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Note

# Inclusion matrices and chains

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**Abstract**

Given integers $t$, $k$, and $v$ such that $0 \leqslant t \leqslant k \leqslant v$, let $W_{tk}(v)$ be the inclusion matrix of $t$-subsets vs. $k$-subsets of a $v$-set. We modify slightly the concept of standard tableau to study the notion of rank of a finite set of positive integers which was introduced by Frankl. Utilizing this, a decomposition of the poset $2^{[v]}$ into symmetric skipless chains is given. Based on this decomposition, we construct an inclusion matrix, denoted by $W_{\bar{t}k}(v)$, which is row-equivalent to $W_{tk}(v)$. Its Smith normal form is determined. As applications, Wilson's diagonal form of $W_{tk}(v)$ is obtained as well as a new proof of the well-known theorem on the necessary and sufficient conditions for existence of integral solutions of the system $W_{tk}\mathbf{x} = \mathbf{b}$ due to Wilson. Finally we present another inclusion matrix with similar properties to those of $W_{\bar{t}k}(v)$ which is in some way equivalent to $W_{tk}(v)$.


*Keywords:* Inclusion matrices; Chains; Smith normal form

## 1. Introduction

To start we fix some notations. For a finite set $F$ and a nonnegative integer $i$ let $\binom{F}{i}$ be the set of all $i$-subsets of $F$, and $2^F$ be the set of all subsets of $F$. For a positive integer $v$, we denote the set $\{1, \ldots, v\}$ by $[v]$ and define $\binom{v}{-1}$ to be 0.

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Let $t, k$, and $v$ be integers satisfying $0 \leqslant t \leqslant k \leqslant v$. The inclusion matrix $W_{tk}(v)$ (denoted briefly as $W_{tk}$) is a 0, 1-matrix whose rows and columns are indexed by $t$-subsets and $k$-subsets of the set $[v]$, respectively, and $W_{tk}(v)(T, K) = 1$ if and only if $T \subseteq K$. These inclusion matrices have applications in many combinatorial problems, particularly in design theory [2] and their properties are studied extensively [5,6,8,9,11,12].

Our motivation in this work relates to the simple but fundamental equation

$$W_{tk}\mathbf{x} = \lambda\mathbf{1}, \tag{1}$$

where $\lambda$ is a positive integer and $\mathbf{1}$ is the all 1's vector. Any integral solution of (1) is called a $t$-$(v, k, \lambda)$ signed design; and any nonnegative integral solution of (1) is a $t$-design. Therefore of importance is having an equivalent system to the above such that its coefficient matrix is of a 'simpler' nature.

Our aim in this paper is to construct such matrices using a notion of 'rank' of a finite set. Our criterion to being simpler is the Smith form. The Smith form of the new matrices is roughly identity while that of $W_{tk}$ is more complicated. See Theorems 1, 2 and Corollary 3.

The paper is organized as follows: In Section 2 we use a slightly modified version of standard tableau to study the notion of the 'rank' of a finite set of positive integers, introduced by Frankl [5]. This notion is useful in Section 3 to describe a decomposition of $2^{[v]}$ into symmetric chains, in which the elements of each chain have the same rank. This decomposition coincides with that of de Bruijn described recursively in [4] and explicitly in [1,7]. In Section 4 we introduce a new inclusion matrix $W_{\bar{t}k}(v)$ based on the above decomposition as follows: Replace each $t$-subset $T$ with the minimal element $\overline{T}$ in the same chain and construct the related inclusion matrix $W_{\bar{t}k}(v)$. The new matrix $W_{\bar{t}k}(v)$ is row-wise equivalent to $W_{tk}$ and has $(I \mid O)$ as the Smith normal form. This is implicit in the work of Bier [3]. Using these results we obtain Wilson's diagonal form of $W_{tk}(v)$ as well as a new proof of the well-known theorem on the existence of signed (integral) $t$-designs due to Wilson and Graver and Jurkat. In Section 5 a similar approach is applied to find another inclusion matrix $W_{t\underline{k}}$ which is somehow equivalent to $W_{tk}(v)$ and has the same Smith form as $W_{\bar{t}k}(v)$.

## 2. Rank of a finite set

The rank of a given set $F$ of positive integers, denoted by $r(F)$, is defined as follows [5]: To each $F \subseteq [v]$ we associate a walk $w(F)$ from the origin to $(v - |F|, |F|)$ by steps of length one such that the $i$th step is to the right (up) if $i \notin F$ $(i \in F)$; the rank of $F$, denoted as $r(F)$, is then $|F| - b$, where $b$ is the largest integer such that the line $y = x + b$ touches $w(F)$ from the above. This definition does not depend on $v$ in the sense that replacing $v$ by any integer $\geqslant \max(F)$, the rank does not change.

A *standard tableau* is a $2 \times m$ array, filled with integers $1, \ldots, 2m$, such that the entries across any row or down any column are ordered increasingly. We modify this object to provide a useful tool which reveals the properties of rank: Consider a $2 \times |F|$ tableau $\mathcal{T}(F)$ and arrange the elements of $F$ on the first row in increasing order. Fill the second row from left to right as follows: Below each element $a$ put the largest positive integer smaller than $a$ not appearing on the left neither in the second row nor in the first; if such an integer does not exist, put the symbol j. After filling the second row, let $r(F) = |F| - \#$j's. In other words if $a_1 < a_2 < \cdots < a_{|F|}$ are the elements of $F$, then

$$\mathcal{T}(F) = \begin{pmatrix} a_1 & a_2 & \ldots & a_{|F|} \\ b_1 & b_2 & \ldots & b_{|F|} \end{pmatrix}.$$

In which, for $i = 1, \dots, |F|$, we have

$$b_i = \max\bigl([a_i] \setminus \bigl(F \cup \{b_1, \dots, b_{i-1}\}\bigr)\bigr), \tag{2}$$

where $\max\emptyset$ is defined to be j. The set of non-j elements in the second row is denoted by $\operatorname{fill}(F)$. The set of corresponding elements in the first row is denoted by $\operatorname{fill}^*(F)$:

$$\operatorname{fill}(F) = \bigl\{b_i\colon\ i = 1, 2, \dots, |F|\bigr\} \setminus \{\mathrm{j}\},$$
$$\operatorname{fill}^*(F) = \bigl\{a_i\colon\ b_i \in \operatorname{fill}(F)\bigr\} = \{a_i\colon\ b_i \neq \mathrm{j}\}.$$

Hence

$$r(F) = \bigl|\operatorname{fill}(F)\bigr| = \bigl|\operatorname{fill}^*(F)\bigr|. \tag{3}$$

If $r(F) = |F|$, then $F$ is said to be *full-rank*.

**Example 1.** Let $F = \{2, 3, 7, 8\}$. Then

$$\mathcal{T}(F) = \begin{pmatrix} 2 & 3 & 7 & 8 \\ 1 & \mathrm{j} & 6 & 5 \end{pmatrix}.$$

Thus $r(F) = 3$ and $F$ is not full-rank. One can consider the walk $w(F)$ corresponding $F$ to calculate the rank of $F$: $w(F) = RUURRUU$ and the corresponding tangent line is $y = x + 1$, so $r(F) = 4 - 1 = 3$.

## 3. A decomposition of the poset $2^{[v]}$

In this section, first we introduce an algorithm to construct a chain of sets of positive integers in which all the sets have the same rank. Then we develop this method to decompose $2^{[v]}$ into symmetric skipless chains for a given value of $v$. The algorithm is extendable in the sense that applying it to $2^{[v+1]}$ and eliminating $v + 1$ yields the same result as applying it to $2^{[v]}$.

To construct a chain which includes a given set $F$ of positive integers, two methods are required: A method to find the successor $F^+$ of $F$ and another one is to find the predecessor $F^-$ of $F$. We simply define $F^+ := F \cup \{a\}$, where $a$ is the minimal positive integer which has not appeared in $\mathcal{T}(F)$. On the other hand, $F^- = F \setminus \{b\}$ where $b$ is the element above the rightmost j in $\mathcal{T}(F)$; if there is no j in the second row of $\mathcal{T}(F)$ then $F$ has full rank and no predecessor. It is easy to prove that $(F^+)^- = F$ and if $F$ does not have full rank, then $(F^-)^+ = F$.

**Example 2.** Consider the set $F$ and the corresponding tableau $\mathcal{T}(F)$ as in Example 1: The number 4 is the least positive integer which has not appeared in $\mathcal{T}(F)$, hence $F^+ = F \cup \{4\}$. Now $\mathcal{T}(F^+)$ is as follows:

$$\mathcal{T}\bigl(F^+\bigr) = \begin{pmatrix} 2 & 3 & 4 & 7 & 8 \\ 1 & \mathrm{j} & \mathrm{j} & 6 & 5 \end{pmatrix}.$$

It is immediately seen that $\operatorname{fill}(F^+) = \operatorname{fill}(F)$, $\operatorname{fill}^*(F^+) = \operatorname{fill}^*(F)$ so by (3), $r(F) = r(F^+)$. In fact the element below 4 is j and adding 4 to $F$, has no effect on how the second row is filled. Thus the rank does not change. Given the set $E = F^+$, to recover $F$ consider the tableau $\mathcal{T}(E)$ and eliminate the element above the rightmost j from $E$ to obtain $E^-$. It is seen that $F = E^-$.

Now we use the above algorithm to find a decomposition of $2^{[v]}$ into symmetric skipless chains for a fixed positive integer $v$. Given a set $F \subseteq [v]$, we would like to construct the chain which contains this subset. Let $q = |F|$, $p = r(F)$ and $A_q = F$. For $p \leqslant i < q$, define $A_i = A_{i+1}^-$ and for $q \leqslant i < v - p$, let $A_{i+1} = A_i^+$. The set $A_p$ has full rank so $A_p^-$ is not defined; and, $A_p$ is the only element of the chain with this property. On the other hand, $\mathcal{T}(A_{v-p})$ contains totally $(v - p) + p = v$ integers, which means that $A_{v-p}^+$ is no more a subset of $[v]$; moreover, $A_{v-p}$ is the only element of the chain with this property. Hence we have the following chain

$$A_p \to \cdots \to A_q \to \cdots \to A_{v-p}$$

in which $A_{i+1} = A_i^+$ for $p \leqslant i < v - p$. Now choose an element $E$ which has not appeared in this chain and similarly construct the corresponding chain. Continue in this way until no more subset of $[v]$ is remained. Obviously by this construction every element of $2^{[v]}$ appears in a unique chain. Each chain is determined by its first element which is a full-rank subset of $[v]$ and denoted as $\overline{F}$ where $F$ is any set of the chain.

**Example 3.** For $v = 6$, the chains are as follow: The only chain of rank 0 is

$$\emptyset \to 1 \to 12 \to 123 \to 1234 \to 12345 \to 123456.$$

Chains with rank 1 are

$$2 \to 23 \to 234 \to 2345 \to 23456,$$
$$3 \to 13 \to 134 \to 1345 \to 13456,$$
$$4 \to 14 \to 124 \to 1245 \to 12456,$$
$$5 \to 15 \to 125 \to 1235 \to 12356,$$
$$6 \to 16 \to 126 \to 1236 \to 12346.$$

Chains with rank 2 are

$$24 \to 245 \to 2456, \qquad 25 \to 235 \to 2356,$$
$$26 \to 236 \to 2346, \qquad 34 \to 345 \to 3456,$$
$$35 \to 135 \to 1356, \qquad 36 \to 136 \to 1346,$$
$$45 \to 145 \to 1456, \qquad 46 \to 146 \to 1246,$$
$$56 \to 156 \to 1256.$$

Finally, chains with rank 3 are those with only one element as the following

$$246, \qquad 256, \qquad 346, \qquad 356, \qquad 456.$$

**Remark 1.** Considering Example 3, given a set $F$ and a positive integer $m$, it is useful to have a straightforward method to obtain elements in the same chain with distance $m$ from $F$. Let $F^{+m}$ (respectively $F^{-m}$) be the set in the same chain as $F$ which has $m$ more (respectively $m$ less) elements. Let $a_1 < \cdots < a_{|F|}$ be the elements of $F$, and

$$B = \bigcup_{i=1}^{|F|+1} \{x \in \mathbb{Z}\colon\ a_{i-1} + 1 \leqslant x \leqslant a_i - 2\},$$

where $a_0 = 0$ and $a_{|F|+1} = +\infty$. By induction on $m$ it is seen that to construct $F^{+m}$ from $F$, it is enough to add $m$ least elements of $B$ to $F$. To construct $F^{-m}$ from $F$, one should delete the elements corresponding to $m$ rightmost j's in $\mathcal{T}(F)$ from $F$.

**Remark 2.** The number of full-rank subsets of $[v]$ with rank $\leqslant r$, for $0 \leqslant r \leqslant v/2$, is $\binom{v}{r}$. To see this it is enough to correspond to each subset $F$ with $r(F) \leqslant r$, the $r$-subset in the same chain. Then it follows that the number of full-rank subsets with rank $r$ (or equivalently the number of chains with rank $r$) is $\binom{v}{r} - \binom{v}{r-1}$.

## 4. The inclusion matrix $W_{\bar{t}k}(v)$

We begin this section by recalling some definitions from matrix theory. A *unimodular matrix* is a square integral matrix with determinant $\pm 1$, or equivalently whose inverse is integral. It is well known that for a given integral matrix $A$ of rank $r$, there exist unimodular matrices $U$ and $V$ such that $UAV = (D \mid O)$, in which $D$ is a diagonal matrix with the positive integers $d_1, \ldots, d_r$ on its diagonal such that $d_1 | \cdots | d_r$. The matrix $(D \mid O)$ is called *Smith normal form* of $A$. Moreover,

$$d_i = f_i / f_{i-1}, \tag{4}$$

where $f_0 = 1$ and $f_k$ is the greatest common divisor of all minors of $A$ of order $k$, $1 \leqslant k \leqslant r$. For more on Smith form see [10, pp. 26–33].

In this section the rank chain which was introduced in the previous section is used to construct a new inclusion matrix from $W_{tk}$. Any row index $T$ of $W_{tk}$ is replaced by $\overline{T}$, the indices of columns are kept, and the new 0, 1-entries, like those of $W_{tk}$, are determined by the inclusion relation. We denote this matrix by $W_{\bar{t}k}$. We shall prove that the row space of $W_{\bar{t}k}(v)$ is the same as that of $W_{tk}(v)$. It is also shown that the Smith form of $W_{\bar{t}k}$ is $(I \mid O)$, where $I$ is the identity matrix.

Let $R_{it}$ be the inclusion matrix whose rows are indexed by the all $i$-subsets of $[v]$ of rank $i$, and the columns by $\binom{[v]}{t}$. Note that by Remark 2, $R_{it}$ has exactly $\binom{v}{i} - \binom{v}{i-1}$ rows. Then

$$W_{\bar{t}k} = \begin{array}{|c|} \hline R_{0k} \\ \hline R_{1k} \\ \hline \vdots \\ \hline R_{tk} \\ \hline \end{array}. \tag{5}$$

We observe that

$$R_{it} W_{tk} = \binom{k-i}{t-i} R_{ik}, \quad i \leqslant t \leqslant k \leqslant v. \tag{6}$$

This holds because for an $i$-subset $S$ and a $k$-subset $K$ of $[v]$,

$$(R_{it} W_{tk})(S, K) = \sum_{T \in \binom{[v]}{t}} R_{it}(S, T) W_{tk}(T, K).$$

The right-hand side is the number of $t$-subsets $T$ such that $S \subseteq T \subseteq K$, and this number is $\binom{k-i}{t-i}$ if $S \subseteq K$, and 0 otherwise.

It follows from (6) that

$$W_{\bar{i}t} W_{tk} = \begin{array}{|c|} \hline \binom{k}{t} R_{0k} \\ \hline \binom{k-1}{t-1} R_{1k} \\ \hline \vdots \\ \hline \binom{k-i}{t-i} R_{ik} \\ \hline \end{array}. \tag{7}$$

Define the $\binom{v}{t} \times \binom{v}{t}$ diagonal matrix $D_{\bar{t}k}(v)$ to be

$$\operatorname{diag}\left(\binom{k-i}{t-i}^{\binom{v}{i}-\binom{v}{i-1}},\ i=0,1,\ldots,t\right),$$

where the exponents indicate the multiplicity. Then by (7),

$$W_{\bar{t}t}W_{tk} = D_{\bar{t}k}W_{\bar{t}k}. \tag{8}$$

**Theorem 1.** *The Smith normal form of $W_{\bar{t}k}$, $t \leqslant k \leqslant v-t$, is $(I \mid O)$, where $I$ is the identity matrix of order $\binom{v}{t}$.*

**Proof.** By (4), it is enough to show that $W_{\bar{t}k}$ has a unimodular submatrix of order $\binom{v}{t}$. By induction on $v+t$ we show that the submatrix of $W_{\bar{t}k}(v)$ consists of the columns indexed by a subset of rank $\leqslant t$, denoted by $A_{tk}(v)$, is the desired one. The assertion is clear for $v \leqslant 2$ with $t \leqslant 1$. Let $v \geqslant 3$, and $t \geqslant 1$. First let $k=t$. In an appropriate ordering we have

$$W_{\bar{t}t} = \begin{array}{|c|c|}\hline A_{t-1,t} & * \\ \hline O & I \\ \hline \end{array},$$

where $I$ is the identity matrix of order $\binom{v}{t} - \binom{v}{t-1}$, whose rows and columns are indexed by subsets of rank $t$. By induction, $A_{t-1,t}$ is unimodular and so is $W_{\bar{t}t}$.

Let $t<k<v-t$. The rows and columns of $W_{\bar{t}k}$ partitioned according to whether or not they contain the element $v$ as well as the following fact

$$\begin{aligned}&\bigl\{T \subseteq [v]:\ r(T) \leqslant t\bigr\}\\ &\quad = \bigl\{T \cup \{v\}:\ T \subseteq [v-1],\ r(T) \leqslant t-1\bigr\} \cup \bigl\{T \subseteq [v-1]:\ r(T) \leqslant t\bigr\}.\end{aligned} \tag{9}$$

This gives us the following decomposition of $W_{\bar{t}k}$

$$W_{\bar{t}k}(v) = \begin{array}{|c|c|}\hline W_{\overline{t-1},k-1}(v-1) & O \\ \hline W_{\bar{t},k-1}(v-1) & W_{\bar{t}k}(v-1) \\ \hline \end{array}. \tag{10}$$

By (9), we observe that

$$A_{tk}(v) = \begin{array}{|c|c|}\hline A_{t-1,k-1}(v-1) & O \\ \hline * & A_{t,k}(v-1) \\ \hline \end{array}.$$

The induction hypothesis implies that the blocks on the diagonal are unimodular, thus $A_{tk}(v)$ is unimodular. Now let $k=v-t$. Note that in this case the number of rows of the matrix $W_{\bar{t}k}(v-1)$ is more than the number of its columns, so Eq. (10) cannot be used directly. But using the partitions

$$W_{\bar{t}k}(v-1) = \begin{array}{|c|}\hline W_{\overline{t-1},k}(v-1) \\ \hline R_{tk}(v-1) \\ \hline \end{array}, \qquad W_{\bar{t},k-1}(v-1) = \begin{array}{|c|}\hline W_{\overline{t-1},k-1}(v-1) \\ \hline R_{t,k-1}(v-1) \\ \hline \end{array},$$

Eq. (10) becomes

$$W_{\bar{t}k}(v) = \begin{array}{|c|c|}\hline W_{\overline{t-1},k-1}(v-1) & O \\ \hline W_{\overline{t-1},k-1}(v-1) & W_{\overline{t-1},k}(v-1) \\ \hline R_{t,k-1}(v-1) & R_{tk}(v-1) \\ \hline \end{array}.$$

Therefore it is easily seen that

$$\begin{aligned}\det W_{\bar{t}k}(v) &= \pm\det\begin{array}{|c|c|}\hline W_{\overline{t-1},k-1}(v-1) & O \\ \hline R_{t,k-1}(v-1) & R_{tk}(v-1) \\ \hline O & W_{\overline{t-1},k}(v-1) \\ \hline \end{array} \\ &= \pm\det\begin{array}{|c|c|}\hline W_{\bar{t},k-1}(v-1) & \dfrac{O}{R_{tk}(v-1)} \\ \hline O & W_{\overline{t-1},k}(v-1) \\ \hline \end{array}.\end{aligned}$$

By induction, being square matrices, $W_{\bar{t},k-1}(v-1)$ and $W_{\overline{t-1},k}(v-1)$ are both unimodular and so is $W_{\bar{t}k}(v)$. This completes the proof. □

The following corollary is an immediate consequence of Theorem 1.

**Corollary 1.** *For any prime $p$, the matrix $W_{\bar{t}k}$ has full $p$-rank.*

**Corollary 2.** *Over any field of characteristic not in the set of primes $\{p\colon\ p \mid \binom{k-i}{t-i},\ i=0,\dots,t\}$, the row space of $W_{\bar{t}k}$ is the same as that of $W_{tk}$.*

**Proof.** Over a field of characteristic not in the set of primes $\{p\colon\ p \mid \binom{k-i}{t-i},\ i=0,\dots,t\}$, the matrix $D_{\bar{t}k}$ is nonsingular. So the result follows from (8). □

The following corollary was first proved by Wilson [12]. Another proof is also given by Bier [3].

**Corollary 3.** *If $t \leqslant k \leqslant v-t$, then $W_{tk}$ has a diagonal form which is the $\binom{v}{t} \times \binom{v}{k}$ diagonal matrix with diagonal entries*

$$\binom{k-i}{t-i} \quad \textit{with multiplicity} \quad \binom{v}{i} - \binom{v}{i-1}, \qquad i = 0, 1, \dots, t.$$

**Proof.** By the proof of Theorem 1, $W_{\bar{t}k} = (A \mid B)$, where $A$ is the unimodular submatrix. We have

$$D_{\bar{t}k} W_{\bar{t}k} = \begin{array}{|c|c|}\hline D_{\bar{t}k} & O \\ \hline \end{array} \begin{array}{|c|c|}\hline A & B \\ \hline O & I \\ \hline \end{array},$$

where $I$ is the identity matrix of order $\binom{v}{k} - \binom{v}{t}$. Therefore by (8),

$$W_{\bar{t}t} W_{tk} \begin{array}{|c|c|}\hline A^{-1} & -A^{-1}B \\ \hline O & I \\ \hline \end{array} = \begin{array}{|c|c|}\hline D_{\bar{t}k} & O \\ \hline \end{array}.$$

Since the matrices $W_{\bar{t}t}$ and $A^{-1}$ are both unimodular, the proof is completed. □

**Remark 3.** We may reformulate the system of (1) in terms of the matrix $W_{\bar{t}k}$. By (8), $\mathbf{x}$ is a solution of (1) if and only if $W_{\bar{t}k}\mathbf{x} = \lambda\mathbf{h}$, in which

$$\mathbf{h} = D_{\bar{t}k}^{-1} W_{\bar{t}t} \mathbf{1} = \left( \frac{\binom{v-i}{t-i}}{\binom{k-i}{t-i}}^{\binom{v}{i}-\binom{v}{i-1}},\ i = 0, \dots, t \right)^{\top},$$

where the exponents indicate the multiplicity.

The following corollary was first proved by Wilson in [11]. An alternative proof is given in [12]. When applied to a constant vector it results in necessary and sufficient conditions on the existence of signed $t$-designs. This case was also proved in [6].

**Corollary 4.** *Let $t \leqslant k \leqslant v-t$. The system*

$$W_{tk}\mathbf{x} = \mathbf{b} \tag{11}$$

*has an integral solution if and only if*

$$\frac{1}{\binom{k-i}{t-i}} R_{it}\mathbf{b} \tag{12}$$

*is integral for $i = 0, \dots, t$.*

**Proof.** By (8), $\mathbf{x}$ is a solution of (11) if and only if $W_{\bar{t}k}\mathbf{x} = \mathbf{b}'$, in which

$$\mathbf{b}' = D_{\bar{t}k}^{-1} W_{\bar{t}t}\mathbf{b} = \begin{array}{|c|} \hline \frac{1}{\binom{k}{t}} R_{0t} \\ \hline \frac{1}{\binom{k-1}{t-1}} R_{1t} \\ \hline \vdots \\ \hline \frac{1}{\binom{k-t}{t-t}} R_{tt} \\ \hline \end{array}\,\mathbf{b}. \tag{13}$$

If (11) has an integral solution, then the right-hand side of (13) has to be integral, this proves the necessity. Now assume $\mathbf{b}$ satisfies (12). So $\mathbf{b}'$ is integral. By the proof of Theorem 1, $W_{\bar{t}k} = (A \mid B)$, where $A$ is a unimodular matrix. So its inverse, $A^{-1}$ is integral. Let $\mathbf{x} = \binom{A^{-1}\mathbf{b}'}{\mathbf{0}}$. Then $\mathbf{x}$ is integral and $W_{\bar{t}k}\mathbf{x} = \mathbf{b}'$. Therefore (11) has an integral solution. □

**Remark 4.** Let $M_{tk}(v)$ denote a matrix which is obtained by stacking the matrices $W_{0k}(v), \dots, W_{tk}(v)$ one on top of the other. The rows of $W_{\bar{t}k}$ are in fact a basis for the row space of $M_{tk}(v)$. The existence of a basis for the row space of $M_{tk}(v)$ with the property that it contains exactly $\binom{v}{i} - \binom{v}{i-1}$ rows from $W_{ik}(v)$, for $i = 0, \dots, t$, was first observed by Wilson [12]. A concrete basis for $M_{tk}(v)$ which is coincident with $W_{\bar{t}k}$ was demonstrated by Bier [3].

**Remark 5.** We may represent the result of Corollary 2 in a more general setting. More precisely if we replace each row index $T$ of $W_{tk}$ with $\operatorname{rank}(T) \leqslant t-1$ by $T^-$ and keep those indices of rank $t$ and form a new inclusion matrix, then the resulting matrix is again row-wise equivalent to $W_{tk}$. To see this, we fix $t$ and $k$ and simply call this new matrix $U$. Then in an appropriate ordering, $U = \binom{W_{t-1,k}}{R_{tk}}$. Since $W_{t-1,k}$ and $W_{\overline{t-1},k}$ have the same row space, the matrix $U$ has the same row space as $\binom{W_{\overline{t-1},k}}{R_{tk}} = W_{\bar{t}k}$. So by Corollary 2, we are done. In the same sense if we replace each row index $T$ of $W_{tk}$ with $\operatorname{rank}(T) \leqslant t-m$ by $T^{-m}$ and each row index $T$ of rank $> t-m$ by $\overline{T}$ and form a new inclusion matrix, then the resulting matrix is again row-wise equivalent to $W_{tk}$.

**Remark 6.** It is known that in any decomposition of $2^{[v]}$ into symmetric skipless chains the number of chains beginning at level $i$ is exactly $\binom{v}{i} - \binom{v}{i-1}$. One may guess that the result of Corollary 1 remains true for any such decomposition (instead of decomposition of rank chains). But here is a counterexample. Consider the following decomposition of the poset $2^{[4]}$ into symmetric skipless chains:

$$\emptyset \to 4 \to 14 \to 124 \to 1234,$$
$$1 \to 13 \to 134, \qquad 2 \to 23 \to 123, \qquad 3 \to 24 \to 234,$$
$$12, \qquad 34.$$

If we replace the row indices of $W_{2,2}(4)$ by the first elements of the chains containing each, we have the following inclusion matrix which is singular:

$$\begin{array}{c|cccccc|} & 12 & 13 & 14 & 23 & 24 & 34 \\ \hline \emptyset & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 1 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 & 1 \\ 12 & 1 & 0 & 0 & 0 & 0 & 0 \\ 34 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array}.$$

## 5. The inclusion matrix $W_{t\underline{k}}(v)$

In this section we construct another inclusion matrix which is somehow equivalent to $W_{tk}(v)$ and have the same Smith form as $W_{\bar{t}k}(v)$. To construct such a matrix, similar to the previous one, we need a set of (suitable) chains which partition the poset of $2^{[v]}$. These chains are obtained as follows. In each chain which is formed by using the notion of rank in Section 3, replace any subset by its complement. In this way we have new chains which partition the poset $2^{[v]}$. Now in $W_{tk}$ replace any column index $K$ by the largest element of the chain containing $K$, namely $\underline{K} := [v] \setminus (\overline{[v] \setminus K})$, keep the indices of the rows and form a new inclusion matrix. We denote this matrix by $W_{t\underline{k}}(v)$. If $k > v/2$ (respectively $k \leqslant v/2$), then for each $k$-subset $K$, $k \leqslant |\underline{K}| \leqslant v$ (respectively $v - k \leqslant |\underline{K}| \leqslant v$). Therefore the minimum size of the indices of the columns of $W_{t\underline{k}}(v)$ is $k^* := \max\{k, v-k\}$. Let $Q_{tj}$, $k^* \leqslant j \leqslant v$, be the submatrix of $W_{t\underline{k}}(v)$, consists of the columns of size $j$. Then we have the following block decomposition

$$W_{t\underline{k}} = \boxed{Q_{tk^*} \,\big|\, Q_{t,k^*+1} \,\big|\, \cdots \,\big|\, Q_{tv}}. \tag{14}$$

By the same reason as (6), we have

$$W_{it} Q_{tj} = \binom{j-i}{t-i} Q_{ij}, \quad i \leqslant t \leqslant k \leqslant v.$$

Therefore it is seen that

$$W_{it} W_{t\underline{k}} = \boxed{\tbinom{k^*-i}{t-i} Q_{ik^*} \,\big|\, \tbinom{k^*+1-i}{t-i} Q_{i,k^*+1} \,\big|\, \cdots \,\big|\, \tbinom{v-i}{t-i} Q_{iv}}.$$

Note that the number of columns of $Q_{tj}$ is equal to the number of rank chains whose smallest elements are of size $v - j$. This number is $\binom{v}{v-j} - \binom{v}{v-j-1} = \binom{v}{j} - \binom{v}{j+1}$. So it follows that

$$W_{tk} W_{k\underline{k}} = W_{t\underline{k}} D_{t\underline{k}}, \tag{15}$$

where $D_{t\underline{k}}$ is the $\binom{v}{k} \times \binom{v}{k}$ diagonal matrix with diagonal entries $\binom{j-t}{k-t}$ with multiplicity $\binom{v}{j} - \binom{v}{j+1}$, $j = k^*, \ldots, v$. Note that

$$\left\{\underline{T}\colon T \in \binom{[v]}{t}\right\} = \left\{[v] \setminus \overline{T}\colon T \in \binom{[v]}{t}\right\}.$$

Let $T_1, T_2 \in \binom{[v]}{t}$, then from the simple relation

$$T_1 \subseteq \underline{T_2} \quad \Leftrightarrow \quad [v] \setminus \underline{T_2} \subseteq [v] \setminus T_1,$$

it follows that

$$W_{t\underline{t}} = \begin{cases} W^{\top}_{\overline{v-t},v-t}, & t > v/2; \\ W^{\top}_{\bar{t},v-t}, & t \leqslant v/2. \end{cases}$$

Being square matrices, $W_{\bar{t},v-t}$ and $W_{\overline{v-t},v-t}$ are unimodular, by Theorem 1, and so is $W_{t\underline{t}}$. If $t \leqslant k \leqslant v-t$, then $k^* \leqslant t^*$. So it follows from (14) that $W_{t\underline{t}}$ is a submatrix of $W_{t\underline{k}}$. Thus $W_{t\underline{k}}$ has a unimodular submatrix of order $\binom{v}{t}$.

We close our paper with the following summary of the results of this section.

**Theorem 2.** *Let $t \leqslant k \leqslant v-t$. Then the following hold.*

(i) *$W_{t\underline{k}}$ has $(I \mid O)$ as Smith form, and consequently has full $p$-rank for any prime $p$.*
(ii) *$W_{t\underline{k}}\mathbf{x} = \lambda\mathbf{1}$ if and only if $W_{k\underline{k}}D^{-1}_{t\underline{k}}\mathbf{x}$ is a solution of* (1).

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