A proof of Price's Law on Schwarzschild black hole manifolds for all angular momenta

Roland Donninger [a,1], Wilhelm Schlag [a,*,2], Avy Soffer [b,3]

[a] *University of Chicago, Department of Mathematics, 5734 South University Avenue, Chicago, IL 60637, USA*
[b] *Rutgers University, Department of Mathematics, 110 Freylinghuysen Road, Piscataway, NJ 08854, USA*

Abstract

Price's Law states that linear perturbations of a Schwarzschild black hole fall off as $t^{-2\ell-3}$ for $t \to \infty$ provided the initial data decay sufficiently fast at spatial infinity. Moreover, if the perturbations are initially static (i.e., their time derivative is zero), then the decay is predicted to be $t^{-2\ell-4}$. We give a proof of $t^{-2\ell-2}$ decay for general data in the form of weighted L^1 to L^∞ bounds for solutions of the Regge–Wheeler equation. For initially static perturbations we obtain $t^{-2\ell-3}$. The proof is based on an integral representation of the solution which follows from self-adjoint spectral theory. We apply two different perturbative arguments in order to construct the corresponding spectral measure and the decay bounds are obtained by appropriate oscillatory integral estimates.
© 2010 Elsevier Inc. All rights reserved.

Keywords: Dispersive estimates for wave equation; Spectral and scattering theory; Schwarzschild black hole

1. Introduction and main result

In General Relativity, the dynamics of spacetime is governed by Einstein's equation which, in the absence of matter, takes the form

$$R_{\mu\nu}(g) = 0$$

where $R_{\mu\nu}(g)$ is the Ricci tensor of the Lorentz metric g. Exact solutions (i.e., solutions which are known in closed form) include the free flat Minkowski spacetime as well as the Schwarzschild metric and, more generally, the Kerr solution. The Schwarzschild solution is spherically symmetric and corresponds to a nonrotating black hole whereas rotating black holes are described by the axially symmetric Kerr spacetime. A fundamental mathematical problem in General Relativity is the understanding of the stability of these solutions. The stability of the flat Minkowski spacetime under small perturbations was shown in the seminal work of Christodoulou and Klainerman [15] in the late 1980's. A simpler proof was later developed by Lindblad and Rodnianski [37]. However, we are very far from understanding the dynamics near a black hole. Yet, latest experimental setups are crucially dependent on such an analysis, in order to observe gravitational waves (see for example [21,23,24,22] and references therein). Most efforts are now focused on understanding the linear dynamics and stability of such solutions, see e.g. [32,7] and references therein, as well as [43]. The mathematical aspects of the problem will be discussed below in more detail. We also refer the reader to the survey [20] which gives an excellent overview of recent developments in the field from the mathematical perspective.

1.1. Wave evolution on the Schwarzschild manifold

As a first approximation to the linear stability problem of a nonrotating black hole one may consider the wave equation on a fixed Schwarzschild background. One is then typically interested in decay estimates for the evolution. To simplify things even more, one restricts the analysis to the exterior region of the black hole which, however, is physically reasonable: such a model describes a black hole subject to a small external perturbation by a scalar field—a situation which, with a more realistic matter model, is certainly relevant in an astrophysical context. In order to formulate the problem we choose coordinates such that the exterior region of the black hole can be written as $(t, r, (\theta, \phi)) \in \mathbb{R} \times (2M, \infty) \times S^2$ with the metric

$$g = -F(r)\,dt^2 + F(r)^{-1}\,dr^2 + r^2\big(d\theta^2 + \sin^2\theta\,d\phi^2\big)$$

where $F(r) = 1 - \frac{2M}{r}$ and, as usual, $M > 0$ denotes the mass. We now introduce the well-known *Regge–Wheeler tortoise coordinate* r_* which (up to an additive constant) is defined by the relation

$$F = \frac{dr}{dr_*}.$$

In this new coordinate system, the outer region is described by $(t, r_*, (\theta, \phi)) \in \mathbb{R} \times \mathbb{R} \times S^2$,

$$g = -F(r)\,dt^2 + F(r)\,dr_*^2 + r^2\big(d\theta^2 + \sin^2\theta\,d\phi^2\big) \tag{1}$$

with F as above and r is now interpreted as a function of r_*. Explicitly, r_* is computed as

$$r_* = r + 2M \log\left(\frac{r}{2M} - 1\right).$$

Generally, the Laplace–Beltrami operator on a manifold with metric g is given by

$$\Box_g = \frac{1}{\sqrt{|\det(g_{\mu\nu})|}} \partial_\mu \left(\sqrt{|\det(g_{\mu\nu})|} g^{\mu\nu} \partial_\nu\right)$$

and thus, for the metric g in (1), we obtain

$$\Box_g = F^{-1}\left(-\partial_t^2 + \frac{1}{r^2} \partial_{r_*}\left(r^2 \partial_{r_*}\right)\right) + \frac{1}{r^2} \Delta_{S^2}.$$

By setting $\psi(t, r_*, \theta, \phi) = r(r_*)\tilde{\psi}(t, r_*, \theta, \phi)$ and writing $x = r_*$, the wave equation $\Box_g \tilde{\psi} = 0$ is equivalent to

$$-\partial_t^2 \psi + \partial_x^2 \psi - \frac{F}{r} \frac{dF}{dr} \psi + \frac{F}{r^2} \Delta_{S^2} \psi = 0. \tag{2}$$

The mathematically rigorous analysis of this equation has been initiated by Wald [58], however, the first complete proof of uniform boundedness of solutions is due to Kay and Wald [34]. Recently, Dafermos and Rodnianski have found a more robust method to prove boundedness of solutions based on vector field multipliers that capture the so-called red-shift effect [18], see also [20] for a survey and generalizations of these results. The goal of our present work is to prove L^1 to L^∞ decay estimates for Eq. (2). Different types of decay estimates have been proved before. Local decay estimates, based on multipliers generalizing the Morawetz estimates, were initiated in [8,10,9]. Later, a similar approach was used in [17,11,18,19,38] to prove both local decay estimates and pointwise decay in time based on conformal type identities. In [42,41,55,39] it is shown how to apply such estimates to obtain Strichartz type decay estimates. We also mention the recent work [3]. After submission of the present paper, Tataru announced a proof of the sharp pointwise t^{-3} decay for general data without symmetry assumptions, see [54]. Moreover, his result also applies to the more complicated case of rotating Kerr black holes. In fact, in the follow-up paper [27] we also obtain pointwise t^{-3} decay on Schwarzschild for general data. We will discuss this below in more detail. Our results differ from the above in certain respects: the methods we use are based on constructing the Green's function and deriving the needed estimates on it. Previous works in this direction include mainly the series of papers [29, 30,28] where the first pointwise decay result for Kerr black holes has been proved, see also [35] and [36] for Schwarzschild. In our approach, we freeze the angular momentum ℓ or, in other words, we project onto a spherical harmonic. More precisely, let $Y_{\ell,m}$ be a spherical harmonic (that is, an eigenfunction of the Laplacian on S^2 with eigenvalue $-\ell(\ell+1)$) and insert the Ansatz $\psi(t, x, \theta, \phi) = \psi_{\ell,m}(t, x)Y_{\ell,m}(\theta, \phi)$ in Eq. (2). This yields the *Regge–Wheeler equation*

$$\partial_t^2 \psi_{\ell,m} - \partial_x^2 \psi_{\ell,m} + V_{\ell,\sigma}(x)\psi_{\ell,m} = 0$$

with $\sigma = 1$ where

$$V_{\ell,\sigma}(x) = \left(1 - \frac{2M}{r(x)}\right)\left(\frac{\ell(\ell+1)}{r^2(x)} + \frac{2M\sigma}{r^3(x)}\right)$$

is known as the *Regge–Wheeler potential*. In the present work, we obtain decay estimates for solutions of this equation. However, before we explain our results in more detail, we further motivate the study of the Regge–Wheeler equation by considering more general black hole perturbations.

1.2. Other types of black hole perturbations

The wave equation on the Schwarzschild manifold describes the time evolution of linearized scalar field perturbations of a black hole. Of course, not all physically relevant situations are covered by this simple model since it ignores the underlying tensorial structure altogether. Eventually, one is interested in perturbing fields of higher spin, in particular gravitational perturbations. However, as a remarkable fact, the Regge–Wheeler equation is also relevant in this context. This follows from a reduction procedure that goes back to Regge and Wheeler [49] as well as Zerilli [59], see also [57] and [14]. We will briefly sketch how this comes about. In order to study gravitational perturbations, one considers a perturbed Schwarzschild metric $\tilde{g}$ of the form

$$\tilde{g} = -e^{2(\nu+\delta\nu)}\,dt^2 + e^{2(\psi+\delta\psi)}(d\phi - \delta\omega\,dt - \delta q_2\,dr - \delta q_3\,d\theta)^2$$
$$+ e^{2(\mu_2+\delta\mu_2)}\,dr^2 + e^{2(\mu_3+\delta\mu_3)}\,d\theta^2$$

where the various coefficients are allowed to depend on t, r, θ and $e^{2\nu} = e^{-2\mu_2} = 1 - \frac{2M}{r}$, $e^{\mu_3} = r$, $e^{\psi} = r\sin\theta$ (we follow the notation of [14]). It can be shown (see [14]) that this Ansatz is sufficiently general. One then requires the metric $\tilde{g}$ to satisfy the linearized Einstein vacuum equations, i.e., one linearizes $R_{\mu\nu}(\tilde{g}) = 0$ with respect to the perturbations $\delta\nu$, $\delta\psi$, etc. It turns out that one has to distinguish between so-called *axial* ($\delta\omega, \delta q_2, \delta q_3$) and *polar* ($\delta\nu, \delta\psi, \delta\mu_1, \delta\mu_2$) perturbations, depending on the behavior of the metric under the reflection $\phi \mapsto -\phi$. After a lengthy calculation and separation of the θ-dependence one arrives at

$$\partial_t^2\psi_\ell - \partial_x^2\psi_\ell + \left(1 - \frac{2M}{r(x)}\right)\left(\frac{\ell(\ell+1)}{r^2(x)} - \frac{6M}{r^3(x)}\right)\psi_\ell = 0$$

where ψ_ℓ is an auxiliary function which completely determines the axial perturbations, see [14] for details. Thus, ψ_ℓ satisfies the Regge–Wheeler equation with $\sigma = -3$. In the case of polar perturbations, Zerilli [59] has derived an analogous equation with a more complicated effective potential. However, Chandrasekhar [13] (see also [14]) has found a transformation involving differential operations that relates this equation to the one for axial perturbations. As a consequence, the Regge–Wheeler equation provides a fairly complete description of gravitational perturbations with a fixed angular momentum parameter ℓ. Moreover, we mention the fact that the Regge–Wheeler equation with parameter $\sigma = 0$ appears in the study of electromagnetic perturbations of Schwarzschild black holes, i.e., if one considers the Einstein–Maxwell system and linearizes around the Reissner–Nordström solution with zero charge. We do not comment on this further but simply refer the reader to the literature, see [14] and references therein. As a consequence,

the study of the Regge–Wheeler equation can provide valuable information on the stability of Schwarzschild black holes under various types of perturbations and it is truly remarkable that such a unified approach is available.

1.3. Decay estimates for the Regge–Wheeler equation

The most salient feature of the Regge–Wheeler potential is that it decays exponentially as $x \to -\infty$ which corresponds to approaching the black hole, whereas for $x \to \infty$, it falls off as x^{-2}. Strictly speaking, this is only true for $\ell > 0$. The case $\ell = 0$ is exceptional and we consider it separately in the companion paper [26] where we obtain the sharp t^{-3} decay as predicted by Price's Law. Consequently, in this paper, we focus on $\ell > 0$ which, unless otherwise stated, will be assumed throughout. For $\ell > 0$ the potential has inverse square decay and it is well known [25] that this fall-off behavior is in some sense critical for the scattering theory. In order to explain this we define the Schrödinger operator $\mathcal{H}_{\ell,\sigma}$ by

$$\mathcal{H}_{\ell,\sigma} f = -f'' + V_{\ell,\sigma} f$$

and recall that the Jost solutions $f_{\pm}(x, \lambda)$ are defined by $\mathcal{H}_{\ell,\sigma} f_{\pm}(\cdot, \lambda) = \lambda^2 f_{\pm}(\cdot, \lambda)$ and $f_{\pm}(x, \lambda) \sim e^{\pm i \lambda x}$ as $x \to \pm\infty$. The property $V_{\ell,\sigma} \in L^1(\mathbb{R})$ is sufficient to guarantee the existence of these solutions, see [25], but the inverse square decay of $V_{\ell,\sigma}$ is critical in the sense that at this power the Jost solutions typically are no longer continuous as $\lambda \to 0$. Nevertheless, following [53], it is possible to perform a detailed spectral and scattering analysis of the Schrödinger operator $\mathcal{H}_{\ell,\sigma}$. However, we emphasize that the present work differs considerably from [53] due to the asymmetric decay properties of the potential $V_{\ell,\sigma}$. Of particular importance is the asymptotic behavior of the resolvent $((\lambda + i0) - \mathcal{H}_{\ell,\sigma})^{-1}$ (and thus, of the Jost solutions and their Wronskian) as $\lambda \to 0$. This is a common feature in dispersive estimates, see [51]. In particular, we are faced with the possibility of a *zero energy resonance*. However, it was already observed earlier [49,16,44,57] that in the physically relevant cases such a zero energy resonance does not occur (see also Lemma 6.3 below). Our approach is detailed enough to show rigorously, for the first time, the decay estimates depending on the angular momentum of the initial data. In his seminal work [44,45], see also [31,46], Price heuristically derived the decay rate in time at a fixed point in space, and concluded that, depending on initial conditions, the decay rate is either $t^{-2\ell-3}$ or $t^{-2\ell-2}$ where ℓ is the angular momentum. This result is now referred to as *Price's Law*. There has been some confusion in the literature concerning the precise prediction of Price's Law. This has been clarified in Price and Burko [46]. If the initial data decay sufficiently fast at spatial infinity then the pointwise decay in time is predicted to be $t^{-2\ell-3}$. In the present paper we give the first proof of an ℓ-dependent decay rate. More precisely, we obtain a $t^{-2\ell-2}$ estimate which is one power off the sharp version of Price's Law. However, we emphasize that our method yields estimates in terms of the initial data and not just a pointwise decay law as is common in the physics literature. To be more precise, we show that

$$\left\| w_\alpha \cos(t\sqrt{\mathcal{H}_{\ell,\sigma}}) f \right\|_{L^\infty(\mathbb{R})} \leqslant C_{\ell,\alpha} \langle t \rangle^{-\alpha} \left(\left\| \frac{f'}{w_\alpha} \right\|_{L^1(\mathbb{R})} + \left\| \frac{f}{w_\alpha} \right\|_{L^1(\mathbb{R})} \right), \qquad (3)$$

$$\left\| w_\alpha \frac{\sin(t\sqrt{\mathcal{H}_{\ell,\sigma}})}{\sqrt{\mathcal{H}_{\ell,\sigma}}} f \right\|_{L^\infty(\mathbb{R})} \leqslant C_{\ell,\alpha} \langle t \rangle^{-\alpha+1} \left\| \frac{f}{w_\alpha} \right\|_{L^1(\mathbb{R})} \qquad (4)$$

for all $t \geqslant 0$ where $w_\alpha(x) := \langle x \rangle^{-\alpha}$ is a polynomial weight and, as usual, $\langle x \rangle := (1 + |x|^2)^{1/2}$. Here one has to require $1 \leqslant \alpha \leqslant 2\ell + 3$ and one needs to exclude[4] $(\ell, \sigma) \in \{(0, 0), (0, -3), (1, -3)\}$ which are exactly those cases where zero energy resonances *do* occur—however, they are physically irrelevant due to a gauge freedom, cf. [14]. Observe that for $\alpha = 2\ell + 3$ we obtain precisely the aforementioned bound. It is also obvious from our approach that the decay of initially static perturbations is better by one power of t as is reflected by our cosine estimate. This is a general effect which is also present in Price's prediction, cf. [46].

The proof of (3) and (4) is based on representing the solution as an oscillatory integral in the energy variable λ, schematically one may write

$$\psi(t, x) = \int U(t, \lambda) \operatorname{Im}\big[G_{\ell, \sigma}(x, x', \lambda)\big] f(x') \, dx' \, d\lambda$$

where $U(t, \lambda)$ is a combination of $\cos(t\lambda)$ and $\sin(t\lambda)$ terms and $G_{\ell, \sigma}(x, x', \lambda)$ is the kernel (Green's function) of the resolvent of the operator $\mathcal{H}_{\ell, \sigma}$. $G_{\ell, \sigma}(x, x', \lambda)$ is constructed in terms of the Jost solutions and we obtain these functions in various domains of the (x, λ) plane by perturbative arguments: for $|x\lambda|$ small we perturb in λ around $\lambda = 0$, whereas for $|x\lambda|$ large we perturb off of Hankel functions. This is done in such a way that there remains a small window where the two different perturbative solutions can be glued together. One of the main technical difficulties of the proof lies with the fact that we need good estimates for arbitrary derivatives of the perturbative solutions. This is necessary in order to control the oscillatory integrals. The most important contributions come from $\lambda \sim 0$ and we therefore need to derive the exact asymptotics of the Green's function and its derivatives in the limit $\lambda \to 0$. For instance, we prove that

$$\operatorname{Im}\big[G_{\ell, \sigma}(0, 0, \lambda)\big] = \lambda P_\ell(\lambda^2) + O(\lambda^{2\ell+1})$$

as $\lambda \to 0+$ where P_ℓ is a polynomial of degree $\ell - 1$ (we set $P_0 \equiv 0$) and the O-term satisfies $O^{(k)}(\lambda^{2\ell+1}) = O(\lambda^{2\ell+1-k})$ for all $k \in \mathbb{N}_0$. Our approach therefore yields further information on the Green's function and the fundamental solution of the wave equation on the Schwarzschild manifold.

1.4. Interpretation of the result and further comments

For the relevant parameter values, i.e., $(\ell, \sigma) \notin \{(0, 0), (0, -3), (1, -3)\}$, the Regge–Wheeler potential $V_{\ell, \sigma}$ is positive, decays as described above, and has a unique nondegenerate maximum at $r = r_0$ which is known as the *photon sphere*. As a helpful analogy, consider a Newtonian particle placed at $r = r_0$ with vanishing kinetic energy but potential energy $V_{\ell, \sigma}|_{r=r_0}$. It will remain at rest, but every slight perturbation will make it lose potential energy and gain kinetic energy; the larger ℓ is, the faster this will occur which reflects itself in the ℓ-dependent decay rates. In the context of the Schwarzschild geometry, the flow of null geodesics near $r = r_0$ is unstable and the dispersion provides a mechanism that spreads out the wave away from the photon sphere. Moreover, the higher the angular momentum ℓ, the faster the geodesics will pull away leading to the accelerated decay provided by Price's Law. Note carefully, however, that this is counteracted by what can be viewed as a stabilizing effect of large ℓ. Technically speaking, this

---

[4] In fact, the case $(\ell, \sigma) = (0, 0)$ is trivial since for these parameter values the Regge–Wheeler equation reduces to the free wave equation on the line.

reflects itself in the constant $C_{\ell,\alpha}$: the larger this constant is, the longer one has to wait before the decay estimates become effective. It is important to note that our approach is essentially blind to the local geometry, that is, the fine structure of the potential is irrelevant—only positivity, the decay properties and the nonexistence of a zero energy resonance are used. This is in contrast to the methods based on Morawetz type estimates. In particular, the phenomenon of trapping does not play a role at this level—it simply enlarges the constants. However, eventually one is interested in the overall decay which is obtained by summing the individual contributions over all ℓ and at this stage, of course, trapping becomes relevant since the ℓ-dependence of the constants is crucial for the summation. As a matter of fact, our proof produces a constant which grows super-exponentially in ℓ. Consequently, in order to be able to sum the estimates, a different approach is necessary for large ℓ. This issue is addressed in our paper [27] where a detailed semiclassical asymptotic analysis is performed. The role of the semiclassical parameter $\hbar$ is played by ℓ^{-1} (simply divide $\mathcal{H}_{\ell,\sigma}$ by ℓ^2). In particular, such an analysis requires a careful study of the spectral measure near the maximum of $V_{\ell,\sigma}$ and it is exactly at this point where the instability of null geodesics at the photon sphere becomes crucial. As a consequence, in [27], we show that the estimates for individual ℓ's can indeed be summed and thereby, we obtain the sharp t^{-3} decay bound for general data with a loss of a finite number of angular derivatives. We also remark that there are various formal approaches in the physics literature to find the ℓ-dependence of the constants, see in particular [4].

Let us finally mention that decay estimates like (3) and (4) play an important role in the current theoretical and numerical analysis of black holes. For instance, they serve as a way to verify various numerical schemes for solving the Einstein equations in the presence of black holes, see e.g. [47,2,48,31,6,5,7,12,4,21,23,24,22] and references therein. For other recent theoretical implications of the angular behavior see for example [4,6,40].

1.5. Notations and conventions

For a given smooth function f we denote by $O(f(x))$ a generic *real-valued* function that satisfies $|O(f(x))| \leqslant |f(x)|$ in a specified range of x which follows from the context. We write $O_{\mathbb{C}}(f(x))$ if the function attains complex values. The symbol $f(x) \sim g(x)$ for $x \to a$, where g is smooth, means $\lim_{x \to a} \frac{f(x)}{g(x)} = 1$. Furthermore, the letter C (possibly with indices) denotes a generic positive constant. We say that $O(x^{\gamma})$, $\gamma \in \mathbb{R}$, *behaves like a symbol*, *is of symbol type*, or *has symbol character*, if the k-th derivative satisfies $O^{(k)}(x^{\gamma}) = O(x^{\gamma-k})$. As usual, we use the abbreviation $\langle x \rangle := \sqrt{1 + |x|^2}$ and the symbol $A \lesssim B$ means that there exists a $C > 0$ such that $A \leqslant CB$. We also note that all of the functions we are going to construct will depend on the parameters ℓ and σ, however, in order to increase readability of the equations, we will omit this dependence in the notation most of the time. The same comment applies to all implicit and explicit constants. Finally, as already mentioned, we assume $\ell > 0$ unless otherwise stated.

2. Solutions of the Regge–Wheeler equation

2.1. Asymptotics of the potential

As explained in the introduction, linear perturbations of the Schwarzschild spacetime are described by the Regge–Wheeler equation

$$\psi_{tt} - \psi_{xx} + V_{\ell,\sigma}(x)\psi = 0$$

where $V_{\ell,\sigma}$ is the Regge–Wheeler potential and x is the tortoise coordinate which is related to the standard r-coordinate by

$$x = r + 2M \log\left(\frac{r}{2M} - 1\right). \tag{5}$$

$V_{\ell,\sigma}$ is given by

$$V_{\ell,\sigma}(x) = \left(1 - \frac{2M}{r(x)}\right)\left(\frac{\ell(\ell+1)}{r^2(x)} + \frac{2M\sigma}{r^3(x)}\right)$$

where $r(x)$ is implicitly defined by Eq. (5). The valid range of the parameters is $\ell \in \mathbb{N}_0$ and $\sigma = -3, 0, 1$. We start by obtaining the asymptotics of the potential $V_{\ell,\sigma}$.

Lemma 2.1. *The function $x \mapsto r(x)$ has the asymptotic behavior $r(x) = x(1 + O(x^{-1+\varepsilon}))$ for $x \to \infty$ and $r(x) = 2M + O(e^{x/(2M)})$ for $x \to -\infty$ where $\varepsilon \in (0, 1)$ is arbitrary and the O-term in the expression for $x \to \infty$ behaves like a symbol.*

Proof. The function $r(x)$ is implicitly defined by $x = r(x) + 2M \log(\frac{r(x)}{2M} - 1)$ and thus, we have $r(x) \to \infty$ as $x \to \infty$. This implies $x \sim r(x)$ and hence, $r(x) \sim x$ as $x \to \infty$. We infer that $x - r(x) = 2M \log(\frac{r(x)}{2M} - 1) \sim 2M \log \frac{r(x)}{2M} \sim 2M \log \frac{x}{2M}$ and this shows $x - r(x) = O(x^\varepsilon)$. For the symbol behavior note that $\frac{dx}{dr}(r) = (1 - \frac{2M}{r})^{-1}$ which implies that $r'(x) = 1 - \frac{2M}{r(x)}$. The claim now follows by induction.

For the case $x \to -\infty$ we have $e^{x/(2M)} = e^{r(x)/(2M)}(\frac{r(x)}{2M} - 1) \sim e(\frac{r(x)}{2M} - 1)$ which shows $r(x) - 2M \sim 2M e^{x/(2M)-1}$ and this implies the claim. $\quad\square$

Corollary 2.1. *The Regge–Wheeler potential $V_{\ell,\sigma}$ has the asymptotic behavior*

$$V_{\ell,\sigma}(x) = \frac{\ell(\ell+1)}{x^2}\left(1 + O\left(x^{-1+\varepsilon}\right)\right)$$

for $x \to \infty$ and $V_{\ell,\sigma}(x) = O(e^{x/(2M)})$ for $x \to -\infty$ where $\varepsilon \in (0, 1)$ is arbitrary and the O-term in the expression for $x \to \infty$ behaves like a symbol.

Proof. Just insert the asymptotic expansions from Lemma 2.1 in the expression for $V_{\ell,\sigma}$. For the symbol behavior apply the Leibniz rule and Lemma A.1. $\quad\square$

2.2. Hilbert space formulation

We define the Schrödinger operator $\mathcal{H}_{\ell,\sigma}$ on $L^2(\mathbb{R})$ with domain $\mathcal{D}(\mathcal{H}_{\ell,\sigma}) := H^2(\mathbb{R})$ by

$$\mathcal{H}_{\ell,\sigma} f := -f'' + V_{\ell,\sigma} f.$$

From the decay properties of $V_{\ell,\sigma}$ it follows that $\mathcal{H}_{\ell,\sigma}$ is self-adjoint (see e.g. [56]). Furthermore, integration by parts shows

$$(\mathcal{H}_{\ell,\sigma} f | f)_{L^2(\mathbb{R})} \geqslant (V_{\ell,\sigma} f | f)_{L^2(\mathbb{R})} \geqslant 0$$

since $V_{\ell,\sigma} \geqslant 0$ for all $\ell \in \mathbb{N}_0$ if $\sigma = 0, 1$. For gravitational perturbations ($\sigma = -3$) we have to assume $\ell \geqslant 2$ to obtain $V_{\ell,\sigma} \geqslant 0$ which we shall do from now on. We conclude that the spectrum of $\mathcal{H}_{\ell,\sigma}$ is purely absolutely continuous and we have $\sigma(\mathcal{H}_{\ell,\sigma}) = \sigma_{ac}(\mathcal{H}_{\ell,\sigma}) = [0, \infty)$ (see [56]) provided that $(\ell, \sigma) \notin \{(0, -3), (1, -3)\}$. An operator formulation of the Regge–Wheeler equation is given by

$$\frac{d^2}{dt^2}\Psi(t) + \mathcal{H}_{\ell,\sigma}\Psi(t) = 0$$

where $\Psi : \mathbb{R} \to L^2(\mathbb{R})$. Applying the functional calculus for self-adjoint operators, the solution Ψ with initial data $\Psi(0) = f$ and $\frac{d\Psi}{dt}(0) = g$ is given by

$$\Psi(t) = \cos(t\sqrt{\mathcal{H}_{\ell,\sigma}})f + \frac{\sin(t\sqrt{\mathcal{H}_{\ell,\sigma}})}{\sqrt{\mathcal{H}_{\ell,\sigma}}}g.$$

Thus, in order to obtain decay estimates for the solution, we have to understand the operators $\cos(t\sqrt{\mathcal{H}_{\ell,\sigma}})$ and $\frac{\sin(t\sqrt{\mathcal{H}_{\ell,\sigma}})}{\sqrt{\mathcal{H}_{\ell,\sigma}}}$.

2.3. The spectral measure

Recall that the spectral theorem for self-adjoint operators asserts the existence of finite complex-valued Borel measures $\mu_{u,v}$ such that, for $u, v \in \mathcal{D}(\mathcal{H}_{\ell,\sigma})$, we have

$$(\mathcal{H}_{\ell,\sigma}u|v)_{L^2(\mathbb{R})} = \int_0^\infty \lambda \, d\mu_{u,v}(\lambda).$$

The solution operator $\cos(t\sqrt{\mathcal{H}_{\ell,\sigma}})$ is then given by

$$\left(\cos(t\sqrt{\mathcal{H}_{\ell,\sigma}})u|v\right)_{L^2(\mathbb{R})} = \int_0^\infty \cos(t\sqrt{\lambda}) \, d\mu_{u,v}(\lambda)$$

for $u, v \in L^2(\mathbb{R})$ and analogous for the sine evolution. The point is that the spectral measure can be calculated in terms of the resolvent $R_{\mathcal{H}_{\ell,\sigma}}(z) = (z - \mathcal{H}_{\ell,\sigma})^{-1}$ of $\mathcal{H}_{\ell,\sigma}$. Indeed, for $u \in L^2(\mathbb{R})$ set

$$F_u(z) := -\left(R_{\mathcal{H}_{\ell,\sigma}}(z)u|u\right)_{L^2(\mathbb{R})} = \int_0^\infty \frac{1}{\lambda - z} \, d\mu_u(\lambda)$$

where $\mu_u := \mu_{u,u}$ and $\operatorname{Im} z > 0$. F_u is the Borel transform of the measure μ_u and, since the measure μ_u is purely absolutely continuous, we have

$$d\mu_u(\lambda) = \frac{1}{\pi} \lim_{\varepsilon \to 0+} \operatorname{Im}\left(F_u(\lambda + i\varepsilon)\right) d\lambda,$$

see [56] for the underlying theory of this. The measure $\mu_{u,v}$ can be reconstructed from μ_u by the polarization identity, i.e., $\mu_{u,v} = \frac{1}{4}(\mu_{u+v} - \mu_{u-v} + i\mu_{u-iv} - i\mu_{u+iv})$. Furthermore, the resolvent is given by

$$R_{\mathcal{H}_{\ell,\sigma}}(z)u(x) = \int_{\mathbb{R}} G_{\ell,\sigma}(x, x', \sqrt{z})u(x')\,dx'$$

where $G_{\ell,\sigma}$ is the Green's function (we always choose the branch of the square root with $\operatorname{Im}\sqrt{z} > 0$ if $\operatorname{Im} z > 0$) and thus, we have

$$d\mu_u(\lambda) = -\frac{1}{\pi} \lim_{\varepsilon \to 0+} \int_{\mathbb{R}} \int_{\mathbb{R}} \operatorname{Im}\big[G_{\ell,\sigma}(x, x', \sqrt{\lambda + i\varepsilon})u(x')\overline{u(x)}\big]\,dx'\,dx\,d\lambda.$$

It is known (and, for the convenience of the reader, will be shown below) that the limit

$$G_{\ell,\sigma}(x, x', \sqrt{\lambda}) := \lim_{\varepsilon \to 0+} G_{\ell,\sigma}(x, x', \sqrt{\lambda + i\varepsilon})$$

exists and satisfies $\sup_{x,x'\in\mathbb{R}} |G_{\ell,\sigma}(x, x', \lambda)| \lesssim 1$ for all $\lambda \geqslant \lambda_0$ where $\lambda_0 > 0$ is arbitrary. Thus, if $u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we have

$$d\mu_u(\lambda) = -\frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \operatorname{Im}\big[G_{\ell,\sigma}(x, x', \sqrt{\lambda})u(x')\overline{u(x)}\big]\,dx'\,dx\,d\lambda$$

by Lebesgue's theorem on dominated convergence and polarization yields

$$d\mu_{u,v}(\lambda) = -\frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \operatorname{Im}\big[G_{\ell,\sigma}(x, x', \sqrt{\lambda})\big]u(x')\overline{v(x)}\,dx'\,dx\,d\lambda$$

for all $u, v \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ since $G_{\ell,\sigma}(x, x', \sqrt{\lambda})$ is symmetric in x and x' for $\lambda > 0$ as follows from the explicit form (see below).

2.4. Pointwise decay estimates

As follows from the discussion above, the functional calculus for self-adjoint operators yields the representation

$$\big(\cos(t\sqrt{\mathcal{H}_{\ell,\sigma}})f\,|\,v\big)_{L^2(\mathbb{R})} = -\frac{2}{\pi} \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \lambda \cos(t\lambda) \operatorname{Im}\big[G_{\ell,\sigma}(x, x', \lambda)\big]f(x')\,dx'\,\overline{v(x)}\,dx\,d\lambda$$

for $f, v \in \mathcal{S}(\mathbb{R})$ (the Schwartz space), where we have changed variables in the integration with respect to λ. Our intention is to obtain an expression for $[\cos(t\sqrt{\mathcal{H}_{\ell,\sigma}})f](x)$ and thus, we have to change the order of integration. However, note carefully that a simple argument based on Fubini's

theorem does not apply here since the integrals cannot be expected to converge absolutely. In order to circumvent this difficulty, first observe that, for any $N \in \mathbb{N}$, we have

$$\int\limits_{1/N}^{N} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \left| \lambda \cos(t\lambda) \operatorname{Im}\left[G_{\ell,\sigma}(x, x', \lambda)\right] f(x') \overline{v(x)} \right| dx' \, dx \, d\lambda \leqslant C_N$$

which follows immediately from $\sup_{x,x' \in \mathbb{R}} |G_{\ell,\sigma}(x, x', \lambda)| \leqslant C_N$ for all $\lambda \geqslant \frac{1}{N}$, see Corollary 3.1 below. Thus, Fubini's theorem yields at least

$$\left(\cos\left(t\sqrt{\mathcal{H}_{\ell,\sigma}}\right) f \big| v\right)_{L^2(\mathbb{R})}$$

$$= -\frac{2}{\pi} \lim_{N \to \infty} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \int\limits_{1/N}^{N} \lambda \cos(t\lambda) \operatorname{Im}\left[G_{\ell,\sigma}(x, x', \lambda)\right] f(x') \, d\lambda \, dx' \, \overline{v(x)} \, dx.$$

Next, we distinguish between high and low energies by introducing a smooth cut-off χ_δ satisfying $\chi_\delta(\lambda) = 1$ for $\lambda \in [0, \frac{\delta}{2}]$ and $\chi(\lambda) = 0$ for $\lambda \geqslant \delta$ where $\delta > 0$ is sufficiently small. In Section 8 below we prove the estimate

$$\sup_{x,x' \in \mathbb{R}} \left| \int\limits_0^\infty \lambda \cos(t\lambda) \operatorname{Im}\left[G_{\ell,\sigma}(x, x', \lambda)\right] \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \chi_\delta(\lambda) \, d\lambda \right| \lesssim \langle t \rangle^{-\alpha} \tag{6}$$

where $\alpha \in \mathbb{N}$ and $\alpha \leqslant 2\ell + 3$. This bound is sufficient to conclude

$$\lim_{N \to \infty} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \int\limits_{1/N}^{N} \lambda \cos(t\lambda) \operatorname{Im}\left[G_{\ell,\sigma}(x, x', \lambda)\right] \chi_\delta(\lambda) f(x') \, d\lambda \, dx' \, \overline{v(x)} \, dx$$

$$= \int\limits_{\mathbb{R}} \lim_{N \to \infty} \int\limits_{\mathbb{R}} \int\limits_{1/N}^{N} \lambda \cos(t\lambda) \operatorname{Im}\left[G_{\ell,\sigma}(x, x', \lambda)\right] \chi_\delta(\lambda) f(x') \, d\lambda \, dx' \, \overline{v(x)} \, dx$$

by dominated convergence since $f, v \in \mathcal{S}(\mathbb{R})$. For the large energy part we show in Section 9 that, for any $\alpha \in \mathbb{N}_0$,

$$\sup_{x \in \mathbb{R}} \left| \lim_{N \to \infty} \int\limits_{\mathbb{R}} \int\limits_{1/N}^{N} \lambda \cos(t\lambda) \operatorname{Im}\left[G_{\ell,\sigma}(x, x', \lambda)\right] \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \phi(x') \left[1 - \chi_\delta(\lambda)\right] d\lambda \, dx' \right|$$

$$\lesssim \langle t \rangle^{-\alpha} \int\limits_{\mathbb{R}} \left(|\phi'(x')| + |\phi(x')| \right) dx' \tag{7}$$

which, by dominated convergence, implies

$$\lim_{N \to \infty} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{1/N}^{N} \lambda \cos(t\lambda) \operatorname{Im}\big[G_{\ell,\sigma}\big(x, x', \lambda\big)\big]\big[1 - \chi_\delta(\lambda)\big] f\big(x'\big) \, d\lambda \, dx' \, \overline{v(x)} \, dx$$

$$= \int_{\mathbb{R}} \lim_{N \to \infty} \int_{\mathbb{R}} \int_{1/N}^{N} \lambda \cos(t\lambda) \operatorname{Im}\big[G_{\ell,\sigma}\big(x, x', \lambda\big)\big]\big[1 - \chi_\delta(\lambda)\big] f\big(x'\big) \, d\lambda \, dx' \, \overline{v(x)} \, dx.$$

By adding up the two contributions and using the density of $\mathcal{S}(\mathbb{R})$ in $L^2(\mathbb{R})$, we arrive at the representation

$$\big[\cos(t\sqrt{\mathcal{H}_{\ell,\sigma}})f\big](x) = -\frac{2}{\pi} \lim_{N \to \infty} \int_{\mathbb{R}} \int_{1/N}^{N} \lambda \cos(t\lambda) \operatorname{Im}\big[G_{\ell,\sigma}\big(x, x', \lambda\big)\big] \, d\lambda \, f\big(x'\big) \, dx'$$

for $f \in \mathcal{S}(\mathbb{R})$ and the estimates (6), (7) imply the bound

$$\big\| w_\alpha \cos(t\sqrt{\mathcal{H}_{\ell,\sigma}})f \big\|_{L^\infty(\mathbb{R})} \lesssim \langle t \rangle^{-\alpha} \bigg(\bigg\| \frac{f'}{w_\alpha} \bigg\|_{L^1(\mathbb{R})} + \bigg\| \frac{f}{w_\alpha} \bigg\|_{L^1(\mathbb{R})} \bigg)$$

for $1 \leqslant \alpha \leqslant 2\ell + 3$ where $w_\alpha(x) := \langle x \rangle^{-\alpha}$. An analogous derivation applies to the sine evolution and therefore, the proof of our result reduces to oscillatory estimates of the type (6) and (7).

2.5. The main theorem

The main result proved in this work is the following.

Theorem 2.1. *Let $(\ell, \sigma) \notin \{(0, 0), (0, -3), (1, -3)\}$, $\alpha \in \mathbb{N}$, $1 \leqslant \alpha \leqslant 2\ell + 3$ and set $w_\alpha(x) := \langle x \rangle^{-\alpha}$. Then the solution operators for the Regge–Wheeler equation satisfy the estimates*

$$\big\| w_\alpha \cos(t\sqrt{\mathcal{H}_{\ell,\sigma}})f \big\|_{L^\infty(\mathbb{R})} \leqslant C_{\ell,\alpha} \langle t \rangle^{-\alpha} \bigg(\bigg\| \frac{f'}{w_\alpha} \bigg\|_{L^1(\mathbb{R})} + \bigg\| \frac{f}{w_\alpha} \bigg\|_{L^1(\mathbb{R})} \bigg)$$

and

$$\bigg\| w_\alpha \frac{\sin(t\sqrt{\mathcal{H}_{\ell,\sigma}})}{\sqrt{\mathcal{H}_{\ell,\sigma}}} g \bigg\|_{L^\infty(\mathbb{R})} \leqslant C_{\ell,\alpha} \langle t \rangle^{-\alpha+1} \bigg\| \frac{g}{w_\alpha} \bigg\|_{L^1(\mathbb{R})}$$

for all $t \geqslant 0$ and initial data f, g such that the right-hand sides are finite.

Remark 2.1. As usual, we prove Theorem 2.1 for Schwartz functions $f, g \in \mathcal{S}(\mathbb{R})$. The general case is then obtained by a standard approximation argument.

Remark 2.2. For the convenience of the reader we make the behavior of the initial data near the horizon more explicit by transforming back to the Schwarzschild r-coordinate. Recall that $x = r + 2M \log(\frac{r}{2M} - 1)$ which implies $\langle x \rangle^\alpha \simeq |\log(\frac{r}{2M} - 1)|^\alpha$ as $r \to 2M+$. Thus, the polynomial weights in x translate into logarithmic weights in r. Moreover, we have $dx = (1 - \frac{2M}{r})^{-1} \, dr$

and this shows that the integrability condition near the horizon for initial data $g(x) = \tilde{g}(r(x))$ transforms as

$$\int\limits_{-\infty}^{0} |g(x)| \langle x \rangle^{\alpha}\, dx \simeq \int\limits_{2M}^{r_0} |\tilde{g}(r)| \left| \log\left(\frac{r}{2M} - 1\right) \right|^{\alpha} \frac{dr}{1 - \frac{2M}{r}}$$

where $r = r_0$ corresponds to $x = 0$.

We remark that our proof actually applies to more general situations like the analogous problem in Hořava–Lifshitz gravity, cf. [33]. The only requirements on the potential are the asymptotics of Corollary 2.1, the nonexistence of bound states and the nonexistence of a zero energy resonance (see Definition 6.1 below).

3. Basic properties of the Green's function

For the convenience of the reader we discuss some well-known properties of the Green's function (cf. [25,56]).

3.1. The Jost solutions

Recall that the Green's function is constructed with the help of the Jost solutions $f_{\pm}(\cdot, z)$ which are defined by $\mathcal{H}_{\ell,\sigma} f_{\pm}(\cdot, z) = z^2 f_{\pm}(\cdot, z)$ and the asymptotic behavior $f_{\pm}(x, z) \sim e^{\pm izx}$ as $x \to \pm\infty$. First we prove that the Jost solutions exist and that they are continuous with respect to z in $\overline{\mathbb{C}_+} \backslash \{0\}$ where $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

Lemma 3.1. *For every $z \in \overline{\mathbb{C}_+} \backslash \{0\}$ there exist smooth functions $f_{\pm}(\cdot, z)$ satisfying*

$$\mathcal{H}_{\ell,\sigma} f_{\pm}(\cdot, z) = z^2 f_{\pm}(\cdot, z)$$

and $f_{\pm}(x, z) \sim e^{\pm izx}$ for $x \to \pm\infty$. Furthermore, for every $x \in \mathbb{R}$, the functions $f_{\pm}(x, \cdot)$ and $f'_{\pm}(x, \cdot)$ are continuous in $\overline{\mathbb{C}_+} \backslash \{0\}$.

Proof. We only prove the assertion for f_+ since the proof for f_- is completely analogous. The variation of constants formula shows that $m_+(x, z) := e^{-izx} f_+(x, z)$, if it exists, satisfies the integral equation

$$m_+(x, z) = 1 + \int\limits_{x}^{\infty} K(x, y, z) m_+(y, z)\, dy \tag{8}$$

where $K(x, y, z) = \frac{1}{2iz}(e^{2iz(y-x)} - 1) V_{\ell,\sigma}(y)$. Conversely, if we can show that Eq. (8) has a smooth solution, we obtain existence of the Jost solution. However, Eq. (8) is a Volterra integral equation with a kernel satisfying

$$\int\limits_{a}^{\infty} \sup_{x \in (a,y)} |K(x, y, z)|\, dy \leqslant \frac{C}{|z|}, \qquad \int\limits_{a}^{\infty} \sup_{x \in (a,y)} |\partial_x K(x, y, z)|\, dy \lesssim 1$$

for all $z \in \overline{\mathbb{C}_+} \setminus \{0\}$ and any fixed $a \in \mathbb{R}$ (see Corollary 2.1) and thus, Lemma B.1 implies the existence of a unique solution $m_+(\cdot, z)$ satisfying $\|m_+(\cdot, z)\|_{L^\infty(a,\infty)} \leqslant e^{C/|z|}$. Furthermore, for fixed $z \in \overline{\mathbb{C}_+} \setminus \{0\}$, we have

$$\int\limits_a^\infty \sup_{x \in (a,y)} \left| \partial_x^k K(x, y, z) \right| dy \leqslant C_k$$

for all $k \in \mathbb{N}_0$ and thus, Lemma B.2 shows that $m_+(\cdot, z)$ is smooth. For the continuity of $m_+(x, \cdot)$ fix $x \in (a, \infty)$, $z \in \overline{\mathbb{C}_+} \setminus \{0\}$ and note that

$$m_+(x, z+h) - m_+(x, z) = g_h(x, z) + \int\limits_x^\infty K(x, y, z+h) \big[m_+(y, z+h) - m_+(y, z) \big] dy$$

where

$$g_h(x, z) := \int\limits_x^\infty \big[K(x, y, z+h) - K(x, y, z) \big] m_+(y, z) \, dy.$$

Now observe that $\|g_h(\cdot, z)\|_{L^\infty(a,\infty)} \to 0$ as $h \to 0$ since $\|m_+(\cdot, z)\|_{L^\infty(a,\infty)} \leqslant e^{C/|z|}$ and hence, Lemma B.1 implies

$$\left| m_+(x, z+h) - m_+(x, z) \right| \leqslant \left\| g_h(\cdot, z) \right\|_{L^\infty(a,\infty)} e^{C/|z|} \to 0 \quad \text{for } h \to 0$$

which shows continuity of $m_+(x, \cdot)$ in $\overline{\mathbb{C}_+} \setminus \{0\}$ as claimed. For the continuity of $m'_+(x, \cdot)$ simply observe that

$$m'_+(x, z) = \int\limits_x^\infty \partial_x K(x, y, z) m_+(y, z) \, dy$$

and the right-hand side of this equation is obviously continuous in z. $\square$

3.2. The Wronskian $W(f_-(\cdot, \sqrt{z}), f_+(\cdot, \sqrt{z}))$

Having established existence of the Jost solutions we can now construct the Green's function and the standard procedure yields

$$G_{\ell,\sigma}(x, x', \sqrt{z}) = \frac{f_-(x', \sqrt{z}) f_+(x, \sqrt{z}) \Theta(x - x') + f_-(x, \sqrt{z}) f_+(x', \sqrt{z}) \Theta(x' - x)}{W(f_-(\cdot, \sqrt{z}), f_+(\cdot, \sqrt{z}))}$$

for $\operatorname{Im} z > 0$ where Θ denotes the Heaviside function. Clearly, $W(f_-(\cdot, \sqrt{z}), f_+(\cdot, \sqrt{z})) \neq 0$ if $\operatorname{Im} z > 0$ since otherwise $f_-(\cdot, \sqrt{z})$ would be an eigenfunction with eigenvalue z contradicting the self-adjointness of $\mathcal{H}_{\ell,\sigma}$. However, it is not a priori clear whether the limit $G_{\ell,\sigma}(x, x', \sqrt{z})$ for $\operatorname{Im} \sqrt{z} \to 0+$ exists. The following observation shows that problems can only occur at $z = 0$.

Lemma 3.2. *Let $\lambda > 0$. Then the limit*

$$W(\sqrt{\lambda}) := \lim_{\varepsilon \to 0+} W\big(f_-(\cdot, \sqrt{\lambda + i\varepsilon}), f_+(\cdot, \sqrt{\lambda + i\varepsilon})\big)$$

exists and is nonzero.

Proof. For brevity we write $W(f_-, f_+)(z)$ instead of $W(f_-(\cdot, \sqrt{z}), f_+(\cdot, \sqrt{z}))$ and likewise for other Wronskians. By Lemma 3.1 we know that $W(f_-, f_+)$ is continuous in $\overline{\mathbb{C}_+} \backslash \{0\}$ and hence, $W(\sqrt{\lambda})$ exists for any $\lambda > 0$. Observe that $f'_\pm(x, \sqrt{\lambda}) \sim \pm i\sqrt{\lambda} e^{\pm i\sqrt{\lambda}x}$ for $x \to \pm\infty$ which follows immediately from the integral representation in the proof of Lemma 3.1. Thus, $W(\overline{f_+}, f_+)(\lambda) = 2i\sqrt{\lambda}$ which shows that $f_+(\cdot, \sqrt{\lambda})$ and $\overline{f_+(\cdot, \sqrt{\lambda})}$ are linearly independent for $\lambda > 0$. Hence, there exist $A(\lambda)$ and $B(\lambda)$ such that $f_-(x, \sqrt{\lambda}) = A(\lambda) f_+(x, \sqrt{\lambda}) + B(\lambda) \overline{f_+(x, \sqrt{\lambda})}$. We conclude

$$2i\sqrt{\lambda} = W(f_-, \overline{f_-})(\lambda) = W(Af_+ + B\overline{f_+}, \overline{A}\overline{f_+} + \overline{B}f_+)(\lambda)$$
$$= -2i\sqrt{\lambda}|A(\lambda)|^2 + 2i\sqrt{\lambda}|B(\lambda)|^2 \tag{9}$$

which implies $|B(\lambda)|^2 \geqslant 1$. However, we have

$$W(\sqrt{\lambda}) = W(f_-, f_+)(\lambda) = W(Af_+ + B\overline{f_+}, f_+)(\lambda) = 2i\sqrt{\lambda}B(\lambda) \tag{10}$$

and thus, $|W(\sqrt{\lambda})| \geqslant 2\sqrt{\lambda}$ which finishes the proof. $\quad\square$

Corollary 3.1. *The limit*

$$G_{\ell,\sigma}\big(x, x', \sqrt{\lambda}\big) = \lim_{\varepsilon \to 0+} G_{\ell,\sigma}\big(x, x', \sqrt{\lambda + i\varepsilon}\big)$$

exists and satisfies

$$\sup_{x,x' \in \mathbb{R}} \big|G_{\ell,\sigma}\big(x, x', \sqrt{\lambda}\big)\big| \leqslant C$$

for all $\lambda \geqslant \lambda_0$ where $\lambda_0 > 0$ is arbitrary.

Proof. From Lemma 3.2 and the asymptotic behavior of the Jost solutions we immediately conclude

$$\sup_{x'<0, x>0} \left| \frac{f_-(x', \sqrt{\lambda}) f_+(x, \sqrt{\lambda})}{W(f_-(\cdot, \sqrt{\lambda}), f_+(\cdot, \sqrt{\lambda}))} \right| \lesssim 1$$

for all $\lambda \geqslant \lambda_0 > 0$. For the remaining cases use reflection and transmission coefficients $A(\lambda)$, $B(\lambda)$ (see the proof of Lemma 3.2) to express $f_\pm$ in terms of $f_\mp$ and $\overline{f_\mp}$. The asymptotic behavior of $A(\lambda)$ and $B(\lambda)$ for $\lambda \to \infty$ is given by Eqs. (10) and (9) and the claim follows. $\quad\square$

4. Perturbative solutions for $|x\lambda|$ small

In this section we obtain approximations to solutions of $\mathcal{H}_{\ell,\sigma} f = \lambda^2 f$ for $|x\lambda|$ small. The solutions are constructed by perturbation in λ around $\lambda = 0$. We closely follow [53].

4.1. Zero energy solutions

We first consider zero energy solutions, i.e., solutions of $\mathcal{H}_{\ell,\sigma} f = 0$. By setting $v(r) := f(x(r))$, the eigenvalue problem $\mathcal{H}_{\ell,\sigma} f = \lambda^2 f$ is equivalent to

$$-\left(1 - \frac{2M}{r}\right)v'' - \frac{2M}{r^2}v' + \left(\frac{\ell(\ell+1)}{r^2} + \frac{2M\sigma}{r^3}\right)v = \lambda^2\left(1 - \frac{2M}{r}\right)^{-1}v \qquad (11)$$

and it turns out that for $\lambda = 0$ this equation can be solved by special functions which will be useful later on. However, the following result describes a fundamental system for $\mathcal{H}_{\ell,\sigma} f = 0$ without making use of explicit solutions.

Lemma 4.1. *There exist smooth functions u_j satisfying $\mathcal{H}_{\ell,\sigma} u_j = 0$ for $j = 0, 1$ with the bounds $u_0(x) = (2\ell+1)^{-1}x^{\ell+1}(1 + O(x^{-1+\varepsilon}))$ and $u_1(x) = x^{-\ell}(1 + O(x^{-1+\varepsilon}))$ for all $x \geqslant 1$ where $\varepsilon \in (0, 1)$ is arbitrary and the O-terms are of symbol type. The Wronskian is $W(u_0, u_1) = -1$.*

Proof. Suppose for the moment that the solution u_1 exists and define the function a by $u_1(x) = x^{-\ell}(1 + a(x))$. Then $\mathcal{H}_{\ell,\sigma} u_1 = 0$ is equivalent to

$$a''(x) - \frac{2\ell}{x}a'(x) = \left[V_{\ell,\sigma}(x) - \frac{\ell(\ell+1)}{x^2}\right](1 + a(x)). \qquad (12)$$

Viewing this equation as an inhomogeneous equation for a', applying the variation of constants formula and integrating by parts, we obtain the integral equation

$$a(x) = \frac{1}{2\ell+1}\int_x^\infty \left(y^{2\ell+1} - x^{2\ell+1}\right)y^{-2\ell}\left[V_{\ell,\sigma}(y) - \frac{\ell(\ell+1)}{y^2}\right](1 + a(y))\,dy. \qquad (13)$$

Therefore, if we can show that Eq. (13) has a smooth solution, we obtain existence of u_1. However, Eq. (13) is a Volterra integral equation of the form

$$a(x) = \int_x^\infty K(x, y)\,dy + \int_x^\infty K(x, y)a(y)\,dy$$

with a kernel K satisfying $|\partial_x^k \partial_y^l K(x, y)| \leqslant C_{k,l}y^{-2+\varepsilon-k-l}$ for $1 \leqslant x \leqslant y$ and an arbitrary $\varepsilon \in (0, 1)$ (cf. Corollary 2.1). Therefore, Lemma B.2 implies the existence of a unique smooth solution $a \in L^\infty(1, \infty)$ and Eq. (13) shows that in fact $|a(x)| \lesssim x^{-1+\varepsilon}$ for $x \geqslant 1$. Furthermore, the first derivative of a is given by

$$a'(x) = \int_x^\infty \partial_x K(x, y)\left(1 + a(y)\right)dy$$

and this implies $|a'(x)| \lesssim x^{-2+\varepsilon}$ for $x \geqslant 1$. The estimates for the higher derivatives follow from Eq. (12), the Leibniz rule and a simple induction.

For the second solution u_0 we use the Wronskian condition

$$1 = W(u_1, u_0) = \left(\frac{u_0}{u_1}\right)' u_1^2$$

which yields

$$u_0(x) = u_1(x) \int\limits_{x_1}^x u_1^{-2}(y)\, dy$$

where $x_1 > 0$ is chosen so large that $u_1(x) > 0$ for all $x \geqslant x_1$. Note that $u_1^{-2}(x) = x^{2\ell}(1 + b(x))$ for a suitable b satisfying $|b^{(k)}(x)| \leqslant C_k \langle x \rangle^{-1+\varepsilon-k}$ (apply the Leibniz rule and Lemma A.1). Inserting the asymptotic expansion for u_1 yields $u_0(x) = (2\ell + 1)^{-1} x^{\ell+1}(1 + O(x^{-1+\varepsilon}))$ where the O-term behaves like a symbol. $\quad\square$

We construct another pair v_0, v_1 of zero energy solutions with specific asymptotic behavior as $x \to -\infty$. This is considerably easier than the above construction for the solutions u_j due to the exponential decay of the Regge–Wheeler potential $V_{\ell,\sigma}(x)$ as $x \to -\infty$.

Lemma 4.2. *There exist smooth functions v_j for $j = 0, 1$ satisfying $\mathcal{H}_{\ell,\sigma} v_j = 0$ and $v_0(x) = x(1 + O(x^{-1}))$ as well as $v_1(x) = 1 + O(x^{-1})$ for all $x \leqslant -1$ where the O-terms behave like symbols under differentiation.*

Proof. For $x \leqslant -1$ consider the Volterra equations

$$\frac{v_0(x)}{x} = 1 - \int\limits_{-\infty}^x \left(\frac{y^2}{x} - y\right) V_{\ell,\sigma}(y) \frac{v_0(y)}{y}\, dy$$

and

$$v_1(x) = 1 - \int\limits_{-\infty}^x (y - x) V_{\ell,\sigma}(y) v_1(y)\, dy$$

which have smooth solutions according to Lemma B.2 since the potential $V_{\ell,\sigma}(y)$ decays exponentially as $y \to -\infty$ (see Corollary 2.1). Obviously, we have the asymptotic behavior $v_0(x) \sim x$, $v_1(x) \sim 1$ as $x \to -\infty$, v_j behave like symbols under differentiation and satisfy $\mathcal{H}_{\ell,\sigma} v_j = 0$, as a straightforward calculation shows. $\quad\square$

Corollary 4.1. *The solutions u_j for $j = 0, 1$ can be uniquely extended to all of $\mathbb{R}$ and we have $u_j(x) = O(x)$ as $x \to -\infty$ where the O-term is of symbol type.*

Proof. Since the potential $V_{\ell,\sigma}$ is smooth on $\mathbb{R}$, the solutions u_j, originally defined on $[1,\infty)$ only, can be uniquely extended to all of $\mathbb{R}$ by solving appropriate initial value problems. Since the solution pair v_0, v_1 forms a fundamental system for the equation $\mathcal{H}_{\ell,\sigma} f = 0$, u_j can be written as a linear combination of v_0, v_1 on $(-\infty, -1]$ and everything follows from Lemma 4.2. □

4.2. Construction of the perturbative solutions

Next, by perturbing in λ around $\lambda = 0$, we obtain useful approximations to solutions of $\mathcal{H}_{\ell,\sigma} f = \lambda^2 f$ for $|x\lambda|$ small.

Lemma 4.3. *There exist constants $x_0, \lambda_0, \delta > 0$ and smooth functions $u_j(\cdot, \lambda)$ satisfying*

$$\mathcal{H}_{\ell,\sigma} u_j(\cdot, \lambda) = \lambda^2 u_j(\cdot, \lambda)$$

for $j = 0, 1$ and $W(u_0(\cdot, \lambda), u_1(\cdot, \lambda)) = -1$ such that $u_j(x, \lambda) = u_j(x)(1 + a_j(x, \lambda))$ where

$$\left| a_j(x, \lambda) \right| \lesssim x^2 \lambda^2$$

for all $\lambda \in (0, \lambda_0)$ and $x \in [x_0, \delta\lambda^{-1}]$ provided that $\ell \geqslant 1$. In the case $\ell = 0$ we have the weaker bounds

$$\left| a_0(x, \lambda) \right| \lesssim x^2 \lambda^2 \quad and \quad \left| a_1(x, \lambda) \right| \lesssim x\lambda$$

in the above ranges of x and λ.

Proof. Let $x_0 > 0$ be so large that $u_0(x) > 0$ for all $x \geqslant x_0$. A straightforward calculation shows that, if the function h solves the integral equation

$$h(x, \lambda) = 1 - \lambda^2 \int_{x_0}^{x} \left[u_0(y)u_1(y) - u_0^2(y)\frac{u_1(x)}{u_0(x)} \right] h(y, \lambda)\, dy, \tag{14}$$

then $u_0(x, \lambda) := u_0(x)h(x, \lambda)$ satisfies $\mathcal{H}_{\ell,\sigma} u_0(\cdot, \lambda) = \lambda^2 u_0(\cdot, \lambda)$. Eq. (14) is a Volterra integral equation

$$h(x, \lambda) = 1 + \int_{x_0}^{x} K(x, y, \lambda)h(y, \lambda)\, dy$$

where the kernel is of the form

$$K(x, y, \lambda) = \lambda^2 \left[y\left(1 + O\left(y^{-1+\varepsilon}\right)\right) - x^{-2\ell-1}y^{2\ell+2}\left(1 + O\left(x^{-1+\varepsilon}\right) + O\left(y^{-1+\varepsilon}\right)\right) \right]$$

for $x, y \geqslant x_0$ and the O-terms are of symbol type (see Lemma 4.1). This shows $|\partial_x^k \partial_y^l K(x, y, \lambda)| \leqslant C_{k,l} y^{1-k-l} \lambda^2$ for $x_0 \leqslant y \leqslant x$ and hence,

$$
\int\limits_{x_0}^{\lambda^{-1}} \sup_{\{x:\, x_0 < y < x\}} \left|\partial_x^k K(x, y, \lambda)\right| dy \lesssim 1
$$

for $\lambda \in (0, \lambda_0)$ and all $k \in \mathbb{N}_0$ where $\lambda_0 > 0$ is arbitrary. Thus, Lemma B.2 shows that Eq. (14) has a unique smooth solution $h(\cdot, \lambda)$ satisfying $\|h(\cdot, \lambda)\|_{L^\infty(x_0, \lambda^{-1})} \leqslant C$ for all $\lambda \in (0, \lambda_0)$. Therefore, Eq. (14) implies $|h(x, \lambda) - 1| \lesssim x^2 \lambda^2$ for $\lambda \in (0, \lambda_0)$ and $x_0 \leqslant x \leqslant \lambda^{-1}$.

Now choose $\delta > 0$ and $\lambda_0 > 0$ so small that $|h(x, \lambda) - 1| \leqslant \frac{1}{2}$ for all $\lambda \in (0, \lambda_0)$ and $x \in [x_0, \delta \lambda^{-1}]$. We use the Wronskian condition

$$
-1 = W\big(u_0(\cdot, \lambda), u_1(\cdot, \lambda)\big) = \left(\frac{u_1(\cdot, \lambda)}{u_0(\cdot, \lambda)}\right)' u_0^2(\cdot, \lambda)
$$

to construct the second solution $u_1(x, \lambda)$, i.e.,

$$
u_1(x, \lambda) = u_0(x, \lambda) \int\limits_x^{\delta \lambda^{-1}} u_0^{-2}(y, \lambda)\, dy
$$

which implies

$$
a_1(x, \lambda) = \frac{u_0(x)}{u_1(x)}\big(1 + a_0(x, \lambda)\big) \int\limits_x^{\delta \lambda^{-1}} u_0^{-2}(y)\big(1 + \tilde{a}_0(y, \lambda)\big)\, dy - 1 \tag{15}
$$

where $\tilde{a}_0(y, \lambda) := (1 + a_0(y, \lambda))^{-2} - 1$ inherits the bound of a_0. Now, by inserting the asymptotics of u_0 from Lemma 4.1, we obtain

$$
u_0(x) \int\limits_x^{\infty} u_0^{-2}(y)\, dy \sim x^{-\ell}
$$

for $x \to \infty$ and this shows

$$
u_1(x) = u_0(x) \int\limits_x^{\infty} u_0^{-2}(y)\, dy
$$

since u_1 is *uniquely* determined by the asymptotic behavior $u_1(x) \sim x^{-\ell}$ for $x \to \infty$ (cf. Lemma 4.1). Using this, Eq. (15) and the asymptotics of $a_0, \tilde{a}_0$, we obtain

$$a_1(x, \lambda) = \frac{u_0(x)}{u_1(x)} \left(- \int_{\delta\lambda^{-1}}^{\infty} u_0^{-2}(y)\, dy + a_0(x, \lambda) \int_x^{\delta\lambda^{-1}} u_0^{-2}(y)\, dy \right)$$

$$+ \frac{u_0(x)}{u_1(x)} \big(1 + a_0(x, \lambda) \big) \int_x^{\delta\lambda^{-1}} u_0^{-2}(y)\tilde{a}_0(y, \lambda)\, dy$$

$$= O(x^{2\ell+1}) \big[O(\lambda^{2\ell+1}) + O(x^2\lambda^2)(O(\lambda^{2\ell+1}) + O(x^{-2\ell-1})) + O(x^{-2\ell-1})O(x^2\lambda^2) \big]$$

$$= O(x^2\lambda^2) + O((x\lambda)^{2\ell+1})$$

which implies the claim. $\quad \square$

4.3. Estimates on the derivatives

Next, we study derivatives of the above constructed solutions.

Proposition 4.1. *The functions a_j for $j = 0, 1$ from Lemma 4.3 are of symbol type, i.e.,*

$$\big| \partial_x^k \partial_\lambda^m a_j(x, \lambda) \big| \leqslant C_{k,m} x^{2-k} \lambda^{2-m}$$

for $k, m \in \mathbb{N}_0$ and $\lambda \in (0, \lambda_0)$, $x \in [x_0, \delta\lambda^{-1}]$ provided that $\ell \geqslant 1$ where $x_0, \lambda_0, \delta > 0$ are constants. In the case $\ell = 0$ we have the weaker bounds

$$\big| \partial_x^k \partial_\lambda^m a_0(x, \lambda) \big| \leqslant C_{k,m} x^{2-k} \lambda^{2-m} \quad and \quad \big| \partial_x^k \partial_\lambda^m a_1(x, \lambda) \big| \leqslant C_{k,m} x^{1-k} \lambda^{1-m}$$

in the above ranges of x and λ.

Proof. We use the notations from the proof of Lemma 4.3 and proceed by induction. We have to consider mixed derivatives and therefore, we order the set $\mathbb{N}_0 \times \mathbb{N}_0$ according to

$$(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), \ldots$$

which defines a bijection $n : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ by $n(0, 0) = 0$, $n(1, 0) = 1$, $n(0, 1) = 2$, etc. Now fix $(k, m) \in \mathbb{N}_0 \times \mathbb{N}_0$ and assume that $|\partial_x^l \partial_\lambda^j (h(x, \lambda) - 1)| \leqslant C_{l,j} x^{2-l} \lambda^{2-j}$ holds for all (l, j) with $n(l, j) \leqslant n(k, m)$ and $\lambda \in (0, \lambda_0)$, $x \in [x_0, \lambda^{-1}]$. We have to show that this implies $|\partial_x^{k'} \partial_\lambda^{m'} h(x, \lambda)| \leqslant C_{k',m'} x^{2-k'} \lambda^{2-m'}$, where $n(k', m') = n(k, m) + 1$. There are two possibilities: Either $(k', m') = (m + 1, 0)$ (if $k = 0$) or $(k', m') = (k - 1, m + 1)$. In the former case we have with $\kappa_l(x, \lambda) := \partial_x^l K(x, y, \lambda)|_{y=x} = O(x^{1-l}\lambda^2)$ (cf. Lemma B.2),

$$\partial_x^{m+1} h(x, \lambda) = \sum_{l=0}^{m} \partial_x^{m-l} \big[\kappa_l(x, \lambda) h(x, \lambda) \big] + \int_{x_0}^{x} \partial_x^{m+1} K(x, y, \lambda) h(y, \lambda)\, dy.$$

Now observe that by assumption $|\partial_x^{m-l} [\kappa_l(x, \lambda) h(x, \lambda)]| \lesssim x^{2-(m+1)} \lambda^2$ for $\lambda \in (0, \lambda_0)$ and $x \in [x_0, \lambda^{-1}]$ and hence, $|\partial_x^{m+1} h(x, \lambda)| \lesssim x^{2-(m+1)} \lambda^2$. In the latter case we have, provided $k \geqslant 2$,

$$\left|\partial_\lambda^{m+1}\partial_x^{k-1}h(x,\lambda)\right| = \left|\sum_{l=0}^{k-2}\partial_\lambda^{m+1}\partial_x^{k-2-l}\left[\kappa_l(x,\lambda)h(x,\lambda)\right] + \int_{x_0}^{x}\partial_\lambda^{m+1}\left[\partial_x^{k-1}K(x,y,\lambda)h(y,\lambda)\right]dy\right|$$

$$\lesssim x^{2-(k-1)}\lambda^{2-(m+1)}$$

by assumption. If $k = 1$ we have

$$\partial_\lambda^{m+1}h(x,\lambda) = \sum_{j=1}^{\min\{m+1,2\}}\binom{m+1}{j}\int_{x_0}^{x}\partial_\lambda^j K(x,y,\lambda)\partial_\lambda^{m+1-j}h(y,\lambda)\,dy$$

$$+ \int_{x_0}^{x}K(x,y,\lambda)\partial_\lambda^{m+1}h(y,\lambda)\,dy$$

and thus, by assumption, the derivative $\partial_\lambda^{m+1}h(x,\lambda)$ satisfies a Volterra equation of the form

$$\partial_\lambda^{m+1}h(x,\lambda) = O\left(x^2\lambda^{2-(m+1)}\right) + \int_{x_0}^{x}K(x,y,\lambda)\partial_\lambda^{m+1}h(y,\lambda)\,dy$$

and the basic estimate from Lemma B.1 yields $|\partial_\lambda^{m+1}h(x,\lambda)| \lesssim x^2\lambda^{2-(m+1)}$ for all $\lambda \in (0,\lambda_0)$ and $x \in [x_0,\lambda^{-1}]$.

For the second solution we use the representation

$$a_1(x,\lambda) = \frac{u_0(x)}{u_1(x)}\left(-\int_{\delta\lambda^{-1}}^{\infty}u_0^{-2}(y)\,dy + a_0(x,\lambda)\int_x^{\delta\lambda^{-1}}u_0^{-2}(y)\,dy\right)$$

$$+ \frac{u_0(x)}{u_1(x)}\left(1 + a_0(x,\lambda)\right)\int_x^{\delta\lambda^{-1}}u_0^{-2}(y)\tilde{a}_0(y,\lambda)\,dy \tag{16}$$

from the proof of Lemma 4.3 where, as before, $\tilde{a}_0(x,\lambda) := (1 + a_0(x,\lambda))^{-2} - 1$. Lemma A.1 and the Leibniz rule show that $\tilde{a}_0$ inherits the bounds of a_0, i.e., $|\partial_x^k\partial_\lambda^m\tilde{a}_0(x,\lambda)| \leqslant C_{k,m}x^{2-k}\lambda^{2-m}$ for all $k, m \in \mathbb{N}_0$ and $\lambda \in (0,\lambda_0)$, $x \in [x_0,\lambda^{-1}]$. Thus, all functions on the right-hand side of Eq. (16) behave like symbols under differentiation with respect to x and λ. Therefore, as in the proof of Lemma 4.3, we have

$$a_1(x,\lambda) = O\left(x^2\lambda^2\right) + O\left((x\lambda)^{2\ell+1}\right)$$

where the O-terms are of symbol type which finishes the proof. $\square$

4.4. Refined bounds for λ-derivatives

As a next step we prove a refinement of the estimates for the solution u_0 which shows that we can effectively trade λ^{-1} for x in the bounds for the λ-derivatives of a_0.

Lemma 4.4. *The function $a_0(\cdot, \lambda)$, defined by $u_0(x, \lambda) = u_0(x)(1 + a_0(x, \lambda))$, satisfies the estimates*

$$\left|\partial_\lambda^{2m} a_0(x, \lambda)\right| \leqslant C_m x^{2m} \quad and \quad \left|\partial_\lambda^{2m+1} a_0(x, \lambda)\right| \leqslant C_m x^{2m+2}\lambda$$

for all $\lambda \in [0, \lambda_0]$, $x \in [x_0, \lambda^{-1}]$ and $m \in \mathbb{N}_0$ where $\lambda_0 > 0$ is a sufficiently small constant.

Proof. We use the notations from the proof of Proposition 4.1 and proceed by induction. The case $m = 0$ has already been proved in Proposition 4.1. Now fix $m \in \mathbb{N}$ and assume that $|\partial_\lambda^{2j}(h(x, \lambda) - 1)| \leqslant C_j x^{2j}$ and $|\partial_\lambda^{2j+1}(h(x, \lambda) - 1)| \leqslant C_j x^{2j+2}\lambda$ for $j < m$ in the above range of λ and x. According to the proof of Proposition 4.1, we have

$$\partial_\lambda^{2m} h(x, \lambda) = \binom{2m}{2} \int_{x_0}^x \partial_\lambda^2 K(x, y, \lambda) \partial_\lambda^{2m-2} h(y, \lambda)\, dy + 2m \int_{x_0}^x \partial_\lambda K(x, y, \lambda) \partial_\lambda^{2m-1} h(y, \lambda)\, dy$$

$$+ \int_{x_0}^x K(x, y, \lambda) \partial_\lambda^{2m} h(y, \lambda)\, dy$$

$$= \int_{x_0}^x O(y) O(y^{2m-2})\, dy + \int_{x_0}^x O(y\lambda) O(y^{2m}\lambda)\, dy + \int_{x_0}^x K(x, y, \lambda) \partial_\lambda^{2m} h(y, \lambda)\, dy$$

$$= O(x^{2m}) + \int_{x_0}^x K(x, y, \lambda) \partial_\lambda^{2m} h(y, \lambda)\, dy$$

by assumption and the estimate in Lemma B.1 implies $|\partial_\lambda^{2m} h(x, \lambda)| \lesssim x^{2m}$ for $\lambda \in [0, \lambda_0]$ and $x \in [x_0, \lambda^{-1}]$. For the odd derivatives we proceed analogously and obtain

$$\partial_\lambda^{2m+1} h(x, \lambda) = \binom{2m+1}{2} \int_{x_0}^x \partial_\lambda^2 K(x, y, \lambda) \partial_\lambda^{2m-1} h(y, \lambda)\, dy$$

$$+ (2m+1) \int_{x_0}^x \partial_\lambda K(x, y, \lambda) \partial_\lambda^{2m} h(y, \lambda)\, dy + \int_{x_0}^x K(x, y, \lambda) \partial_\lambda^{2m+1} h(y, \lambda)\, dy$$

$$= \int_{x_0}^x O(y) O(y^{2m}\lambda)\, dy + \int_{x_0}^x O(y\lambda) O(y^{2m})\, dy + \int_{x_0}^x K(x, y, \lambda) \partial_\lambda^{2m+1} h(y, \lambda)\, dy$$

$$= O(x^{2m+2}\lambda) + \int_{x_0}^x K(x, y, \lambda) \partial_\lambda^{2m+1} h(y, \lambda)\, dy$$

by assumption and again, Lemma B.1 yields the claim. □

Note that, by construction, we have $u_0(x_0, \lambda) = u_0(x_0)$ and $u_0'(x_0, \lambda) = u_0'(x_0)$ (see the proof of Lemma 4.3) which shows in particular that $u_0(x_0, \lambda)$ and $u_0'(x_0, \lambda)$ are smooth functions of λ. Next, we prove similar bounds for the function a_1 but unfortunately, the situation here is a bit more complicated.

Lemma 4.5. *Let $u_1(x, \lambda) = u_1(x)(1 + a_1(x, \lambda))$ and $x_0, \lambda_0, \delta > 0$ be as in Lemma 4.3. Then, for all $\lambda \in (0, \lambda_0)$ and all $x \in [x_0, \delta\lambda^{-1}]$, we have the estimates*

$$\left| \partial_\lambda^{2m} a_1(x, \lambda) \right| \leqslant C_m x^{2m} \quad and \quad \left| \partial_\lambda^{2m+1} a_1(x, \lambda) \right| \leqslant C_m x^{2m+2} \lambda$$

provided that $m \leqslant \ell - 1$. Furthermore, for higher derivatives we have the bounds

$$\left| \partial_\lambda^{2\ell+m} a_1(x, \lambda) \right| \leqslant C_m x^{2\ell} \lambda^{-m}$$

for $m \in \mathbb{N}_0$ in the above ranges of x and λ.

Proof. The function $a_1(x, \lambda)$ is given by

$$a_1(x, \lambda) = \frac{u_0(x)}{u_1(x)} \left(1 + a_0(x, \lambda) \right) \int\limits_x^{\delta\lambda^{-1}} u_0^{-2}(y) \left(1 + \tilde{a}_0(y, \lambda) \right) dy - 1$$

with $\tilde{a}_0(x, \lambda) = (1 + a_0(x, \lambda))^{-2} - 1$, see Eq. (15). Thus, in view of Lemma 4.4 it suffices to prove the claimed bounds for

$$h(x, \lambda) := \frac{u_0(x)}{u_1(x)} \int\limits_x^{\delta\lambda^{-1}} u_0^{-2}(y) \left(1 + \tilde{a}_0(y, \lambda) \right) dy - 1.$$

For $k \in \mathbb{N}$ we have

$$\partial_\lambda^k h(x, \lambda) = -\frac{u_0(x)}{u_1(x)} \sum_{j=0}^{k-1} \frac{d^j}{d\lambda^j} \left(u_0^{-2}(\delta\lambda^{-1}) \partial_\lambda^{k-1-j} \left(1 + \tilde{a}_0(y, \lambda) \right) \big|_{y=\delta\lambda^{-1}} \delta\lambda^{-2} \right)$$

$$+ \frac{u_0(x)}{u_1(x)} \int\limits_x^{\delta\lambda^{-1}} u_0^{-2}(y) \partial_\lambda^k \left(1 + \tilde{a}_0(y, \lambda) \right) dy$$

and, by using the symbol behavior of $\tilde{a}_0$, we infer

$$\frac{u_0(x)}{u_1(x)} \sum_{j=0}^{k-1} \frac{d^j}{d\lambda^j} \left(u_0^{-2}(\delta\lambda^{-1}) \partial_\lambda^{k-1-j} \left(1 + \tilde{a}_0(y, \lambda) \right) \big|_{y=\delta\lambda^{-1}} \delta\lambda^{-2} \right) = O(x^{2\ell+1}) O(\lambda^{2\ell+1-k}).$$

If $k = 2m$ we have $O(x^{2\ell+1}) O(\lambda^{2\ell+1-k}) = O(x^{2m})$ provided that $m \leqslant \ell$. If $k = 2m + 1$ we obtain

$$O(x^{2\ell+1}) O(\lambda^{2\ell+1-k}) = O(x^{2\ell+1-2\ell+2m+1} \lambda) = O(x^{2m+2} \lambda)$$

provided that $m \leqslant \ell - 1$. In order to estimate the integral term note that $\tilde{a}_0$ inherits the bounds of a_0 from Lemma 4.4 as a consequence of the Leibniz rule and Lemma A.1. For even derivatives we therefore have

$$\frac{u_0(x)}{u_1(x)} \int_x^{\delta\lambda^{-1}} u_0^{-2}(y) \partial_\lambda^{2m}\left(1 + \tilde{a}_0(y, \lambda)\right) dy = O\left(x^{2\ell+1}\right) \int_x^{\delta\lambda^{-1}} O\left(y^{-2\ell-2+2m}\right) dy$$

$$= O\left(x^{2\ell+1}\right)\left(O\left(\lambda^{2\ell+1-2m}\right) + O\left(x^{-2\ell-1+2m}\right)\right)$$

and, provided that $m \leqslant \ell$, we obtain $O(x^{2\ell+1})O(\lambda^{2\ell+1-2m}) = O(x^{2m})$ as before. For odd derivatives we use $\partial_\lambda^{2m+1}\tilde{a}_0(y, \lambda) = O(y^{2m+2}\lambda)$ and infer

$$\frac{u_0(x)}{u_1(x)} \int_x^{\delta\lambda^{-1}} u_0^{-2}(y) \partial_\lambda^{2m+1}\left(1 + \tilde{a}_0(y, \lambda)\right) dy = O\left(x^{2\ell+1}\right) \int_x^{\delta\lambda^{-1}} O\left(y^{-2\ell-2+2m+2}\lambda\right) dy$$

$$= O\left(x^{2\ell+1}\right)\left(O\left(\lambda^{2\ell-2m}\right) + O\left(x^{-2\ell-1+2m+2}\lambda\right)\right)$$

and $O(x^{2\ell+1})O(\lambda^{2\ell-2m}) = O(x^{2m+2}\lambda)$ provided that $m \leqslant \ell - 1$. The claim for the higher derivatives follows directly from the symbol behavior of the above O-terms. $\quad\square$

At this point it is convenient to introduce a new notation.

Definition 4.1. For $N \in \mathbb{N}_0$ we write $f(x) = O_{2N}(1)$ if, for a constant $a > 0$,

(1) $f : (0, a) \to \mathbb{R}$ is smooth,
(2) $|f^{(k)}(x)| \leqslant C_k$ for $k \leqslant 2N$ and all $x \in (0, a)$,
(3) $|f^{(2N+k)}(x)| \leqslant C_k x^{-k}$ for all $k \in \mathbb{N}_0$ and $x \in (0, a)$,
(4) $\lim_{x\to 0+} f^{(2k-1)}(x) = 0$ for all $1 \leqslant k \leqslant N$.

Similarly, we write $f(x) = O_{2N+1}(x)$ if, for a constant $a > 0$,

(1) $f : (0, a) \to \mathbb{R}$ is smooth,
(2) $|f^{(k)}(x)| \leqslant C_k$ for $k \leqslant 2N + 1$ and all $x \in (0, a)$,
(3) $|f^{(2N+1+k)}(x)| \leqslant C_k x^{-k}$ for all $k \in \mathbb{N}_0$ and $x \in (0, a)$,
(4) $\lim_{x\to 0+} f^{(2k)}(x) = 0$ for all $0 \leqslant k \leqslant N$.

Note carefully that the crucial difference between O_{2N} and O_{2N+1} is in condition (4). We also use the symbols O_{2N+1} and O_{2N} to denote *generic real-valued* functions with the respective properties.

Corollary 4.2. *Let $u_1(x, \lambda)$ and $x_0, \lambda_0 > 0$ be as in Lemma* 4.3. *Then we have $u_1(x_0, \lambda) = O_{2\ell}(1)$ and $u_1'(x_0, \lambda) = O_{2\ell}(1)$ for all $\lambda \in (0, \lambda_0)$.*

Proof. The first assertion $u_1(x_0, \lambda) = O_{2\ell}(1)$ follows immediately from Lemma 4.5. For the second one note that $u_0(x_0) \neq 0$ and by construction we have $-1 = u_0(x_0)u_1'(x_0, \lambda) - u_0'(x_0)u(x_0, \lambda)$ which implies

$$u_1'(x_0, \lambda) = \frac{u_0'(x_0)u_1(x_0, \lambda) - 1}{u_0(x_0)} = O_{2\ell}(1)$$

by the first part. □

4.5. Extension of $u_j(x, \lambda)$ to negative values of x

Finally, we extend the solutions $u_j(x, \lambda)$ to negative values of x and prove appropriate estimates.

Lemma 4.6. *The functions $u_j(x, \lambda)$, $j = 0, 1$, from Lemma 4.3 can be smoothly extended to $x \in [-\lambda^{-1}, x_0]$ for $\lambda \in (0, \lambda_0)$ where $\lambda_0 > 0$ is a constant. Furthermore, in the above ranges of x and λ, the function u_0 satisfies the bounds*

$$\left| \partial_\lambda^m u_0(x, \lambda) \right| \leqslant C_m \langle x \rangle^{m+1},$$

$m \in \mathbb{N}_0$, whereas for u_1 we have the estimates

$$\left| \partial_\lambda^m u_1(x, \lambda) \right| \leqslant C_m \langle x \rangle^{m+1}$$

if $m \leqslant 2\ell$ and

$$\left| \partial_\lambda^{2\ell+m} u_1(x, \lambda) \right| \leqslant C_m \langle x \rangle^{2\ell+1} \lambda^{-m}$$

for $m \in \mathbb{N}_0$.

Proof. The coefficients of the equation $\mathcal{H}_{\sigma,\ell} f = \lambda^2 f$ are smooth on $\mathbb{R}$ and thus, any solution of $\mathcal{H}_{\sigma,\ell} f = \lambda^2 f$ can be smoothly extended to all of $\mathbb{R}$ by solving an appropriate initial value problem. Applying the variation of constants formula and noting that $u_0(x_0, \lambda) = u_0(x_0)$, $u_0'(x_0, \lambda) = u_0'(x_0)$ shows that the solution $u_0(\cdot, \lambda)$ satisfies the integral equation

$$u_0(x, \lambda) = u_0(x) + \lambda^2 \int\limits_x^{x_0} \left[v_0(x)v_1(y) - v_0(y)v_1(x) \right] u_0(y, \lambda) \, dy,$$

see also Corollary 4.1, where v_0, v_1 are the smooth extensions to $(-\infty, x_0]$ of the functions constructed in Lemma 4.2. This is a Volterra equation with a kernel $K(x, y, \lambda) := \lambda^2(v_0(x)v_1(y) - v_0(y)v_1(x))$. According to Lemma 4.2 we have the bound $|K(x, y, \lambda)| \lesssim \lambda^2(\langle x \rangle + \langle y \rangle)$ which implies

$$\int\limits_{-\lambda^{-1}}^{x_0} \sup_{x \in (-\lambda^{-1}, x_0)} \left| K(x, y, \lambda) \right| dy \lesssim 1$$

and hence, Lemma B.1 and Corollary 4.1 show that $|u_0(x, \lambda)| \lesssim \langle x \rangle$ for all $\lambda \in (0, \lambda_0)$ and $x \in [-\lambda^{-1}, x_0]$ where $\lambda_0 > 0$ is the constant from Lemma 4.3. We proceed by induction. Fix $m \in \mathbb{N}_0$ and assume that we have $|\partial_\lambda^l u_0(x, \lambda)| \leqslant C_l \langle x \rangle^{l+1}$ for all $l \leqslant m$ in the above range of λ and x. This implies

$$\partial_\lambda^{m+1} u_0(x, \lambda) = \sum_{l=1}^{\min\{m+1,2\}} \binom{m+1}{l} \int_x^{x_0} \partial_\lambda^l K(x, y, \lambda) \partial_\lambda^{m+1-l} u_0(y, \lambda) \, dy$$

$$+ \int_x^{x_0} K(x, y, \lambda) \partial_\lambda^{m+1} u_0(y, \lambda) \, dy$$

$$= \sum_{l=1}^{\min\{m+1,2\}} \underbrace{O\left(\lambda^{2-l} \langle x \rangle^2\right)}_{O(\langle x \rangle^l)} O\left(\langle x \rangle^{1+m+1-l}\right) + \int_x^{x_0} K(x, y, \lambda) \partial_\lambda^{m+1} u_0(y, \lambda) \, dy$$

and the estimate from Lemma B.1 yields $|\partial_\lambda^{m+1} u_0(x, \lambda)| \lesssim \langle x \rangle^{m+2}$ as claimed. Note carefully that the index l in the last sum is at most equal to 2 and therefore, we only estimate nonnegative powers of λ here.

For the second assertion we proceed similarly and note that a straightforward calculation as well as the variation of constants formula show that $u_1(\cdot, \lambda)$ satisfies the Volterra equation

$$u_1(x, \lambda) = \frac{u_1(x_0, \lambda) v_1'(x_0) - u_1'(x_0, \lambda) v_1(x_0)}{W(v_0, v_1)} v_0(x) - \frac{u_1(x_0, \lambda) v_0'(x_0) - u_1'(x_0, \lambda) v_0(x_0)}{W(v_0, v_1)} v_1(x)$$

$$+ \int_x^{x_0} K(x, y, \lambda) u_1(y, \lambda) \, dy.$$

According to Corollary 4.2 we have

$$u_1(x, \lambda) = O_{2\ell}(1) v_0(x) + O_{2\ell}(1) v_1(x) + \int_x^{x_0} K(x, y, \lambda) u_1(y, \lambda) \, dy$$

and, since $v_j(x) = O(\langle x \rangle)$, we obtain $|u_1(x, \lambda)| \lesssim \langle x \rangle$ for all $x \in [-\lambda^{-1}, x_0]$ and $\lambda \in (0, \lambda_0)$ by Lemma B.1. Again, we proceed by induction and first we consider the case $m \leqslant 2\ell$. If $\ell = 0$ there is nothing left to prove, so we restrict ourselves to $\ell \geqslant 1$. Assuming that $|\partial_\lambda^l u_1(x, \lambda)| \lesssim \langle x \rangle^{l+1}$ holds for all $l \leqslant m$ and a fixed $m \leqslant 2\ell - 1$, we infer

$$\partial_\lambda^{m+1} u_1(x, \lambda) = O\left(\langle x \rangle\right) + \sum_{l=1}^{\min\{m+1,2\}} \binom{m+1}{l} \int_x^{x_0} \underbrace{\partial_\lambda^l K(x, y, \lambda) \partial_\lambda^{m+1-l} u_1(y, \lambda)}_{O(\langle x \rangle) O(\langle y \rangle^m)} \, dy$$

$$+ \int_x^{x_0} K(x, y, \lambda) \partial_\lambda^{m+1} u_1(y, \lambda) \, dy$$

$$= O\big(\langle x\rangle^{m+2}\big) + \int_x^{x_0} K(x, y, \lambda)\partial_\lambda^{m+1} u_1(y, \lambda)\, dy$$

and Lemma B.1 yields $|\partial_\lambda^{m+1} u_1(x, \lambda)| \lesssim \langle x\rangle^{m+2}$ for all $x \in [\lambda^{-1}, x_0]$ and $\lambda \in (0, \lambda_0)$. The claim for the higher derivatives follows by a similar induction. $\square$

5. Perturbative solutions for $|x\lambda|$ large

The solutions $u_j(\cdot, \lambda)$ obtained by perturbing in energy cannot directly be matched with the Jost solution $f_+(\cdot, \lambda)$ since the approximations in Lemma 4.3 are valid for $|x\lambda|$ small whereas the behavior of $f_+(x, \lambda)$ is known only for fixed λ and $x \to \infty$. Thus, we construct another set of solutions to $\mathcal{H}_{\ell,\sigma} f = \lambda^2 f$ by perturbing the potential.

5.1. Construction of the perturbative solutions

To do so, we first rescale the equation $\mathcal{H}_{\ell,\sigma} f = \lambda^2 f$ by introducing a new independent variable $z := \lambda x$. Setting $\tilde{f}(z) := f(\lambda^{-1} z)$, the equation $\mathcal{H}_{\ell,\sigma} f = \lambda^2 f$ is equivalent to

$$\tilde{f}'' + \left(1 - \frac{\ell(\ell+1)}{z^2}\right)\tilde{f} = \lambda^{-2} U_{\ell,\sigma}\big(\lambda^{-1} z\big)\tilde{f} \tag{17}$$

where $U_{\ell,\sigma}(x) := V_{\ell,\sigma}(x) - \frac{\ell(\ell+1)}{x^2}$. Now observe that the equation

$$\tilde{f}'' + \left(1 - \frac{\ell(\ell+1)}{z^2}\right)\tilde{f} = 0$$

has the fundamental system $\{\sqrt{z}J_{\ell+1/2}(z), \sqrt{z}Y_{\ell+1/2}(z)\}$ where $J_{\ell+1/2}$ and $Y_{\ell+1/2}$ are the Bessel functions [1]. Thus, for a small right-hand side of Eq. (17) we expect to obtain solutions of Eq. (17) by perturbing the Bessel functions. According to Corollary 2.1, the right-hand side satisfies the estimate $\lambda^{-2} U_{\ell,\sigma}(\lambda^{-1} z) \lesssim \lambda^{-2}(\lambda^{-1} z)^{-3+\varepsilon} = \lambda^{-2} x^{-3+\varepsilon}$ for $x \to \infty$. Thus, our approximation is expected to be good if $\lambda^{-2} x^{-3+\varepsilon}$ is small. Smallness can be achieved by fixing $\lambda > 0$ and letting $x \to \infty$ which is required for the matching with the Jost solution f_+. However, we can also enforce smallness by setting $x = \lambda^{-1+\varepsilon}$ (for a small $\varepsilon > 0$) and letting $\lambda \to 0$. For $\lambda > 0$ sufficiently small, we have $|x\lambda| < \delta$ and the matching with the solutions $u_j(\cdot, \lambda)$ can be done as well. As a result, we obtain a good approximation to the Jost solution $f_+(x, \lambda)$ at a finite x for $\lambda \to 0$.

Lemma 5.1. *There exists a smooth solution $\phi_\ell(\cdot, \lambda)$ of Eq. (17) such that*

$$\phi_\ell(z, \lambda) = \beta_\ell \sqrt{z} H_{\ell+1/2}^+(z)\big(1 + b_\ell(z, \lambda)\big)$$

where $H_{\ell+1/2}^+ := J_{\ell+1/2} + iY_{\ell+1/2}$ is the Hankel function and $\beta_\ell := i\sqrt{\frac{\pi}{2}}e^{i\ell\frac{\pi}{2}}$. For all $\lambda \in (0, 1)$, the function b_ℓ satisfies the bounds

$$\big|b_\ell(z, \lambda)\big| \leqslant C\lambda^{1-\varepsilon(2\ell+3)}$$

for all $z \in [\lambda^\varepsilon, 1]$ and

$$\left|b_\ell(z, \lambda)\right| \leqslant C z^{-2+\varepsilon} \lambda^{1-\varepsilon}$$

for all $z \in [1, \infty)$ where $\varepsilon \in (0, \frac{1}{2\ell+3})$ is arbitrary.

Proof. Set $\varphi_\ell(z) := \beta_\ell \sqrt{z} H^+_{\ell+1/2}(z)$ and observe that $|\varphi_\ell(z)| > 0$ for all $z > 0$. Furthermore, we have $W(\varphi_\ell, \overline{\varphi_\ell}) = -2i$ which follows by noting that $\varphi_\ell(z) \sim e^{iz}$ and $\varphi'_\ell(z) \sim i e^{iz}$ for $z \to \infty$. A straightforward calculation shows that, if h satisfies the integral equation

$$h(z, \lambda) = 1 - \frac{1}{2i} \int\limits_z^\infty \left[\varphi_\ell(y)\overline{\varphi_\ell(y)} - \varphi_\ell^2(y)\frac{\overline{\varphi_\ell(z)}}{\varphi_\ell(z)} \right] \lambda^{-2} U_{\ell,\sigma}(\lambda^{-1}y) h(y, \lambda) \, dy, \qquad (18)$$

then $\phi_\ell(z, \lambda) := \varphi_\ell(z) h(z, \lambda)$ is a solution to Eq. (17). Eq. (18) is of the form

$$h(z, \lambda) = 1 + \int\limits_z^\infty K(z, y, \lambda) h(y, \lambda) \, dy.$$

Recall the asymptotic behavior $\varphi_\ell(z) = c_1 \beta_\ell z^{\ell+1}(1 + O(z)) + i c_2 \beta_\ell z^{-\ell}(1 + O(z))$ for $z \to 0$ where c_1, c_2 are nonzero real constants and the O-terms are smooth (cf. [1]). Furthermore, we have $\varphi_\ell(z) \sim e^{iz}$ as $z \to \infty$ and $|\lambda^{-2} U_{\ell,\sigma}(\lambda^{-1}z)| \lesssim \lambda^{1-\varepsilon} z^{-3+\varepsilon}$ for all $\lambda, z > 0$ with, say, $\lambda^{-1}z \geqslant 1$ (see Corollary 2.1). This shows that, for $k \in \mathbb{N}_0$,

$$\left| \left(\frac{\overline{\varphi_\ell}}{\varphi_\ell} \right)^{(k)}(z) \right| \leqslant C_k$$

for all $z \geqslant 0$ and hence, $|\partial_z^k K(z, y, \lambda)| \lesssim \lambda^{1-\varepsilon} y^{-3+\varepsilon}(1 + y^{-2\ell})$ for all $\lambda \in (0, 1)$ and all y, z with $\lambda^\varepsilon \leqslant z \leqslant y$. Thus, we have

$$\mu(\lambda) := \int\limits_{\lambda^\varepsilon}^\infty \sup_{z \in (\lambda^\varepsilon, y)} \left| K(z, y, \lambda) \right| dy \lesssim \lambda^{1-\varepsilon(2\ell+3)}$$

for all $\lambda \in (0, 1)$ and hence, $\mu := \sup_{\lambda \in (0,1)} \mu(\lambda) < \infty$ provided that $\varepsilon \leqslant \frac{1}{2\ell+3}$. Applying Lemma B.2 we conclude that Eq. (18) has a unique smooth solution $h(\cdot, \lambda)$ satisfying $\|h(\cdot, \lambda)\|_{L^\infty(\lambda^\varepsilon, \infty)} \leqslant e^\mu$ for all $\lambda \in (0, 1)$. Thus, Eq. (18) implies

$$\left| h(z, \lambda) - 1 \right| \lesssim \int\limits_z^\infty \left| K(z, y, \lambda) \right| dy \lesssim \int\limits_z^\infty \lambda^{1-\varepsilon} y^{-3+\varepsilon} \, dy \lesssim \lambda^{1-\varepsilon} z^{-2+\varepsilon}$$

for all $\lambda \in (0, 1)$ and all $z \in [1, \infty)$. Similarly, for $z \in [\lambda^\varepsilon, 1]$, we have

$$\left|h(z,\lambda)-1\right| \lesssim \int_z^\infty \left|K(z,y,\lambda)\right| dy \lesssim \int_z^\infty \lambda^{1-\varepsilon} y^{-3+\varepsilon}\left(1+y^{-2\ell}\right) dy$$

$$\lesssim \lambda^{1-\varepsilon}\left(z^{-2+\varepsilon}+z^{-2-2\ell+\varepsilon}\right) \lesssim \lambda^{1-\varepsilon(2\ell+3)}$$

as claimed. $\square$

5.2. Estimates for the derivatives

Lemma 5.2. *For all* $\lambda \in (0,1)$ *and* $m,k \in \mathbb{N}_0$, *the function* b_ℓ *from Lemma* 5.1 *satisfies the estimates*

$$\left|\partial_z^k \partial_\lambda^m b_\ell(z,\lambda)\right| \leqslant C_{k,m} z^{-k}\lambda^{1-\varepsilon(2\ell+3)-m}$$

for all $z \in [\lambda^\varepsilon, 1]$ *and*

$$\left|\partial_z^k \partial_\lambda^m b_\ell(z,\lambda)\right| \leqslant C_{k,m} z^{-2+\varepsilon-k}\lambda^{1-\varepsilon-m}$$

for all $z \in [1,\infty)$ *where* $\varepsilon \in (0,\frac{1}{2\ell+3})$ *is arbitrary.*

Proof. The function $h := 1 + b_\ell$ satisfies the equation

$$h(z,\lambda) = 1 + \int_z^\infty K(z,y,\lambda)h(y,\lambda)\,dy$$

where

$$K(z,y,\lambda) := -\frac{1}{2i}\left[\varphi_\ell(y)\overline{\varphi_\ell(y)} - \varphi_\ell^2(y)\frac{\overline{\varphi_\ell(z)}}{\varphi_\ell(z)}\right]\lambda^{-2}U_{\ell,\sigma}\left(\lambda^{-1}y\right)$$

and $\varphi_\ell(z) := \beta_\ell \sqrt{z} H_{\ell+1/2}^+(z)$, see the proof of Lemma 5.1. Recall the asymptotic behavior $\varphi_\ell(z) = e^{iz}(1 + O_{\mathbb{C}}(z^{-1}))$ for $z \to \infty$ where the $O_{\mathbb{C}}$-term is of symbol type. On the other hand, we have $\varphi_\ell(z) = c_1\beta_\ell z^{\ell+1}(1 + O(z)) + ic_2\beta_\ell z^{-\ell}(1 + O(z))$ for $z \to 0$ where c_1, c_2 are nonzero real constants and the O-terms are smooth (see [1]). Thus, by the Leibniz rule and Lemma A.1, we have $K(z,y,\lambda) = (1 - e^{2i(y-z)})O_{\mathbb{C}}(y^{-3+\varepsilon}\lambda^{1-\varepsilon})$ for all $1 \leqslant z \leqslant y$ and $K(z,y,\lambda) = O_{\mathbb{C}}(y^{-3-2\ell+\varepsilon}\lambda^{1-\varepsilon}) = O_{\mathbb{C}}(y^{-1}\lambda^{1-\varepsilon(2\ell+3)})$ for $\lambda^\varepsilon \leqslant z \leqslant y \leqslant 1$ where all O-terms are of symbol type. Let $n : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ denote the bijection from the proof of Proposition 4.1. As before, we proceed by induction. Fix $(k,m) \in \mathbb{N}_0 \times \mathbb{N}_0$ and suppose we have $|\partial_z^l \partial_\lambda^j (h(z,\lambda) - 1)| \leqslant C_{l,j} z^{-l}\lambda^{1-\varepsilon(2\ell+3)-j}$ for all (l,j) with $n(l,j) \leqslant n(k,m)$ and $\lambda \in (0,1)$, $z \in [\lambda^\varepsilon, 1]$. We have to show that this implies $|\partial_z^{k'} \partial_\lambda^{m'} (h(z,\lambda) - 1)| \leqslant C_{k',m'} z^{-k'}\lambda^{1-\varepsilon(2\ell+3)-m'}$ where $n(k',m') = n(k,m) + 1$. If $k = 0$ we have $(k',m') = (m+1,0)$ and, with $\kappa_l(z,\lambda) := \partial_z^l K(z,y,\lambda)|_{y=z}$,

$$\partial_z^{m+1}h(z,\lambda) = -\sum_{l=0}^{m} \partial_z^{m-l}\left[\kappa_l(z,\lambda)h(z,\lambda)\right] + \int_z^\infty \partial_z^{m+1}K(z,y,\lambda)h(y,\lambda)\,dy,$$

see Lemma B.2. By assumption we have $|\partial_z^{m-l}[\kappa_l(z,\lambda)h(z,\lambda)]| \lesssim z^{-(m+1)}\lambda^{1-\varepsilon(2\ell+3)}$ for $\lambda \in (0,1)$, $z \in [\lambda^\varepsilon, 1]$ and this implies $|\partial_z^{m+1}h(z,\lambda)| \lesssim z^{-(m+1)}\lambda^{1-\varepsilon(2\ell+3)}$. If $k \geqslant 1$ we have $(k', m') = (k-1, m+1)$ and

$$\partial_\lambda^{m+1}\partial_z^{k-1}h(x,\lambda) = -\sum_{l=0}^{k-2}\partial_\lambda^{m+1}\partial_z^{k-2-l}\big[\kappa_l(z,\lambda)h(z,\lambda)\big]$$
$$+ \int_z^\infty \partial_\lambda^{m+1}\big[\partial_z^{k-1}K(z,y,\lambda)h(y,\lambda)\big]\,dy. \qquad (19)$$

If $k \geqslant 2$, Eq. (19) and the assumption shows that $|\partial_\lambda^{m+1}\partial_z^{k-1}h(z,\lambda)| \lesssim z^{-(k-1)}\lambda^{1-\varepsilon(2\ell+3)-(m+1)}$ for $\lambda \in (0,1)$, $z \in [\lambda^\varepsilon, 1]$. If $k = 1$, Eq. (19) is of the form

$$\partial_\lambda^{m+1}h(z,\lambda) = O_\mathbb{C}\big(\lambda^{1-\varepsilon(2\ell+3)-(m+1)}\big) + \int_z^\infty K(z,y,\lambda)\partial_\lambda^{m+1}h(y,\lambda)\,dy$$

by assumption and therefore, Lemma B.1 yields $|\partial_\lambda^{m+1}h(z,\lambda)| \lesssim \lambda^{1-\varepsilon(2\ell+3)-(m+1)}$ for $\lambda \in (0,1)$ and $z \in [\lambda^\varepsilon, 1]$. This proves the first estimate.

For the second estimate we proceed by an analogous induction and write

$$\int_z^\infty K(z,y,\lambda)h(y,\lambda)\,dy = \int_0^\infty K(z,\eta+z,\lambda)h(\eta+z,\lambda)\,d\eta$$
$$= \int_0^\infty (1-e^{2i\eta})O_\mathbb{C}\big((\eta+z)^{-3+\varepsilon}\lambda^{1-\varepsilon}\big)h(\eta+z,\lambda)\,d\eta$$

for $z \geqslant 1$ where the $O_\mathbb{C}$-term is of symbol type. Thus, the assumption yields

$$\partial_z^{m+1}h(z,\lambda) = O_\mathbb{C}\big(z^{-2+\varepsilon-(m+1)}\lambda^{1-\varepsilon}\big) + \int_z^\infty K(z,y,\lambda)\partial_y^{m+1}h(y,\lambda)\,dy$$

and Lemma B.1 implies $|\partial_z^{m+1}h(z,\lambda)| \lesssim z^{-2+\varepsilon-(m+1)}\lambda^{1-\varepsilon}$ for $\lambda \in (0,1)$ and $z \geqslant 1$. Analogously, we obtain

$$\partial_\lambda^{m+1}\partial_z^{k-1}h(z,\lambda) = O_\mathbb{C}\big(z^{-2+\varepsilon-(k-1)}\lambda^{1-\varepsilon-(m+1)}\big) + \int_z^\infty K(z,y,\lambda)\partial_\lambda^{m+1}\partial_y^{k-1}h(y,\lambda)\,dy$$

and again, Lemma B.1 yields $|\partial_z^{k-1}\partial_\lambda^{m+1}h(z,\lambda)| \lesssim z^{-2+\varepsilon-(k-1)}\lambda^{1-\varepsilon-(m+1)}$ for $\lambda \in (0,1)$ and $z \geqslant 1$ which finishes the proof. $\square$

6. Matching with the Jost solutions

In this section we match the Jost functions $f_\pm(\cdot, \lambda)$ with the solutions $u_j(\cdot, \lambda)$ which allows us to calculate the asymptotic behavior of $f_\pm(\cdot, \lambda)$ and $W(f_-(\cdot, \lambda), f_+(\cdot, \lambda))$ in the limit $\lambda \to 0+$.

6.1. Matching with $f_+(\cdot, \lambda)$

Note that the solution ϕ_ℓ constructed in Lemma 5.1 has the asymptotic behavior $\phi_\ell(z, \lambda) \sim e^{iz}$ for $z \to \infty$ which shows that $f_+(x, \lambda) = \phi_\ell(\lambda x, \lambda)$. Thus, we have found a representation of the Jost solution $f_+(x, \lambda)$ which is valid for all $\lambda \in (0, 1)$ and all x with $\lambda^{-1+\varepsilon} \leqslant x < \infty$ with a sufficiently small $\varepsilon > 0$. For given $\varepsilon, \delta > 0$ we can always accomplish $\lambda^{-1+\varepsilon} \leqslant \delta \lambda^{-1}$ for all $\lambda \in (0, \lambda_0)$ provided $\lambda_0 > 0$ is chosen small enough. Thus, at $x = \lambda^{-1+\varepsilon}$ for λ sufficiently close to 0, we can match the solutions $f_+(\cdot, \lambda)$ and $u_j(\cdot, \lambda)$ (see Lemma 4.3).

Lemma 6.1. *The Wronskians $c_j^+(\lambda) := W(f_+(\cdot, \lambda), u_j(\cdot, \lambda))$ for $j = 0, 1$ have the asymptotic behavior*

$$c_0^+(\lambda) = i\alpha_0\beta_\ell\lambda^{-\ell}\left(1 + O\left(\lambda^\varepsilon\right) + iO\left(\lambda^{\varepsilon(2\ell+2)}\right)\right)$$

and

$$c_1^+(\lambda) = \alpha_1\beta_\ell\lambda^{\ell+1}\left(1 + O\left(\lambda^\varepsilon\right) + iO\left(\lambda^{-2\ell\varepsilon}\right)\right)$$

as $\lambda \to 0+$ for a sufficiently small $\varepsilon > 0$ where α_j are real nonzero constants and all O-terms are of symbol type.

Proof. We have $f_+(x, \lambda) = \beta_\ell\sqrt{\lambda x}H_{\ell+1/2}^+(\lambda x)(1 + b_\ell(\lambda x, \lambda))$ by Lemma 5.1. Note that $\partial_x b_\ell(\lambda x, \lambda) = \lambda\partial_z b_\ell(\lambda x, \lambda)$ as well as $\partial_\lambda b_\ell(\lambda x, \lambda) = x\partial_z b_\ell(\lambda x, \lambda) + \partial_\lambda b_\ell(\lambda x, \lambda)$ and hence, by Lemma 5.2 and the chain rule, we infer

$$\left|\partial_x^k\partial_\lambda^m b_\ell(\lambda x, \lambda)\right| \leqslant C_{k,m}x^{-k}\lambda^{1-\varepsilon(2\ell+3)-m}$$

for all $k, m \in \mathbb{N}_0$. By the same reasoning we obtain the symbol character (with respect to differentiation in x and λ) of the O-terms in

$$\beta_\ell\sqrt{\lambda x}H_{\ell+1/2}^+(\lambda x) = \alpha_1\beta_\ell(\lambda x)^{\ell+1}\left(1 + O(\lambda x)\right) + i\alpha_0\beta_\ell(\lambda x)^{-\ell}\left(1 + O(\lambda x)\right)$$

where α_0, α_1 are nonzero real constants. This shows that

$$f_+(x, \lambda) = \left(\alpha_1\beta_\ell(\lambda x)^{\ell+1} + i\alpha_0\beta_\ell(\lambda x)^{-\ell}\right)\left(1 + O_{\mathbb{C}}\left(\lambda^{1-\varepsilon(2\ell+3)}\right)\right)\left(1 + O(\lambda x)\right)$$

where the O-terms are of symbol type and the representation is valid for all $\lambda \in (0, 1)$ and all $x \in [\lambda^{-1+\varepsilon}, \lambda^{-1}]$. Differentiating this expression with respect to x and using the symbol character of the O-terms we obtain

$$f_+'(x, \lambda) = \left((\ell+1)\alpha_1\beta_\ell\lambda(\lambda x)^\ell - i\ell\alpha_0\beta_\ell\lambda(\lambda x)^{-\ell-1}\right)\left(1 + O_{\mathbb{C}}\left(\lambda^{1-\varepsilon(2\ell+3)}\right)\right)\left(1 + O(\lambda x)\right).$$

Evaluation at $x = \lambda^{-1+\varepsilon}$ yields

$$f_+\big(\lambda^{-1+\varepsilon}, \lambda\big) = \big(\alpha_1\beta_\ell\lambda^{\varepsilon(\ell+1)} + i\alpha_0\beta_\ell\lambda^{-\varepsilon\ell}\big)\big(1 + O\big(\lambda^\varepsilon\big) + i O\big(\lambda^{1-\varepsilon(2\ell+3)}\big)\big)$$

and

$$f_+'\big(\lambda^{-1+\varepsilon}, \lambda\big) = \big((\ell+1)\alpha_1\beta_\ell\lambda^{1+\varepsilon\ell} - i\ell\alpha_0\beta_\ell\lambda^{1-\varepsilon(\ell+1)}\big)\big(1 + O\big(\lambda^\varepsilon\big) + i O\big(\lambda^{1-\varepsilon(2\ell+3)}\big)\big)$$

for $\varepsilon > 0$ sufficiently small. Furthermore, by Lemma 4.3, we have

$$u_0(x, \lambda) = u_0(x)\big(1 + O\big(x^2\lambda^2\big)\big), \qquad u_0'(x, \lambda) = u_0'(x)\big(1 + O\big(x^2\lambda^2\big)\big)$$

and

$$u_1(x, \lambda) = u_1(x)\big(1 + O(x\lambda)\big), \qquad u_1'(x, \lambda) = u_1'(x)\big(1 + O(x\lambda)\big)$$

for all $\lambda \in (0, \lambda_0)$ and all $x \in [x_0, \delta\lambda^{-1}]$ where $\lambda_0, \delta > 0$ are sufficiently small and $x_0 > 0$ is sufficiently large. Thus, by choosing λ sufficiently close to 0, we obtain $\lambda^{-1+\varepsilon} \in [x_0, \delta\lambda^{-1}]$ and we can evaluate the above expressions at $x = \lambda^{-1+\varepsilon}$ which yields

$$u_0\big(\lambda^{-1+\varepsilon}, \lambda\big) = (2\ell+1)^{-1}\lambda^{-\ell-1+\varepsilon(\ell+1)}\big(1 + O\big(\lambda^{2\varepsilon}\big)\big),$$
$$u_0'\big(\lambda^{-1+\varepsilon}, \lambda\big) = (\ell+1)(2\ell+1)^{-1}\lambda^{-\ell+\varepsilon\ell}\big(1 + O\big(\lambda^{2\varepsilon}\big)\big),$$
$$u_1\big(\lambda^{-1+\varepsilon}, \lambda\big) = \lambda^{\ell-\varepsilon\ell}\big(1 + O\big(\lambda^\varepsilon\big)\big),$$
$$u_1'\big(\lambda^{-1+\varepsilon}, \lambda\big) = -\ell\lambda^{\ell+1-\varepsilon(\ell+1)}\big(1 + O\big(\lambda^\varepsilon\big)\big)$$

by Lemma 4.1 and all O-terms are of symbol type. The claim now follows from a straightforward computation. $\square$

6.2. The Jost solution $f_-(\cdot, \lambda)$ in the limit $\lambda \to 0+$

The Jost solution $f_-(\cdot, \lambda)$ satisfies the Volterra integral equation

$$f_-(x, \lambda) = e^{-i\lambda x} - \int_{-\infty}^{x} \frac{\sin(\lambda(y-x))}{\lambda} V_{\ell,\sigma}(y) f_-(y, \lambda)\, dy$$

as can be seen from the definition and the variation of constants formula. The decay properties of the potential $V_{\ell,\sigma}$ are crucial for the behavior of $f_\pm$. Since $V_{\ell,\sigma}$ decays exponentially as $x \to -\infty$ (Corollary 2.1), the situation for f_- is much simpler. In fact, f_- behaves essentially as in the free case $V_{\ell,\sigma} = 0$.

Lemma 6.2. *Let $a \in \mathbb{R}$ and $\lambda_0 > 0$. Then the Jost solution $f_-(x, \lambda) = e^{-i\lambda x}m_-(x, \lambda)$ exists for all $\lambda \in [-\lambda_0, \lambda_0]$ and, for $m \in \mathbb{N}_0$, we have the bounds $|\partial_\lambda^m m_-(x, \lambda)| \leqslant C_m$ as well as $|\partial_\lambda^m m_-'(x, \lambda)| \leqslant C_m$ for all $x \in (-\infty, a]$ and all $\lambda \in [-\lambda_0, \lambda_0]$.*

Proof. According to Lemma 3.1, the function $m_-(x, \lambda) = e^{i\lambda x} f_-(x, \lambda)$ satisfies the Volterra equation

$$m_-(x, \lambda) = 1 + \int_{-\infty}^{x} K(x, y, \lambda) m_-(y, \lambda) \, dy \tag{20}$$

where $K(x, y, \lambda) := \frac{1}{2i\lambda}(e^{2i\lambda(y-x)} - 1)V_{\ell,\sigma}(y)$. Note the bound

$$\left| \partial_\lambda^m K(x, y, \lambda) \right| \leqslant C_m |y - x|^{m+1} \left| V_{\ell,\sigma}(y) \right|$$

and thus, Lemma B.3 shows that the solution of Eq. (20) satisfies $\|\partial_\lambda^m m_-(\cdot, \lambda)\|_{L^\infty(-\infty,a)} \leqslant C_m$ for all $\lambda \in [-\lambda_0, \lambda_0]$ since $V_{\ell,\sigma}(y)$ decays exponentially as $y \to -\infty$ (Corollary 2.1). The estimate for the derivative $m_-'(x, \lambda)$ follows by differentiating Eq. (20) and using the bounds for $\partial_\lambda^m m_-(x, \lambda)$. $\square$

6.3. Zero energy resonances

Next we discuss the occurrence of resonances. First we give the precise definition of a zero energy resonance.

Definition 6.1. We say that the operator $\mathcal{H}_{\ell,\sigma}$ has a *zero energy resonance* if there exists a function $f \in L^\infty(\mathbb{R})$ such that $\mathcal{H}_{\ell,\sigma} f = 0$.

Recall that the equation $\mathcal{H}_{\ell,\sigma} f = 0$ is equivalent to

$$-\left(1 - \frac{2M}{r}\right)v'' - \frac{2M}{r^2}v' + \left(\frac{\ell(\ell+1)}{r^2} + \frac{2M\sigma}{r^3}\right)v = 0$$

where $v(r) = f(x(r))$ (see Eq. (11)). As already mentioned, solutions of this equation can be given in terms of special functions and therefore, we even know the behavior of $f_-(x, 0)$ for $x \to \infty$. This is crucial to see whether the operator $\mathcal{H}_{\ell,\sigma}$ has a zero energy resonance or not. As the following lemma shows, no resonances occur for scalar perturbations. However, in the case of electromagnetic or gravitational perturbations one has to require $\ell \geqslant 1$ (which we do anyway) or $\ell \geqslant 2$, respectively, in order to avoid resonances.

Lemma 6.3. *Let $(\ell, \sigma) \in \mathbb{N}_0 \times \{-3, 0, 1\} \backslash \{(0, 0), (0, -3), (1, -3)\}$. Then the zero energy Jost solution has the asymptotic behavior $f_-(x, 0) \sim cx^{\ell+1}$ for $x \to \infty$ where c is a nonzero constant. In particular, there does not exist a function $f \in L^\infty(\mathbb{R})$ that satisfies $\mathcal{H}_{\ell,\sigma} f = 0$.*

Proof. As already mentioned, the equation $\mathcal{H}_{\ell,\sigma} f = 0$ is equivalent to

$$-\left(1 - \frac{2M}{r}\right)v'' - \frac{2M}{r^2}v' + \left(\frac{\ell(\ell+1)}{r^2} + \frac{2M\sigma}{r^3}\right)v = 0$$

where $v(r) := f(x(r))$. Set $u(z) := z^{-(1+s)} v(2Mz)$ where $s := \sqrt{1 - \sigma}$ (note that s is the spin of the perturbing field, i.e., $s \in \{0, 1, 2\}$). Then the above equation is equivalent to

$$z(1 - z)u'' + [c - (a + b + 1)z]u' - abu = 0$$

where $a = -\ell + s$, $b = \ell + 1 + s$ and $c = 1 + 2s$. This is the hypergeometric differential equation and we have the solution $u(z) = {}_2F_1(a, b; c; z) = {}_2F_1(-\ell + s, \ell + 1 + s; 1 + 2s; z)$ which reduces to a polynomial of order $\ell - s$ provided that $\ell - s \in \mathbb{N}_0$ (see [1]) and this is equivalent to $(\ell, \sigma) \notin \{(0, 0), (0, -3), (1, -3)\}$. The Frobenius indices for the hypergeometric differential equation at the regular singular point $z = 1$ (which corresponds to $r = 2M$ and hence, $x \to -\infty$) are $(0, c - a - b) = (0, 0)$ (see [1]) which shows that $u(1) \neq 0$ and hence, there exists a nonzero constant c_0 such that $f_-(x, 0) = c_0 r(x)^{1+s} u(\frac{r(x)}{2M})$. Since u is a polynomial of order $\ell - s$, we obtain $f_-(x, 0) \sim c_1 x^{\ell+1}$ for $x \to \infty$ by Lemma 2.1 where c_1 is a nonzero constant. $\quad\square$

Remark 6.1. Note that Lemma 6.3 is sharp in the sense that the operator $\mathcal{H}_{\ell,\sigma}$ does indeed have zero energy resonances if $(\ell, \sigma) \in \{(0, 0), (0, -3), (1, -3)\}$. The resonance functions $f_{\ell,\sigma}$ are given by $f_{0,0}(x) = 1$, $f_{0,-3}(x) = 1 - \frac{3M}{r(x)}$ and $f_{1,-3}(x) = \frac{1}{r(x)}$ as can be checked immediately.

Remark 6.2. In the scalar case ($\sigma = 1$), the hypergeometric function in the proof of Lemma 6.3 reduces to the Legendre polynomial P_ℓ and we have

$$f_-(x, 0) = \frac{r(x)}{2M} P_\ell \left(\frac{r(x)}{M} - 1 \right).$$

6.4. Matching with $f_-(\cdot, \lambda)$

The above results are sufficient to match the Jost solution $f_-(\cdot, \lambda)$ to the solutions $u_j(\cdot, \lambda)$ obtained in Lemma 4.3 by perturbing in energy. In what follows we will always assume that we are in the nonresonant regime, i.e.,

$$(\ell, \sigma) \notin \big\{(0, 0), (0, -3), (1, -3)\big\}.$$

Lemma 6.4. *The Wronskians $c_j^-(\lambda) := W(f_-(\cdot, \lambda), u_j(\cdot, \lambda))$ for $j = 0, 1$ are of the form*[5]

$$c_j^-(\lambda) = O_{2\ell}(1) + i \, O_{2\ell+1}(\lambda)$$

for $\lambda \in (0, \lambda_0)$ where $\lambda_0 > 0$ is a constant. Furthermore, we have $c_1^-(0) \neq 0$.

Proof. According to Lemma 6.2, $f_-(x, \lambda)$ is smooth in λ around $\lambda = 0$ and by definition we have $f_-(x, \lambda) = \overline{f_-(x, -\lambda)}$ for $\lambda \in \mathbb{R}$. In particular, this implies $\operatorname{Re} f_-(x_0, \lambda) = O_{2\ell}(1)$ and $\operatorname{Im} f_-(x_0, \lambda) = O_{2\ell+1}(\lambda)$ where $x_0 > 0$ is the constant from Lemma 4.3. Repeating these arguments for the derivative $f_-'(x, \lambda)$, we similarly obtain $\operatorname{Re} f_-'(x_0, \lambda) = O_{2\ell}(1)$ and $\operatorname{Im} f_-'(x_0, \lambda) = O_{2\ell+1}(\lambda)$. By construction (cf. Lemma 4.3), we have $u_0(x_0, \lambda) = u_0(x_0)$, $u_0'(x_0, \lambda) = u_0'(x_0)$.

---

[5] See Definition 4.1.

Combining this with Corollary 4.2 we obtain $u_j(x_0, \lambda) = O_{2\ell}(1)$ and $u'_j(x_0, \lambda) = O_{2\ell}(1)$ for $j = 0, 1$. This shows

$$c_j^-(\lambda) = O_{2\ell}(1)\big(O_{2\ell}(1) + i\,O_{2\ell+1}(\lambda)\big) = O_{2\ell}(1) + i\,O_{2\ell+1}(\lambda).$$

Suppose $c_1^-(0) = W(f_-(\cdot, 0), u_1(\cdot)) = 0$. This is equivalent to $f_-(\cdot, 0)$ and u_1 being linearly dependent which implies that $f_-(x, 0) = O(x^{-\ell})$ for $x \to \infty$, a contradiction to Lemma 6.3.   □

7. The spectral measure at zero energy

Recall that we are interested in estimating the integral

$$\int_0^\infty \lambda \cos(t\lambda) \operatorname{Im}\big[G_{\ell,\sigma}(x, x', \lambda)\big]\langle x\rangle^{-\alpha}\langle x'\rangle^{-\alpha}\, d\lambda$$

and thus, we have to study the expressions

$$\operatorname{Im} \frac{f_-(x', \lambda)f_+(x, \lambda)}{W(f_-(\cdot, \lambda), f_+(\cdot, \lambda))}.$$

In this section we obtain estimates for $\lambda \to 0+$ and, as will be clear afterwards, the decay properties for solutions of the Regge–Wheeler equation are completely determined by this asymptotic behavior. In view of this, the following lemma is in fact the central result of our work.

We have to consider different ranges of x, x' and λ separately and we start with estimates for $|\lambda|$, $|x\lambda|$ and $|x'\lambda|$ small which turns out to be the most important case. For all $\lambda \in (0, \lambda_0)$ with a sufficiently small constant $\lambda_0 > 0$, we have the representation

$$f_\pm(x, \lambda) = -c_1^\pm(\lambda)u_0(x, \lambda) + c_0^\pm(\lambda)u_1(x, \lambda)$$

where[6] $c_j^\pm(\lambda) = W(f_\pm, u_j)(\lambda)$ (see Lemmas 6.1 and 6.4). Note carefully the slightly inconvenient fact that $c_0^\pm(\lambda)$ is the coefficient of $u_1(\cdot, \lambda)$ and not $u_0(\cdot, \lambda)$! It follows that

$$W(f_-, f_+)(\lambda) = c_1^-(\lambda)c_0^+(\lambda) - c_0^-(\lambda)c_1^+(\lambda).$$

We abbreviate

$$A_{jk}(\lambda) := \operatorname{Im}\left[\frac{c_j^-(\lambda)c_k^+(\lambda)}{c_1^-(\lambda)c_0^+(\lambda) - c_0^-(\lambda)c_1^+(\lambda)}\right]$$

and, since $u_j(\cdot, \lambda)$ are real-valued, we have to study expressions of the form

$$A_{00}(\lambda)u_1(x, \lambda)u_1(x', \lambda), \quad A_{10}(\lambda)u_0(x, \lambda)u_1(x', \lambda), \quad \text{etc.}$$

---

[6] From now on we write $W(f_\pm, u_j)(\lambda)$ instead of $W(f_\pm(\cdot, \lambda), u_j(\cdot, \lambda))$.

Lemma 7.1. *The function A_{jk} is of the form*

$$A_{00}(\lambda) = O_{2\ell+1}(\lambda) \quad and \quad A_{jk}(\lambda) = O\big(\lambda^{2\ell+1}\big) \quad if \ j + k \geqslant 1$$

for $\lambda \in (0, \lambda_0)$ where $\lambda_0 > 0$ is a sufficiently small constant and the O-term behaves like a symbol.

Proof. We have to distinguish four cases.

(1) For A_{00} we write

$$A_{00}(\lambda) = \operatorname{Im} \frac{\frac{c_0^-}{c_1^-}(\lambda)}{1 - \frac{c_0^-}{c_1^-}\frac{c_1^+}{c_0^+}(\lambda)}.$$

According to Lemma 6.4 we have

$$\frac{c_0^-}{c_1^-}(\lambda) = \frac{c_0^-(\lambda)\overline{c_1^-(\lambda)}}{|c_1^-(\lambda)|^2} = \frac{O_{2\ell}(1) + i\,O_{2\ell+1}(\lambda)}{|c_1^-(\lambda)|^2}$$

and, since $|c_1^-(\lambda)|^2 = O_{2\ell}(1) + O_{2\ell+2}(\lambda^2) = O_{2\ell}(1)$ as well as $|c_1^-(0)| \neq 0$, we infer $\frac{c_0^-}{c_1^-}(\lambda) = O_{2\ell}(1) + i\,O_{2\ell+1}(\lambda)$ with the help of Lemma A.1. Furthermore, Lemma 6.1 and Lemma A.1 yield

$$\frac{c_1^+}{c_0^+}(\lambda) = -i\frac{\alpha_1}{\alpha_0}\lambda^{2\ell+1}\big(1 + O\big(\lambda^\varepsilon\big) + i\,O\big(\lambda^{-2\ell\varepsilon}\big)\big) = O\big(\lambda^{2\ell+1-2\ell\varepsilon}\big) + i\,O\big(\lambda^{2\ell+1}\big)$$

where the O-terms are of symbol type. Applying Lemma A.1 again, we conclude

$$A_{00}(\lambda) = \operatorname{Im} \frac{(O_{2\ell}(1) + i\,O_{2\ell+1}(\lambda))(1 + O(\lambda^{2\ell+1-2\ell\varepsilon}) + i\,O(\lambda^{2\ell+1}))}{1 + O(\lambda^{2\ell+1-2\ell\varepsilon})}$$

$$= O_{2\ell+1}(\lambda) + O\big(\lambda^{2\ell+1}\big)$$

for a sufficiently small ε and the O-term is of symbol type.

(2) For A_{10} we use the representation

$$A_{10}(\lambda) = \operatorname{Im} \frac{1}{1 - \frac{c_0^-}{c_1^-}\frac{c_1^+}{c_0^+}(\lambda)} = \operatorname{Im} \frac{1}{1 + O(\lambda^{2\ell+1-2\ell\varepsilon}) + i\,O(\lambda^{2\ell+1})}$$

$$= \operatorname{Im} \frac{1 + O(\lambda^{2\ell+1-2\ell\varepsilon}) + i\,O(\lambda^{2\ell+1})}{1 + O(\lambda^{2\ell+1-2\ell\varepsilon})} = O\big(\lambda^{2\ell+1}\big)$$

where all O-terms are of symbol type (use Lemma A.1).

(3) In order to estimate A_{01} first note that

$$W(f_-, f_+)(\lambda) = c_1^-(\lambda)c_0^+(\lambda) - c_0^-(\lambda)c_1^+(\lambda) = i\alpha_0\beta_\ell c\lambda^{-\ell}\big(1 + O\big(\lambda^\varepsilon\big) + i\,O\big(\lambda^{\varepsilon(2\ell+2)}\big)\big)$$

as follows straightforward from Lemmas 6.1 and 6.4 where c is a nonzero real constant and all O-terms are of symbol type. However, this implies

$$A_{01}(\lambda) = \text{Im} \frac{i\frac{\alpha_1}{\alpha_0 c}\lambda^{2\ell+1}(O(1) + iO(\lambda))(1 + O(\lambda^\varepsilon) + iO(\lambda^{-2\ell\varepsilon}))}{1 + O(\lambda^\varepsilon) + iO(\lambda^{\varepsilon(2\ell+2)})}$$

$$= \text{Im} \frac{i\frac{\alpha_1}{\alpha_0 c}\lambda^{2\ell+1}(O(1) + O(\lambda^\varepsilon) + iO(\lambda^{-2\ell\varepsilon}))(1 + O(\lambda^\varepsilon) + iO(\lambda^{\varepsilon(2\ell+2)}))}{1 + O(\lambda^\varepsilon)}$$

$$= \frac{O(\lambda^{2\ell+1})}{1 + O(\lambda^\varepsilon)} = O(\lambda^{2\ell+1})$$

where all O-terms are of symbol type (see Lemma A.1).

(4) Finally, for A_{11} we proceed exactly as above and obtain

$$A_{11}(\lambda) = \text{Im} \frac{i\frac{\alpha_1}{\alpha_0 c}\lambda^{2\ell+1}(O(1) + iO(\lambda))(1 + O(\lambda^\varepsilon) + iO(\lambda^{-2\ell\varepsilon}))}{1 + O(\lambda^\varepsilon) + iO(\lambda^{\varepsilon(2\ell+2)})} = O(\lambda^{2\ell+1})$$

where the O-term behaves like a symbol. $\square$

Remark 7.1. The fact that $A_{00}(\lambda)$ is somewhat exceptional is a direct consequence of the asymmetric decay properties of the Regge–Wheeler potential. This phenomenon is not present in [52] or [53].

8. Oscillatory integral estimates for small energies

In this section we obtain bounds for the oscillatory integrals that describe the time evolution of solutions to the Regge–Wheeler equation. We distinguish different regimes, depending on the ranges of x, x' and λ and in this section we only consider the case $|\lambda|$ small. As already mentioned, the most important contribution comes from the regime $|x\lambda|$ and $|x'\lambda|$ small which yields the decay rates stated in Theorem 2.1. The remaining cases can be treated very similar to [53], however, for the sake of completeness we give explicit proofs for all of the following statements.

8.1. Estimates for $|x\lambda|$ and $|x'\lambda|$ small

We will need the following result on oscillatory integrals.

Lemma 8.1. For an $N \in \mathbb{N}_0$ let $\omega(\lambda) = O_{2N+1}(\lambda)$ and suppose there exists a constant $\lambda_0 > 0$ such that $\omega(\lambda) = 0$ for all $\lambda \geqslant \lambda_0$. Then we have the estimates

$$\left| \int_0^\infty \lambda \cos(t\lambda)\omega(\lambda) \, d\lambda \right| \leqslant C(\omega)\langle t \rangle^{-(2N+3)},$$

$$\left| \int_0^\infty \sin(t\lambda)\omega(\lambda) \, d\lambda \right| \leqslant C(\omega)\langle t \rangle^{-(2N+2)}$$

for all $t \geqslant 0$ where $C(\omega)$ can be estimated as

$$C(\omega) \leqslant C \max\left\{ \|\omega\|_{L^\infty(0,\infty)}, \sup_{\lambda>0}\left|\lambda^j \omega^{(2N+1+j)}(\lambda)\right| : j = 1, 2, 3\right\}$$

for an absolute constant $C > 0$.

Proof. We only prove the sine estimate since the proof for the cosine estimate is completely analogous. It suffices to consider $t \geqslant 1$. $(2N + 2)$-fold integration by parts yields

$$\left|\int_0^\infty \sin(t\lambda)\omega(\lambda)\,d\lambda\right| = \left|\frac{1}{t^{2N+2}}\int_0^\infty \sin(t\lambda)\omega^{(2N+2)}(\lambda)\,d\lambda\right|$$

since the boundary terms vanish thanks to $\omega^{(2m)}(0) = 0$ for $m \leqslant N$ and the fact that $\omega(\lambda) = 0$ for all $\lambda \geqslant \lambda_0$. Thus, it suffices to show that

$$\left|\int_0^\infty \sin(t\lambda)\omega^{(2N+2)}(\lambda)\,d\lambda\right| \leqslant C$$

for a constant C independent of t. Let χ be a smooth cut-off satisfying $\chi(\lambda) = 1$ for $0 \leqslant \lambda \leqslant \frac{1}{2}$ and $\chi(\lambda) = 0$ for $\lambda \geqslant 2$. Then we have

$$\left|\int_0^\infty \sin(t\lambda)\omega^{(2N+2)}(\lambda)\chi(t\lambda)\,d\lambda\right| = \left|\int_0^\infty \sin(\eta)\omega^{(2N+2)}\left(\frac{\eta}{t}\right)\chi(\eta)\frac{d\eta}{t}\right| \lesssim \int_0^\infty \left|\frac{\sin(\eta)}{\eta}\chi(\eta)\right|d\eta \lesssim 1$$

for all $t \geqslant 1$. Furthermore, by an additional integration by parts we obtain

$$\left|\int_0^\infty \sin(t\lambda)\omega^{(2N+2)}(\lambda)\big[1 - \chi(t\lambda)\big]\,d\lambda\right| \leqslant \left|\frac{1}{t}\int_0^\infty \cos(t\lambda)\omega^{(2N+3)}(\lambda)\big[1 - \chi(t\lambda)\big]\,d\lambda\right|$$

$$+ \left|\frac{1}{t}\int_0^\infty \cos(t\lambda)\omega^{(2N+2)}(\lambda)t\chi'(t\lambda)\,d\lambda\right|$$

where the boundary term vanishes thanks to the cut-off and $\omega^{(2N+2)}(\lambda) = 0$ for all $\lambda \geqslant \lambda_0$. However, we have

$$\left|\frac{1}{t}\int_0^\infty \cos(t\lambda)\omega^{(2N+3)}(\lambda)\big[1 - \chi(t\lambda)\big]\,d\lambda\right| = \left|\frac{1}{t}\int_0^\infty \cos(\eta)\omega^{(2N+3)}\left(\frac{\eta}{t}\right)\big[1 - \chi(\eta)\big]\frac{d\eta}{t}\right|$$

$$\lesssim \frac{1}{t^2}\int_0^\infty \left|\cos(\eta)\frac{t^2}{\eta^2}\big[1 - \chi(\eta)\big]\right|d\eta \lesssim 1$$

as well as

$$\left| \frac{1}{t} \int_0^\infty \cos(t\lambda) \omega^{(2N+2)} t \chi'(t\lambda) \, d\lambda \right| = \left| \int_0^\infty \cos(\eta) \omega^{(2N+2)} \left(\frac{\eta}{t} \right) \chi'(\eta) \frac{d\eta}{t} \right|$$

$$\lesssim \int_0^\infty \left| \frac{\cos \eta}{\eta} \chi'(\eta) \right| d\eta \lesssim 1$$

since $\mathrm{supp}(\chi') \subset [\frac{1}{2}, 2]$. $\square$

Now we are ready to prove the first oscillatory integral estimate, valid for small λ and $|x\lambda| \leqslant \delta$, $|x'\lambda| \leqslant \delta$ where $\delta > 0$ is sufficiently small. In what follows we denote by χ_δ a smooth cut-off function supported in a δ-neighborhood of the origin, i.e.,

$$\chi_\delta(x) = \begin{cases} 1 & \text{if } |x| \leqslant \frac{\delta}{2}, \\ 0 & \text{if } |x| \geqslant \delta. \end{cases}$$

Lemma 8.2. *Let $\alpha \geqslant 2\ell + 1$ and $\delta > 0$ be sufficiently small. Then we have the estimates*

$$\sup_{x,x'\in\mathbb{R}} \left| \int_0^\infty \lambda \cos(t\lambda) \, \mathrm{Im}\left[\frac{f_-(x',\lambda) f_+(x,\lambda)}{W(f_-(\cdot,\lambda), f_+(\cdot,\lambda))} \right] \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \chi_\delta(\lambda) \chi_\delta(x\lambda) \chi_\delta(x'\lambda) \, d\lambda \right|$$

$$\lesssim \langle t \rangle^{-(2\ell+3)}$$

and

$$\sup_{x,x'\in\mathbb{R}} \left| \int_0^\infty \sin(t\lambda) \, \mathrm{Im}\left[\frac{f_-(x',\lambda) f_+(x,\lambda)}{W(f_-(\cdot,\lambda), f_+(\cdot,\lambda))} \right] \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \chi_\delta(\lambda) \chi_\delta(x\lambda) \chi_\delta(x'\lambda) \, d\lambda \right| \lesssim \langle t \rangle^{-(2\ell+2)}$$

for all $t \geqslant 0$.

Proof. We set $\omega(x, x', \lambda) := A_{00}(\lambda) u_1(x, \lambda) u_1(x', \lambda) \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \chi_\delta(\lambda) \chi_\delta(x\lambda) \chi_\delta(x'\lambda)$. According to Lemmas 7.1 and 4.5, we have $\omega(x, x', \lambda) = O_{2\ell+1}(\lambda)$ for fixed x, x'. Combining Lemmas 4.3, 4.5 and 4.6, we obtain $|\partial_\lambda^{2\ell+m} u_1(x, \lambda)| \leqslant C_m \langle x \rangle^{2\ell+1} \lambda^{-m}$ for $m \in \mathbb{N}_0$, $\lambda \in (0, \delta)$ and $x \in [-\delta\lambda^{-1}, \delta\lambda^{-1}]$. This implies

$$\left| \partial_\lambda^{2\ell+2} \omega(x, x', \lambda) \right| \lesssim \langle x \rangle^{2\ell+1-\alpha} \langle x' \rangle^{2\ell+1-\alpha} \lambda^{-1},$$

and, analogously,

$$\left| \partial_\lambda^{2\ell+3} [\omega(x, x', \lambda)] \right| \lesssim \langle x \rangle^{2\ell+1-\alpha} \langle x' \rangle^{2\ell+1-\alpha} \lambda^{-2},$$

$$\left| \partial_\lambda^{2\ell+4} [\omega(x, x', \lambda)] \right| \lesssim \langle x \rangle^{2\ell+1-\alpha} \langle x' \rangle^{2\ell+1-\alpha} \lambda^{-3}$$

for all $\lambda > 0$ and $x, x' \in \mathbb{R}$. Finally, $\omega(\lambda) = 0$ for $\lambda \geqslant \delta$. Thus, Lemma 8.1 yields

$$\sup_{x,x'\in\mathbb{R}}\left|\int_0^\infty \lambda \cos(t\lambda)\omega(x, x', \lambda)\, d\lambda\right| \lesssim \langle t\rangle^{-(2\ell+3)} \quad \text{and}$$

$$\sup_{x,x'\in\mathbb{R}}\left|\int_0^\infty \sin(t\lambda)\omega(x, x', \lambda)\, d\lambda\right| \lesssim \langle t\rangle^{-(2\ell+2)}$$

for all $t \geqslant 0$. For the remaining cases assume $j + k \geqslant 1$ and set

$$\omega(x, x', \lambda) := A_{jk}(\lambda)u_{j'}(x, \lambda)u_{k'}(x', \lambda)\langle x\rangle^{-\alpha}\langle x'\rangle^{-\alpha}\chi_\delta(\lambda)\chi_\delta(x\lambda)\chi_\delta(x'\lambda).$$

According to Lemmas 7.1 and 4.3, we have $\omega(x, x', \lambda) = O(\lambda^{2\ell+1})O(\langle x\rangle^{\ell+1})O(\langle x'\rangle^{\ell+1})$ where the O-terms behave like symbols (use Proposition 4.1 and Lemma 4.6). In particular, $\omega(x, x', \lambda) = O_{2\ell+1}(\lambda)$ for fixed x, x' and

$$\left|\partial_\lambda^{2\ell+2}\omega(x, x', \lambda)\right| \lesssim \langle x\rangle^{\ell+1-\alpha}\langle x'\rangle^{\ell+1-\alpha}\lambda^{-1},$$

as well as

$$\left|\partial_\lambda^{2\ell+3}\left[\omega(x, x', \lambda)\right]\right| \lesssim \langle x\rangle^{\ell+1-\alpha}\langle x'\rangle^{\ell+1-\alpha}\lambda^{-2},$$

$$\left|\partial_\lambda^{2\ell+4}\left[\omega(x, x', \lambda)\right]\right| \lesssim \langle x\rangle^{\ell+1-\alpha}\langle x'\rangle^{\ell+1-\alpha}\lambda^{-3}$$

for all $\lambda > 0$ and $x, x' \in \mathbb{R}$. Thus, as before, applying Lemma 8.1 yields the claim. $\square$

Remark 8.1. Obviously, by performing fewer integrations by parts (cf. the proof of Lemma 8.1), one may obtain weaker decay bounds (in t). By doing so, however, one can relax the decay requirements (in x) of the data, that is, the index α in Lemma 8.2 can be chosen smaller. To be more precise, one obtains the additional bounds, valid for $\alpha \in \mathbb{N}$, $\alpha \leqslant 2\ell + 1$,

$$\sup_{x,x'\in\mathbb{R}}\left|\int_0^\infty \lambda \cos(t\lambda)\, \mathrm{Im}\left[\frac{f_-(x', \lambda)f_+(x, \lambda)}{W(f_-(\cdot, \lambda), f_+(\cdot, \lambda))}\right]\langle x\rangle^{-\alpha}\langle x'\rangle^{-\alpha}\chi_\delta(\lambda)\chi_\delta(x\lambda)\chi_\delta(x'\lambda)\, d\lambda\right| \lesssim \langle t\rangle^{-\alpha-2}$$

and

$$\sup_{x,x'\in\mathbb{R}}\left|\int_0^\infty \sin(t\lambda)\, \mathrm{Im}\left[\frac{f_-(x', \lambda)f_+(x, \lambda)}{W(f_-(\cdot, \lambda), f_+(\cdot, \lambda))}\right]\langle x\rangle^{-\alpha}\langle x'\rangle^{-\alpha}\chi_\delta(\lambda)\chi_\delta(x\lambda)\chi_\delta(x'\lambda)\, d\lambda\right| \lesssim \langle t\rangle^{-\alpha-1}$$

for all $t \geqslant 0$.

Remark 8.2. Note that the sine estimate from Lemma 8.2 is the main obstacle to proving better decay. The remaining oscillatory estimates of Sections 8 and 9 below are consistent with faster decay.

8.2. Estimates for $|x\lambda|$ and $|x'\lambda|$ large

For the remaining small energy contributions it is useful to note that, for $\lambda \in \mathbb{R}$, $f_\pm(x, -\lambda) = \overline{f_\pm(x, \lambda)}$ by definition of the Jost solutions. This implies $G_{\ell,\sigma}(x, x', -\lambda) = \overline{G_{\ell,\sigma}(x, x', \lambda)}$ and hence, the real part $\mathrm{Re}[G_{\ell,\sigma}(x, x', \lambda)]$ is an even function of λ whereas the imaginary part $\mathrm{Im}[G_{\ell,\sigma}(x, x', \lambda)]$ is odd. Thus, we have

$$\int_0^\infty \lambda \cos(t\lambda) \, \mathrm{Im}\big[G_{\ell,\sigma}(x, x', \lambda)\big] \, d\lambda = \frac{1}{2} \int_\mathbb{R} \lambda \cos(t\lambda) G_{\ell,\sigma}(x, x', \lambda) \, d\lambda$$

and similarly for the sine evolution. This shows that we can replace the imaginary part of $G_{\ell,\sigma}(x, x', \lambda)$ by $G_{\ell,\sigma}(x, x', \lambda)$ itself in the oscillatory integrals and change the domain of integration from $\lambda > 0$ to $\lambda \in \mathbb{R}$. Furthermore, recall

$$W(f_-, f_+)(\lambda) = c_1^-(\lambda) c_0^+(\lambda) - c_0^-(\lambda) c_1^+(\lambda) = c\lambda^{-\ell}\big(1 + O_\mathbb{C}(\lambda^\varepsilon)\big) \qquad (21)$$

for a nonzero constant c where the O-term is of symbol type. This has been shown in the proof of Lemma 7.1.

In order to deal with terms that involve $f_-(x', \lambda)$ for $x' \geqslant 0$ and $f_+(x, \lambda)$ for $x \leqslant 0$ we have to consider reflection and transmission coefficients. For $\lambda \neq 0$, the functions $f_+(\cdot, \lambda)$ and $\overline{f_+(\cdot, \lambda)}$ are linearly independent which shows that there exist coefficients $a(\lambda)$ and $b(\lambda)$[7] such that $f_-(x, \lambda) = a(\lambda) f_+(x, \lambda) + b(\lambda) \overline{f_+(x, \lambda)}$. This representation implies $|b(\lambda)|^2 - |a(\lambda)|^2 = 1$ (cf. the proof of Lemma 3.2) and thus, $f_+(x, \lambda) = -\overline{a(\lambda)} f_-(x, \lambda) + b(\lambda) \overline{f_-(x, \lambda)}$. Furthermore, we have $W(f_-, f_+)(\lambda) = b(\lambda) W(\overline{f_+}, f_+)(\lambda) = 2i\lambda b(\lambda)$ which is equivalent to

$$\frac{b(\lambda)}{W(f_-, f_+)(\lambda)} = \frac{1}{2i\lambda}. \qquad (22)$$

Similarly, we obtain $W(f_-, \overline{f_+})(\lambda) = -2i\lambda a(\lambda)$ and therefore,

$$\frac{a(\lambda)}{W(f_-, f_+)(\lambda)} = -\frac{W(f_-, \overline{f_+})(\lambda)}{2i\lambda W(f_-, f_+)(\lambda)}.$$

However, from Lemma 6.1 and $W(f_-, \overline{f_+})(\lambda) = c_1^-(\lambda) \overline{c_0^+(\lambda)} - c_0^-(\lambda) \overline{c_1^+(\lambda)}$ it follows that

$$\frac{W(f_-, \overline{f_+})(\lambda)}{W(f_-, f_+)(\lambda)} = c + O_\mathbb{C}(\lambda^\varepsilon)$$

where c is a nonzero constant and the O-term is of symbol type and hence,

$$\frac{a(\lambda)}{W(f_-, f_+)(\lambda)} = c\lambda^{-1}\big(1 + O_\mathbb{C}(\lambda^\varepsilon)\big). \qquad (23)$$

---

[7] In order to avoid confusion, we remark that the coefficients a and b are not exactly the same as A and B in the proof of Lemma 3.2 but they are related by $a(\lambda) = A(\lambda^2)$ and $b(\lambda) = B(\lambda^2)$.

Lemma 8.3. *Let $\alpha \in \mathbb{N}_0$ and $\delta > 0$ sufficiently small. Then we have the estimates*

$$\sup_{x,x' \in \mathbb{R}} \left| \int_{\mathbb{R}} \lambda \cos(t\lambda) \frac{f_-(x',\lambda) f_+(x,\lambda)}{W(f_-,f_+)(\lambda)} \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \chi_\delta(\lambda) \big(1 - \chi_\delta(x\lambda)\big) \big(1 - \chi_\delta(x'\lambda)\big) \, d\lambda \right| \lesssim \langle t \rangle^{-\alpha}$$

and

$$\sup_{x,x' \in \mathbb{R}} \left| \int_{\mathbb{R}} \sin(t\lambda) \frac{f_-(x',\lambda) f_+(x,\lambda)}{W(f_-,f_+)(\lambda)} \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \chi_\delta(\lambda) \big(1 - \chi_\delta(x\lambda)\big) \big(1 - \chi_\delta(x'\lambda)\big) \, d\lambda \right| \lesssim \langle t \rangle^{-\alpha+1}$$

for all $t \geqslant 0$.

Proof. Let $|\lambda| \leqslant \delta$, $x \geqslant 0$, $x' \leqslant 0$ and $|\lambda x|, |\lambda x'| \geqslant \frac{\delta}{2}$. We set $m_\pm(x,\lambda) := e^{\mp i\lambda x} f_\pm(x,\lambda)$. According to Lemma 6.2, we have the bound $|\partial_\lambda^m m_-(x,\lambda)| \leqslant C_m$ for all $x \leqslant 0$ and $m \in \mathbb{N}_0$. Furthermore, since λ is small, we have

$$m_+(x,\lambda) = e^{-i\lambda x} \phi_\ell(\lambda x, \lambda) = \big(1 + O_\mathbb{C}\big((\lambda x)^{-1}\big)\big)\big(1 + b_\ell(\lambda x, \lambda)\big)$$

by Lemma 5.1 and the asymptotics of the Hankel function where the O-term is of symbol type. By Lemma 5.2 and the chain rule, we have the estimate $|\partial_\lambda^m b_\ell(\lambda x, \lambda)| \leqslant C_m \langle x \rangle^m$ since $|\lambda|^{-1} \lesssim \langle x \rangle$ and this implies $|\partial_\lambda^m m_+(x,\lambda)| \leqslant C_m \langle x \rangle^m$ for all $m \in \mathbb{N}_0$. Set

$$\omega(x,x',\lambda) := \lambda \frac{m_-(x',\lambda) m_+(x,\lambda)}{W(f_-,f_+)(\lambda)} \chi_\delta(\lambda) \big(1 - \chi_\delta(x\lambda)\big)\big(1 - \chi_\delta(x'\lambda)\big).$$

Then we have

$$\big| \partial_\lambda^m \omega(x,x',\lambda) \big| \lesssim \langle x \rangle^m \langle x' \rangle^m$$

for all $m \in \mathbb{N}_0$ since $W(f_-,f_+)(\lambda)$ is of symbol type (cf. Eq. (21)). We have to estimate the integral

$$\int_{\mathbb{R}} e^{i\lambda(\pm t + x - x')} \omega(x,x',\lambda) \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \, d\lambda.$$

If $|\pm t + x - x'| \geqslant \frac{1}{2}t$ we integrate by parts α-times to obtain

$$\left| \int_{\mathbb{R}} e^{i\lambda(\pm t + x - x')} \omega(x,x',\lambda) \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \, d\lambda \right| \lesssim |\pm t + x - x'|^{-\alpha} \lesssim \langle t \rangle^{-\alpha}$$

and, if $|\pm t + x - x'| \leqslant \frac{1}{2}t$, we have $\langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \lesssim \langle t \rangle^{-\alpha}$ as $t \to \infty$ and thus,

$$\left| \int_{\mathbb{R}} e^{i\lambda(\pm t + x - x')} \omega(x,x',\lambda) \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \, d\lambda \right| \lesssim \langle t \rangle^{-\alpha}.$$

If $x' \geq 0$ or $x \leq 0$ we use the representations $m_-(x', \lambda) = a(\lambda)e^{2i\lambda x'} m_+(x', \lambda) + b(\lambda)\overline{m_+(x', \lambda)}$ or $m_+(x, \lambda) = -\overline{a(\lambda)}e^{-2i\lambda x} m_-(x, \lambda) + b(\lambda)\overline{m_-(x, \lambda)}$ and with the help of Eqs. (22) and (23) the corresponding integrals can be estimated as above.

For the sine evolution note that we are missing one λ and thus, for instance, if

$$\omega(x, x', \lambda) := \frac{a(\lambda)m_+(x', \lambda)m_+(x, \lambda)}{W(f_-, f_+)(\lambda)} \chi_\delta(\lambda)\big(1 - \chi_\delta(x\lambda)\big)\big(1 - \chi_\delta(x'\lambda)\big)$$

we have $|\omega(x, x', \lambda)| \lesssim |\lambda|^{\ell-1}$ (cf. Eq. (21)) which, in the case $\ell = 0$, only yields the weaker bound $|\partial_\lambda^m \omega(x, x', \lambda)| \lesssim \langle x \rangle^{m+1} \langle x' \rangle^m$. $\quad \square$

8.3. Estimates for $|x\lambda|$ small and $|x'\lambda|$ large

The next regime to be considered is $|x\lambda| \leq \delta$ and $|x'\lambda| \geq \frac{\delta}{2}$. We distinguish the two cases $x' \leq 0$ and $x' \geq 0$.

Lemma 8.4. *Let $\alpha \in \mathbb{N}$ and $\delta > 0$ sufficiently small. Then we have the estimates*

$$\sup_{x \in \mathbb{R},\, x' < 0} \left| \int_{\mathbb{R}} \lambda \cos(t\lambda) \frac{f_-(x', \lambda) f_+(x, \lambda)}{W(f_-, f_+)(\lambda)} \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \chi_\delta(\lambda)\chi_\delta(x\lambda)\big(1 - \chi_\delta(x'\lambda)\big) \, d\lambda \right| \lesssim \langle t \rangle^{-\alpha}$$

and

$$\sup_{x \in \mathbb{R},\, x' < 0} \left| \int_{\mathbb{R}} \sin(t\lambda) \frac{f_-(x', \lambda) f_+(x, \lambda)}{W(f_-, f_+)(\lambda)} \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \chi_\delta(\lambda)\chi_\delta(x\lambda)\big(1 - \chi_\delta(x'\lambda)\big) \, d\lambda \right| \lesssim \langle t \rangle^{-\alpha+1}$$

for all $t \geq 0$.

Proof. Let $x \in \mathbb{R}$, $x' \leq 0$, $|\lambda| \leq \delta$, $|\lambda x| \leq \delta$ and $|\lambda x'| \geq \frac{\delta}{2}$. As before, we write $f_\pm(x, \lambda) = e^{\mp i\lambda x} m_\pm(x, \lambda)$ and recall the bound $|\partial_\lambda^m m_-(x', \lambda)| \leq C_m$ for all $m \in \mathbb{N}_0$ (Lemma 6.2). For m_+ we use the representation

$$m_+(x, \lambda) = e^{-i\lambda x}\big(-c_1^+(\lambda)u_0(x, \lambda) + c_0^+(\lambda)u_1(x, \lambda)\big)$$

where we extend $c_j^+(\lambda)$ and $u_j(x, \lambda)$ to negative λ according to $c_j^+(-\lambda) = \overline{c_j^+(\lambda)}$ and $u_j(x, -\lambda) = \overline{u_j(x, \lambda)}$. Applying Proposition 4.1 and Lemmas 4.6, 6.1, we obtain the bounds

$$\big|\partial_\lambda^m c_1^+(\lambda)u_0(x, \lambda)\big| \lesssim \langle x \rangle^{\ell+1} |\lambda|^{\ell-m} \lesssim \langle x \rangle |\lambda|^{-m} \lesssim \langle x' \rangle^m |\lambda|^{-1}$$

and

$$\big|\partial_\lambda^m c_0^+(\lambda)u_1(x, \lambda)\big| \lesssim \langle x \rangle |\lambda|^{-\ell-m} \lesssim \langle x' \rangle^m |\lambda|^{-\ell-1}$$

for $m \in \mathbb{N}_0$ which implies $|\partial_\lambda^m m_+(x, \lambda)| \lesssim \langle x' \rangle^m |\lambda|^{-\ell-1}$. We also have $|\partial_\lambda^m \chi_\delta(x\lambda)| \lesssim \langle x \rangle^m \lesssim |\lambda|^{-m} \lesssim \langle x' \rangle^m$ and, putting all this together, we arrive at

$$\big|\partial_\lambda^m \omega(x, x', \lambda)\big| \lesssim \langle x' \rangle^m$$

where

$$\omega(x, x', \lambda) := \lambda \frac{m_-(x', \lambda) m_+(x, \lambda)}{W(f_-, f_+)(\lambda)} \chi_\delta(\lambda) \chi_\delta(x\lambda) \big(1 - \chi_\delta(x'\lambda)\big),$$

see also Eq. (21). Thus, the claim follows by appropriate integration by parts as in the proof of Lemma 8.3. The proof for the sine evolution goes along the same lines but one loses one power of λ. $\quad\square$

Lemma 8.5. *Let* $\alpha \in \mathbb{N}$ *and* $\delta > 0$ *sufficiently small. Then we have the estimates*

$$\sup_{x \in \mathbb{R}, \, x' > 0} \left| \int_{\mathbb{R}} \lambda \cos(t\lambda) \frac{f_-(x', \lambda) f_+(x, \lambda)}{W(f_-, f_+)(\lambda)} \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \chi_\delta(\lambda) \chi_\delta(x\lambda) \big(1 - \chi_\delta(x'\lambda)\big) \, d\lambda \right| \lesssim \langle t \rangle^{-\alpha}$$

and

$$\sup_{x \in \mathbb{R}, \, x' > 0} \left| \int_{\mathbb{R}} \sin(t\lambda) \frac{f_-(x', \lambda) f_+(x, \lambda)}{W(f_-, f_+)(\lambda)} \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \chi_\delta(\lambda) \chi_\delta(x\lambda) \big(1 - \chi_\delta(x'\lambda)\big) \, d\lambda \right| \lesssim \langle t \rangle^{-\alpha+1}$$

for all $t \geq 0$.

Proof. Let $x \in \mathbb{R}$, $x' \geq 0$, $|\lambda| \leq \delta$, $|x\lambda| \leq \delta$ and $|x'\lambda| \geq \frac{\delta}{2}$. As always, we write $f_\pm(x, \lambda) = e^{\mp i\lambda x} m_\pm(x, \lambda)$. Again, by Lemmas 4.3, 6.1, 4.6 and the representation

$$m_+(x, \lambda) = e^{-i\lambda x} \big(-c_1^+(\lambda) u_0(x, \lambda) + c_0^+(\lambda) u_1(x, \lambda)\big)$$

we obtain the bound

$$\big| m_+(x, \lambda) \big| \lesssim \langle x \rangle^{\ell+1} |\lambda|^\ell + \langle x \rangle |\lambda|^{-\ell} \lesssim \langle x \rangle |\lambda|^{-\ell}$$

and, by using the symbol behavior of the involved terms (see Proposition 4.1 and Lemma 6.1), this implies

$$\big| \partial_\lambda^m m_+(x, \lambda) \big| \lesssim \langle x \rangle |\lambda|^{-\ell-m} \lesssim \langle x \rangle \langle x' \rangle^m |\lambda|^{-\ell}.$$

For $m_-(x', \lambda)$ we use reflection and transmission coefficients, i.e.,

$$m_-(x', \lambda) = a(\lambda) e^{2i\lambda x'} m_+(x', \lambda) + b(\lambda) \overline{m_+(x', \lambda)}$$

and, from the proof of Lemma 8.3, we have $|\partial_\lambda^m m_+(x', \lambda)| \lesssim \langle x' \rangle^m$. Hence, Eqs. (23) and (22) show that

$$\big| \partial_\lambda^m m_-(x', \lambda) \big| \lesssim |\lambda|^{-1-m} + \langle x' \rangle^m |\lambda|^{-1} \lesssim \langle x' \rangle^m |\lambda|^{-1}.$$

Setting

$$\omega(x, x', \lambda) := \lambda \frac{m_-(x', \lambda) m_+(x, \lambda)}{W(f_-, f_+)(\lambda)} \chi_\delta(\lambda) \chi_\delta(x\lambda) \big(1 - \chi_\delta(x'\lambda)\big)$$

the above estimates and Eq. (21) imply $|\partial_\lambda^m \omega(x, x', \lambda)| \lesssim \langle x \rangle \langle x' \rangle^m$ and the cosine estimate follows by appropriate integration by parts as in the proof of Lemma 8.3. For the sine estimate we set

$$\omega(x, x', \lambda) := \frac{m_-(x', \lambda) m_+(x, \lambda)}{W(f_-, f_+)(\lambda)} \chi_\delta(\lambda) \chi_\delta(x\lambda) \big(1 - \chi_\delta(x'\lambda)\big)$$

and we only have the weaker bound $|\partial_\lambda^m \omega(x, x', \lambda)| \lesssim \langle x \rangle \langle x' \rangle^m |\lambda|^{-1} \lesssim \langle x \rangle \langle x' \rangle^{m+1}$. □

8.4. Estimates for $|x'\lambda|$ small and $|x\lambda|$ large

Due to the asymmetric decay of the Regge–Wheeler potential, this case is slightly different from the above considered $|x\lambda|$ small and $|x'\lambda|$ large. Thus, it has to be studied separately and does not follow from symmetry arguments as in [53]. Similar as above, we distinguish $x \geqslant 0$ and $x \leqslant 0$.

Lemma 8.6. *Let $\alpha \in \mathbb{N}$ and $\delta > 0$ sufficiently small. Then we have the estimates*

$$\sup_{x > 0, x' \in \mathbb{R}} \left| \int_\mathbb{R} \lambda \cos(t\lambda) \frac{f_-(x', \lambda) f_+(x, \lambda)}{W(f_-, f_+)(\lambda)} \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \chi_\delta(\lambda) \big(1 - \chi_\delta(x\lambda)\big) \chi_\delta(x'\lambda) \, d\lambda \right| \lesssim \langle t \rangle^{-\alpha}$$

and

$$\sup_{x > 0, x' \in \mathbb{R}} \left| \int_\mathbb{R} \sin(t\lambda) \frac{f_-(x', \lambda) f_+(x, \lambda)}{W(f_-, f_+)(\lambda)} \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \chi_\delta(\lambda) \big(1 - \chi_\delta(x\lambda)\big) \chi_\delta(x'\lambda) \, d\lambda \right| \lesssim \langle t \rangle^{-\alpha+1}$$

for all $t \geqslant 0$.

Proof. Let $x \geqslant 0$, $x' \in \mathbb{R}$, $|\lambda| \leqslant \delta$, $|x'\lambda| \leqslant \delta$ and $|x\lambda| \geqslant \frac{\delta}{2}$. As in the proof of Lemma 8.3, we have the bounds $|\partial_\lambda^m m_+(x, \lambda)| \lesssim \langle x \rangle^m$. For $m_-(x', \lambda)$ we use the representation

$$m_-(x', \lambda) = e^{i\lambda x'} \big(-c_1^-(\lambda) u_0(x', \lambda) + c_0^-(\lambda) u_1(x', \lambda)\big)$$

since $|x'\lambda|$ is small. Lemmas 4.3, 4.6 and 6.4 imply the bound

$$\big|m_-(x', \lambda)\big| \lesssim \langle x' \rangle^{\ell+1} + \langle x' \rangle \lesssim |\lambda|^{-\ell-1}$$

and, by using the symbol behavior (see Proposition 4.1 and Lemmas 4.4, 6.4) and $\langle x' \rangle \lesssim |\lambda|^{-1}$, we infer

$$\big|\partial_\lambda^m m_-(x', \lambda)\big| \lesssim |\lambda|^{-\ell-1-m} \lesssim \langle x \rangle^m |\lambda|^{-\ell-1}.$$

We set

$$\omega(x, x', \lambda) := \lambda \frac{m_-(x', \lambda) m_+(x, \lambda)}{W(f_-, f_+)(\lambda)} \chi_\delta(\lambda) \big(1 - \chi_\delta(x\lambda)\big) \chi_\delta(x'\lambda)$$

and the above bounds as well as Eq. (21) imply the estimate $|\partial_\lambda^m \omega(x, x', \lambda)| \lesssim \langle x \rangle^m$. Thus, as before, the claim follows by appropriate integration by parts. $\quad\square$

Lemma 8.7. *Let $\alpha \in \mathbb{N}$ and $\delta > 0$ sufficiently small. Then we have the estimates*

$$\sup_{x<0,\, x'\in\mathbb{R}} \left| \int_{\mathbb{R}} \lambda \cos(t\lambda) \frac{f_-(x', \lambda) f_+(x, \lambda)}{W(f_-, f_+)(\lambda)} \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \chi_\delta(\lambda) \big(1 - \chi_\delta(x\lambda)\big) \chi_\delta(x'\lambda) \, d\lambda \right| \lesssim \langle t \rangle^{-\alpha}$$

and

$$\sup_{x<0,\, x'\in\mathbb{R}} \left| \int_{\mathbb{R}} \sin(t\lambda) \frac{f_-(x', \lambda) f_+(x, \lambda)}{W(f_-, f_+)(\lambda)} \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \chi_\delta(\lambda) \big(1 - \chi_\delta(x\lambda)\big) \chi_\delta(x'\lambda) \, d\lambda \right| \lesssim \langle t \rangle^{-\alpha+1}$$

for all $t \geq 0$.

Proof. Let $x \leq 0$, $x' \in \mathbb{R}$, $|\lambda| \leq \delta$, $|x'\lambda| \leq \delta$ and $|x\lambda| \geq \frac{\delta}{2}$. Like in the proof of Lemma 8.7 we have

$$\left| \partial_\lambda^m m_-(x', \lambda) \right| \lesssim \langle x' \rangle^{\ell+1} |\lambda|^{-m} \lesssim \langle x' \rangle |\lambda|^{-\ell-m}$$

for all $m \in \mathbb{N}_0$. Since $x \leq 0$ we use reflection and transmission coefficients to obtain the representation

$$m_+(x, \lambda) = -\overline{a(\lambda)} e^{-2i\lambda x} m_-(x, \lambda) + b(\lambda) \overline{m_-(x, \lambda)}$$

which immediately implies the bound $|m_+(x, \lambda)| \lesssim |\lambda|^{-1}$ by Eqs. (23), (22) and Lemma 6.2. Thus, from the symbol behavior of $a(\lambda)$, $b(\lambda)$ and $|\partial_\lambda^m m_-(x, \lambda)| \leq C_m$ (Lemma 6.2), we infer

$$\left| \partial_\lambda^m m_+(x, \lambda) \right| \lesssim \langle x \rangle^m |\lambda|^{-1}$$

for all $m \in \mathbb{N}_0$ since $|\lambda|^{-1} \lesssim \langle x \rangle$. Thus, Eq. (21) implies

$$\left| \partial_\lambda^m \omega(x, x', \lambda) \right| \lesssim \langle x \rangle^m \langle x' \rangle$$

for all $m \in \mathbb{N}_0$ where

$$\omega(x, x', \lambda) := \lambda \frac{m_-(x', \lambda) m_+(x, \lambda)}{W(f_-, f_+)(\lambda)} \chi_\delta(\lambda) \big(1 - \chi_\delta(x\lambda)\big) \chi_\delta(x'\lambda).$$

As a consequence, by appropriate integration by parts (cf. the proof of Lemma 8.3), we obtain

$$\left| \int_{\mathbb{R}} e^{i\lambda(\pm t + x - x')} \omega(x, x', \lambda) \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \, d\lambda \right| \lesssim \langle t \rangle^{-\alpha}$$

provided that $\alpha \geq 1$. For the sine evolution we set

$$\omega(x, x', \lambda) := \frac{m_-(x', \lambda) m_+(x, \lambda)}{W(f_-, f_+)(\lambda)} \chi_\delta(\lambda) \big(1 - \chi_\delta(x\lambda)\big) \chi_\delta(x'\lambda)$$

and use the bounds

$$\left|\partial_\lambda^m m_+(x,\lambda)\right| \lesssim \langle x\rangle^m |\lambda|^{-1} \lesssim \langle x\rangle^{m+1}$$

to obtain

$$\left|\partial_\lambda^m \omega(x,x',\lambda)\right| \lesssim \langle x\rangle^{m+1}\langle x'\rangle$$

for all $m \in \mathbb{N}_0$. Hence, as before, the claim follows from

$$\left|\int_{\mathbb{R}} e^{i\lambda(\pm t+x-x')}\omega(x,x',\lambda)\langle x\rangle^{-\alpha}\langle x'\rangle^{-\alpha}\,d\lambda\right| \lesssim \langle t\rangle^{-\alpha+1}$$

which can be obtained by appropriate integration by parts similar to the proof of Lemma 8.3.   □

9. Oscillatory integral estimates for large energies

9.1. The Jost solutions at large energies

In order to estimate the contributions from large energies, we need the behavior of the Jost solutions for $\lambda \to \infty$. As usual, we write $m_\pm(x,\lambda) = e^{\mp i\lambda x} f_\pm(x,\lambda)$.

Lemma 9.1. *Let $\lambda_0 > 0$. Then, for $k, m \in \mathbb{N}_0$, the function $m_+(\cdot,\lambda)$ satisfies the estimates*

$$\left|\partial_x^k \partial_\lambda^m \big(m_+(x,\lambda) - 1\big)\right| \leqslant C_{k,m}\langle x\rangle^{-1-k}\lambda^{-1-m}$$

for all $\lambda \geqslant \lambda_0$ and all $x \geqslant 0$. The same bounds hold for $m_-(x,\lambda)$ if $x \leqslant 0$.

Proof. As already discussed (see Lemma 3.1), the function $m_+(\cdot,\lambda)$ satisfies the Volterra equation

$$m_+(x,\lambda) = 1 + \frac{1}{2i\lambda}\int_x^\infty \big(e^{2i\lambda(y-x)} - 1\big)V_{\ell,\sigma}(y)m_+(y,\lambda)\,dy$$

$$= 1 + \int_0^\infty (e^{2i\eta} - 1)\left[V_{\ell,\sigma}\left(\frac{\eta}{\lambda} + x\right)\frac{m_+(\frac{\eta}{\lambda} + x,\lambda)}{2i\lambda^2}\right]d\eta$$

and thus, the lemma is obviously true for $k = m = 0$. Let $n : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ denote the bijection from Proposition 4.1. Fix $(k,m) \in \mathbb{N}_0 \times \mathbb{N}_0$ and suppose the assertion is true for all (j,l) with $n(j,l) \leqslant n(k,m)$. We need to show that this implies the claim for (k',m') where $n(k',m') = n(k,m) + 1$. There are two possibilities: Either $(k',m') = (m+1,0)$ (if $k = 0$) or $(k',m') = (k-1, m+1)$. In the former case we have

$$\partial_x^{m+1} m_+(x, \lambda) = \frac{1}{2i\lambda^2} \sum_{j=0}^{m} \binom{m+1}{j} \int_0^\infty (e^{2i\eta} - 1) \partial_x^{m+1-j} V_{\ell,\sigma}\left(\frac{\eta}{\lambda} + x\right) \partial_x^j m_+\left(\frac{\eta}{\lambda} + x, \lambda\right) d\eta$$

$$+ \frac{1}{2i\lambda^2} \int_0^\infty (e^{2i\eta} - 1) V_{\ell,\sigma}\left(\frac{\eta}{\lambda} + x\right) \partial_x^{m+1} m_+\left(\frac{\eta}{\lambda} + x, \lambda\right) d\eta$$

$$= O_{\mathbb{C}}\big(\langle x \rangle^{-1-(m+1)} \lambda^{-1}\big) + \frac{1}{2i\lambda} \int_x^\infty (e^{2i\lambda(y-x)} - 1) V_{\ell,\sigma}(y) \partial_y^{m+1} m_+(y, \lambda) \, dy$$

by assumption and thus, the estimate from Lemma B.1 yields $|\partial_x^{m+1} m_+(x, \lambda)| \lesssim \langle x \rangle^{-1-(m+1)} \lambda^{-1}$. For the latter case observe that

$$\partial_\lambda \partial_x^j m_+\left(\frac{\eta}{\lambda} + x, \lambda\right) \lesssim \left\langle \frac{\eta}{\lambda} + x \right\rangle^{-1-j-1} \frac{\eta}{\lambda} \lambda^{-1} + \left\langle \frac{\eta}{\lambda} + x \right\rangle^{-1-j} \lambda^{-2} \lesssim \left\langle \frac{\eta}{\lambda} + x \right\rangle^{-1-j} \lambda^{-1}$$

and, more generally,[8]

$$\partial_\lambda^l \partial_x^j m_+\left(\frac{\eta}{\lambda} + x, \lambda\right) \lesssim \left\langle \frac{\eta}{\lambda} + x \right\rangle^{-1-l-j} \frac{\eta^l}{\lambda^l} \lambda^{-l} + \cdots + \left\langle \frac{\eta}{\lambda} + x \right\rangle^{-1-j} \lambda^{-1-l} \lesssim \left\langle \frac{\eta}{\lambda} + x \right\rangle^{-1-j} \lambda^{-l}$$

for all (j, l) with $n(j, l) \leqslant n(k, m)$ and $(j, l) \neq (0, 0)$ by assumption. This shows that

$$\partial_x^{k-1} \partial_\lambda^{m+1} m_+(x, \lambda)$$

$$= \sum_{j=0}^m \binom{m+1}{j} \int_0^\infty (e^{2i\eta} - 1) \partial_x^{k-1} \left[\partial_\lambda^{m+1-j} \frac{V_{\ell,\sigma}(\frac{\eta}{\lambda} + x)}{2i\lambda^2} \partial_\lambda^j m_+\left(\frac{\eta}{\lambda} + x, \lambda\right) \right] d\eta$$

$$+ \int_0^\infty (e^{2i\eta} - 1) \partial_x^{k-1} \left[\frac{V_{\ell,\sigma}(\frac{\eta}{\lambda} + x)}{2i\lambda^2} \partial_\lambda^{m+1} m_+\left(\frac{\eta}{\lambda} + x, \lambda\right) \right] d\eta$$

$$= O_{\mathbb{C}}\big(\langle x \rangle^{-1-(k-1)} \lambda^{-1-(m+1)}\big) + \frac{1}{2i\lambda} \int_x^\infty (e^{2i\lambda(y-x)} - 1) V_{\ell,\sigma}(y) \partial_y^{k-1} \partial_\lambda^{m+1} m_+(y, \lambda) \, dy$$

and Lemma B.1 yields the claim. The proof for m_- is (*mutatis mutandis*) identical. □

Corollary 9.1. *Let $\lambda_0 > 0$. Then the Wronskian $W(f_-, f_+)(\lambda)$ has the behavior*

$$\frac{1}{W(f_-, f_+)(\lambda)} = \frac{1}{2i\lambda}\big(1 + O_{\mathbb{C}}(\lambda^{-1})\big)$$

for all $\lambda \geqslant \lambda_0$ where the O-term behaves like a symbol.

---

[8] One may apply Faà di Bruno's formula (cf. Lemma A.1) to obtain a completely explicit expression for the higher λ-derivatives.

Proof. With $m_\pm(x, \lambda) = e^{\mp i\lambda x} f_\pm(x, \lambda)$ we have

$$W(f_-, f_+)(\lambda) = 2i\lambda m_-(0, \lambda)m_+(0, \lambda) + O_\mathbb{C}(\lambda^{-1}) = 2i\lambda(1 + O_\mathbb{C}(\lambda^{-1}))$$

by Lemma 9.1 where the O-term is of symbol type. Thus, the claim follows from Lemma A.1. $\square$

Before proceeding to the final oscillatory integral estimate, we need the large λ behavior of the reflection and transmission coefficients, i.e., the coefficients $a(\lambda)$ and $b(\lambda)$ satisfying $f_-(x, \lambda) = a(\lambda)f_+(x, \lambda) + b(\lambda)\overline{f_+(x, \lambda)}$. The behavior of b is given by Eq. (22). By Lemma 9.1 we have

$$W(f_-, \overline{f_+})(\lambda) = m_-(0, \lambda)\overline{m'_+(0, \lambda)} - m'_-(0, \lambda)\overline{m_+(0, \lambda)} = O_\mathbb{C}(\lambda^{-1})$$

where the O-term behaves like a symbol and therefore,

$$\frac{a(\lambda)}{W(f_-, f_+)(\lambda)} = -\frac{W(f_-, \overline{f_+})(\lambda)}{2i\lambda W(f_-, f_+)(\lambda)} = O_\mathbb{C}(\lambda^{-3}). \tag{24}$$

We also remark that, by symmetry, the above considerations extend to large negative λ. Now we are ready to prove the final oscillatory integral estimate.

9.2. The cosine estimate

We distinguish between the cosine and the sine estimate since in the former case we obtain a bound involving the derivative of the data.

Proposition 9.1. *Let $\alpha \in \mathbb{N}_0$ and $\delta > 0$ sufficiently small. Then we have the estimate*

$$\sup_{x \in \mathbb{R}} \left| \lim_{N \to \infty} \int_\mathbb{R} \int_{-N}^{N} \lambda e^{\pm it\lambda} G_{\ell, \sigma}(x, x', \lambda)(1 - \chi_\delta(\lambda))\langle x \rangle^{-\alpha}\langle x' \rangle^{-\alpha}\phi(x')\, dx'\, d\lambda \right|$$

$$\lesssim \langle t \rangle^{-\alpha} \int_\mathbb{R} (|\phi'(x')| + |\phi(x')|)\, dx'$$

for all $t \geqslant 0$ and any $\phi \in \mathcal{S}(\mathbb{R})$.

Proof. We split the integral according to

$$\int_\mathbb{R} \int_{-N}^{N} \ldots d\lambda\, dx' = \int_{-\infty}^{x} \int_{-N}^{N} \ldots d\lambda\, dx' + \int_{x}^{\infty} \int_{-N}^{N} \ldots d\lambda\, dx'$$

and only consider the first summand since the proof for the second one is completely analogous. In the domain $x' \leqslant x$, which we study now, the Green's function is given by

$$G_{\ell, \sigma}(x, x', \lambda) = \frac{f_-(x', \lambda)f_+(x, \lambda)}{W(f_-, f_+)(\lambda)}.$$

We distinguish between $x \leqslant 0$ and $x \geqslant 0$ and start with $x \leqslant 0$. Using reflection and transmission coefficients we obtain

$$
\begin{aligned}
f_-\bigl(x', \lambda\bigr) f_+(x, \lambda) &= e^{-i\lambda x'} m_-\bigl(x', \lambda\bigr) \bigl[-\overline{a(\lambda)} e^{-i\lambda x} m_-(x, \lambda) + b(\lambda) e^{i\lambda x} \overline{m_-(x, \lambda)} \bigr] \\
&= -e^{-i\lambda(x+x')} \overline{a(\lambda)} m_-\bigl(x', \lambda\bigr) m_-(x, \lambda) + e^{i\lambda(x-x')} b(\lambda) m_-\bigl(x', \lambda\bigr) \overline{m_-(x, \lambda)},
\end{aligned}
$$

and consider each term separately. We define

$$
\omega\bigl(x, x', \lambda\bigr) := \frac{\lambda b(\lambda)(1 - \chi_\delta(\lambda))}{W(f_-, f_+)(\lambda)} m_-\bigl(x', \lambda\bigr) \overline{m_-(x, \lambda)}
$$

and by Lemma 9.1 and Corollary 9.1 as well as Eqs. (22), (24), we obtain the estimates

$$
\bigl| \partial_\lambda^m \omega\bigl(x, x', \lambda\bigr) \bigr| \leqslant C_m |\lambda|^{-m}
$$

for all $|\lambda| \geqslant \frac{\delta}{2}$ and $x, x' \leqslant 0$ (recall that $1 - \chi_\delta(\lambda) \equiv 1$ for $|\lambda| \geqslant \delta$). Note that, by Fubini, we can freely interchange the order of integration and thus, integration by parts with respect to x' yields

$$
\int_{-\infty}^{x} \int_{-N}^{N} e^{i\lambda(\pm t + x - x')} \omega\bigl(x, x', \lambda\bigr) \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \phi\bigl(x'\bigr) \, d\lambda \, dx'
$$

$$
= -\phi(x) \langle x \rangle^{-2\alpha} \int_{-N}^{N} \frac{e^{\pm i\lambda t}}{i\lambda} \omega(x, x, \lambda) \, d\lambda
$$

$$
+ \int_{-\infty}^{x} \int_{-N}^{N} \frac{e^{i\lambda(\pm t + x - x')}}{i\lambda} \langle x \rangle^{-\alpha} \partial_{x'} \bigl[\omega\bigl(x, x', \lambda\bigr) \langle x' \rangle^{-\alpha} \phi\bigl(x'\bigr) \bigr] \, d\lambda \, dx'. \tag{25}
$$

We first claim that

$$
\sup_{x < 0} \left| \int_{\mathbb{R}} e^{\pm i\lambda t} \lambda^{-1} \omega(x, x, \lambda) \, d\lambda \right| \lesssim 1. \tag{26}
$$

Indeed, set

$$
\tilde{\omega}(x, \lambda) := \lambda^{-1} \omega(x, x, \lambda) = \frac{b(\lambda)(1 - \chi_\delta(\lambda))}{W(f_-, f_+)(\lambda)} m_-(x, \lambda) \overline{m_-(x, \lambda)}
$$

and observe that $\tilde{\omega}(x, \lambda) = \frac{1}{2i\lambda}(1 + O_{\mathbb{C}}(|\lambda|^{-1}))$ for $|\lambda| \to \infty$ by Eq. (22) and Lemma 9.1. This shows that $\tilde{\omega}(x, -\lambda) = -\tilde{\omega}(x, \lambda) + O_{\mathbb{C}}(|\lambda|^{-2})$. Thus, we have

$$\left| \int\limits_{\mathbb{R}} e^{\pm i\lambda t} \tilde\omega(x,\lambda)\, d\lambda \right| = \left| \int\limits_{\frac{\delta}{2}}^{\infty} \left[e^{\pm i\lambda t} \tilde\omega(x,\lambda) + e^{\mp i\lambda t} \tilde\omega(x,-\lambda) \right] d\lambda \right|$$

$$\lesssim \left| \int\limits_{\frac{\delta}{2}}^{\infty} \sin(\lambda t)\tilde\omega(x,\lambda)\, d\lambda \right| + \left| \int\limits_{\frac{\delta}{2}}^{\infty} e^{\pm i\lambda t} O_{\mathbb{C}}(|\lambda|^{-2})\, d\lambda \right|$$

$$\lesssim \left| \int\limits_{\frac{\delta}{2}}^{\infty} \sin(\lambda t)(\lambda^{-1} + O_{\mathbb{C}}(|\lambda|^{-2}))\, d\lambda \right| + 1 \lesssim 1$$

for all $x \leqslant 0$ and this proves Eq. (26). Therefore, for $N \to \infty$, we can estimate the first term in Eq. (25) as

$$\left| \phi(x)\langle x \rangle^{-2\alpha} \int\limits_{\mathbb{R}} e^{\pm i\lambda t} \lambda^{-1} \omega(x,x,\lambda)\, d\lambda \right| \lesssim \sup_{x<0} |\phi(x)| \frac{1}{t^{\alpha}} \int\limits_{\mathbb{R}} \left| e^{\pm i\lambda t} \partial_{\lambda}^{\alpha} \left[\lambda^{-1} \omega(x,x,\lambda) \right] \right| d\lambda$$

$$\lesssim \langle t \rangle^{-\alpha} \int\limits_{\mathbb{R}} \left(|\phi'(x')| + |\phi(x')| \right) dx'$$

for all $t \geqslant 1$ and $x \leqslant 0$ by α-fold integration by parts and Sobolev embedding. By Eq. (26) this inequality is in fact valid for all $t \geqslant 0$.

For the second term we similarly claim that

$$\sup_{x<0} \int\limits_{-\infty}^{x} \left| \int\limits_{\mathbb{R}} e^{i\lambda(\pm t + x - x')} \lambda^{-1} \langle x \rangle^{-\alpha} \partial_{x'} \left[\omega(x,x',\lambda)\langle x' \rangle^{-\alpha} \phi(x') \right] d\lambda \right| dx'$$

$$\lesssim \langle x \rangle^{-\alpha} \int\limits_{\mathbb{R}} \langle x' \rangle^{-\alpha} \left(|\phi'(x')| + |\phi(x')| \right) dx'. \tag{27}$$

Indeed, we have

$$\left| \int\limits_{\mathbb{R}} e^{i\lambda(\pm t + x - x')} \lambda^{-1} \langle x \rangle^{-\alpha} \partial_{x'} \omega(x,x',\lambda)\langle x' \rangle^{-\alpha} \phi(x')\, d\lambda \right|$$

$$\leqslant \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \int\limits_{\mathbb{R}} \left| \lambda^{-1} \partial_{x'} \omega(x,x',\lambda)\phi(x') \right| d\lambda$$

$$\lesssim \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} |\phi(x')|$$

for all $x \leqslant 0$ since $|\lambda^{-1}\partial_{x'}\omega(x,x',\lambda)| \lesssim |\lambda|^{-2}$ by Lemma 9.1. Moreover,

$$\left| \int\limits_{\mathbb{R}} e^{i\lambda(\pm t + x - x')} \lambda^{-1} \omega(x,x',\lambda)\langle x \rangle^{-\alpha} \partial_{x'} \left[\langle x' \rangle^{-\alpha} \phi(x') \right] d\lambda \right| \lesssim \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \left(|\phi'(x')| + |\phi(x')| \right)$$

for all $x \leqslant 0$ which can be shown by exploiting exactly the same cancellation that led to Eq. (26). This proves Eq. (27). Note in particular that Eq. (27) implies

$$\lim_{N \to \infty} \int_{-\infty}^{x} \int_{-N}^{N} \frac{e^{i\lambda(\pm t + x - x')}}{i\lambda} \langle x \rangle^{-\alpha} \partial_{x'} \big[\omega(x, x', \lambda) \langle x' \rangle^{-\alpha} \phi(x') \big] d\lambda \, dx'$$

$$= \int_{-\infty}^{x} \int_{\mathbb{R}} \frac{e^{i\lambda(\pm t + x - x')}}{i\lambda} \langle x \rangle^{-\alpha} \partial_{x'} \big[\omega(x, x', \lambda) \langle x' \rangle^{-\alpha} \phi(x') \big] d\lambda \, dx'$$

by dominated convergence. Now we distinguish two cases. If $| \pm t + x - x'| \geqslant \frac{1}{2}t$, we integrate by parts α-times to obtain

$$\left| \int_{-\infty}^{x} \int_{\mathbb{R}} e^{i\lambda(\pm t + x - x')} \lambda^{-1} \langle x \rangle^{-\alpha} \partial_{x'} \big[\omega(x, x', \lambda) \langle x' \rangle^{-\alpha} \phi(x') \big] d\lambda \, dx' \right|$$

$$\lesssim | \pm t + x - x'|^{-\alpha} \int_{-\infty}^{x} \int_{\mathbb{R}} \big| \langle x \rangle^{-\alpha} \partial_{x'} \partial_{\lambda}^{\alpha} \big[\lambda^{-1} \omega(x, x', \lambda) \langle x' \rangle^{-\alpha} \phi(x') \big] \big| d\lambda \, dx'$$

$$\lesssim \langle t \rangle^{-\alpha} \int_{\mathbb{R}} \int_{|\lambda| \geqslant \frac{\delta}{2}} |\lambda|^{-(\alpha+1)} \langle x \rangle^{-\alpha} \big| \partial_{x'} \big[\langle x' \rangle^{-\alpha} \phi(x') \big] \big| d\lambda \, dx'$$

$$\lesssim \langle t \rangle^{-\alpha} \int_{\mathbb{R}} \big(|\phi'(x')| + |\phi(x')| \big) dx'$$

for all $t \geqslant 0$ and all $x \leqslant 0$. If $| \pm t + x - x'| \leqslant \frac{1}{2}t$, we have $\langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \lesssim \langle t \rangle^{-\alpha}$ as $t \to \infty$ and Eq. (27) implies

$$\sup_{x<0} \int_{-\infty}^{x} \left| \int_{\mathbb{R}} e^{i\lambda(\pm t + x - x')} \lambda^{-1} \langle x \rangle^{-\alpha} \partial_{x'} \big[\omega(x, x', \lambda) \langle x' \rangle^{-\alpha} \phi(x') \big] d\lambda \right| dx'$$

$$\lesssim \langle t \rangle^{-\alpha} \int_{\mathbb{R}} \big(|\phi'(x')| + |\phi(x')| \big) dx'$$

for all $t \geqslant 0$. The remaining cases are treated in a completely analogous fashion. Note that the terms involving the coefficient $a(\lambda)$ are even simpler due to the stronger decay given by Eq. (24). For terms that contain no reflection and transmission coefficients, use Corollary 9.1 for the cancellation argument. $\quad \square$

9.3. The sine estimate

The sine estimate is slightly stronger since it does not require derivatives of the data.

Corollary 9.2. *Let $\alpha \in \mathbb{N}_0$ and $\delta > 0$ sufficiently small. Then we have the estimate*

$$\sup_{x \in \mathbb{R}} \left| \lim_{N \to \infty} \int_{\mathbb{R}} \int_{-N}^{N} e^{\pm it\lambda} G_{\ell,\sigma}(x, x', \lambda)(1 - \chi_\delta(\lambda)) \langle x \rangle^{-\alpha} \langle x' \rangle^{-\alpha} \phi(x') \, dx' \, d\lambda \right|$$

$$\lesssim \langle t \rangle^{-\alpha} \int_{\mathbb{R}} |\phi(x')| \, dx'$$

for all $t \geqslant 0$ and any $\phi \in \mathcal{S}(\mathbb{R})$.

Proof. Just repeat the arguments from the proof of Proposition 9.1. However, note that we are lacking one factor of λ compared to Proposition 9.1 which makes the integration by parts with respect to x' unnecessary. This explains why no term containing ϕ' appears on the right-hand side of the estimate. $\square$

Appendix A. Symbol behavior

Lemma A.1. *Let $I \subset \mathbb{R}$ and suppose f is smooth on I and satisfies $|f(x)| \leqslant C < 1$ for all $x \in I$. Then, for all $x \in I$, we have the estimate*

$$\left| \left(\frac{1}{1+f} \right)^{(k)}(x) \right| \leqslant C_k \sum \prod_{j=1}^{k} |f^{(j)}(x)|^{m_j}$$

for all $k \in \mathbb{N}$ where the sum runs over all possible k-tuples $(m_1, m_2, \ldots, m_k) \in \mathbb{N}_0^k$ satisfying $\sum_{j=1}^{k} j m_j = k$.

Proof. This follows from the identity

$$\left(\frac{1}{1+f} \right)^{(k)} = \sum_{\sum_{j=1}^{k} j m_j = k} a_{m_1, m_2, \ldots, m_k} \left(\frac{1}{1+f} \right)^{1 + \sum_{j=1}^{k} m_j} \prod_{j=1}^{k} (f^{(j)})^{m_j}$$

which is known as Faà di Bruno's formula (see e.g. [50], the explicit form of the coefficients $a_{m_1, m_2, \ldots, m_k}$ is irrelevant for our purposes) and the fact that $|(1+f)^{-1}| \lesssim 1$ on I. $\square$

Appendix B. Volterra integral equations

In this section we establish some well-known facts about Volterra integral equations which are frequently used throughout this work.

Lemma B.1. *Let $a \in \mathbb{R}$, $g \in L^\infty(a, \infty)$ and suppose the integral kernel K satisfies*

$$\mu := \int_{a}^{\infty} \sup_{x \in (a, y)} |K(x, y)| \, dy < \infty.$$

Then the Volterra equation

$$f(x) = g(x) + \int_x^\infty K(x, y) f(y) \, dy$$

has a unique solution f satisfying

$$\|f\|_{L^\infty(a,\infty)} \leqslant e^\mu \|g\|_{L^\infty(a,\infty)}.$$

Proof. See e.g. [25] or [52]. □

The next lemma states differentiability properties of solutions of Volterra integral equations.

Lemma B.2. *If, in addition to the assumptions of Lemma B.1, $g \in C^\infty(a, \infty)$ and the kernel K is smooth in both variables on (a, ∞) and satisfies*

$$\int_a^\infty \left| \partial_x^k K(x, y) \right| dy < \infty$$

for any $x \geqslant a$ and all $k \in \mathbb{N}$ then the solution f is smooth on (a, ∞). Furthermore, the derivatives can be calculated by formal differentiation, i.e.,

$$f^{(k)}(x) = g^{(k)}(x) - \sum_{j=0}^{k-1} (\kappa_j f)^{(k-1-j)}(x) + \int_x^\infty \partial_x^k K(x, y) f(y) \, dy$$

where $\kappa_j(x) := \partial_x^j K(x, y)|_{y=x}$.

Proof. The claim follows from a straightforward application of Lebesgue's theorem on dominated convergence and an induction. □

The next lemma shows how the dependence of the kernel K on a parameter λ carries over to the solution of the corresponding Volterra equation.

Lemma B.3. *Let $I \subset \mathbb{R}$ be open and suppose*

$$\int_a^\infty \sup_{x \in (a, y)} \left| \partial_\lambda^m K(x, y, \lambda) \right| dy < \infty$$

as well as $\partial_\lambda^m g(\cdot, \lambda) \in L^\infty(a, \infty)$ for all $m \in \mathbb{N}_0$ and $\lambda \in I$. Then the Volterra equation

$$f(x, \lambda) = g(x, \lambda) + \int_x^\infty K(x, y, \lambda) f(y, \lambda) \, dy$$

has a unique solution $f(x, \lambda)$ for all $x \geqslant a$ and $\lambda \in I$ which is smooth in λ. Furthermore, we have $\partial_\lambda^m f(\cdot, \lambda) \in L^\infty(a, \infty)$ for all $m \in \mathbb{N}_0$ and the derivatives are given by

$$\partial_\lambda^m f(x, \lambda) = \partial_\lambda^m g(x, \lambda) + \sum_{j=0}^m \binom{m}{j} \int_x^\infty \partial_\lambda^j K(x, y, \lambda) \partial_\lambda^{m-j} f(y, \lambda) \, dy.$$

Proof. According to Lemma B.1, the solution f exists, is unique and satisfies $\|f(\cdot, \lambda)\|_{L^\infty(a, \infty)} < \infty$. Now consider the integral equation

$$h(x, \lambda) = \partial_\lambda g(x, \lambda) + \tilde{h}(x, \lambda) + \int_x^\infty K(x, y, \lambda) h(y, \lambda) \, dy \qquad (28)$$

where

$$\tilde{h}(x, \lambda) := \int_x^\infty \partial_\lambda K(x, y, \lambda) f(y, \lambda) \, dy.$$

We have $\|\tilde{h}(\cdot, \lambda)\|_{L^\infty(a, \infty)} < \infty$ and thus, by Lemma B.1, Eq. (28) has a unique solution $h(\cdot, \lambda) \in L^\infty(a, \infty)$ for all $\lambda \in I$. However, by dominated convergence we conclude

$$\lim_{\nu \to 0} \left| \frac{f(x, \lambda + \nu) - f(x, \lambda)}{\nu} - h(x, \lambda) \right| = 0$$

and hence, $\partial_\lambda f$ exists and equals h. Existence of the higher derivatives follows by the Leibniz rule and an induction. $\quad\square$

We finally remark that all of the above lemmas have counterparts for Volterra equations of the form

$$f(x, \lambda) = g(x, \lambda) + \int_a^x K(x, y, \lambda) f(y, \lambda) \, dy$$

with almost identical proofs.

References

[1] Milton Abramowitz, Irene A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Nat. Bureau Standards Appl. Math. Ser., vol. 55, U.S. Government Printing Office, Washington, D.C., 1964, for sale by the superintendent of documents.

[2] Peter C. Aichelburg, Piotr Bizoń, Tabor Zbislaw, Bifurcation and fine structure phenomena in critical collapse of a self-gravitating σ-field, Classical Quantum Gravity 23 (16) (2006) S299–S306.

[3] Lars Andersson, Pieter Blue, Hidden symmetries and decay for the wave equation on the Kerr spacetime, Preprint, arXiv:0908.2265, 2009.

[4] Leor Barack, Amos Ori, Late-time decay of scalar perturbations outside rotating black holes, Phys. Rev. Lett. 82 (22) (1999) 4388–4391.

[5] Piotr Bizoń, Tadeusz Chmaj, Andrzej Rostworowski, Late-time tails of a Yang–Mills field on Minkowski and Schwarzschild backgrounds, Classical Quantum Gravity 24 (13) (2007) F55–F63.

[6] Piotr Bizoń, Tadeusz Chmaj, Andrzej Rostworowski, Stanislaw Zajac, Late-time tails of wave maps coupled to gravity, Preprint, arXiv:0906.2919, 2009.

[7] Carl J. Blaksley, Lior M. Burko, Late-time tails in the Reissner–Nordström spacetime revisited, Phys. Rev. D (Particles, Fields, Gravitation and Cosmology) 76 (10) (2007) 104035.

[8] P. Blue, A. Soffer, Semilinear wave equations on the Schwarzschild manifold. I. Local decay estimates, Adv. Differential Equations 8 (5) (2003) 595–614.

[9] Pieter Blue, Avy Soffer, A space-time integral estimate for a large data semi-linear wave equation on the Schwarzschild manifold, Lett. Math. Phys. 81 (3) (2007) 227–238.

[10] P. Blue, A. Soffer, Phase space analysis on some black hole manifolds, J. Funct. Anal. 256 (1) (2009) 1–90.

[11] Pieter Blue, Jacob Sterbenz, Uniform decay of local energy and the semi-linear wave equation on Schwarzschild space, Comm. Math. Phys. 268 (2) (2006) 481–504.

[12] Lior M. Burko, Gaurav Khanna, Late-time Kerr tails revisited, Classical Quantum Gravity 26 (1) (2009) 015014, 28.

[13] S. Chandrasekhar, On the equations governing the perturbations of the Schwarzschild black hole, Proc. R. Soc. Lond. Ser. A 343 (1975) 289–298.

[14] S. Chandrasekhar, The Mathematical Theory of Black Holes, Oxf. Class. Texts Phys. Sci., The Clarendon Press, Oxford University Press, New York, 1998, reprint of the 1992 edition.

[15] Demetrios Christodoulou, Sergiu Klainerman, The Global Nonlinear Stability of the Minkowski Space, Princeton Math. Ser., vol. 41, Princeton Univ. Press, Princeton, NJ, 1993.

[16] Jeffrey M. Cohen, Robert M. Wald, Point charge in the vicinity of a Schwarzschild black hole, J. Math. Phys. 12 (1971) 1845–1849.

[17] Mihalis Dafermos, Igor Rodnianski, A proof of Price's law for the collapse of a self-gravitating scalar field, Invent. Math. 162 (2) (2005) 381–457.

[18] Mihalis Dafermos, Igor Rodnianski, The red-shift effect and radiation decay on black hole spacetimes, Preprint, arXiv:gr-qc/0512119, 2005.

[19] Mihalis Dafermos, Igor Rodnianski, A note on energy currents and decay for the wave equation on a Schwarzschild background, Preprint, arXiv:0710.0171, 2007.

[20] Mihalis Dafermos, Igor Rodnianski, Lectures on black holes and linear waves, Preprint, arXiv:0811.0354, 2008.

[21] Thibault Damour, Alessandro Nagar, Comparing effective-one-body gravitational waveforms to accurate numerical data, Phys. Rev. D (Particles, Fields, Gravitation and Cosmology) 77 (2) (2008) 024043.

[22] Thibault Damour, Alessandro Nagar, Improved analytical description of inspiralling and coalescing black-hole binaries, Phys. Rev. D (Particles, Fields, Gravitation and Cosmology) 79 (8) (2009) 081503.

[23] Thibault Damour, Alessandro Nagar, Ernst Nils Dorband, Denis Pollney, Luciano Rezzolla, Faithful effective-one-body waveforms of equal-mass coalescing black-hole binaries, Phys. Rev. D (Particles, Fields, Gravitation and Cosmology) 77 (8) (2008) 084017.

[24] Thibault Damour, Alessandro Nagar, Mark Hannam, Sascha Husa, Bernd Brügmann, Accurate effective-one-body waveforms of inspiralling and coalescing black-hole binaries, Phys. Rev. D (Particles, Fields, Gravitation and Cosmology) 78 (4) (2008) 044039.

[25] P. Deift, E. Trubowitz, Inverse scattering on the line, Comm. Pure Appl. Math. 32 (2) (1979) 121–251.

[26] Roland Donninger, Wilhelm Schlag, Decay estimates for the one-dimensional wave equation with an inverse power potential, Int. Math. Res. Not. (2010), in press.

[27] Roland Donninger, Wilhelm Schlag, Avy Soffer, On pointwise decay of linear waves on a Schwarzschild black hole background, Preprint, arXiv:0911.3179, 2009.

[28] F. Finster, N. Kamran, J. Smoller, S.-T. Yau, Decay of solutions of the wave equation in the Kerr geometry, Comm. Math. Phys. 264 (2) (2006) 465–503.

[29] F. Finster, N. Kamran, J. Smoller, S.-T. Yau, Linear waves in the Kerr geometry: A mathematical voyage to black hole physics, Bull. Amer. Math. Soc. 46 (2009) 635–659.

[30] F. Finster, N. Kamran, J. Smoller, S.-T. Yau, A rigorous treatment of energy extraction from a rotating black hole, Comm. Math. Phys. 287 (3) (2009) 829–847.

[31] Carsten Gundlach, Richard H. Price, Jorge Pullin, Late-time behavior of stellar collapse and explosions. I. Linearized perturbations, Phys. Rev. D (Particles, Fields, Gravitation and Cosmology) 49 (2) (1994) 883–889.

[32] Shahar Hod, How pure is the tail of gravitational collapse? Classical Quantum Gravity 26 (2) (2009) 028001, 4.

[33] Petr Hořava, Quantum gravity at a Lifshitz point, Phys. Rev. D (Particles, Fields, Gravitation and Cosmology) 79 (8) (2009) 084008.

[34] Bernard S. Kay, Robert M. Wald, Linear stability of Schwarzschild under perturbations which are nonvanishing on the bifurcation 2-sphere, Classical Quantum Gravity 4 (4) (1987) 893–898.

[35] Johann Kronthaler, The Cauchy problem for the wave equation in the Schwarzschild geometry, J. Math. Phys. 47 (4) (2006) 042501, 29.

[36] Johann Kronthaler, Decay rates for spherical scalar waves in the Schwarzschild geometry, Preprint, arXiv:0709.3703, 2007.

[37] Hans Lindblad, Igor Rodnianski, Global existence for the Einstein vacuum equations in wave coordinates, Comm. Math. Phys. 256 (1) (2005) 43–110.

[38] Jonathan Luk, Improved decay for solutions to the linear wave equation on a Schwarzschild black hole, Preprint, arXiv:0906.5588, 2009.

[39] Jeremy Marzuola, Jason Metcalfe, Daniel Tataru, Mihai Tohaneanu, Strichartz estimates on Schwarzschild black hole backgrounds, Preprint, arXiv:0802.3942, 2008.

[40] G.E.A. Matsas, M. Richartz, A. Saa, A.R.R. da Silva, D.A.T. Vanzella, Can quantum mechanics fool the cosmic censor? Phys. Rev. D (Particles, Fields, Gravitation and Cosmology) 79 (10) (2009) 101502.

[41] J. Metcalfe, D. Tataru, Global parametrices and dispersive estimates for variable coefficient wave equations, Preprint, arXiv:0707.1191, 2007.

[42] J. Metcalfe, D. Tataru, Decay estimates for variable coefficient wave equations in exterior domains, Preprint, arXiv:0806.3409, 2008.

[43] Eric Poisson, Radiative falloff of a scalar field in a weakly curved spacetime without symmetries, Phys. Rev. D (Particles, Fields, Gravitation and Cosmology) 66 (4) (2002) 044008.

[44] Richard H. Price, Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations, Phys. Rev. D (3) 5 (1972) 2419–2438.

[45] Richard H. Price, Nonspherical perturbations of relativistic gravitational collapse. II. Integer-spin, zero-rest-mass fields, Phys. Rev. D (3) 5 (1972) 2439–2454.

[46] Richard H. Price, Lior M. Burko, Late time tails from momentarily stationary, compact initial data in Schwarzschild spacetimes, Phys. Rev. D 70 (8) (Oct. 2004) 084039.

[47] Michael Pürrer, Peter C. Aichelburg, Tails for the Einstein–Yang–Mills system, Classical Quantum Gravity 26 (3) (2009) 035004, 10.

[48] Michael Pürrer, Sascha Husa, Peter C. Aichelburg, News from critical collapse: Bondi mass, tails, and quasinormal modes, Phys. Rev. D (Particles, Fields, Gravitation and Cosmology) 71 (10) (2005) 104005.

[49] Tullio Regge, John A. Wheeler, Stability of a Schwarzschild singularity, Phys. Rev. 108 (4) (1957) 1063–1069.

[50] Steven Roman, The formula of Faà di Bruno, Amer. Math. Monthly 87 (10) (1980) 805–809.

[51] W. Schlag, Dispersive estimates for Schrödinger operators: a survey, in: Mathematical Aspects of Nonlinear Dispersive Equations, in: Ann. of Math. Stud., vol. 163, Princeton Univ. Press, Princeton, NJ, 2007, pp. 255–285.

[52] Wilhelm Schlag, Avy Soffer, Wolfgang Staubach, Decay for the wave and Schrödinger evolutions on manifolds with conical ends. I, Trans. Amer. Math. Soc. 362 (1) (2010) 19–52.

[53] Wilhelm Schlag, Avy Soffer, Wolfgang Staubach, Decay for the wave and Schrödinger evolutions on manifolds with conical ends. II, Trans. Amer. Math. Soc. 362 (1) (2010) 289–318.

[54] Daniel Tataru, Local decay of waves on asymptotically flat stationary space-times, Preprint, arXiv:0910.5290, 2009.

[55] Daniel Tataru, Mihai Tohaneanu, Local energy estimate on Kerr black hole backgrounds, Preprint, arXiv:0810.5766, 2008.

[56] Gerald Teschl, Mathematical Methods in Quantum Mechanics, Grad. Stud. Math., vol. 99, American Mathematical Society, Providence, RI, 2009, with applications to Schrödinger operators.

[57] C.V. Vishveshwara, Stability of the Schwarzschild metric, Phys. Rev. D 1 (10) (1970) 2870–2879.

[58] Robert M. Wald, Note on the stability of the Schwarzschild metric, J. Math. Phys. 20 (6) (1979) 1056–1058.

[59] Frank J. Zerilli, Effective potential for even-parity Regge–Wheeler gravitational perturbation equations, Phys. Rev. Lett. 24 (13) (1970) 737–738.