lambda\int_0^1 |u|^{p-2}uv\,dx \tag{1.3}$$

holds for all $v \in W_0^{1,p}(0,1)$, where $W_0^{1,p}(0,1)$ is the Sobolev space (see [1] for Sobolev spaces). Here, the parameters (w,k,λ) are called eigen-parameters and the associated non-trivial function $u \in W_0^{1,p}(0,1)$ is called the eigenfunction. First, we observe that searching for $0 \neq u \in W_0^{1,p}(0,1)$ satisfying (1.3) is equivalent to finding critical points of the functional

$$F(u) = -\frac{w}{2}\int_0^1 u^2\,dx + \frac{1}{p}\int_0^1 |\nabla_k u|^p\,dx - \frac{\lambda}{p}\int_0^1 |u|^p\,dx.$$

For this we just note that by using the definition of Gateaux derivative, we obtain

$$\begin{aligned}\frac{d}{dt}\left[\frac{1}{p}\int_0^1 \left|\nabla_k(u+tv)\right|^p dx\right]\Bigg|_{t=0} &= \frac{1}{p}\int_0^1 \frac{p}{2}\left(k^2u^2+u'^2\right)^{\frac{p-2}{2}}\left(2k^2uv+2u'v'\right)dx\\ &= k^2\int_0^1 |\nabla_k u|^{p-2}uv\,dx + \int_0^1 |\nabla_k u|^{p-2}u'v'\,dx.\end{aligned}$$

In what follows, we denote $X := W_0^{1,p}(0,1)$, which is a Banach space. $F' : X \to X^*$ will denote the Fréchet derivative of F, where X^* is the dual of the space X. Hence, $u \in X$ is a solution to (1.3) if and only if u is a free critical point for $F(u)$, i.e., $\langle F'(u), v\rangle = 0$, for all $v \in X$, where $\langle F'(u), v\rangle$ denotes the value of the functional $F'(u)$ at $v \in X$. Moreover, by Sobolev's embedding theorem $W_0^{1,p}(0,1)$ is compactly embedded in $C[0,1]$ and consequently, $W_0^{1,p}(0,1)$ is compactly embedded in $L_q(0,1)$ for all $q \in [1,\infty)$ (see [1]). Therefore, the functional $F(u)$ is well defined for all $u \in W_0^{1,p}(0,1)$.

The rest of this paper will be organized as follows. In Section 2 we present some general results about the structure of the eigen-parameters (w,k,λ), including the $\lambda(k)$ dependence, the existence of positive eigenfunctions and the localization of eigen-parameters (w,k,λ). In this section we use mainly variational techniques. Section 3 is devoted to the problems with cubic nonlinearities. In this case we construct analytical solutions and relying on these solutions we study dispersion relations $\lambda(k)$ and $k(\lambda)$ (Theorem 3.1). Finally, in Section 4 we study the same problems in the most general case. Here, our approach is based on constructing implicit solutions which allow us to analyze the spectral problems and establish dispersion relations in the most general case (Theorem 4.2). In Appendix A we present full details of the reducing the order of the main spectral problem.

2. The structure of the eigen-parameters (w,k,λ): general results

In this section we present some general results, which are obtained by using variational techniques. We study separately two cases for the eigenvalue problem (1.3): $w = 0$ and $w \neq 0$.

The case: $w = 0$.

In this case problem (1.3) reduces to the form:

$$k^2\int_0^1 |\nabla_k u|^{p-2}uv\,dx + \int_0^1 |\nabla_k u|^{p-2}u'v'\,dx = \lambda\int_0^1 |u|^{p-2}uv\,dx, \tag{2.1}$$

where $u := u(x)$ and $|\nabla_k u| = (k^2u^2 + u'^2)^{1/2}$.

In the case of $k = 0$ problem (2.1) coincides with the typical one-dimensional eigenvalue problem for p-Laplacians:

$$\begin{aligned}&-\left(\left|u'\right|^{p-2}u'\right)' = \lambda|u|^{p-2}u, \quad p > 1,\\ &u(0) = u(1) = 0.\end{aligned} \tag{2.2}$$

The eigenvalue problems for p-Laplacians have been studied by many authors (see [3–5,13,14] and references therein). An alternative and interesting method for studying the existence of eigenfunctions of the one-dimensional p-Laplacian was suggested in [2]. However, when $k \neq 0$ then we deal with (2.1) which is different from the typical one-dimensional p-Laplacian eigenvalue problem (2.2).

Our first observation is given in Theorem 2.1. Since many facts in this theorem are proved in the same way as those in the classical p-Laplacian eigenvalue problems we present only a sketch of the proof.

Theorem 2.1. *Let us fix $k \in \mathbb{R}$. Then,*

(a) *there exists an infinite sequence of variational eigenvalues $\lambda_n(k)$ for problem* (2.1)*, arranged as*

$$0 < \lambda_1(k) < \lambda_2(k) \leqslant \cdots \leqslant \lambda_n(k) \leqslant \cdots \quad \text{and} \quad \lambda_n(k) \to +\infty \quad \text{as } n \to \infty,$$

(b) $0 < \lambda_1(\Delta_p) \leqslant \lambda_1(k)$ *and* $|k|^p \leqslant \lambda_1(k)$*,*

where $\lambda_1(\Delta_p)$ denotes the first eigenvalue of problem (2.2)*.*

A sketch of the proof. (a) The proof is based on the Ljusternik–Schnirelman critical point theory (see [19, Chapter 44]). Let us define $G_k(u) := \frac{1}{p}\int_0^1 |\nabla_k u|^p\,dx$ and $\Phi(u) := \frac{1}{p}\int_0^1 |u|^p\,dx$. Consider the following eigenvalue problem:

$$\Phi'(u) = \mu G_k'(u), \quad u \in S_{G_k},\ \mu \in \mathbb{R}, \tag{2.3}$$

where $S_{G_k} = \{u \in X \mid G_k(u) = 1\}$. Clearly, Eq. (2.3) is the same as Eq. (2.1) with $\mu = \frac{1}{\lambda}$. It is well known that to find a sequence μ_n of the eigenvalues of problem (2.3), it is sufficient to check the following basic conditions (see [19, p. 325 and p. 328]):

$H1$. Let X be a reflexive Banach space. F and G are the even functionals such that $F, G \in C^1(X, \mathbb{R})$ and $F(0) = G(0) = 0$.

$H2$. $F' : X \to X^*$ is strongly continuous (i.e. $u_n \rightharpoonup u$ implies $F'(u_n) \to F'(u)$) and $\langle F'(u), u\rangle = 0$, $u \in \overline{co} S_G$ implies $F(u) = 0$, where $\overline{co} S_G$ denotes the closure of the convex hull of the set S_G.

$H3$. $G' : X \to X^*$ is continuous, bounded and satisfies the following condition:

$$u_n \rightharpoonup u, \qquad G'(u_n) \rightharpoonup v, \qquad \big\langle G'(u_n), u_n\big\rangle \to \langle v, u\rangle$$

implies $u_n \to u$ as $n \to \infty$, where $u_n \rightharpoonup u$ denotes the weak convergence in X.

$H4$. The level set S_G is bounded and $u \neq 0$ implies,

$$\big\langle G'(u), u\big\rangle > 0, \qquad \lim_{t \to +\infty} G(tu) = +\infty, \qquad \inf_{u \in S_G} \big\langle G(u), u\big\rangle > 0.$$

Note that in our case, $F = \Phi$ and $G = G_k$. It has been shown in [13] that all conditions $H1$–$H4$ are satisfied for the functionals $\Phi(u)$ and $G_0(u)$, which is the norm of the space $X = W_0^{1,p}(0,1)$. If $k \neq 0$, then it is enough to note that for a fixed $k \in \mathbb{R}$ the functional $G_k(u) := \frac{1}{p}\int_0^1 |\nabla_k u|^p\,dx$ forms an equivalent norm in X.

Now, we denote by $\mathcal{K}_n(k)$ the class of all compact, symmetric subsets K of G_k, such that $\operatorname{gen} K \geqslant n$ (see [19, Chapter 44]). Thus, for a fixed $k \in \mathbb{R}$, according to the Ljusternik–Schnirelman variational principle [19, p. 326, Theorem 44.A] there exists a sequence of eigenvalues of problem (2.1), depending on k and arranged as:

$$0 < \lambda_1(k) \leqslant \lambda_2(k) \leqslant \cdots \leqslant \lambda_n(k) \leqslant \cdots$$

which are characterized by

$$\frac{1}{\lambda_n(k)} = \mu_n(k) = \sup_{K \subset \mathcal{K}_n(k)} \inf_{u \in K} \Phi(u).$$

Moreover, for all $k \in \mathbb{R}$

$$0 < \lambda_1(k) \quad \text{and} \quad \lambda_n(k) \to +\infty \quad \text{as } n \to \infty.$$

(b) It follows from

$$0 < \lambda_1(\Delta_p) = \inf_{0 \neq u \in W_0^{1,p}(0,1)} \frac{\int_0^1 |u'|^p\,dx}{\int_0^1 |u|^p\,dx} \leqslant \inf_{0 \neq u \in W_0^{1,p}(0,1)} \frac{\int_0^1 (k^2u^2 + u'^2)^{\frac{p}{2}}\,dx}{\int_0^1 |u|^p\,dx} = \lambda_1(k)$$

and

$$|k|^p \int_0^1 |u|^p\,dx \leqslant \int_0^1 \big(k^2u^2 + u'^2\big)^{\frac{p}{2}}\,dx$$

that $0 < \lambda_1(\Delta_p) \leqslant \lambda_1(k)$ and $|k|^p \leqslant \lambda_1(k)$. $\square$

Remark. Evidently, the Ljusternik–Schnirelman critical point theory can be applied not only to one-dimensional problem (2.1) but also to the n-dimensional problem

$$k^2 \int_\Omega |\nabla_k u|^{p-2} uv\,dx + \int_\Omega |\nabla_k u|^{p-2} u'v'\,dx = \lambda \int_\Omega |u|^{p-2} uv\,dx,$$

and we obtain that the same results, given in Theorem 2.1 are also valid for this problem. But, in the n-dimensional case we can only state that: $\lambda_1(k)$ is simple, isolated and $\lambda_2(k)$ is the second eigenvalue of the problem and the sequence

$$0 < \lambda_1(k) < \lambda_2(k) \leqslant \cdots \leqslant \lambda_n(k) \leqslant \cdots \quad \text{and} \quad \lambda_n(k) \to +\infty \quad \text{as } n \to \infty,$$

may not be complete (see [3,4,14]).

However, for the one-dimensional case we completely solve this problem in the following sections (see Theorem 3.1 and Theorem 4.2) and show that the spectrum is discrete, i.e., it consists of a sequence of isolated eigenvalues.

The case: $w \neq 0$.

In this case our main concern is the existence of positive solutions to problem (1.3). By using the scaling property, a solution to problem (1.3) can be obtained by a constrained minimization problem for the functional

$$E_{k,\lambda}(u) = \int_0^1 |\nabla_k u|^p\,dx - \lambda \int_0^1 |u|^p\,dx$$

on the Banach space $W_0^{1,p}(0,1)$, restricted to the set

$$M = \left\{ u \in W_0^{1,p}(0,1) \,\middle|\, \int_0^1 u^2\,dx = 1 \right\}.$$

Our method is based on some regularity ideas, the fact that $W_0^{1,p}(0,1)$ is compactly embedded in $L_q(0,1)$ for all $q \in [1,\infty)$ and the following theorem.

Theorem 2.2. *(See [17, Theorem 1.2].) Suppose X is a reflexive Banach space with norm $\|.\|$, and let $M \subset X$ be a weakly closed subset of X. Suppose $E : M \to \mathbb{R} \cup +\infty$ is coercive on M with respect to X, that is*

(1) $E(u) \to \infty$ *as* $\|u\| \to \infty$, $u \in M$

and it is (sequentially) weakly lower semi-continuous on M with respect to X, that is
(2) *for any $u \in M$, any sequence (u_n) in M such that $u_n \rightharpoonup u$ (weakly) in X there holds:*

$$E(u) \leqslant \lim_{n\to\infty} \inf E(u_n).$$

Then E is bounded from below on M and attains its infimum in M.

Now we formulate and prove our main result in this section.

Theorem 2.3. *If either $(w,k,\lambda) \in \mathbb{R}_+ \times \mathbb{R} \times (-\infty, \lambda_1(k))$ or $(w,k,\lambda) \in \mathbb{R}_- \times \mathbb{R} \times (\lambda_1(k), +\infty)$, then problem (1.3) has a positive solution.*

Proof. Let us consider the condition $(w,k,\lambda) \in \mathbb{R}_+ \times \mathbb{R} \times (-\infty, \lambda_1(k))$. By this condition we have to prove that for a fixed $k \in \mathbb{R}$ and $\lambda < \lambda_1(k)$ problem (1.3) has a positive solution for any $w > 0$. We set in Theorem 2.2: $X = W_0^{1,p}(0,1)$, $E(u) := E_{k,\lambda}(u)$ and $M = \{u \in W_0^{1,p}(0,1) \mid \int_0^1 u^2\,dx = 1\}$. Evidently, all conditions of Theorem 2.2 are satisfied. Particularly, by the Sobolev's embedding theorem M is a weakly closed set. Now, the existence of a non-trivial solution to problem (1.3) immediately follows from this theorem. The existence of a non-negative solution is obtained if we replace u by $|u|$. Let us prove that a non-negative solution to (1.3) is positive. One can use the Moser's iteration technique to show that a solution to (1.3) belongs to $L^\infty(0,1)$ (see [10] or [13, pp. 1070–1073]). Then, $u \in C^{1,\alpha}(0,1)$-Hölder continuously differentiable function with the exponent $0 \leqslant \alpha \leqslant 1$. This property follows from the following fact (here we restrict ourselves to the interval $(0,1)$): Let $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ be a Carathédory function (see [13, p. 1074]). Then if $g(x) := f(x,u(x)) \in L^\infty(0,1)$ then a result of DiBenedetto [7] and Tolksdorf [18] states that a weak solution of the equation

$$-\Delta_p u(x) = f\big(x, u(x)\big) \tag{2.4}$$

is a $C^{1,\alpha}(0,1)$ function.

Finally, we use the following Harnack type inequality due to Trudinger ([11] and [13, p. 1075]) to prove that $u > 0$.

Harnack inequality: Let $u \in W^{1,p}(0,1)$ be a weak solution of (2.4) and for all $M < \infty$ and for all $(x,s) \in (0,1) \times (-M, M)$ the condition

$$\left|f(x,s)\right| \leqslant b_1(x)|s|^{p-1} + b_2(x)$$

holds, where b_1, b_2 are non-negative functions in $L^\infty(0,1)$. Then if $0 \leqslant u(x) < M$ in a subinterval of $(0,1)$, there exists a constant C such that

$$\max_{K(r)} u(x) \leqslant C \min_{K(r)} u(x).$$

We also note that the condition $\max_{K(r)} u(x) \leqslant C \min_{K(r)} u(x)$ means either: $u = 0$ or $u > 0$ in $(0,1)$. Since $\|u\|_{L_2(0,1)} = 1$ we obtain that $u > 0$ in $(0,1)$. □

3. The cubic nonlinearity: analytical solutions and dispersion relations

In this section we separately study the analytical solutions and dispersion relations between k and λ of the following problem in the case of $p = 4$ (the problem with the cubic nonlinearity):

$$\begin{aligned} &k^2|\nabla_k u|^{p-2}u - \left(|\nabla_k u|^{p-2}u'\right)' = \lambda |u|^{p-2}u, \quad p > 1, \\ &u(0) = u(1) = 0, \end{aligned} \tag{3.1}$$

where $u := u(x)$ and $|\nabla_k u| = (k^2u^2 + u'^2)^{1/2}$.

Problems with the cubic nonlinearities have important applications and that is why we study this case separately. Moreover, it turns out that in this case it is possible to construct analytical solutions.

First we note that in the case $k = 0$ we deal with the following classical one-dimensional p-Laplacian eigenvalue problem which was fully studied by P. Drábek [9] (see also a paper of M. Del Pino, M. Elgueta, and R. Manasevich [16]):

$$\begin{aligned} &-\left(|u'|^{p-2}u'\right)' = \lambda |u|^{p-2}u, \quad p > 1, \\ &u(0) = u(1) = 0. \end{aligned} \tag{3.2}$$

All eigenfunctions and eigenvalues of problem (3.2) are given by $u_n(x) = c\lambda_n^{-\frac{1}{p}} \sin_p(\lambda_n^{\frac{1}{p}} x)$ and $\lambda_n = (n\pi_p)^p$, respectively. Here,

$$\pi_p := 2 \int_0^{(p-1)^{\frac{1}{p}}} \frac{ds}{(1 - \frac{s^p}{p-1})^{1/p}}$$

and $\sin_p(x)$ is defined as an implicit function $\sin_p : [0, \frac{\pi_p}{2}] \to [0, (p-1)^{1/p}]$ by

$$\int_0^{\sin_p(x)} \frac{ds}{(1 - \frac{s^p}{p-1})^{1/p}} = x,$$

then it is extended by setting: $\widetilde{\sin}_p(x) := \sin_p(\pi_p - x)$, $x \in [\frac{\pi_p}{2}, \pi_p]$ and $\widetilde{\sin}_p(x) := -\widetilde{\sin}_p(-x)$ for $x \in [-\pi_p, 0]$. Finally, $\sin_p : \mathbb{R} \to \mathbb{R}$ is defined as the $2\pi_p$-periodic extension of $\widetilde{\sin}_p(x)$ to all of $\mathbb{R}$ (see [9] and [16], and also the recent paper [6] for more interesting properties of $\sin_p(x)$).

To construct analytical solutions to problem (3.3) we also apply methods similar to those applied in the above mentioned papers. However, we have to modify many techniques of these papers which are not applicable to our problems. Namely, this modified method is applied below to find analytical solutions to the following problem:

$$\begin{aligned} &k^2\left(k^2u^2 + u'^2\right)u - \left(\left(k^2u^2 + u'^2\right)u'\right)' = \lambda u^3, \\ &u(0) = u(1) = 0, \end{aligned} \tag{3.3}$$

where $u = u(x)$. We let $k \geqslant 0$.

First, we reduce the order of the equation by the substitution $u' = v(u)$. Then, $u'' = v' = \frac{dv}{du}\frac{du}{dx} = \frac{dv}{du}v$ and we have

$$\left(3v^3 + k^2u^2v\right)\frac{dv}{du} = k^4u^3 - k^2uv^2 - \lambda u^3.$$

This is a homogeneous ordinary differential equation and it may be integrated by changing from $(u,v) \to (u,w)$ by the standard substitution $v = uw(u)$. This then gives a separable equation

$$\left(3w^3 + k^2w\right)uw' = k^4 - \lambda - 2k^2w^2 - 3w^4,$$

which integrates to

$$k^4 - \lambda - 2k^2 w^2 - 3w^4 = \frac{c}{u^4}.$$

Putting the inverse substitution $w = \frac{u'}{u}$ into the above equation we obtain

$$k^4 - \lambda - 2k^2\left(\frac{u'}{u}\right)^2 - 3\left(\frac{u'}{u}\right)^4 = \frac{c}{u^4}$$

or

$$3u'^4 + 2k^2u^2u'^2 + \left(\lambda - k^4\right)u^4 - c = 0$$

through the multiplication by $-u^4$. By Theorem 2.1 we have $\lambda \geqslant k^4$. Hence, in order to get a non-trivial solution to (3.3) the condition $c > 0$ must be satisfied. Let $u = \pm(\frac{c}{3})^{\frac{1}{4}} y$. Then we can rewrite the above given equation in the following form

$$y'^4 + \frac{2}{3}k^2y^2y'^2 + \frac{\lambda - k^4}{3}y^4 - 1 = 0.$$

Finally, by solving this quadratic equation in y'^2 we get

$$y'^2 = -\frac{k^2}{3}y^2 + \sqrt{\frac{k^4}{9}y^4 + 1 - \frac{\lambda - k^4}{3}y^4}.$$

Evidently, this formula makes sense if $y^4 \leqslant \frac{3}{\lambda - k^4}$. Moreover $y' > 0$ or $y' < 0$ iff $|y| < (\frac{3}{\lambda - k^4})^{\frac{1}{4}}$. Consequently, in a neighborhood of zero one can write

$$y' = \left[-\frac{k^2}{3}y^2 + \sqrt{\frac{k^4}{9}y^4 + 1 - \frac{\lambda - k^4}{3}y^4}\right]^{\frac{1}{2}} > 0.$$

Clearly, $[0, \varepsilon)$, $\varepsilon = \min\{x \mid x > 0 \text{ and } y(x) = (\frac{3}{\lambda - k^4})^{\frac{1}{4}}\}$ is the maximal interval of the form $[0, a)$, where $y' > 0$.

Note. At this point we have to note that if $y' > 0$ on $[0, 1]$, then we get a trivial solution. However, in general the condition $y'(x) > 0$, $x \in [0, 1]$ is not satisfied for all (λ, k). Actually, these cases allow us to construct non-trivial solutions. One can see from the arguments provided below that we construct (for some λ and k) a solution satisfying $u(0) = 0$, on the interval $[0, \varepsilon)$, where $\varepsilon < 1$ and then extend it to get a non-trivial solution to problem (3.3) on the interval $[0, 1]$.

Integrating the last equation and using the initial condition $y(0) = 0$ gives

$$\int_0^{y(x)} \frac{ds}{[-\frac{k^2}{3}s^2 + \sqrt{\frac{k^4}{9}s^4 + 1 - \frac{\lambda - k^4}{3}s^4}]^{1/2}} = x. \tag{3.4}$$

Let us define

$$F_{\lambda,k}(x) := \int_0^{x} \frac{ds}{[-\frac{k^2}{3}s^2 + \sqrt{\frac{k^4}{9}s^4 + 1 - \frac{\lambda - k^4}{3}s^4}]^{1/2}}. \tag{3.5}$$

The function $F_{\lambda,k}(x)$ is well defined if $1 - \frac{\lambda - k^4}{3}s^4 > 0$. Therefore, $s < (\frac{3}{\lambda - k^4})^{1/4}$ and

$$F_{\lambda,k} : \left[0, \left(\frac{3}{\lambda - k^4}\right)^{1/4}\right] \to \left[0, \frac{\pi_4(\lambda, k)}{2}\right], \quad k^4 < \lambda,$$

where

$$\pi_4(\lambda, k) = 2\int_0^{(\frac{3}{\lambda - k^4})^{1/4}} \frac{ds}{[-\frac{k^2}{3}s^2 + \sqrt{\frac{k^4}{9}s^4 + 1 - \frac{\lambda - k^4}{3}s^4}]^{1/2}}. \tag{3.6}$$

Since $F'_{\lambda,k}(x) > 0$, then there exists an inverse function

$$G_{\lambda,k} : \left[0, \frac{\pi_4(\lambda, k)}{2}\right] \to \left[0, \left(\frac{3}{\lambda - k^4}\right)^{1/4}\right]$$

defined by

$$\int_0^{G_{\lambda,k}(x)} \frac{ds}{[-\frac{k^2}{3}s^2+\sqrt{\frac{k^4}{9}s^4+1-\frac{\lambda-k^4}{3}s^4}]^{1/2}} = x. \tag{3.7}$$

$G_{\lambda,k}(x)$ is extended to $\mathbb{R}$ by the following way: $\widetilde{G}_{\lambda,k}(x) := G_{\lambda,k}(\pi_4(\lambda,k)-x)$, $x \in [\frac{\pi_4(\lambda,k)}{2}, \pi_4(\lambda,k)]$ and $\widetilde{G}_{\lambda,k}(x) := -\widetilde{G}_{\lambda,k}(-x)$ for $x \in [-\pi_4(\lambda,k), 0]$. Finally, $G_{\lambda,k}(x) : \mathbb{R} \to \mathbb{R}$ is defined as the $2\pi_4(\lambda,k)$-periodic extension of $\widetilde{G}_{\lambda,k}(x)$ to all of $\mathbb{R}$.

By this extension we have $G_{\lambda,k}(x) = 0 \Leftrightarrow x = n\pi_4(\lambda,k)$. Now, it follows from (3.4) and (3.7) that $y(x) = G_{\lambda,k}(x)$ and therefore $u(x) = cG_{\lambda,k}(x)$ is the general solution for the following equation

$$k^2\big(k^2u^2+u'^2\big)u - \big(\big(k^2u^2+u'^2\big)u'\big)' = \lambda u^3, \qquad u(0)=0. \tag{3.8}$$

Thus, the main question is: for what values of k and λ are there non-trivial solutions, among $u(x) = cG_{\lambda,k}(x)$, satisfying the condition $u(1) = 0$? The answer to this question has almost been given in the above discussion. We summarize these in the following theorem.

Theorem 3.1. *Let $p = 4$. In this case we have*:

(a) *All eigen-parameters (λ, k) for problem* (3.3) *lie in the parabola $k^4 < \lambda$*;

(b) *For every $k \in \mathbb{R}$, the set of all eigenvalues λ of problem* (3.3) *consists of a sequence of positive numbers $\lambda_n(k)$ such that $k^4 < \lambda_n(k) \to \infty$ for $n \to \infty$*;

(c) *There is a number $\lambda^* > 0$ such that for each λ satisfying $\lambda^* \leqslant \lambda$, the set of all eigen-parameters k of problem* (3.3) *consists of a finite number of eigen-parameters $k_1(\lambda), k_2(\lambda), \ldots, k_{n(\lambda)}(\lambda)$ which belong to the interval $(-\lambda^{1/4}, \lambda^{1/4})$. Moreover, $n(\lambda) \to \infty$ as $\lambda \to \infty$*;

(d) *In the case of $\lambda < \lambda^*$ problem* (3.3) *has only trivial solution.*

Proof. (a) Actually, the weaker inequality $k^4 \leqslant \lambda$ has been established in Theorem 2.1 by using variational techniques. Here we prove the stronger inequality: $k^4 < \lambda$. Let $\lambda \leqslant k^4$. Then we have $-\frac{k^2}{3}s^2 + \sqrt{\frac{k^4}{9}s^4 + 1 - \frac{\lambda-k^4}{3}s^4} > 0$ for all $s \in \mathbb{R}$. Hence, it follows from (3.5) that the function $F_{\lambda,k}(x)$ is well defined, positive function on $(0, +\infty)$ and by (3.7) so is the function $G_{\lambda,k}(x)$. Since the solution of (3.8) is given by $u(x) = cG_{\lambda,k}(x)$, then $u(1) = 0$ implies $c = 0$. Thus, all eigen-parameters (λ, k) for problem (3.3) lie in the parabola $k^4 < \lambda$ (see [12] for more facts on the localization of the eigen-parameters).

(b) (See also Theorem 2.1(a) and Remark after this theorem.) By (3.6), for a fixed k the function $\pi_4(\lambda,k)$ satisfies the following conditions: $\lim_{\lambda\to k^{4+}} \pi_4(\lambda,k) = +\infty$, $\lim_{\lambda\to+\infty} \pi_4(\lambda,k) = 0$ and $\pi_4(\lambda,k)$ is a decreasing function on $(k^4, +\infty)$, with respect to λ. Indeed, we can use the inequality $\sqrt{a^2+b^2} \leqslant a+b$, $a \geqslant 0$, $b \geqslant 0$ to get

$$0 < -\frac{k^2}{3}s^2 + \sqrt{\frac{k^4}{9}s^4 + 1 - \frac{\lambda-k^4}{3}s^4} \leqslant -\frac{k^2}{3}s^2 + \sqrt{\frac{k^4}{9}s^4+1} \leqslant -\frac{k^2}{3}s^2 + \frac{k^2}{3}s^2 + 1 = 1.$$

Hence $\pi_4(\lambda,k) \geqslant 2(\frac{3}{\lambda-k^4})^{1/4}$ and consequently $\lim_{\lambda\to k^{4+}} \pi_4(\lambda,k) = +\infty$.

Let us show that $\lim_{\lambda\to+\infty} \pi_4(\lambda,k) = 0$. We need to check this property, since $\lambda \to +\infty$ implies $s \to 0$ in (3.6). By using the substitution $s = (\frac{3}{\lambda-k^4})^{1/4}t$ we can rewrite $\pi_4(\lambda,k)$ in the form:

$$\pi_4(\lambda,k) = 2\int_0^1 \frac{dt}{[-\frac{k^2}{3}t^2 + \sqrt{\frac{k^4}{9}t^4 + \frac{\lambda-k^4}{3}(1-t^4)}]^{1/2}}.$$

Thus, the property $\lim_{\lambda\to+\infty} \pi_4(\lambda,k) = 0$ follows from the Lebesgue's dominated convergence theorem.

By using these properties we obtain that, for each $n \in \mathbb{N}$, the equation $\pi_4(\lambda,k) = \frac{1}{n}$ has a unique solution $\lambda_n(k)$ and $\lambda_n(k) \to +\infty$ as $n \to \infty$.

(c) Now, let us fix $\lambda > 0$. Then $\pi_4(\lambda, .)$ is defined on the interval $(-\lambda^{1/4}, \lambda^{1/4})$. Moreover, $\lim_{k\to-\lambda^{1/4}+} \pi_4(\lambda,k) = +\infty$ and $\lim_{k\to\lambda^{1/4}-} \pi_4(\lambda,k) = +\infty$. Let λ^* be the solution of the equation $\pi_4(\lambda,0) = 1$. It follows from

$$\pi_4(\lambda,0) = 2\left(\frac{3}{\lambda}\right)^{\frac{1}{4}} \int_0^1 \frac{dt}{\sqrt[4]{1-t^4}}$$

that $\pi_4(\lambda,0)$ is a decreasing function and $\pi_4(\lambda,0) \to 0$ as $\lambda \to +\infty$. Now, clearly the equation $\pi_4(\lambda,k) = \frac{1}{n}$ has a solution if and only if $\lambda^* \leqslant \lambda$ and $\frac{1}{n} \in [\pi_4(\lambda,0), 1]$. These facts prove (c) and (d). $\square$

4. The general case: implicit solutions and a dispersion analysis

In this section we analyze the existence of non-trivial solutions and dispersion relations between λ and k in the general case, i.e. for arbitrary $p > 1$. We suggest new techniques to construct solutions in the implicit form. Relying on these implicit solutions we provide a complete dispersion analysis.

The problem is

$$\begin{aligned}&k^2\left(k^2u^2+u'^2\right)^{\frac{p-2}{2}}u-\left(\left(k^2u^2+u'^2\right)^{\frac{p-2}{2}}u'\right)'=\lambda|u|^{p-2}u,\quad p>1,\\&u(0)=u(1)=0,\end{aligned}\tag{4.1}$$

where $u := u(x)$. By special substitutions we can reduce the order of (4.1). Full details are given in Appendix A. The final reduced 1st order differential equation is

$$\begin{aligned}&\left(u'^2+k^2u^2\right)^{\frac{p-2}{2}}\left[(p-1)u'^2-k^2u^2\right]+\lambda|u|^p-c=0,\\&u(0)=u(1)=0.\end{aligned}\tag{4.2}$$

Lemma 4.1. *For all differentiable u and $p > 1$ the following inequality*

$$0\leqslant\left(u'^2+k^2u^2\right)^{\frac{p-2}{2}}\left[(p-1)u'^2-k^2u^2\right]+|k|^p|u|^p$$

holds.

Proof. Indeed

$$\begin{aligned}0&\leqslant|k|^{p-2}|u|^{p-2}(p-1)u'^2=|k|^{p-2}|u|^{p-2}\left[(p-1)u'^2-k^2u^2\right]+|k|^p|u|^p\\&\leqslant\left(u'^2+k^2u^2\right)^{\frac{p-2}{2}}\left[(p-1)u'^2-k^2u^2\right]+|k^p|u|^p.\qquad\square\end{aligned}$$

For convenience we rewrite (4.2) in the form

$$\begin{aligned}&\left[\left(u'^2+k^2u^2\right)^{\frac{p-2}{2}}\left[(p-1)u'^2-k^2u^2\right]+|k|^p|u|^p\right]+\left(\lambda-|k|^p\right)|u|^p-c=0,\\&u(0)=u(1)=0.\end{aligned}\tag{4.3}$$

By Theorem 2.1 we have $|k|^p \leqslant \lambda$. Then, by using Lemma 4.1 we get: (4.3) (or the same (4.2)) has a non-trivial solution if $c > 0$.

On the other hand by setting directly the substitution $u = \pm(\frac{c}{p-1})^{\frac{1}{p}} y$ into (4.2) we can cancel c and write

$$\begin{aligned}&\left(y'^2+k^2y^2\right)^{\frac{p-2}{2}}\left[y'^2-\frac{k^2}{p-1}y^2\right]+\frac{\lambda}{p-1}|y|^p-1=0,\\&y(0)=0,\qquad y(1)=0.\end{aligned}\tag{4.4}$$

In the same way, setting the substitution $u = \pm(\frac{c}{p-1})^{\frac{1}{p}} y$ into (4.3) we can cancel c and write

$$\begin{aligned}&\left[\left(y'^2+k^2y^2\right)^{\frac{p-2}{2}}\left[y'^2-\frac{k^2}{p-1}y^2\right]+\frac{|k|^p}{p-1}|y|^p\right]+\frac{\lambda-|k|^p}{p-1}|y|^p-1=0,\\&y(0)=0,\qquad y(1)=0.\end{aligned}\tag{4.5}$$

Next, we analyze multi-parameter spectral problem (4.4) (or its modified form (4.5)). For this, we first study the following intermediate problem

$$\begin{aligned}&\left[\left(y'^2+k^2y^2\right)^{\frac{p-2}{2}}\left[y'^2-\frac{k^2}{p-1}y^2\right]+\frac{|k|^p}{p-1}|y|^p\right]+\frac{\lambda-|k|^p}{p-1}|y|^p-1=0,\quad x\in[0,+\infty)\\&y(0)=0.\end{aligned}\tag{4.6}$$

Lemma 4.2. *Let y be a solution of* (4.6)*.*

(i) *If $\lambda \leqslant |k|^p$, then $y'(x) > 0$ or $y'(x) < 0$, for all $x \in [0,\infty)$,*

(ii) *If* $|k|^p < \lambda$, *then* $|y(x)| \leqslant (\frac{p-1}{\lambda-|k|^p})^{\frac{1}{p}}$, $x \in [0, +\infty)$ *and there exists a point* $x_0 \in [0, 1]$ *such that*

$$\left|y(x_0)\right| < \left(\frac{p-1}{\lambda-|k|^p}\right)^{\frac{1}{p}}.$$

Moreover, $y(x_0) = (\frac{p-1}{\lambda-|k|^p})^{\frac{1}{p}}$ *iff* $y'(x_0) = 0$.

Proof. (i) Let $\lambda \leqslant |k|^p$. We show that in this case $y'(x) > 0$ or $y'(x) < 0$ for all $x \in [0, +\infty)$. Indeed, if there exists a number $x_0 \in [0, +\infty)$ such that $y'(x_0) = 0$, then setting $x = x_0$ into (4.6) we obtain the contradiction

$$1 = \frac{\lambda-|k|^p}{p-1}\left|y(x_0)\right|^p \leqslant 0.$$

(ii) Let $|k|^p < \lambda$. By Lemma 4.1 we have $0 \leqslant (y'^2 + k^2y^2)^{\frac{p-2}{2}}[y'^2 - \frac{k^2}{p-1}y^2] + \frac{|k|^p}{p-1}|y|^p$. Then, we need the conditions: $\frac{\lambda-|k|^p}{p-1}|y|^p - 1 \leqslant 0$ and $\frac{\lambda-|k|^p}{p-1}|y|^p - 1 \not\equiv 0$ in order to get a solution of (4.6). Clearly, $\frac{\lambda-|k|^p}{p-1}|y|^p - 1 \equiv 0$ would imply $y(x) \equiv (\frac{p-1}{\lambda-|k|^p})^{\frac{1}{p}}$, which does not satisfy the initial condition $y(0) = 0$. □

Corollary 4.1. *If* $\lambda \leqslant |k|^p$ *then* (4.5) *has no solution and consequently,* (4.1) *has only a trivial solution.*

Now we are going to construct a solution to problem (4.6) in the implicit form, in a neighborhood of 0. Then we use this solution to construct the general solution to problem (4.1). By Corollary 4.1 we can always suppose that $|k|^p < \lambda$.

Theorem 4.1. *For all* λ *and* k *such that* $|k|^p < \lambda$ *there exists the interval* $[0, \varepsilon(\lambda, k))$ *of the maximal length such that*

(i) *the problem*

$$\left(y'^2 + k^2y^2\right)^{\frac{p-2}{2}}\left[y'^2 - \frac{k^2}{p-1}y^2\right] + \frac{\lambda}{p-1}|y|^p - 1 = 0, \quad x \in \left[0, \varepsilon(\lambda, k)\right], \qquad y(0) = 0$$

has unique solution $y(x)$,

(ii) $y'(x) > 0$, $x \in [0, \varepsilon(\lambda, k))$.

Proof. Let us define $L(y', y, \lambda, k) := (y'^2 + k^2y^2)^{\frac{p-2}{2}}[y'^2 - \frac{k^2}{p-1}y^2] + \frac{\lambda}{p-1}|y|^p - 1$. Evidently, $y = 0$ implies $y' = \pm 1$. Now we apply the standard implicit function theorem to $L(y', y, \lambda, k)$ with respect to y', around $(0, 1)$. We have

(i) $L(1, 0, \lambda, k) = 0$,

(ii) $L_{y'} = (y'^2 + k^2y^2)^{\frac{p-4}{2}} y'[py'^2 + \frac{p}{p-1}k^2y^2] > 0$ iff $y' > 0$ (or $L_{y'} < 0$ iff $y' < 0$).

Hence, as long as $(y, y') \in (-\infty, \infty) \times (0, +\infty)$, there is a unique implicit function, we denote by $H(y, \lambda, k)$, such that

$$\begin{aligned} &y'(x) = H\big(y(x), \lambda, k\big), \\ &y(0) = 0. \end{aligned} \tag{4.7}$$

Let $\varepsilon(\lambda, k) = \min\{x \mid x > 0 \text{ and } y'(x) = 0\}$. Evidently, $[0, \varepsilon(\lambda, k))$ is the maximal interval, where the condition $y'(x) \in (0, +\infty)$ is satisfied and the implicit function theorem can be applied.

From the point of view of the case $p = 2$ and $p = 4$ it is natural to introduce the number

$$\frac{\pi_p(\lambda, k)}{2} := \varepsilon(\lambda, k).$$

Hence

$$\frac{\pi_p(\lambda, k)}{2} = \min\left\{x \,\middle|\, x > 0 \text{ and } y'(x) = 0\right\}. \tag{4.8}$$

By Lemma 4.2

$$\frac{\pi_p(\lambda, k)}{2} := \min\left\{x \,\middle|\, x > 0 \text{ and } y(x) = \left(\frac{p-1}{\lambda-|k|^p}\right)^{\frac{1}{p}}\right\}. \tag{4.9}$$

Integrating (4.7) and using the initial condition $y(0) = 0$ yields

$$\int_0^{y(x)} \frac{ds}{H(s, \lambda, k)} = x, \quad x \in \left[0, \frac{\pi_p(\lambda, k)}{2}\right]. \tag{4.10}$$

Let

$$F^p_{\lambda,k}(x) := \int_0^x \frac{ds}{H(s,\lambda,k)}. \tag{4.11}$$

By (4.9) $y(\frac{\pi_p(\lambda,k)}{2}) = (\frac{p-1}{\lambda-k^p})^{\frac{1}{p}}$. Consequently, it follows from (4.10) that

$$\pi_p(\lambda,k) = 2\int_0^{(\frac{p-1}{\lambda-k^p})^{\frac{1}{p}}} \frac{ds}{H(s,\lambda,k)}. \tag{4.12}$$

Hence

$$F^p_{\lambda,k} : \left[0, \left(\frac{p-1}{\lambda-|k|^p}\right)^{\frac{1}{p}}\right] \to \left[0, \frac{\pi_p(\lambda,k)}{2}\right].$$

By (4.11), $\frac{d}{dx}F^p_{\lambda,k}(x) = \frac{1}{H(x,\lambda,k)} > 0$. Consequently, there exists an inverse function for $F^p_{\lambda,k}(x)$ denoted by $G^p_{\lambda,k}(x)$. Finally, comparing (4.10) and (4.11) we obtain that $F^p_{\lambda,k}(y(x)) = x$ and $y(x) = G^p_{\lambda,k}(x)$ is the needed non-trivial solution. □

Note. If $p = 4$ then

$$H(y,\lambda,k) = \left[-\frac{k^2}{3}y^2 + \sqrt{\frac{k^4}{9}y^4 + 1 - \frac{\lambda-k^4}{3}y^4}\right]^{\frac{1}{2}}.$$

Therefore, $F^4_{\lambda,k}(x) = F_{\lambda,k}(x)$ and $G^4_{\lambda,k}(x) = G_{\lambda,k}(x)$. Moreover, it follows from (4.12) that

$$\pi_4(\lambda,k) = 2\int_0^{(\frac{3}{\lambda-k^4})^{1/4}} \frac{ds}{[-\frac{k^2}{3}s^2 + \sqrt{\frac{k^4}{9}s^4 + 1 - \frac{\lambda-k^4}{3}s^4}]^{1/2}},$$

which coincides with (3.6) in the case of $p = 4$.

Finally, we construct the general solution of the main problem (4.1) in the same way as in the case $p = 4$. To this end, $G^p_{\lambda,k}(x)$ is extended to $\mathbb{R}$ in the following way: $\widetilde{G}_{\lambda,k}(x) := G^p_{\lambda,k}(\pi_p(\lambda,k) - x)$, $x \in [\frac{\pi_p(\lambda,k)}{2}, \pi_p(\lambda,k)]$ and $\widetilde{G}_{\lambda,k}(x) := -\widetilde{G}_{\lambda,k}(-x)$ for $x \in [-\pi_p(\lambda,k), 0]$. Finally, $G^p_{\lambda,k}(x) : \mathbb{R} \to \mathbb{R}$ is defined as the $2\pi_p(\lambda,k)$-periodic extension of $\widetilde{G}_{\lambda,k}(x)$ to all of $\mathbb{R}$. The extended function $y = G^p_{\lambda,k}(x)$ satisfies the following equation

$$\begin{aligned}&\left(y'^2 + k^2y^2\right)^{\frac{p-2}{2}}\left[y'^2 - \frac{k^2}{p-1}y^2\right] + \frac{\lambda}{p-1}|y|^p - 1 = 0, \quad x \in \mathbb{R},\\ &y(0) = 0\end{aligned}$$

because it is invariant under the transformations: $x \mapsto \pi_p(\lambda,k) - x$ and $y(x) \mapsto -y(-x)$. Consequently, $u(x) = cG^p_{\lambda,k}(x)$ is the general solution to the problem

$$\begin{aligned}&k^2\left(k^2u^2 + u'^2\right)^{\frac{p-2}{2}}u - \left(\left(k^2u^2 + u'^2\right)^{\frac{p-2}{2}}u'\right)' = \lambda|u|^{p-2}u, \quad x \in \mathbb{R},\\ &u(0) = 0.\end{aligned}$$

It remains to check the conditions $u(1) = 0$ and $u \neq 0$. These are possible iff $G^p_{\lambda,k}(1) = 0$. By the construction of $G^p_{\lambda,k}(x)$ we have

$$G^p_{\lambda,k}(1) = 0 \quad \Leftrightarrow \quad n\pi_p(\lambda,k) = 1.$$

Thus the general eigenvalue problem (4.1) reduces to the algebraic problem:

$$n\pi_p(\lambda,k) = 1. \tag{4.13}$$

Further results are based on the properties of $\pi_p(\lambda,k)$.

Finally, we give an analog of Theorem 3.1 in the general case. Here we summarize the results about the dispersion relations, which follow from (4.13) and the above given constructions.

Theorem 4.2. (a) (*See also Theorem* 2.1(a) *and Remark after this theorem.*) *For every* $k \in \mathbb{R}$, *the set of all eigenvalues* λ *of problem* (4.1) *consists of a sequence of positive numbers* $\lambda_n(k)$ *such that* $\lambda_n(k) \to \infty$ *for* $n \to \infty$;

(b) *There is a number* $\lambda^* > 0$ *such that for each* λ *satisfying* $\lambda^* \leqslant \lambda$, *the set of all eigen-parameters* k *of problem* (4.1) *consists of a finite number of eigen-parameters* $k_1(\lambda), k_2(\lambda), \ldots, k_{n(\lambda)}(\lambda)$ *which belong to the interval* $(-\lambda^{1/p}, \lambda^{1/p})$. *Moreover,* $n(\lambda) \to \infty$ *as* $\lambda \to \infty$;

(c) *In the case of* $\lambda < \lambda^*$ *problem* (4.1) *has only trivial solution.*

Proof. By (4.13) we have: (λ, k) are eigen-parameter for (4.1) if and only if

$$n\pi_p(\lambda, k) = 1.$$

We note the following properties of the function $\pi_p(\lambda, k)$ (the proof repeats the same arguments from Theorem 3.1), which follow from the definition of $\pi_p(\lambda, k)$ and related properties of the function $H(s, \lambda, k)$:

(i) for a fixed k, $\lim_{\lambda \to +\infty} \pi_p(\lambda, k) = 0$,

(ii) for a fixed $\lambda > 0$ we have $k \in (-\lambda^{\frac{1}{4}}, \lambda^{\frac{1}{4}})$ and

$$\lim_{k \to -\lambda^{\frac{1}{p}+}} \pi_p(\lambda, k) = \lim_{k \to \lambda^{\frac{1}{p}-}} \pi_p(\lambda, k) = +\infty,$$

(iii) $\lim_{\lambda \to +\infty} \pi_p(\lambda, 0) = 0$.

Hence, (a) follows from (4.13) and (i). The property (b) follows from (4.13), (ii) and (iii), where λ^* is the solution of the equation

$$\pi_p(\lambda, 0) = 1. \qquad \square$$

Appendix A

In this appendix we give the full details of reducing the order of general multi-parameter eigenvalue problem (4.1). We apply some techniques, similar to those applied in the case of $p = 4$, to reduce the order of (4.1).

For the convenience of computations we set $\frac{p-2}{2} = q$. Then (4.1) reduces to the following form

$$\begin{aligned} &k^2\big(k^2u^2 + u'^2\big)^q u - \big(\big(k^2u^2 + u'^2\big)^q u'\big)' = \lambda |u|^{2q} u, \\ &u(0) = u(1) = 0. \end{aligned} \tag{A.1}$$

By using the substitution $u'(x) = v(u)$ in (A.1) we obtain

$$\begin{aligned} &k^2\big(k^2u^2 + v^2\big)^q u - \big(\big(k^2u^2 + v^2\big)^q v\big)' = \lambda |u|^{2q} u, \\ &k^2\big(k^2u^2 + v^2\big)^q u - q\big(k^2u^2 + v^2\big)^{q-1}\big(2k^2uu' + 2vv'u'\big)v - \big(k^2u^2 + v^2\big)^q v'u' = \lambda |u|^{2q} u, \end{aligned}$$

where $v' = \frac{dv}{du}$. After simplification we get

$$k^2\big(k^2u^2 + v^2\big)^q u - 2k^2q\big(k^2u^2 + v^2\big)^{q-1} uv^2 - 2q\big(k^2u^2 + v^2\big)^{q-1} v^3 v' - \big(k^2u^2 + v^2\big)^q vv' = \lambda |u|^{2q} u. \tag{A.2}$$

This is a homogeneous ordinary differential equation of degree $2q + 1$. Let us divide both sides of (A.2) by $|u|^{2q}u$. Then

$$\begin{aligned} &k^2\left(k^2 + \left(\frac{v}{u}\right)^2\right)^q - 2k^2q\left(k^2 + \left(\frac{v}{u}\right)^2\right)^{q-1}\left(\frac{v}{u}\right)^2 - 2q\left(k^2 + \left(\frac{v}{u}\right)^2\right)^{q-1}\left(\frac{v}{u}\right)^3 v' \\ &\quad - \left(k^2 + \left(\frac{v}{u}\right)^2\right)^q \frac{v}{u} v' = \lambda. \end{aligned} \tag{A.3}$$

The change of variable $v = uw(u)$ transforms Eq. (A.3) to the following separable equation

$$\begin{aligned} &k^2\big(k^2 + w^2\big)^q - 2k^2q\big(k^2 + w^2\big)^{q-1} w^2 - 2q\big(k^2 + w^2\big)^{q-1} w^3\big(w + uw'\big) \\ &\quad - \big(k^2 + w^2\big)^q w\big(w + uw'\big) = \lambda. \end{aligned}$$

Simplifications yield

$$\begin{aligned} &k^2\big(k^2 + w^2\big)^q - 2k^2q\big(k^2 + w^2\big)^{q-1} w^2 - 2q\big(k^2 + w^2\big)^{q-1} w^4 \\ &\quad - 2q\big(k^2 + w^2\big)^{q-1} w^3uw' - \big(k^2 + w^2\big)^q w^2 - \big(k^2 + w^2\big)^q wuw' = \lambda \end{aligned}$$

and

$$\begin{aligned}&\left(k^2+w^2\right)^{q-1}\left[k^2\left(k^2+w^2\right)-2k^2qw^2-2qw^4-\left(k^2+w^2\right)w^2\right]-\lambda\\&\quad=\left(k^2+w^2\right)^{q-1}\left[2qw^3+\left(k^2+w^2\right)w\right]uw'.\end{aligned}\tag{A.4}$$

Since

$$k^2\left(k^2+w^2\right)-2k^2qw^2-2qw^4-\left(k^2+w^2\right)w^2=\left(k^2+w^2\right)\left(k^2-(2q+1)w^2\right),$$

we obtain from (A.4) that

$$\left(k^2+w^2\right)^q\left(k^2-(2q+1)w^2\right)-\lambda=\left(k^2+w^2\right)^{q-1}\left[(2q+1)w^3+k^2w\right]uw'.$$

Finally, we have the following separable equation

$$\frac{(k^2+w^2)^{q-1}[(2q+1)w^3+k^2w]}{(k^2+w^2)^q(k^2-(2q+1)w^2)-\lambda}\,dw=\frac{du}{u}.\tag{A.5}$$

A straightforward calculation gives

$$\left[\left(k^2+w^2\right)^q\left(k^2-(2q+1)w^2\right)-\lambda\right]'=-(2q+2)\left(k^2+w^2\right)^{q-1}\left[(2q+1)w^3+k^2w\right].$$

Then (A.5) integrates to

$$\ln\left|\left(k^2+w^2\right)^q\left(k^2-(2q+1)w^2\right)-\lambda\right|=-(2q+2)\ln|u|+c$$

or the same

$$\left(k^2+w^2\right)^q\left(k^2-(2q+1)w^2\right)-\lambda=\frac{c}{|u|^{2q+2}},\quad c\neq 0.$$

Putting the inverse substitution $w=\frac{u'}{u}$ into this equation we have

$$\left(k^2+\left(\frac{u'}{u}\right)^2\right)^q\left(k^2-(2q+1)\left(\frac{u'}{u}\right)^2\right)-\lambda=\frac{c}{|u|^{2q+2}}.$$

Hence,

$$\left(u'^2+k^2u^2\right)^q\left[(2q+1)u'^2-k^2u^2\right]+\lambda|u|^{2q+2}-c=0$$

(here we replaced c by $-c$). Thus, the final reduced 1st order differential equation is

$$\left(u'^2+k^2u^2\right)^{\frac{p-2}{2}}\left[(p-1)u'^2-k^2u^2\right]+\lambda|u|^p-c=0.\tag{A.6}$$

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