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Journal of Algebra 320 (2008) 2328–2348

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

L-algebras, self-similarity, and l-groups

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Received 19 October 2007

Available online 23 July 2008

Communicated by Efim Zelmanov

Dedicated to B. V. M.

Abstract

Every set X with a binary operation satisfying $(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$ corresponds to a solution of the quantum Yang–Baxter equation if the left multiplication is bijective [W. Rump, A decomposition theorem for square-free unitary solutions of the quantum Yang–Baxter equation, Adv. Math. 193 (2005) 40–55]. The same equation becomes a true statement of propositional logic if the binary operation is interpreted as implication. In the present paper, *L-algebras*, essentially defined by the mentioned equation, are introduced and studied. For example, Hilbert algebras, locales, (left) hoops, (pseudo) MV-algebras, and l-group cones, are L-algebras. The main result states that every L-algebra admits a self-similar closure. In a further step, a structure group $G(X)$ is associated to any L-algebra X. One more equation implies that the structure group $G(X)$ is lattice-ordered. As an application, we characterize the L-algebras with a natural embedding into the negative cone of an l-group. In particular, this implies Mundici's equivalence between MV-algebras and unital abelian l-groups, and Dvurečenskij's non-commutative generalization.

Keywords: L-algebra; Self-similar; Cycle set; Hoop; l-Group; Yang–Baxter equation

Introduction

For a set X with a binary operation $X \times X \xrightarrow{\cdot} X$, let $M(X)$ denote the free monoid over X (with product written as juxtaposition). We show (Theorem 2) that the binary operation $\cdot$ admits a unique extension to $M(X)$ such that the equations

$$ab \cdot c = a \cdot (b \cdot c), \tag{A}$$

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.05.033

$$a \cdot bc = \big((c \cdot a) \cdot b\big)(a \cdot c) \tag{S}$$

and $1 \cdot a = a$ hold for all $a, b, c \in M(X)$. We prove that if the equation

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z) \tag{L}$$

holds for all $x, y, z \in X$, it carries over to $M(X)$.

Eq. (L) occurs in two rather different contexts. In [34], we call X a *cycle set* if it satisfies (L) so that the left multiplication $y \mapsto x \cdot y$ is bijective for all $x \in X$. By [34], Proposition 1, cycle sets correspond to a class of solutions of the quantum Yang–Baxter equation. On the other hand, Eq. (L) arose in algebraic logic [7,35]. If the elements of X are viewed as propositions, and $x \cdot y$ is interpreted as implication $x \to y$, Eq. (L) becomes a true statement in classical or intuitionistic logic, as well as in other logical systems like Łukasiewicz' infinite-valued propositional logic [10,11].

Algebraically, the logic of Łukasiewicz is represented by MV-algebras [10,22]. D. Mundici [29] proved that the category of MV-algebras is equivalent to that of abelian l-groups (the "l" stands for "lattice ordered") with a strong order unit. An extension to non-commutative l-groups was pursued by several authors [9,16,19,20,26,32,33] (see also Bosbach [7] who studied similar questions two decades earlier). These papers deal with two negations (or two types of logical implication) which causes some complication. We will show that one implication suffices, and that few concepts are well-suited to characterize pseudo MV-algebras and other logical systems embeddable into an l-group. In this sense, the occurrence of two implications signifies that an l-group includes a dual pair of logical systems with implication according to the natural connection between its positive and its negative cone. An analogous phenomenon arises in the theory of cycle sets (see [34], Proposition 2).

Assume that X satisfies (L). We call an element $1 \in X$ a *logical unit* if $1 \cdot x = x$ and $x \cdot x = x \cdot 1 = 1$ holds for all $x \in X$. Such an element 1 is necessarily unique and represents a true proposition. If $1 \in X$ exists and $x \cdot y = y \cdot x = 1$ implies $x = y$ (i.e. equivalent propositions are equal), we call X an *L-algebra*. It will be our main concern in this article to associate a structure group $G(X)$ to any L-algebra X and analyse the case where $G(X)$ is an l-group. Moreover, it will turn out that the concept of L-algebra is fundamental in the sense that various algebraic structures, even with several operations like Heyting algebras, (one-sided) hoops, (pseudo) MV-algebras or l-group cones, are definable as L-algebras (so that the other operations are uniquely determined by the implication).

In Section 1, we show that surjective morphisms between L-algebras can be described by ideals. We call an L-algebra X *self-similar* if every left multiplication $y \mapsto x \cdot y$ maps the downset $\downarrow x$ bijectively onto the whole X. For a self-similar L-algebra X, these bijections $\downarrow x \xrightarrow{\sim} X$ are isotone with respect to the partial order $x \leqslant y \ :\Leftrightarrow\ x \cdot y = 1$. We show that self-similarity of an L-algebra can be described by equations, e.g., by the above Eqs. (A) and (S) together with the hoop equation

$$(a \cdot b)a = (b \cdot a)b. \tag{H}$$

This means that a self-similar L-algebra $(X, \cdot)$ has a unique operation $(x, y) \mapsto xy$ such that (A), (S), and (H) hold (Theorem 1).

We prove that every L-algebra X admits a minimal embedding into a self-similar L-algebra $S(X)$, the *self-similar closure* of X. Furthermore, $S(X)$ is a $\wedge$-semilattice with $a \wedge b = (a \cdot b)a$.

We obtain $S(X)$ from $M(X)$ by factoring out a certain congruence relation. Since by virtue of (H), $S(X)$ satisfies the left Ore condition, it has a quotient group $G(X)$, and there is a natural map

$$q : X \to G(X).$$

We call $G(X)$ the *structure group* of X.

Thus every L-algebra X is related to a group $G(X)$. The map q has a logical meaning, namely, $q(x) = q(y)$ holds if and only if x and y are *equipollent* in $S(X)$. If X has a smallest element 0 (= “false”), equipollence in X means that x and y have the same negation: $x \cdot 0 = y \cdot 0$. Note, however, that $S(X)$ cannot have a smallest element unless $X = \{1\}$.

Besides the self-similar closure, a second main concept is necessary to relate L-algebras to l-groups. We call an L-algebra X *semiregular* if the equation

$$\big((x \cdot y) \cdot z\big) \cdot \big((y \cdot x) \cdot z\big) = \big((x \cdot y) \cdot z\big) \cdot z \tag{R}$$

holds in X, or equivalently, if $S(X)$ satisfies

$$a(b \wedge c) = ab \wedge ac.$$

We show that the structure group $G(X)$ of a semiregular L-algebra X is an l-group (Theorem 4), so that q induces an L-algebra morphism $X \to G(X)_-$ into the negative cone of $G(X)$. In this way, we obtain a characterization of L-algebras isomorphic to or embeddable into the negative cone of an l-group. In particular, we characterize L-algebras equivalent to pseudo MV-algebras [16,20].

Using the concept of self-similarity, the commutative case becomes particularly simple. First, the inequality $\leqslant$ in (R) is equivalent to

$$x \cdot (y \cdot x) = 1 \tag{K}$$

and implies that $G(X)$ is a partially ordered group. Now $S(X)$ is commutative if and only if X satisfies the identities (K) and

$$(x \cdot y) \cdot y = (y \cdot x) \cdot x$$

known from the theory of BCK-algebras [25]. Such L-algebras X coincide with those which can be embedded into the negative cone of an abelian l-group, and then $G(X)_-$ coincides with $S(X)$. In particular, this gives a new method to relate an MV-algebra X to an abelian l-group (cf. [29]). If X is Boolean, the embedding $X \hookrightarrow G(X)$ is uniquely determined by the l-group $G(X)$ which is a Specker group in this case.

We briefly discuss Hilbert algebras as special L-algebras. For example, the open sets of a topological space X form a Hilbert algebra $\mathcal{O}(X)$. We characterize the semiregular Hilbert algebras, i.e. those for which the structure group is an l-group.

1. L-Algebras

Let $(X, \to)$ be a *magma*, i.e. a set X with a binary operation $\to$. An element $e \in X$ will be called a *logical unit* if

$$x \to x = x \to e = e; \qquad e \to x = x \tag{1}$$

holds for all $x \in X$. If e, e' are logical units, then $e = e \to e = e'$. Thus a logical unit is necessarily unique. We henceforth denote it by 1. If we interpret $x \to y$ as "x implies y," Eq. (1) characterizes the truth value $e = 1 =$ "true." We call $(X, \to)$ a *cycloid* if it satisfies

$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z). \tag{L}$$

This equation holds in classical and intuitionistic logic, and also in logics based on a continuous t-norm [17,23], among which the Łukasiewicz infinite-valued logic [10] forms an important special case. On the other hand, cycloids with bijective left multiplication correspond to a class of solutions of the quantum Yang–Baxter equation [34].

Definition 1. Let $(X, \to)$ be a *unital* cycloid, i.e. a cycloid with a logical unit. We call $I \subset X$ an *ideal* if the following hold for all $x, y \in X$:

$$1 \in I, \tag{I0}$$

$$x,\ x \to y \in I \quad \Rightarrow \quad y \in I, \tag{I1}$$

$$x \in I \quad \Rightarrow \quad (x \to y) \to y \in I, \tag{I2}$$

$$x \in I \quad \Rightarrow \quad y \to x,\ y \to (x \to y) \in I. \tag{I3}$$

Property (I3) can be dropped if X satisfies

$$x \to (y \to x) = 1 \tag{K}$$

which holds for every BCK-algebra [12,25]. In fact, assume that (K) holds, and $x \in I$. Then $y \to x \in I$ follows by (I1) since $x \to (y \to x) = 1 \in I$.

Proposition 1. *Let $(X, \to)$ be a unital cycloid. Every ideal I defines a congruence*

$$x \sim y \quad :\Leftrightarrow \quad x \to y,\ y \to x \in I. \tag{2}$$

Conversely, every congruence $\sim$ defines an ideal $I := \{x \in X \mid x \sim 1\}$.

Proof. For a given ideal I, we have to show that (2) is an equivalence such that $x \sim y$ implies $(z \to x) \sim (z \to y)$ and $(x \to z) \sim (y \to z)$. From (I0) we get $x \sim x$, while the symmetry of $\sim$ follows by definition. Assume that $x \sim y \sim z$. Then (I3) gives $(x \to y) \to (x \to z) = (y \to x) \to (y \to z) \in I$. Hence (I1) yields $x \to z \in I$. Thus $\sim$ is an equivalence.

Assume that $x \sim y$. Then $(z \to x) \to (z \to y) = (x \to z) \to (x \to y) \in I$, whence $(z \to x) \sim (z \to y)$ by symmetry. To prove $(x \to z) \sim (y \to z)$, we show first that

$$u,\ (u \to v) \to w \in I \quad \Rightarrow \quad v \to w \in I \tag{3}$$

holds for $u, v, w \in X$. By (L) and (I3), the premise of (3) gives $(v \to (u \to v)) \to (v \to w) = ((u \to v) \to v) \to ((u \to v) \to w) \in I$. Therefore, (I1) and (I3) together with $u \in I$ yields $v \to w \in I$. This proves (3). Using (I2), we get $((x \to y) \to (x \to z)) \to (y \to z) = ((y \to x) \to (y \to z)) \to (y \to z) \in I$, and thus (3) gives $(x \to z) \to (y \to z) \in I$. So $\sim$ is a congruence.

Conversely, let $\sim$ be a congruence. Then (I0) trivially holds. Assume that $x, x \to y \in I$. Then $1 \sim x$ and therefore, $y \sim (1 \to y) \sim (x \to y) \sim 1$. This gives (I1). To prove (I2) and (I3), let $x \in I$. Then $(x \to y) \sim (1 \to y) \sim y$, and thus $((x \to y) \to y) \sim (y \to y) \sim 1$. Furthermore, we have $(y \to x) \sim (y \to 1) \sim 1$ and $(x \to y) \sim (1 \to y) \sim y$, whence $(y \to (x \to y)) \sim (y \to y) \sim 1$. $\square$

In particular, the ideal $\{1\}$ defines the congruence

$$x \sim y \quad \Leftrightarrow \quad x \to y = y \to x = 1. \tag{4}$$

If this congruence (4) reduces to equality for a unital cycloid X, we call X an *L-algebra*. Thus L-algebras provide the semantics for a rather general type of logic. We introduce the relation

$$x \leqslant y \quad :\Leftrightarrow \quad x \to y = 1. \tag{5}$$

Proposition 2. *Let $(X, \to)$ be an L-algebra. Then* (5) *is a partial order on X, and the implication*

$$y \leqslant z \quad \Rightarrow \quad x \to y \leqslant x \to z \tag{6}$$

holds for all $x, y, z \in X$. Furthermore, X satisfies (K) *if and only if*

$$x \leqslant y \quad \Rightarrow \quad y \to z \leqslant x \to z \tag{7}$$

holds for $x, y, z \in X$.

Proof. Assume $y \leqslant z$, i.e. $y \to z = 1$. Then $(x \to y) \to (x \to z) = (y \to x) \to (y \to z) = 1$, which yields (6). In particular, this implies that (5) is transitive, and thus a partial order. If (K) holds, then $x \leqslant y$ yields $y \to z \leqslant (y \to x) \to (y \to z) = (x \to y) \to (x \to z) = x \to z$. Conversely, let (7) be satisfied. Then $y = 1$ gives $z \leqslant x \to z$, whence (K) follows. $\square$

Corollary 1. *For an L-algebra X, there is a bijective correspondence between ideals and congruences $\sim$ for which $X/{\sim}$ is an L-algebra.*

Proof. According to Proposition 1, each ideal comes from the congruence it defines. Conversely, any congruence $\sim$ defines an ideal which leads to the congruence $x \approx y \Leftrightarrow (x \to y) \sim (y \to x) \sim 1$. Thus $\sim$ and $\approx$ coincide if and only if $X/{\sim}$ is an L-algebra. $\square$

For an ideal I of an L-algebra X, the L-algebra $X/{\sim}$ of Corollary 1 will be denoted by X/I.

Corollary 2. *Let X be an L-algebra, and $x, y \in X$. Then $x = y$ if and only if $x \to z = y \to z$ for all $z \in X$.*

Proof. The latter condition implies $x \to y = y \to y = 1$ and $y \to x = x \to x = 1$, hence $x = y$. $\square$

From now on, we write $x \cdot y$ instead of $x \to y$. For an element x of an L-algebra $(X, \cdot)$, we define the downset

$$\downarrow x := \{y \in X \mid y \leqslant x\}. \tag{8}$$

Definition 2. We call an L-algebra $(X, \cdot)$ *self-similar* if for each $x \in X$, the left multiplication $y \mapsto x \cdot y$ induces a bijection $\downarrow x \xrightarrow{\sim} X$.

Note. For a self-similar L-algebra X, the bijections $\downarrow x \xrightarrow{\sim} X$ are isotone. In fact, Proposition 2 shows that it is monotone increasing. Conversely, if $y, z \leqslant x$ and $x \cdot y \leqslant x \cdot z$, then $y \cdot z = (y \cdot x) \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z) = 1$.

In Section 3, we will show that every L-algebra admits a natural embedding into a self-similar one.

2. Self-similarity

We show first that self-similar L-algebras admit a natural monoid structure.

Definition 3. A monoid H with an additional binary operation $\cdot$ will be called a *left hoop* if the following are satisfied for $a, b, c \in H$.

$$a \cdot a = 1, \tag{E}$$

$$ab \cdot c = a \cdot (b \cdot c), \tag{A}$$

$$(a \cdot b)a = (b \cdot a)b. \tag{H}$$

The monoid operation is expressed by juxtaposition. If this operation is commutative, H turns into a *hoop*, a structure introduced and studied by Büchi and Owens [6], and further investigated by several authors [4,5,17]. One-sided hoops were studied earlier by Bosbach [8] under a different name.

Proposition 3. *Every left hoop H gives rise to an L-algebra $(H, \cdot)$.*

Proof. From (H) and (A), we get (L), and (H) also implies that the relation $a \leqslant b \;:\Leftrightarrow\; a \cdot b = 1$ is antisymmetric. Thus we only have to show that 1 is a logical unit. First, we have $(1 \cdot a) \cdot a = (1 \cdot a)1 \cdot a = (1 \cdot a) \cdot (1 \cdot a) = 1$. Hence $1 \cdot a = ((1 \cdot a) \cdot a)(1 \cdot a) = (a \cdot (1 \cdot a))a = (a1 \cdot a)a = a$ for all $a \in H$. Therefore, we get $a \cdot 1 = (1 \cdot a) \cdot 1 = (1 \cdot a)1 \cdot 1 = (a \cdot 1)a \cdot 1 = (a \cdot 1) \cdot (a \cdot 1) = 1$. $\square$

Axiom (A) of a left hoop H shows that

$$ab \leqslant c \quad \Leftrightarrow \quad a \leqslant b \cdot c \tag{9}$$

holds for $a, b, c \in H$. Therefore, H is completely determined by its underlying L-algebra structure. In other words, left hoops form a special class of L-algebras. Our next result shows that self-similar L-algebras belong to this class.

Theorem 1. *Every self-similar L-algebra is a left hoop.*

Proof. Let H be a self-similar L-algebra, and let $a, b \in L$. Then the restriction of $c \mapsto b \cdot c$ to $\downarrow b$ is a bijection $\downarrow b \xrightarrow{\sim} H$. We define $ab \in \downarrow b$ to be the inverse image of a. Thus ab is uniquely determined by

$$ab \leqslant b; \qquad b \cdot ab = a. \tag{10}$$

We show first that this product is associative. For $a, b, c \in H$, we have $a(bc) \leqslant bc \leqslant c$ and $(ab)c \leqslant c$. Therefore, $a(bc) = (ab)c$ is equivalent to $c \cdot a(bc) = c \cdot (ab)c$, i.e. $c \cdot a(bc) = ab$. By Proposition 2, we have $c \cdot a(bc) \leqslant c \cdot bc = b$. Hence $c \cdot a(bc) = ab$ is equivalent to $b \cdot (c \cdot a(bc)) = a$. Now $b \cdot (c \cdot a(bc)) = (c \cdot bc) \cdot (c \cdot a(bc)) = (bc \cdot c) \cdot (bc \cdot a(bc)) = 1 \cdot a = a$. This proves that H is a semigroup. Furthermore, $a \cdot 1a = 1$ implies that $a \leqslant 1a \leqslant a$, whence $1a = a$. Since $a1 = 1 \cdot a1 = a$, we infer that H is a monoid.

Now (A) follows by $ab \cdot c = (ab \cdot b) \cdot (ab \cdot c) = (b \cdot ab) \cdot (b \cdot c) = a \cdot (b \cdot c)$. Therefore, we get $(a \cdot b)a \cdot c = (a \cdot b) \cdot (a \cdot c) = (b \cdot a) \cdot (b \cdot c) = (b \cdot a)b \cdot c$. By Corollary 2 of Proposition 2, this yields (H). $\square$

For $a, b \in H$, we define

$$a \wedge b := (a \cdot b)a. \tag{11}$$

Like in the commutative case [3], this is an infimum.

Proposition 4. *For a left hoop H, every pair $a, b \in H$ admits an infimum* (11)*, and*

$$a \cdot (b \wedge c) = (a \cdot b) \wedge (a \cdot c) \tag{12}$$

holds for $a, b, c \in H$.

Proof. By (A), we have $(a \cdot b)a \leqslant a, b$. Suppose that $c \leqslant a, b$. Then $c = (c \cdot a)c = (c \cdot b)c$. Hence $c = (c \cdot a)c = (c \cdot a)(c \cdot b)c = ((c \cdot b)c \cdot a)(c \cdot b)c = ((b \cdot c)b \cdot a)(c \cdot b)c = ((b \cdot c) \cdot (b \cdot a))(b \cdot c)b = ((b \cdot a) \cdot (b \cdot c))(b \cdot a)b = ((a \wedge b) \cdot c)(a \wedge b) \leqslant a \wedge b$.

By symmetry, the inequality $\leqslant$ of (12) reduces to $a \cdot (b \wedge c) \leqslant a \cdot b$, which follows by Proposition 2. The inequality $\geqslant$ of (12) can be put into $((a \cdot b) \wedge (a \cdot c))a \leqslant b \wedge c$. Hence, by symmetry, it suffices to show that $((a \cdot b) \wedge (a \cdot c))a \leqslant b$. If we switch a to the right-hand side, this becomes $(a \cdot b) \wedge (a \cdot c) \leqslant a \cdot b$, which is obvious. $\square$

Our next result yields a characterization of self-similarity.

Proposition 5. *For a left hoop H, and $a \in H$, the following are equivalent.*

(a) *The map $b \mapsto ba$ is injective.*
(b) *The map $b \mapsto a \cdot b$ is surjective.*
(c) *$a \cdot ba = b$ for all $b \in H$.*
(d) *$ba \cdot ca = b \cdot c$ for all $b, c \in H$.*
(e) *$a \cdot bc = ((c \cdot a) \cdot b)(a \cdot c)$ holds for all $b, c \in H$.*

Proof. (a) $\Rightarrow$ (c): By (H), we have $(a \cdot ba)a = (ba \cdot a)ba = (b \cdot (a \cdot a))ba = ba$.

(c) $\Rightarrow$ (d): $ba \cdot ca = b \cdot (a \cdot ca) = b \cdot c$.

(d) $\Rightarrow$ (b): This follows if we set $b = 1$.

(b) $\Rightarrow$ (a): Assume that $ba = ca$. Then $ba \cdot d = ca \cdot d$, and thus $b \cdot (a \cdot d) = c \cdot (a \cdot d)$ for all $d \in H$. Hence $b \cdot e = c \cdot e$ for all $e \in H$. By Corollary 2 of Proposition 2, this implies that $b = c$.

So we have shown that (a)–(d) are pairwise equivalent. To include (e), assume that (b) holds. Then $(a \cdot bc) \cdot (a \cdot d) = (bc \cdot a) \cdot (bc \cdot d) = (b \cdot (c \cdot a)) \cdot (b \cdot (c \cdot d)) = ((c \cdot a) \cdot b) \cdot ((c \cdot a) \cdot (c \cdot d)) = ((c \cdot a) \cdot b) \cdot ((a \cdot c) \cdot (a \cdot d)) = ((c \cdot a) \cdot b)(a \cdot c) \cdot (a \cdot d)$ holds for $b, c, d \in H$, whence (e) follows. Conversely, (e) implies (c) if we set $c = a$. □

Corollary 1. *A left hoop H is self-similar if and only if the equivalent conditions of Proposition* 5 *hold for all $a \in H$.*

Proof. By definition, self-similarity implies (b) of Proposition 5. Conversely, let the equivalent statements of Proposition 5 be satisfied. Then Proposition 5(c) shows that the map $\rho_a : H \to \downarrow a$ with $\rho_a(b) := ba$ has a retraction $\lambda_a : \downarrow a \to H$ with $\lambda_a(b) := a \cdot b$. Therefore, Eq. (11) yields $\rho_a = \lambda_a^{-1}$. □

Corollary 2. *For a self-similar left hoop H,*

$$(a \wedge b)c = ac \wedge bc \tag{13}$$

holds for $a, b, c \in H$.

Proof. This follows immediately by Proposition 5(d). □

Corollary 3. *Let I be an ideal of a self-similar L-algebra H. Then H/I is again self-similar, and the natural map $p : H \to H/I$ is a monoid homomorphism.*

Proof. By Theorem 1, H is a left hoop. For $a, b \in H$, we have $p(a) = p(b \cdot ab) = p(b) \cdot p(ab)$. Hence H/I is self-similar by Proposition 5(b). So we get $p(a) = p(b) \cdot p(a)p(b)$. Since $p(ab) \leqslant p(b)$, this gives $p(ab) = p(a)p(b)$. □

For a self-similar left hoop H, Proposition 5(d) yields, in particular,

$$a \leqslant b \quad \Leftrightarrow \quad ac \leqslant bc \tag{14}$$

for $a, b, c \in H$. Here, the implication "$\Rightarrow$" even holds for arbitrary left hoops. The left-hand version of (14) is more delicate. We start with

Proposition 6. *A left hoop H satisfies* (K) *if and only if*

$$b \leqslant c \quad \Rightarrow \quad ab \leqslant ac \tag{15}$$

holds for $a, b, c \in H$.

Proof. Assume (K) and $b \leqslant c$. Then (9) and Proposition 2 imply $a \leqslant c \cdot ac \leqslant b \cdot ac$, and thus $ab \leqslant ac$. Conversely, let (15) be satisfied. Then $c = 1$ yields $ab \leqslant a$, whence $a \leqslant b \cdot a$. $\square$

The converse of (15) will be treated in Section 4.

3. The self-similar closure

The following construction is based on the equations

$$ab \cdot c = a \cdot (b \cdot c), \tag{A}$$

$$a \cdot bc = \big((c \cdot a) \cdot b\big)(a \cdot c) \tag{S}$$

which occurred in the previous section. The 1-cocycle condition (S) characterizes self-similarity by Proposition 5.

Let $(X, \cdot)$ be an arbitrary magma. We denote the free monoid over X by $M(X)$.

Theorem 2. *Let $(X, \cdot)$ be a magma. The binary operation of X admits a unique extension to $M(X)$ such that* (A)*,* (S)*, and $1 \cdot a = a$ hold for all $a, b, c \in M(X)$.*

Proof. For $b = c = 1$, condition (S) turns into $a \cdot 1 = ((1 \cdot a) \cdot 1)(a \cdot 1)$, i.e. $(1 \cdot a) \cdot 1 = 1$. Thus if $1 \cdot a = a$, this gives $a \cdot 1 = 1$. By induction, the binary operation of X uniquely extends to a map $M(X) \times X \overset{\cdot}{\to} X$ which satisfies (A). Using this map, we inductively define a map $X \times M(X) \overset{\cdot}{\to} M(X)$ by $x \cdot 1 := 1$ and

$$x \cdot ya := \big((a \cdot x) \cdot y\big)(x \cdot a) \tag{16}$$

for $x, y \in X$ and $a \in M(X)$. Substituting $a = 1$, we infer that (16) extends the binary operation of X. Let us show that

$$x \cdot ab := \big((b \cdot x) \cdot a\big)(x \cdot b) \tag{17}$$

holds for $x \in X$ and $a, b \in M(X)$. We proceed by induction. For $a = 1$, Eq. (17) is trivial. So let us assume that (17) holds for a given a. For any $y \in X$, it follows that $x \cdot (ya)b = x \cdot y(ab) = ((ab \cdot x) \cdot y)(x \cdot ab) = ((a \cdot (b \cdot x)) \cdot y)((b \cdot x) \cdot a)(x \cdot b) = ((b \cdot x) \cdot ya)(x \cdot b)$. Thus (17) is verified.

Now the map $X \times M(X) \overset{\cdot}{\to} M(X)$ uniquely extends to an operation

$$M(X) \times M(X) \overset{\cdot}{\to} M(X)$$

such that (A) holds in $M(X)$. By construction, this operation on $M(X)$ is compatible with $M(X) \times X \overset{\cdot}{\to} X$. So we have proved that the claimed extension $M(X) \times M(X) \overset{\cdot}{\to} M(X)$ is unique, and it remains to be shown that our construction of it satisfies (S).

We start with the case $c \in X$. Then (S) trivially holds for $a = 1$. Assume that (S) has been verified for some $a \in M(X)$. For any $x \in X$, we then get

$$\begin{aligned} xa \cdot bc = x \cdot (a \cdot bc) = x \cdot \big((c \cdot a) \cdot b\big)(a \cdot c) &= \Big(\big((a \cdot c) \cdot x\big) \cdot \big((c \cdot a) \cdot b\big)\Big)\big(x \cdot (a \cdot c)\big) \\ &= \Big(\big((a \cdot c) \cdot x\big)(c \cdot a) \cdot b\Big)(xa \cdot c) = \big((c \cdot xa) \cdot b\big)(xa \cdot c) \end{aligned}$$

by virtue of (17). So we have verified (S) for $a, b \in M(X)$ and $c \in X$. Furthermore, (S) holds for $c = 1$. For our final induction, assume that (S) holds for a given $c \in M(X)$. Then $a \cdot b(cy) = a \cdot (bc)y = ((y \cdot a) \cdot bc)(a \cdot y) = ((c \cdot (y \cdot a)) \cdot b)((y \cdot a) \cdot c)(a \cdot y) = ((cy \cdot a) \cdot b)(a \cdot cy)$. This completes the proof of (S). $\square$

Note. Equations which hold in X, need not extend to $M(X)$. For example, if $X \neq \emptyset$, then $(M(X), \cdot)$ is never commutative since $1 \cdot x = x \neq 1 = x \cdot 1$ holds for any $x \in X$. The next result shows that Eq. (L) is exceptional.

Proposition 7. *If X is a cycloid, then $M(X)$ is a cycloid.*

Proof. Assume that X is a cycloid. We show first that

$$(x \cdot y) \cdot (x \cdot c) = (y \cdot x) \cdot (y \cdot c) \tag{18}$$

holds for $x, y \in X$ and $c \in M(X)$. The case $c = 1$ is trivial. Assume that Eq. (18) holds for some $c \in M(X)$. For any $z \in X$, we then have $(x \cdot y) \cdot (x \cdot cz) = (x \cdot y) \cdot ((z \cdot x) \cdot c)(x \cdot z) = (((x \cdot z) \cdot (x \cdot y)) \cdot ((z \cdot x) \cdot c))((x \cdot y) \cdot (x \cdot z))$, where $((x \cdot z) \cdot (x \cdot y)) \cdot ((z \cdot x) \cdot c) = ((z \cdot x) \cdot (z \cdot y)) \cdot ((z \cdot x) \cdot c) = ((z \cdot y) \cdot (z \cdot x)) \cdot ((z \cdot y) \cdot c) = ((y \cdot z) \cdot (y \cdot x)) \cdot ((z \cdot y) \cdot c)$. This shows that $(x \cdot y) \cdot (x \cdot cz)$ is symmetric in x and y. Thus (18) is proved.

Next we show that

$$(a \cdot y) \cdot (a \cdot c) = (y \cdot a) \cdot (y \cdot c) \tag{19}$$

holds for $y \in X$ and $a, c \in M(X)$. For $a = 1$, this is true. Thus let (19) be satisfied for some $a \in M(X)$, and let $x \in X$. Then $(xa \cdot y) \cdot (xa \cdot c) = (x \cdot (a \cdot y)) \cdot (x \cdot (a \cdot c)) = ((a \cdot y) \cdot x) \cdot ((a \cdot y) \cdot (a \cdot c)) = ((a \cdot y) \cdot x) \cdot ((y \cdot a) \cdot (y \cdot c)) = ((a \cdot y) \cdot x)(y \cdot a) \cdot (y \cdot c) = (y \cdot xa) \cdot (y \cdot c)$. This proves (19).

Finally, we prove

$$(a \cdot b) \cdot (a \cdot c) = (b \cdot a) \cdot (b \cdot c) \tag{20}$$

for $a, b, c \in M(X)$, which trivially holds for $b = 1$. Assume that (20) holds for a particular $b \in M(X)$, and let $y \in X$. Using (19), this gives $(a \cdot yb) \cdot (a \cdot c) = ((b \cdot a) \cdot y)(a \cdot b) \cdot (a \cdot c) = ((b \cdot a) \cdot y) \cdot ((a \cdot b) \cdot (a \cdot c)) = ((b \cdot a) \cdot y) \cdot ((b \cdot a) \cdot (b \cdot c)) = (y \cdot (b \cdot a)) \cdot (y \cdot (b \cdot c)) = (yb \cdot a) \cdot (yb \cdot c)$. The proof is complete. $\square$

Let $(X, \cdot)$ be a magma. By Theorem 2, the unit element $1 \in M(X)$ satisfies $a \cdot 1 = 1$ and $1 \cdot a = a$ for all a, but it is not a logical unit. To make this possible, we introduce the following equivalence in $M(X)$.

$$a \approx b \quad :\Leftrightarrow \quad \forall\, c, d \in M(X){:}\ (c \cdot a) \cdot d = (c \cdot b) \cdot d. \tag{21}$$

Proposition 8. *The equivalence relation* (21) *is a congruence* (*with respect to both operations*) *in $M(X)$.*

Proof. Assume that $a \approx b$, and let $e \in M(X)$ be arbitrary. Note first that with $c = 1$, Eq. (21) implies $a \cdot d = b \cdot d$ for all d. In particular, $a \cdot e = b \cdot e$. Furthermore, $(c \cdot (e \cdot a)) \cdot d = (ce \cdot a) \cdot d = (ce \cdot b) \cdot d = (c \cdot (e \cdot b)) \cdot d$ yields $e \cdot a \approx e \cdot b$. Next, we have $(c \cdot ae) \cdot d = ((e \cdot c) \cdot a)(c \cdot e) \cdot d = ((e \cdot c) \cdot a) \cdot ((c \cdot e) \cdot d) = ((e \cdot c) \cdot b) \cdot ((c \cdot e) \cdot d)$, which yields $ae \approx be$ by symmetry. Finally, $(c \cdot ea) \cdot d = ((a \cdot c) \cdot e)(c \cdot a) \cdot d = ((a \cdot c) \cdot e) \cdot ((c \cdot a) \cdot d) = ((b \cdot c) \cdot e) \cdot ((c \cdot b) \cdot d)$, which gives $ea \approx eb$. $\square$

For a magma $(X, \cdot)$, we define

$$S(X) := M(X)/\approx. \tag{22}$$

Proposition 9. *Let $(X, \cdot)$ be a magma with a logical unit e. Then 1 is a logical unit of $S(X)$, and the natural map $X \to S(X)$ carries e to 1.*

Proof. We show first that $a \cdot e = e$ and $e \cdot a = a$ holds in $M(X)$. For $a = 1$, these equations are trivial. Proceeding by induction, we assume that they hold for some $a \in M(X)$. For any $x \in X$, we then have $ax \cdot e = a \cdot (x \cdot e) = a \cdot e = e$, and $e \cdot xa = ((a \cdot e) \cdot x)(e \cdot a) = xa$, which proves both equations. For $c, d \in M(X)$, this yields $(c \cdot e) \cdot d = (c \cdot 1) \cdot d$, whence $e \approx 1$. It remains to be shown that $a \cdot a = 1$ holds in $S(X)$. For $a = 1$, this is obvious. Assume that the equation is true for some $a \in S(X)$, and let $x \in X$ be given. Then $xa \cdot xa = x \cdot (a \cdot xa) = x \cdot ((a \cdot a) \cdot x)(a \cdot a) = x \cdot x = 1$. Hence 1 is a logical unit in $S(X)$. $\square$

Definition 4. We define a *morphism* $f : X \to Y$ between L-algebras X, Y to be a map which satisfies $f(1) = 1$ and $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in X$. If f is an inclusion $X \hookrightarrow Y$, we call X an *L-subalgebra* of Y. In case H is a self-similar L-algebra with an L-subalgebra X which generates H as a monoid, we call H a *self-similar closure* of X.

Theorem 3. *Let X be an L-algebra. Then the natural morphism $X \to S(X)$ is injective, and up to isomorphism of left hoops, $S(X)$ is the unique self-similar closure of X.*

Proof. By Propositions 7 and 9, $S(X)$ is a cycloid with a logical unit. Let us show that $S(X)$ satisfies the hoop equation (H). Thus let $a, b \in M(X)$ be given. We have to verify that $(c \cdot (a \cdot b)a) \cdot d = (c \cdot (b \cdot a)b) \cdot d$ holds for $c, d \in M(X)$. Now $(c \cdot (a \cdot b)a) \cdot d = ((a \cdot c) \cdot (a \cdot b))(c \cdot a) \cdot d = ((c \cdot a) \cdot (c \cdot b))(c \cdot a) \cdot d = ((c \cdot a) \cdot (c \cdot b)) \cdot ((c \cdot a) \cdot d) = ((c \cdot b) \cdot (c \cdot a)) \cdot ((c \cdot b) \cdot d)$, which exhibits the symmetry in a and b. Thus (H) is verified. Hence $S(X)$ is a left hoop.

From Proposition 5(e), we infer that $S(X)$ is self-similar. Corollary 2 of Proposition 2 implies that the natural map $X \to S(X)$ is injective. This proves that $S(X)$ is a self-similar closure of X.

Suppose that there is another self-similar closure H of X. Then $X \hookrightarrow H$ has a unique extension $e : M(X) \to H$ to the free monoid $M(X)$. Since X generates H as a monoid, e is surjective with $e(1) = 1$. By Theorem 1 and Proposition 5, H satisfies (A) and (S). Let us show by induction that $e(x \cdot b) = e(x) \cdot e(b)$ holds for $x \in X$ and $b \in M(X)$. For $b = 1$, this is trivial. Thus assume that it holds for some b and let $y \in X$. Then $e(x \cdot by) = e((y \cdot x) \cdot b)e(x \cdot y) = (e(y \cdot x) \cdot e(b))e(x \cdot y) = ((y \cdot x) \cdot e(b))(x \cdot y) = x \cdot e(b)y = e(x) \cdot e(by)$. Next we show that $e(a \cdot b) = e(a) \cdot e(b)$ holds for arbitrary $a, b \in M(X)$. While this is trivial for $a = 1$, assume that it holds for a particular a, and let $x \in X$. Then $e(ax \cdot b) = e(a \cdot (x \cdot b)) = e(a) \cdot e(x \cdot b) = e(a) \cdot (e(x) \cdot e(b)) = e(a)e(x) \cdot e(b) = e(ax) \cdot e(b)$. Thus $e(a \cdot b) = e(a) \cdot e(b)$ holds for

$a, b \in M(X)$. Therefore, Corollary 2 of Proposition 2 implies that the map e induces a surjective morphism $f : S(X) \to H$. By Corollary 1 of Proposition 2, it remains to be shown that $\operatorname{Ker} f := \{a \in S(X) \mid f(a) = 1\}$ is trivial. Suppose that $a \in \operatorname{Ker} f$ and $a \neq 1$. Then $a = xb$ for some $x \neq 1$ in X. So we get $f(x) = f(b) \cdot f(x)f(b) = f(b) \cdot f(xb) = f(b) \cdot 1 = 1$, a contradiction. Thus f is an isomorphism of L-algebras. □

In the sequel, we will regard an L-algebra X as an L-subalgebra of $S(X)$. So we can use the monoid structure of $S(X)$ to form expressions of elements in X.

Let us define a *KL-algebra* to be an L-algebra which satisfies (K). In the next section, we will use the following

Proposition 10. *Let X be a KL-algebra. Then $S(X)$ is a KL-algebra.*

Proof. We show first that $x \cdot (b \cdot x) = 1$ holds for $x \in X$ and $b \in S(X)$. For $b = 1$, this is obvious. Suppose that it holds for some $b \in S(X)$. For any $x \in X$, this implies that $x \cdot (by \cdot x) = x \cdot (b \cdot (y \cdot x)) = (x \cdot (y \cdot x)) \cdot (x \cdot (b \cdot (y \cdot x))) = ((y \cdot x) \cdot x) \cdot ((y \cdot x) \cdot (b \cdot (y \cdot x))) = 1$. This proves the claim.

Now let us prove $a \cdot (b \cdot a) = 1$ for all $a, b \in S(X)$. While this is trivial for $a = 1$, let us assume that it holds for a particular a. For any $x \in X$, we then get

$$\begin{aligned} ax \cdot (b \cdot ax) &= ax \cdot \big((x \cdot b) \cdot a\big)(b \cdot x) = a \cdot \big(x \cdot \big((x \cdot b) \cdot a\big)(b \cdot x)\big) \\ &= a \cdot \big(\big((b \cdot x) \cdot x\big) \cdot \big((x \cdot b) \cdot a\big)\big)\big(x \cdot (b \cdot x)\big) = a \cdot \big(\big((b \cdot x) \cdot x\big)(x \cdot b) \cdot a\big) \\ &= 1. \qquad \square \end{aligned}$$

4. The structure group of an L-algebra

Recall that a monoid M is said to fulfil the *left Ore condition* if for each pair of elements $a, b \in M$, there are $c, d \in M$ with $ca = db$. Thus every left hoop is left Ore. If, in addition, $ac = bc$ implies that there is some $d \in M$ with $da = db$, we can form a *quotient group* $Q(M)$ consisting of formal fractions $a^{-1}b$ with $a, b \in M$ (see, e.g., [18, Chapter I]). For $a, b, c \in M$, the equation $a^{-1}b = (ca)^{-1}(cb)$ holds in $Q(M)$, and the left Ore condition guarantees that in this way, two arbitrary fractions can be transformed into those with a common denominator. They are defined to be equal in $Q(M)$ if the common denominator can be chosen such that the numerators become equal in M.

By (14), every self-similar left hoop H admits a quotient group $Q(H)$.

Definition 5. Let X be an L-algebra. We define the *structure group* of X to be the quotient group $G(X) := QS(X)$ of the self-similar closure $S(X)$.

The morphism $X \hookrightarrow S(X) \to QS(X)$ gives a natural map

$$q : X \to G(X) \tag{23}$$

with

$$q(x) = q(y) \quad \Leftrightarrow \quad \exists c \in S(X)\colon\ cx = cy. \tag{24}$$

Definition 6. Let X be an L-algebra. We call $x, y \in X$ *equipollent* if $x \cdot z = y \cdot z$ holds for some $z \leqslant x, y$. If x is equipollent to 1, we say that x is *dense*. The set of dense elements of X will be denoted by $D(X)$.

Proposition 11. *Let X be an L-algebra. Assume that $x \cdot z = y \cdot z$ with $z \leqslant x, y$. Then $x \cdot t = y \cdot t$ holds for any $t \leqslant z$.*

Proof. We use the embedding $X \hookrightarrow S(X)$. Thus it suffices to show that $x \cdot t = (z \cdot t)(x \cdot z)$. In fact, $x \cdot t = x \cdot (z \wedge t) = x \cdot (z \cdot t)z = ((z \cdot x) \cdot (z \cdot t))(x \cdot z) = (z \cdot t)(x \cdot z)$. $\square$

Remark. Proposition 11 shows that if X has a smallest element 0 (= “false”), equipollence of x and y just means that x and y have the same negation $x \cdot 0 = y \cdot 0$. Definition 6 expresses this property without reference to 0. In particular, dense elements $x \in X$ can be conceived as “essentially true” statements, i.e. those whose double negation is “true.”

Now we can determine the fibers of the map (23).

Proposition 12. *Let X be an L-algebra, and $x, y \in X$. Then $q(x) = q(y)$ if and only if x and y are equipollent in $S(X)$.*

Proof. Assume that $q(x) = q(y)$. Then $cx = cy$ for some $c \in S(X)$. Hence $d := cx \leqslant x, y$ satisfies $x \cdot d = c = y \cdot d$ by Proposition 5(c). Conversely, let x, y be equipollent in $S(X)$. Thus $x \cdot d = y \cdot d$ for some $d \in S(X)$ with $d \leqslant x, y$. Therefore, $c := x \cdot d$ satisfies $cx = d = cy$. $\square$

For a KL-algebra X, we define a partial order on $G(X)$ as follows.

$$a^{-1}b \leqslant a^{-1}c \quad :\Leftrightarrow \quad \exists e \in S(X)\colon\ eb \leqslant ec. \tag{25}$$

Proposition 13. *Let X be a KL-algebra. With* (25)*, the structure group $G(X)$ becomes a partially ordered group such that the map* (23) *is monotone.*

Proof. Assume that $a^{-1}b \leqslant a^{-1}c$ with $a, b, c \in S(X)$. Then $eb \leqslant ec$ for some $e \in S(X)$. For any $d \in S(X)$, Proposition 6 implies that $(e \cdot d)eb \leqslant (e \cdot d)ec$, whence $(d \cdot e)db \leqslant (d \cdot e)dc$. This shows that (25) is a well-defined relation on $G(X)$. Using the Ore condition, it is readily seen that (25) gives a partial order on $G(X)$.

Next we show that our assumption $a^{-1}b \leqslant a^{-1}c$ implies that $a^{-1}bu^{-1}v \leqslant a^{-1}cu^{-1}v$ for any $u, v \in S(X)$. Using (H), we find $r, s, t \in S(X)$ with $rb = su$ and $rc = tu$, which gives $bu^{-1} = r^{-1}s$ and $cu^{-1} = r^{-1}t$. Thus we have to show that $(ra)^{-1}sv \leqslant (ra)^{-1}tv$. By assumption, $(ra)^{-1}(rb) \leqslant (ra)^{-1}(rc)$. So there exists some $e \in S(X)$ with $erb \leqslant erc$. Hence $esu \leqslant etu$, and thus $es \leqslant et$ by Proposition 5(d). Consequently, $esv \leqslant etv$, which proves the claim.

It remains to verify $u^{-1}va^{-1}b \leqslant u^{-1}va^{-1}c$. If $u^{-1}va^{-1} = d^{-1}e$, this means that $d^{-1}eb \leqslant d^{-1}ec$. Since $(ea)^{-1}(eb) \leqslant (ea)^{-1}(ec)$, this follows immediately. $\square$

Definition 7. We call an L-algebra X *semiregular* if the equation

$$\bigl((x \cdot y) \cdot z\bigr) \cdot \bigl((y \cdot x) \cdot z\bigr) = \bigl((x \cdot y) \cdot z\bigr) \cdot z \tag{R}$$

holds for $x, y, z \in X$.

Proposition 14. *The inequality* $\geqslant$ *in* (R) *is equivalent to* (K).

Proof. If (K) is satisfied, then $z \leqslant (y \cdot x) \cdot z$. Hence $\geqslant$ in (R) follows by Proposition 2. The converse follows if we set $y = 1$ in (R). $\square$

Note that (R) implies the linearity condition for BCK-algebras [31]:

$$(y \cdot x) \cdot z \leqslant \big((x \cdot y) \cdot z\big) \cdot z.$$

In terms of $S(X)$, Eq. (R) can be stated as follows.

Proposition 15. *An L-algebra X is semiregular if and only if*

$$x(y \wedge z) = xy \wedge xz \tag{26}$$

holds in $S(X)$ for all $x, y, z \in X$.

Proof. First, we have

$$zx \cdot zy = z \cdot (x \cdot zy) = z \cdot \big((y \cdot x) \cdot z\big)(x \cdot y) = \Big(\big((x \cdot y) \cdot z\big) \cdot \big((y \cdot x) \cdot z\big)\Big)\big(z \cdot (x \cdot y)\big).$$

Using (H), we get $zx \wedge zy = (zx \cdot zy)zx = (((x \cdot y) \cdot z) \cdot ((y \cdot x) \cdot z))(z \cdot (x \cdot y))zx = (((x \cdot y) \cdot z) \cdot ((y \cdot x) \cdot z))((x \cdot y) \cdot z)(x \cdot y)x = (((x \cdot y) \cdot z) \wedge ((y \cdot x) \cdot z))(x \wedge y)$. Therefore, Proposition 5(a) implies that Eq. (26) is equivalent to

$$\big((x \cdot y) \cdot z\big) \wedge \big((y \cdot x) \cdot z\big) = z.$$

Assuming (K), this reduces to $(((x \cdot y) \cdot z) \cdot ((y \cdot x) \cdot z))((x \cdot y) \cdot z) \leqslant z$, which is equivalent to the inequality $\leqslant$ in (R). Together with Proposition 14, this completes the proof. $\square$

Next we show that Eq. (26) carries over to $S(X)$.

Proposition 16. *An L-algebra X is semiregular if and only if $S(X)$ is semiregular.*

Proof. By Theorem 3, the left hoops $S(X)$ and $S(S(X))$ coincide. Therefore, we have to show that in case X is semiregular, Eq. (26) holds for all $x, y, z \in S(X)$. Thus let (26) be satisfied for $x, y, z \in X$. By induction, this immediately implies that (26) holds for arbitrary $x \in S(X)$. Using this, we show that

$$a(b \wedge z) = ab \wedge az \tag{27}$$

holds for $a, b \in S(X)$ and $z \in X$. For $b = 1$, this follows by Propositions 10 and 14. Thus assume that (27) holds for a particular b, and assume that $y \in X$. Then Corollary 2 of Proposition 5 yields $a(bx \wedge z) = a(z \cdot bx)z = a((x \cdot z) \cdot b)(z \cdot x)z = a((x \cdot z) \cdot b)(x \cdot z)x = a(b \wedge (x \cdot z))x = (ab \wedge a(x \cdot z))x = abx \wedge a(x \cdot z)x = abx \wedge a(x \wedge z) = abx \wedge ax \wedge az = (ab \wedge a)x \wedge az = abx \wedge az$, where $ab \leqslant a$ holds by (K). This proves (27). Now the same argument, applied to z instead of b, shows that Eq. (26) holds for all $x, y, z \in S(X)$. $\square$

Proposition 17. *Let H be a semiregular self-similar L-algebra. Then $D(H)$ is an ideal, and two elements of H are mapped to the same element in $H/D(H)$ if and only if they are equipollent.*

Proof. We show first that $D(H)$ is an ideal. By Proposition 14, it suffices to verify (I0)–(I2). Now (I0) is trivial. Thus let a and $a \cdot b$ be dense. By Proposition 12 and (H), we can assume that $ca = c(a \cdot b) = c$ for some $c \in H$. Therefore, Proposition 15 yields $(ca \cdot cb)ca = ca \wedge cb = c(a \wedge b) = c(a \cdot b)a = ca$, whence $ca \cdot cb = 1$ by Proposition 5(a). Thus $c = ca \leqslant cb$. By Proposition 14, we have $c \leqslant b \cdot c$, which gives $cb \leqslant c$. So we get $cb = c$, which proves (I1).

To verify (I2), it is enough to show that $ca = c$ implies $c((a \cdot b) \cdot b) = c$. From (26) and (K), we get $c((a \cdot b) \cdot b)(a \wedge b) = c((a \cdot b) \cdot b)(a \cdot b)a = c((a \cdot b) \wedge b)a = cba = c(1 \wedge b)a = (c \wedge cb)a = (c \cdot cb)ca = (ca \cdot cb)ca = ca \wedge cb = c(a \wedge b)$. Now Proposition 5(a) proves the claim.

For $a, b, c \in H$, Proposition 15 gives $c(a \cdot b)a = c(a \wedge b) = ca \wedge cb$. Hence $ca \leqslant cb \Leftrightarrow c(a \cdot b)a = ca \Leftrightarrow c(a \cdot b) = c$ by Proposition 5(a). Using (H), it follows that a, b are equipollent if and only if $a \cdot b$ and $b \cdot a$ are dense. □

Now we are ready to characterize the converse of the implication (15).

Proposition 18. *Let H be a semiregular self-similar L-algebra. Then $D(H) = 1$ if and only if*

$$b \leqslant c \quad \Leftrightarrow \quad ab \leqslant ac \tag{28}$$

holds for $a, b, c \in H$.

Proof. Assume first that $D(H) = 1$ and $ab \leqslant ac$. By Propositions 5 and 15, this yields $b \cdot c = 1 \Leftrightarrow a(b \cdot c) = a \Leftrightarrow a(b \cdot c)b = ab \Leftrightarrow a(b \wedge c) = ab \Leftrightarrow ab \wedge ac = ab \Leftrightarrow ab \leqslant ac$. Conversely, let (28) be satisfied. If $a \in D(H)$, then Proposition 12 implies that $ca = c$ for some $c \in H$. Therefore, $c1 \leqslant ca$ gives $1 \leqslant a$, whence $a = 1$. □

5. *l*-Group cones

A partially ordered group G is said to be an *l-group* if its underlying poset is a lattice. Since left and right multiplication are isotone, the equations

$$a(b \wedge c) = ab \wedge ac; \qquad (a \wedge b)c = ac \wedge bc \tag{29}$$

hold in every l-group G. As $a \mapsto a^{-1}$ is antitone, we have

$$(a \vee b)^{-1} = a^{-1} \wedge b^{-1} \tag{30}$$

for all $a, b \in G$. The submonoids $G_+ := \{a \in G \mid a \geqslant 0\}$ and $G_- := \{a \in G \mid a \leqslant 0\}$ are called the *positive cone* and the *negative cone*, respectively. For the basic theory of l-groups, we refer the reader to monographs like [1,2,14,21].

The negative cone G_- of an l-group G can be regarded as a left hoop with

$$a \cdot b := ba^{-1} \wedge 1. \tag{31}$$

Furthermore, G_- is self-similar since $a \cdot ba = b$, and G_- is semiregular by Proposition 15. Thus $S(G_-) = G_-$, and the structure group of G_- is G itself. As G_- is left cancellable, $D(G_-) = 1$.

Definition 8. We call an L-algebra X *regular* if X is semiregular and for elements $x \leqslant y$ in X, there exists some $z \geqslant x$ in X with $z \cdot x = y$.

For example, the negative cone G_- of an l-group G is regular: For $x \leqslant y$ in G_-, the element $z := y^{-1}x \in G_-$ satisfies $z \geqslant x$ and $z \cdot x = y$. Therefore, G_- is self-similar and regular with $D(X) = 1$. Now we are able to prove the converse.

Theorem 4. *Let X be a semiregular L-algebra. Then the structure group $G(X)$ is an l-group. The map $q : X \to G(X)$ induces a morphism $q^- : X \to G(X)_-$ of L-algebras, and q^- is injective if and only if $D(X) = 1$.*

Proof. By Proposition 16, $S(X)$ is semiregular. Proposition 17 implies that $D(S(X))$ is an ideal of $S(X)$ such that $S(X)/D(S(X))$ consists of the equipollence classes of $S(X)$, i.e. the fibers of $S(X) \to G(X)$ by Proposition 12. Corollary 3 of Proposition 5 implies that $S(X)/D(S(X))$ is a semiregular self-similar left hoop. Since $S(X)/D(S(X))$ is isomorphic to a submonoid of $G(X)$, we have $D(S(X)/D(S(X))) = 1$. Hence $S(X)/D(S(X))$ can be regarded as a subposet of $G(X)$ by Proposition 18. From Proposition 15, we thus infer that $G(X)$ is a $\wedge$-semilattice with

$$a^{-1}b \wedge a^{-1}c = a^{-1}(b \wedge c)$$

for $a, b, c \in S(X)$. Therefore, Proposition 13 implies that $G(X)$ is an l-group. For $a, b \in S(X)/D(S(X))$, the equation $(a \cdot b)a = a \wedge b$ yields $a \cdot b = (a \wedge b)a^{-1} = ba^{-1} \wedge 1$. Hence (23) induces an L-algebra morphism $q^- : X \to G(X)_-$.

If q^- is injective, then $D(X) \subset D(S(X)) = 1$. Conversely, assume that $D(X) = 1$. We have to show that $D(S(X)) = 1$. Now every $a \in D(S(X))$ satisfies $ca = c$ for some $c \in S(X)$. Suppose that $a \neq 1$. Then $c \neq 1$. So we have $c = xd$ for some $x \in X$ and $d \in S(X)$. By Proposition 11, this gives $x = d \cdot xd = d \cdot xda = ((da \cdot d) \cdot x)(d \cdot da) = x(d \cdot da)$. Thus if $d \cdot da \neq 1$, the equation $ca = c$ can be replaced by $x(d \cdot da) = x$. Otherwise, $d \cdot da = 1$, which yields $d \leqslant da \leqslant d$. Hence, by induction, we can assume without loss of generality that $c = x$. So we get an equation $xyb = x$ for some $x, y \in X$, $y \neq 1$, and $b \in S(X)$. This gives $1 = x \cdot xyb = ((b \cdot x) \cdot xy)(x \cdot b) \leqslant x \cdot b$. Therefore, $x \cdot b = 1$, and thus $1 = (b \cdot x) \cdot xy = ((y \cdot (b \cdot x)) \cdot x)((b \cdot x) \cdot y) \leqslant (b \cdot x) \cdot y$, which yields $(b \cdot x) \cdot y = 1$ and $(y \cdot (b \cdot x)) \cdot x = 1$. So we obtain $b \cdot x \leqslant y$ and $y \cdot (b \cdot x) \leqslant x \leqslant b$. Hence $x \leqslant b \cdot x \leqslant y$ and $x \leqslant y \cdot x \leqslant y \cdot (b \cdot x) \leqslant x$. Thus $y \in D(X)$, contrary to our assumption $y \neq 1$. □

Corollary 1. *An L-algebra X is isomorphic to the negative cone of an l-group if and only if X is self-similar and regular with $D(X) = 1$.*

Proof. The necessity is obvious. Conversely, let X be self-similar and regular with $D(X) = 1$. Then we have an embedding of L-algebras $X = S(X) \hookrightarrow G(X)_-$, and every element of $G(X)_-$ is of the form $y^{-1}x$ with $x, y \in X$. Hence $x \leqslant y$. Since X is regular, there is some $z \geqslant x$ in X with $z \cdot x = y$. Therefore, $x = z \wedge x = (z \cdot x)z = yz$, which gives $y^{-1}x = z \in X$. □

For an element $u < 1$ of an l-group G, the interval $[u, 1]$ in G is an L-subalgebra of G_-. The following result characterizes pseudo MV-algebras (cf. [16]) as a particular class of L-algebras.

Corollary 2. *An L-algebra X is regular with a smallest element* 0 *and* $D(X) = 1$ *if and only if X is isomorphic to an interval* $[u, 1]$ *in an l-group.*

Proof. Assume first that $X = [u, 1]$ in an l-group G. Then X is semiregular with smallest element u and $D(X) = 1$. For $x \leqslant y$ in X, the element $y^{-1}x$ belongs to X and satisfies $y^{-1}x \geqslant x$ and $y^{-1}x \cdot x = y$. Hence X is regular. Conversely, let X be regular with a smallest element 0 and $D(X) = 1$. By Theorem 4, the structure group $G(X)$ is an l-group, and X can be regarded as an L-subalgebra of $[0, 1] \subset G(X)$. Suppose that $a \in [0, 1] \smallsetminus X$. Then $a = bx$ for some $x \in X$ and $b \in S(X)$. Hence $1 = 0 \cdot bx = ((x \cdot 0) \cdot b)(0 \cdot x) = (x \cdot 0) \cdot b$, which gives $0 \leqslant x \cdot 0 \leqslant b$. So by induction, we can assume that $b \in X$. Therefore, since X is regular, there exists some $z \in X$ with $z \cdot 0 = b \cdot (x \cdot 0)$, i.e. $0z^{-1} = b \cdot 0x^{-1} = 0x^{-1}b^{-1} \wedge 1 = 0a^{-1} \wedge 1 = 0a^{-1}$. Thus $a = z \in X$. $\square$

In accordance with [33], we call an L-algebra X satisfying the equivalent conditions of Corollary 2 a *GMV-algebra*. Note that in [19,20] and [32,33], such an algebra X is defined to be a monoid with two negations (or equivalently, two types of implication), while our concept requires only one implication. The monoid structure of a GMV-algebra is obtained by the following

Corollary 3. *Every GMV-algebra X is a left hoop with bijective negation $x \mapsto x \cdot 0$.*

Proof. We use the embedding $X = [0, 1] \subset G(X)$ and define a binary operation $X \times X \xrightarrow{*} X$ by

$$x * y := xy \vee 0. \tag{32}$$

Then $(x * y) * z = (xy \vee 0)z \vee 0 = xyz \vee 0z \vee 0 = xyz \vee 0$ holds for $x, y, z \in X$. Furthermore, $(x * y) \cdot z = z(xy \vee 0)^{-1} \wedge 1 = z(y^{-1}x^{-1} \wedge 0^{-1}) \wedge 1 = zy^{-1}x^{-1} \wedge 1 = (zy^{-1} \wedge 1)x^{-1} \wedge 1 = x \cdot (y \cdot z)$, and $(x \cdot y) * x = (yx^{-1} \wedge 1)x \vee 0 = (y \wedge x) \vee 0 = y \wedge x$. Hence $(X, *)$ is a left hoop. Since X is regular with $D(X) = 1$, the map $x \mapsto x \cdot 0$ is bijective. $\square$

Let us briefly consider the commutative case. Note that an L-algebra which satisfies

$$(x \cdot y) \cdot y = (y \cdot x) \cdot x \tag{33}$$

can be regarded as a *commutative* BCK-algebra [25]. Our next result gives a new interpretation of this terminology.

Proposition 19. *Let X be an L-algebra. Then $S(X)$ is commutative if and only if X satisfies* (K) *and Eq.* (33). *If $S(X)$ is commutative, $S(X) \cong G(X)_-$, and*

$$a \vee b = (a \cdot b) \cdot b \tag{34}$$

holds for $a, b \in S(X)$.

Proof. In $S(X)$ we have

$$xy \cdot yx = x \cdot (y \cdot yx) = x \cdot \big((x \cdot y) \cdot y\big)(y \cdot x) = \big(\big((y \cdot x) \cdot x\big) \cdot \big((x \cdot y) \cdot y\big)\big)\big(x \cdot (y \cdot x)\big).$$

This proves the first statement. Now let $S(X)$ be commutative. Then $S(X)$ is semiregular by Proposition 15 and Corollary 2 of Proposition 5, and $D(S(H)) = 1$ by Proposition 5(a). Therefore, Theorem 4 implies that $S(X)$ can be identified with an L-subalgebra of $G(X)_-$. For $a, b \in S(X)$, this yields $(a \cdot b) \cdot b = b(ba^{-1} \wedge 1)^{-1} \wedge 1 = b(ab^{-1} \vee 1) \wedge 1 = a \vee b$. Thus if

$a \leqslant b$, then $b \cdot a \geqslant a$ and $(b \cdot a) \cdot a = b$, which proves that $S(X)$ is regular. By Corollary 1 of Theorem 4, it follows that $S(X) \cong G(X)_-$. $\square$

Remark. By Gispert and Mundici [22], an MV-algebra is a commutative monoid X with an involutive map $x \mapsto x^*$ (the *negation*) such that $0 := 1^*$ satisfies $x0 = 0$, and

$$x(xy^*)^* = y(yx^*)^* \tag{35}$$

holds for all $x, y \in X$. We define

$$x \cdot y := (xy^*)^*. \tag{36}$$

Then (35) turns into the hoop equation (H), and $xx^* = x(x0^*)^* = 0(0x^*)^* = 0$, which gives $x \cdot x = 1$. Furthermore, $xy \cdot z = (xyz^*)^* = (x(y \cdot z)^*)^* = x \cdot (y \cdot z)$. Thus X is a hoop with smallest element 0 and $x^* = x \cdot 0$. Using Proposition 19, it is easily checked that an MV-algebra is the same as an L-algebra with smallest element 0 such that $S(X)$ is commutative. If restricted to this special case, Corollary 2 of Theorem 4 yields an alternative proof of Mundici's equivalence [29] between MV-algebras and unital abelian l-groups.

Definition 9. We call an element $s \leqslant 1$ of an l-group G *singular* (cf. [2, 11.2]) if $s = ab$ with $a, b \in G_-$ implies that $a \vee b = 1$. We say that $u \leqslant 1$ is a *strong unit* if for any $a \in G$, there is some $n \in \mathbb{N}$ with $u^n \leqslant a$.

A Boolean algebra X can be regarded as an MV-algebra with idempotent multiplication. Our next theorem shows that for a Boolean algebra X, the embedding $X \hookrightarrow G(X)$ is uniquely determined by the abelian l-group $G(X)$.

Theorem 5. *For an l-group G, there is at most one singular strong unit $s \leqslant 1$. If such an s exists, G is archimedean, and the interval $[s, 1]$ is a Boolean algebra with structure group G. Conversely, every Boolean algebra arises in this way.*

Proof. Let $s \leqslant 1$ be singular, and let $u \leqslant 1$ be a strong unit in G. Assume that $u^n \leqslant s$ with $n \geqslant 2$. Then $s = (s \vee u)a$ with $a = (s \vee u)^{-1}s \leqslant 1$. Hence $s \vee u \vee a = 1$, and thus $u^{n-1}s \vee u^n \vee u^{n-1}a = u^{n-1}$. Since $u^{n-1}a = u^{n-1}(s^{-1} \wedge u^{-1})s \leqslant u^{n-2}s \leqslant s$, we get $u^{n-1} \leqslant s$. Hence, by induction, $u \leqslant s$. This proves that a singular strong unit is unique.

Now let $s \leqslant 1$ be a singular strong unit. Since every conjugate of s is also a singular strong unit, we infer that s belongs to the center of G. To show that G is archimedean, let $a, b \in G_-$ be such that $b \leqslant a^n$ for all $n \in \mathbb{N}$. Then there is some $m \in \mathbb{N}$ with $s^m \leqslant a^n$ for all n. We choose m to be a power of 2. For $m = 2k$, this gives $s^k(s^k \wedge a^n) = s^m \wedge s^k a^n \leqslant a^{2n} \wedge s^k a^n = a^n(a^n \wedge s^k)$, hence $s^k \leqslant a^n$ for all n. Thus, by induction, we obtain $s \leqslant a^n$ for all $n \in \mathbb{N}$. Now for any $a \in [s, 1]$, we have $s = a(a^{-1}s)$ with $a^{-1}s \leqslant 1$. Hence $a \vee a^{-1}s = 1$, and thus $a^2 \vee s = a$. Therefore, $s \leqslant a^2$ implies that $a^2 = a$, i.e. $a = 1$. This proves that G is archimedean. Since archimedean l-groups are commutative, we infer that $B := [s, 1]$ is an MV-algebra. By Eq. (32), the above equation $a^2 \vee s = a$ for $a \in B$ means that the elements of B are idempotent, i.e. B is Boolean.

Finally, let B be a Boolean algebra. As an MV-algebra, $B = [0, 1]$ in $G(B)$, and it remains to show that $0 \in G(B)$ is singular. Thus let $0 = ab$ with $a, b \leqslant 1$ hold in $G(B)$. Then $a = 0b^{-1} \wedge 1 = b \cdot 0$, whence $a \vee b = 1$. $\square$

Remark. The structure group $G(B)$ of a Boolean algebra B is a *Specker group*, i.e. an l-group generated as a group by its singular elements (see, e.g., [13,27]). By Nöbeling's theorem [30], $G(B)$ is free as an abelian group, i.e. $G(B) \cong \mathbb{Z}^{(C)}$ for a suitable subset C of B.

We conclude with some further examples of L-algebras.

Example 1. Let Ω be a partially ordered set, and $\widehat{\Omega} := \Omega \cup \{1\}$, where $x < 1$ for all $x \in \Omega$. We can regard $\widehat{\Omega}$ as a KL-algebra by defining

$$x \cdot y := \begin{cases} 1 & \text{if } x \leqslant y, \\ y & \text{if } x \nleqslant y \end{cases} \tag{37}$$

for all $x, y \in \widehat{\Omega}$. In fact, (37) implies that 1 is a logical unit, and it is easily verified that $\widehat{\Omega}$ is a KL-algebra. Recall that Ω is said to be a *tree* if $\downarrow x$ is a chain for any $x \in \Omega$.

Proposition 20. *Let Ω be a partially ordered set. Then*

(a) *$\widehat{\Omega}$ is self-similar $\Leftrightarrow$ $\Omega = \emptyset$.*
(b) *$\widehat{\Omega}$ is semiregular $\Leftrightarrow$ Ω is a tree.*
(c) *$\widehat{\Omega}$ is a left hoop $\Leftrightarrow$ Ω is linear.*
(d) *$D(\widehat{\Omega}) = 1$ $\Leftrightarrow$ Ω is an antichain.*

Proof. (a) For any $x \in \Omega$, there is no $y \in \widehat{\Omega}$ with $x \cdot y = x$. Hence Ω cannot be self-similar unless $\Omega = \emptyset$.

(b) Suppose that $x, y \in \Omega$ are incomparable, and $x, y < z$ for some $z \in \Omega$. Then (R) is violated for these x, y, z. Conversely, let Ω be a tree, and $x, y, z \in \widehat{\Omega}$. If x and y are incomparable, then (R) becomes $(y \cdot z) \cdot (x \cdot z) = (y \cdot z) \cdot z$, which holds if $z = 1$ or $x \nleqslant z < 1$. If $x \leqslant z$, then $y \nleqslant z$, and the equation is valid. In case $x \leqslant y$, the Eq. (R) holds by virtue of (K), while (R) is trivial for $x > y$.

(c) Suppose that $x, y \in \Omega$ are incomparable, and $\widehat{\Omega}$ is a left hoop. Then $xy < x$ and $xy < y$, which yields $x \nleqslant xy = y \cdot xy$, a contradiction. Conversely, let Ω be linear. Then $xy := x \wedge y$ makes $\widehat{\Omega}$ into a hoop.

(d) An element $x \in \widehat{\Omega}$ is dense if and only if there exists some $y \leqslant x$ with $x \cdot y = y$, i.e. $y < x$. Thus x is dense if and only if x is not minimal. $\square$

By Theorem 4, the map $q : \widehat{\Omega} \to G(\widehat{\Omega})$ is injective if and only if Ω is an antichain. If Ω is an antichain, then $S(\widehat{\Omega})$ is commutative, and $G(\widehat{\Omega})$ is the free abelian group generated by Ω.

Example 2. It is easily checked that an L-algebra X is a (dual) BCK-algebra [25] if and only if

$$x \cdot (y \cdot z) = (y \cdot x) \cdot z \tag{C}$$

holds for all $x, y, z \in X$. Such algebras have been studied by Traczyk [35]. For any $x \in X$, it then follows [35] that the L-subalgebra $\{y \cdot x \mid y \in X\}$ of X is an MV-algebra.

Example 3. By Example 2 and [28], a *Hilbert algebra* [15] is an L-algebra X which is left self-distributive, i.e. for which

$$x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z) \tag{38}$$

holds for all $x, y, z \in X$. As logical systems, Hilbert algebras characterize the implications which satisfy the deduction theorem (see [24]).

For a poset Ω, the L-algebra $\widehat{\Omega}$ of Example 1 is a Hilbert algebra. Furthermore, the set $\mathcal{O}(X)$ of open sets of a topological space X is a complete Heyting algebra, hence a Hilbert algebra, with

$$U \cdot V := X \smallsetminus \overline{U \smallsetminus V}. \tag{39}$$

Proposition 21. *A Hilbert algebra X is semiregular if and only if*

$$(x \cdot y) \vee (y \cdot x) = 1 \tag{40}$$

holds for all $x, y \in X$.

Proof. Assume that X is semiregular, and $x \cdot y, y \cdot x \leqslant z$. Then (R) yields $z = 1$. Conversely, let (40) hold. For $x, y, z \in X$, we have $(x \cdot y) \cdot z \leqslant ((y \cdot x) \cdot z) \cdot ((x \cdot y) \cdot z) = (x \cdot y) \cdot (((y \cdot x) \cdot z) \cdot z)$, which gives $x \cdot y \leqslant ((x \cdot y) \cdot z) \cdot (((y \cdot x) \cdot z) \cdot z) =: t$. Furthermore, $y \cdot x \leqslant ((y \cdot x) \cdot z) \cdot z \leqslant t$. Hence (40) yields $t = 1$. Therefore, with $a := (x \cdot y) \cdot z$ and $b := (y \cdot x) \cdot z$, Eq. (38) gives $(a \cdot b) \cdot (a \cdot z) = a \cdot (b \cdot z) = t = 1$. Whence (R) follows. □

Remark. In the self-similar closure $S(X)$, the defining equation (38) for a Hilbert algebra X can be written as $xy \cdot z = (x \wedge y) \cdot z$. While z is restricted to X, this does not imply that $S(X)$ is commutative. By Proposition 19, $S(X)$ is commutative if and only if X satisfies Eq. (33). Such Hilbert algebras X are equivalent to *implicative* BCK-algebras [25]. As in the preceding remark, the structure group $G(X)$ of an implicative BCK-algebra X is a Specker l-group, and X then consists of the singular elements in the negative cone of $G(X)$.

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