Computation of generalized equivariant cohomologies of Kac–Moody flag varieties [☆]

Megumi Harada[a],[*], André Henriques[b], Tara S. Holm[c]

[a]*Department of Mathematics, University of Toronto, Toronto, Ont. M5S 3G3 Canada*
[b]*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*
[c]*Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA*

Abstract

In 1998, Goresky, Kottwitz, and MacPherson showed that for certain projective varieties X equipped with an algebraic action of a complex torus T, the equivariant cohomology ring $H_T^*(X)$ can be described by combinatorial data obtained from its orbit decomposition. In this paper, we generalize their theorem in three different ways. First, our group G need not be a torus. Second, our space X is an equivariant stratified space, along with some additional hypotheses on the attaching maps. Third, and most important, we allow for generalized equivariant cohomology theories E_G^* instead of H_T^*. For these spaces, we give a combinatorial description of $E_G^*(X)$ as a subring of $\prod E_G^*(F_i)$, where the F_i are certain invariant subspaces of X. Our main examples are the flag varieties $\mathscr{G}/\mathscr{P}$ of Kac–Moody groups $\mathscr{G}$, with the action of the torus of $\mathscr{G}$. In this context, the F_i are the T-fixed points and E_G^* is a T-equivariant complex oriented cohomology theory, such as H_T^*, K_T^* or MU_T^*. We detail several explicit examples.
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1. Introduction and background

The goal of this paper is to give a combinatorial description of certain generalized equivariant cohomologies of stratified spaces. The important examples to which our main theorems apply include T-equivariant cohomology, K-theory, and complex cobordism of Kac–Moody flag varieties. Although the examples that motivate us come from the theory of algebraic groups, our proofs rely heavily on techniques from algebraic topology. Indeed, we state the results of Sections 2–4 in the following context.

Let G be a topological group and E_G^* a G-equivariant cohomology theory (see [May96, Chapter XIII] for a definition) with a commutative cup product. Let X be a stratified G-space such that successive quotients X_i / X_{i-1} are homeomorphic to Thom spaces $Th(V_i)$ of E-orientable G-vector bundles $V_i \to F_i$. In this setting, and with the assumption that the Euler classes $e(V_i)$ are not zero divisors, we show that the restriction map

$$\imath^* : E_G^*(X) \to \prod_i E_G^*(F_i)$$

is injective. Moreover, when X and the G-action satisfy additional technical assumptions, we identify the image of $\imath^*$ as a subring of $\prod_i E_G^*(F_i)$ defined by explicit compatibility conditions involving divisibility by certain Euler classes. We also construct free E_G^*-module generators of $E_G^*(X)$.

Our theorems generalize known results in algebraic and symplectic geometry. When X is a projective variety, G a complex torus, and E_G^* ordinary equivariant cohomology, then we recover a theorem of Goresky, Kottwitz, and MacPherson [GKM98] that computes $H_T^*(X; \mathbb{C})$. They assume that X has finitely many 0- and 1-dimensional T-orbits, and then consider the graph Γ whose vertices are the fixed points X^T and edges are the 1-dimensional orbits. An edge (v, w) in Γ is decorated with the weight $\alpha_{(v,w)}$ of the T-action on the corresponding orbit. They provide a combinatorial description of $H_T^*(X)$ as a subring of $H_T^*(X^T)$ in terms of this graph. Each edge of Γ gives a condition as follows. Let $x(v)$ denote the restriction of a class $x \in H_T^*(X)$ to $v \in X^T$. Then the condition reads

$$\alpha_{(v,w)} \;\Big|\; x(v) - x(w). \tag{1.1}$$

We illustrate an example in Fig. 1.

This article is organized as follows. In Section 2 we prove the injectivity of the map

$$\imath^* : E_G^*(X) \to \prod_i E_G^*(F_i).$$

Next, in Section 3, we identify the image of $\imath^*$, giving combinatorial conditions similar to those in (1.1). In Section 4, we give a description of module generators for $E_G^*(X)$. Finally, in Sections 5 and 6, we return to our motivating examples, which
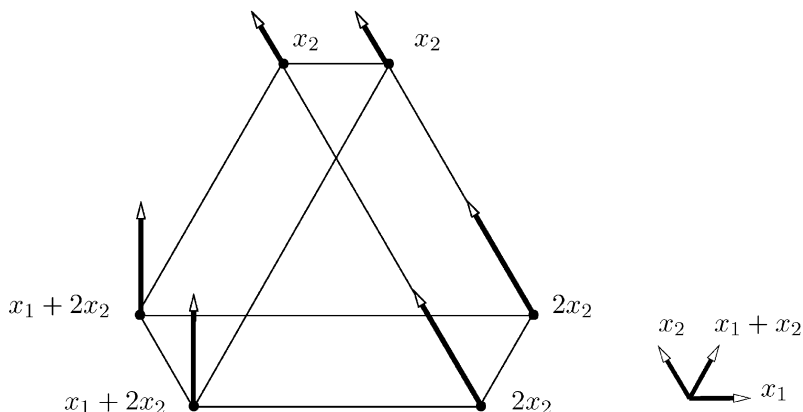
Fig. 1. This shows the graph Γ for a flag variety $SL(3, \mathbb{C})/\mathcal{B}$. The weight $\alpha_{(v,w)}$ is exactly the direction of the edge (v, w), as explained in Section 5. There is a linear polynomial attached to each vertex, also depicted as a vector. The polynomials satisfy the compatibility conditions, so this does represent an equivariant cohomology class in $H_T^2(SL(3, \mathbb{C})/\mathcal{B})$.

are homogeneous spaces $\mathcal{G}/\mathcal{P}$ for Kac–Moody groups $\mathcal{G}$, equipped with the action of a torus T. For these spaces, our theory applies when E_T^* is any complex oriented T-equivariant cohomology theory. We make explicit computations for three examples: a homogeneous space of G_2, the based loop space $\Omega SU(2)$, and a homogeneous space of $\widehat{LSL(3, \mathbb{C})}^{\mathbb{Z}/2\mathbb{Z}} \rtimes \mathbb{C}^*$.

2. The injectivity theorem for stratified spaces

Let G be a topological group and E_G^* a G-equivariant cohomology theory with commutative cup product. We consider stratified G-spaces

$$X = \bigcup_{i \geqslant 1} X_i, \quad X_1 \subseteq X_2 \subseteq X_3 \ldots, \tag{2.1}$$

where the successive quotients X_i/X_{i-1} are homeomorphic to the Thom spaces $Th(V_i)$ of some G-vector bundles $V_i \to F_i$. Moreover, we require that the above vector bundles be E-orientable (see [May96, p. 177]). In other words, X is built by successively attaching disc bundles $D(V_i)$ via equivariant attaching maps $\varphi_i : S(V_i) \to X_{i-1}$. This should be compared to the way one builds CW complexes by successively attaching discs.

We recall that an E-orientation, or Thom class, of a G-vector bundle $V \to F$ is an element $u \in E_G^*(Th(V))$. For each closed subgroup $H < G$ and point $x \in F^H$, the restriction of u to $V|_{G \cdot x}$ is a generator of the free E_H^*-module

$$E_G^*(Th(V|_{G \cdot x})) \simeq E_H^*(D(V_x), S(V_x)).$$

The Euler class $e(V)$ is the restriction of the Thom class u to the base F via the zero section map.

Remark 2.1. As with CW complexes, the stratification is often more naturally indexed by a poset I rather than $\mathbb{N}$. In that case, one should replace the expression X_i/X_{i-1} by $X_i/\bigcup_{j<i} X_j$. The poset I is required to satisfy the condition that $\{j \in I : j < i\}$ is finite for all $i \in I$, which makes the inductive proofs work. In the proofs, we ignore this fact and pretend that $I = \mathbb{N}$. The only thing that we need is that for each $i \in I$, the subspace X_i is obtained by a finite sequence of gluings, and that $X = \varinjlim X_i$.

Remark 2.2. In the examples in Sections 5 and 6, the group $G = T$ is a finite-dimensional torus, the T-spaces F_i are single points and the V_i are complex T-representations. The stratification (2.1) expresses X as a cell complex with even dimensional cells.

The main theorem of this section establishes the injectivity of the restriction map $E_G^*(X) \to E_G^*(\coprod F_i) \cong \prod E_G^*(F_i)$ when the Euler classes are not zero divisors.

Theorem 2.3. *Let X be a stratified G-space and let E_G^* be a multiplicative cohomology theory as above. Assume that the Euler classes $e(V_i) \in E_G^*(F_i)$ of the vector bundles $V_i \to F_i$ are not zero divisors. Then the inclusion $\imath : \coprod F_i \hookrightarrow X$ induces an injection*

$$\imath^* : E_G^*(X) \to \prod_i E_G^*(F_i). \tag{2.2}$$

Moreover, let $E_G^*(X)$ be given the induced filtration under the above inclusion. Then the associated graded E_G^*-module $QE_G^*(X)$ is isomorphic to (the direct product of) the ideals generated by the Euler classes in the $E_G^*(F_i)$. Explicitly,

$$QE_G^*(X) \cong \prod_i e(V_i)E_G^*(F_i). \tag{2.3}$$

Proof. We first prove the theorem when the stratification of X is finite. This is done by induction on the length of the stratification.

We first consider the assertion that (2.2) is injective. If the length of the stratification is 0, then X is empty, both sides of (2.2) are zero, and the result trivially holds. We now argue the inductive step. Assume that the stratification of X has length i (i.e. $X = X_i$) and consider the cofiber sequence

$$X_{i-1} \longrightarrow X_i \xrightarrow{p} Th(V_i). \tag{2.4}$$

It follows from the assumption on the Euler class that the long exact sequence in E-cohomology associated to (2.4) splits into short exact sequences

$$0 \longrightarrow E_G^*(Th(V_i)) \xrightarrow{p^*} E_G^*(X_i) \longrightarrow E_G^*(X_{i-1}) \longrightarrow 0. \tag{2.5}$$

To see this, we prove that p^* is an injection. Indeed, the composition

$$E^*_G(F_i) \xrightarrow[\cong]{\cdot u} E^*_G(Th(V_i)) \xrightarrow{p^*} E^*_G(X_i) \longrightarrow E^*_G(F_i)$$

is multiplication by the Euler class $e(V_i)$, and is therefore injective. The first map is the Thom isomorphism (see [May96, Theorem 9.2]), so the middle map p^* must be injective.

Now consider the map of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E^*_G(Th(V_i)) & \longrightarrow & E^*_G(X_i) & \longrightarrow & E^*_G(X_{i-1}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E^*_G(F_i) & \longrightarrow & \prod_{j \leqslant i} E^*_G(F_j) & \longrightarrow & \prod_{j < i} E^*_G(F_j) & \longrightarrow & 0.
\end{array}
\tag{2.6}
$$

The left vertical map is injective by the assumption on $e(V_i)$, with image $e(V_i)E^*_G(F_i)$. The right vertical map is injective by induction. By the Five Lemma, the central map is also injective. This proves (2.2) when the filtration of X is finite.

We now prove (2.3). Again, the base case is trivial, since both sides of (2.3) are zero when the stratification has length zero. We now argue the inductive step. The associated graded $QE^*_G(X_i)$ is isomorphic to $E^*_G(Th(V_i)) \oplus QE^*_G(X_{i-1})$. The image of $QE^*_G(X_{i-1})$ under the rightmost vertical map in (2.6) is $\prod_{j<i} e(V_j)E^*_G(F_j)$ by the induction hypothesis. So, the image of $QE^*_G(X_i)$ under the center vertical map is

$$QE^*_G(X_i) \cong e(V_i)E^*_G(F_i) \oplus \prod_{j<i} e(V_j)E^*_G(F_j) = \prod_{j \leqslant i} e(V_j)E^*_G(F_j)$$

as claimed in (2.3).

For both statements (2.2) and (2.3), the general case $X = \varinjlim X_i$ follows directly from the finite case since

$$E^*_G(X) = \varprojlim E^*_G(X_i).$$

Note that there is no Milnor $\varprojlim^1$ term here because the maps $E^*_G(X_i) \to E^*_G(X_{i-1})$ are all surjective. $\square$

3. The combinatorial description of $E_G^*(X)$

We now identify the image of $E_G^*(X)$ in $\prod E_G^*(F_i)$: it is specified by simple combinatorial restrictions. This is the content of Theorem 3.1. In order to make this computation, we must make some additional assumptions on *X*. We formalize our hypotheses on *X* below.

Assumption 1. The space *X* is equipped with a *G*-invariant stratification

$$X = \bigcup_{i \in I} X_i$$

and each successive quotient $X_i / X_{<i}$ is homeomorphic to the Thom space of a *G*-equivariant vector bundle $\pi_i : V_i \to F_i$. Here $X_{<i}$ denotes the subspace $\bigcup_{j<i} X_j \subset X_i$.

Assumption 2. The bundles $V_i \to F_i$ are *E*-orientable and admit *G*-equivariant direct sum decompositions

$$(\pi_i : V_i \to F_i) \cong \bigoplus_{j<i} (\pi_{ij} : V_{ij} \to F_i)$$

into *E*-orientable vector bundles V_{ij}. We allow the case $V_{ij} = 0$.

Assumption 3. There exist *G*-equivariant maps $f_{ij} : F_i \to F_j$ such that the attaching maps $\varphi_i : S(V_i) \to X_{i-1}$, when restricted to $S(V_{ij})$, are given by

$$\varphi_i |_{S(V_{ij})} = f_{ij} \circ \pi_{ij}.$$

Here, we identify the F_j with their images in X_{i-1}.

Assumption 4. The Euler classes $e(V_{ij})$ are not zero divisors and are pairwise relatively prime in $E_G^*(F_i)$. Namely, for any class $x \in E_G^*(F_i)$, we have that

$$(\forall j)\ e(V_{ij}) \,\Big|\, x \;\Leftrightarrow\; e(V_i) \,\Big|\, x.$$

With these assumptions, we may now formulate our main theorem.

Theorem 3.1. *Let X be a G-space satisfying Assumptions 1–4. Then the map*

$$\imath^* : E_G^*(X) \to \prod_i E_G^*(F_i)$$

is injective with image

$$R := \left\{ (x_i) \in \prod_i E_G^*(F_i) \; \middle| \; e(V_{ij}) \mid x_i - f_{ij}^*(x_j) \; \text{for all} \; j < i \right\}. \tag{3.1}$$

When $V_{ij} = 0$ in the theorem above, the relation $e(V_{ij}) \mid x_i - f_{ij}^*(x_j)$ is vacuous because $e(0) = 1$. We introduce a decorated graph Γ that carries all the information from X necessary to compute the image R of $E_G^*(X)$. Each edge of Γ corresponds to a non-vacuous relation.

Definition 3.2. The GKM graph Γ associated to X is the graph with one vertex v_i for each subspace F_i and an edge (v_i, v_j) whenever V_{ij} is non-zero. Each edge is labeled with the bundle V_{ij} and the map $f_{ij} : F_i \rightarrow F_j$.

Remark 3.3. In Sections 5 and 6, the description of Γ simplifies greatly. In those examples, all the F_i are single points, and the maps $f_{ij} : F_i \rightarrow F_j$ are the only possible ones. Moreover, the bundles V_{ij} are all 1-dimensional complex T-representations. Hence, Γ is a graph with a character $\alpha \in \Lambda := \text{Hom}(T, S^1)$ attached to each edge.

Remark 3.4. Theorem 3.1 generalizes many results found in the literature. We survey some of these results here.

(A) Suppose that X is a projective variety equipped with an algebraic action of a complex torus, with finitely many 0- and 1-dimensional orbits. Let E_G^* be ordinary T-equivariant cohomology. In this setting, Theorem 3.1 is precisely the result of Goresky, Kottwitz, and MacPherson [GKM98].
(B) Theorem 3.1 recovers the main theorem of [GH04] when X is a compact Hamiltonian T-space with possibly non-isolated fixed points, and generalizes this result to equivariant K-theory.
(C) When E_G^* is T-equivariant K-theory with complex coefficients and X is a GKM manifold, then Theorem 3.1 is identical to [KR03, Corollary A.5].
(D) If X is a Kac–Moody flag variety and E_G^* is T-equivariant K-theory, then Theorem 3.1 is closely related to a result of Kostant–Kumar [KK87]. Indeed, their Theorem 3.13 identifies $K_T^*(\mathcal{G}/\mathcal{B})$ with the subring of elements of $\prod_W K_T^*$ that are mapped to K_T^* by certain operators, which include the divided difference operators

$$(\delta_w - \delta_{w r_\alpha}) \frac{1}{1 - e^\alpha}$$

for all $w \in W$ and reflections r_α. These are exactly the same conditions as in (3.1). Their Corollary 3.20 determines $K_T^*(\mathcal{G}/\mathcal{P})$ in a similar fashion.

Before proving Theorem 3.1, we give a Lemma which computes $E_G^*(X)$ when the stratification of X has length 2.

Lemma 3.5. *Let* $Y = F_1 \cup_\varphi D(V)$ *be obtained by gluing the sphere bundle of* $\pi : V \to F_2$ *onto* F_1, *where* $\varphi = f \circ \pi$ *for a map* $f : F_2 \to F_1$. *Assume that* $e(V)$ *is not a zero divisor. Then the images of the restriction maps* $\imath^* : E_G^*(Y, F_1) \to E_G^*(F_2)$ *and* $\jmath^* : E_G^*(Y) \to E_G^*(F_1) \oplus E_G^*(F_2)$ *are*

$$\imath^*(E_G^*(Y, F_1)) = \left\{ g \in E_G^*(F_2) \mid e(V) \mid g \right\} \tag{3.2}$$

and

$$\jmath^*(E_G^*(Y)) = \left\{ (g_1, g_2) \in E_G^*(F_1) \oplus E_G^*(F_2) \mid e(V) \mid g_2 - f^*(g_1) \right\}, \tag{3.3}$$

respectively.

Proof. Clearly $E_G^*(Y, F_1) \cong E_G^*(Th(V)) \cong E_G^*(F_2)$ via the Thom isomorphism. The map

$$E_G^*(F_2) \cong E_G^*(Th(V)) \xrightarrow{\imath^*} E_G^*(F_2)$$

is multiplication by $e(V)$, so $\mathrm{Im}(\imath^*)$ is $e(V)E_G^*(F_2)$ as claimed in (3.2).

The space Y retracts onto F_1 via the map $f \circ \pi$, so the long exact sequence associated to the pair (Y, F_1) splits. Now consider the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E_G^*(Y, F_1) & \longrightarrow & E_G^*(Y) & \longrightarrow & E_G^*(F_1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \imath^*} & & \downarrow{\scriptstyle \jmath^*} & & \downarrow & & \\
0 & \longrightarrow & E_G^*(F_2) & \longrightarrow & E_G^*(F_1 \sqcup F_2) & \longrightarrow & E_G^*(F_1) & \longrightarrow & 0.
\end{array}
$$

Both rows split, and we get $\mathrm{Im}(\jmath) = E_G^*(F_1) \oplus \mathrm{Im}(\imath^*)$, where $E_G^*(F_1)$ is mapped via the diagonal inclusion $(1, f^*) : E_G^*(F_1) \to E_G^*(F_1) \oplus E_G^*(F_2)$. It is now straightforward to check that $\{(g_1, f^*(g_1))\} \oplus \{(0, g_2) : e(V) \mid g_2\}$ is the same group as described in (3.3). $\square$

We now have the technical tool to prove our main theorem.

Proof of Theorem 3.1. The map $\imath^*$ is injective by Theorem 2.3, so we must show that its image $\mathrm{Im}(\imath^*)$ equals the ring R of (3.1).

We first show that $\mathrm{Im}(\imath^*) \subseteq R$. Let Y_{ij} be the subspace of X given by

$$Y_{ij} := F_j \cup_{f_{ij} \circ \pi_{ij}} D(V_{ij}).$$

Consider a class $x \in E_G^*(X)$, and let x_i denote its restriction to F_i. Since (x_j, x_i) is the image of $x|_{Y_{ij}} \in E_G^*(Y_{ij})$ under the restriction map $E_G^*(Y_{ij}) \to E_G^*(F_j) \oplus E_G^*(F_i)$, we know by Lemma 3.5 that

$$e(V_{ij}) \mid x_i - f_{ij}^*(x_j). \tag{3.4}$$

The conditions (3.4) characterize R, so we conclude $(x_i) \in R$.

We now have a map $E_G^*(X) \to R$ and want to show that it is surjective. Following Remark 2.1, we are using $I = \mathbb{N}$. We argue by induction on the length of the stratification. If the length is zero, then $X = \emptyset$ and there is nothing to show. We now assume that $X = X_i$ and that surjectivity holds for

$$E_G^*(X_j) \to R_j := \left\{ (x_k) \in \prod_{k \leqslant j} E_G^*(F_k) \ \middle| \ e(V_{k\ell}) \mid x_k - f_{k\ell}^*(x_\ell) \text{ for all } \ell < k \right\}$$

for all $j < i$.

Let $r_i : R_i \to R_{i-1}$ be the restriction map. By Assumption 4, its kernel can be written

$$ker(r_i) = \left\{ (x_j) \in \prod_{j \leqslant i} E_G^*(F_j) \ \middle| \ \begin{array}{l} x_j = 0 \text{ for } j < i \\ e(V_{ij}) \mid x_i \text{ for all } j < i \end{array} \right\} \simeq e(V_i) E_G^*(F_i). \tag{3.5}$$

We now consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E_G^*(X_i, X_{i-1}) & \longrightarrow & E_G^*(X_i) & \longrightarrow & E_G^*(X_{i-1}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & ker(r_i) & \longrightarrow & R_i & \xrightarrow{\ r_i\ } & R_{i-1}. & &
\end{array} \tag{3.6}$$

The top sequence comes from the long exact sequence of the pair, which splits into short exact sequences as shown in the proof of Theorem 2.3. By the induction hypothesis, we know that the right vertical arrow is an isomorphism. By comparing (3.2) and (3.5), the left vertical arrow is also an isomorphism. It is now an easy diagram chase to verify that r_i is surjective and that $E_G^*(X_i) \simeq R_i$.

Finally, we note that

$$E_G^*(X) = \varprojlim E_G^*(X_i) = \varprojlim R_i = R,$$

completing the proof. $\quad\square$

4. Module generators

The second part of Theorem 2.3 gives us a lot of information about the structure of $E_G^*(X)$ as an E_G^*-module. When the spaces F_i consist of isolated fixed points, we can say more. With this assumption, (2.3) tells us that as an E_G^*-module, $E_G^*(X)$ is (non-canonically) a product of principal ideals of E_G^*:

$$E_G^*(X) \cong \prod_{v \in F} e(V_v) E_G^*,$$

where $F = \cup F_i$ and V_v is the fiber over v. Moreover, given a collection of classes $x_v \in E_G^*(X)$, one for each $v \in F$, it is very easy to check whether they form a set of free generators [1] for $E_G^*(X)$.

We write $v < w$ when $v \in F_i$, $w \in F_j$ and $i < j$. We write $v \leqslant w$ if $v < w$ or $v = w$. Let $x_v(w)$ denote $x_v|_w$. We then have:

Proposition 4.1. *Suppose X satisfies Assumptions* 1–4 *and that the spaces F_i consist of isolated fixed points. Let $x_v \in E_G^*(X)$ be classes satisfying*

$$x_v(w) = 0 \quad \text{for } w \ngeqslant v;$$

and

$$x_v(v) \text{ is a generator of the ideal } e(V_v) E_G^*. \tag{4.1}$$

Then $\{x_v\}$ is a set of free topological E_G^*-module generators.

It might happen that a space X with G-action satisfies the Assumptions 1–4 for some cohomology theory E_G^*, but that Assumption 4 fails for some closely related cohomology theory $\widetilde{E}_G^*$. For example, this can happen when $\widetilde{E}_G^*$ is non-equivariant E-cohomology $E^*(X) := E_G^*(X \times G)$, or when $E_G^* = H_G^*(-; \mathbb{Z})$ and $\widetilde{E}_G^* = H_G^*(-; \mathbb{Z}/2)$. In that case we have:

Proposition 4.2. *Suppose X satisfies Assumptions* 1–4 *for the cohomology theory E_G^*, and that the F_i consist of isolated fixed points. Let $\widetilde{E}_G^*$ be a module cohomology theory over the ring cohomology theory E_G^*. Then one can recover $\widetilde{E}_G^*(X)$ by tensoring*

$$\widetilde{E}_G^*(X) = E_G^*(X) \widehat{\otimes}_{E_G^*} \widetilde{E}_G^*.$$

Here $E_G^*(X)$ is viewed as a topological E_G^*-module and $\widehat{\otimes}$ denotes the completed tensor product.

---

[1] Here, $E_G^*(X)$ should be viewed as a topological E_G^*-module, and the word 'generator' should be interpreted in the topological sense.

In particular, if $\widetilde{E}_G^*$ is an E_G^*-algebra and $x_v \in E_G^*(X)$ satisfy (4.1), then $x_v \otimes 1$ are free $\widetilde{E}_G^*$-module generators of $\widetilde{E}_G^*(X)$.

Proof. We argue by induction on the length of the stratification.

Without loss of generality, we may assume the F_i are single points. The short exact sequence (2.5) consists of free E_G^*-modules. Therefore, the functor $- \otimes_{E_G^*} \widetilde{E}_G^*$ preserves exactness, and we get the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E_G^*(Th(V_i)) \otimes_{E_G^*} \widetilde{E}_G^* & \longrightarrow & E_G^*(X_i) \otimes_{E_G^*} \widetilde{E}_G^* & \longrightarrow & E_G^*(X_{i-1}) \otimes_{E_G^*} \widetilde{E}_G^* & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \widetilde{E}_G^*(Th(V_i)) & \xrightarrow{\ \alpha\ } & \widetilde{E}_G^*(X_i) & \xrightarrow{\ \beta\ } & \widetilde{E}_G^*(X_{i-1}). & &
\end{array}
$$

The right vertical arrow is an isomorphism by induction. The left vertical arrow is an isomorphism since

$$
E_G^*(Th(V_i)) \otimes_{E_G^*} \widetilde{E}_G^* \cong E_G^*(F_i) \otimes_{E_G^*} \widetilde{E}_G^* \cong \widetilde{E}_G^* \cong \widetilde{E}_G^*(Th(V_i)),
$$

where the first and last isomorphisms are the equivariant suspension isomorphisms.

A diagram chase shows that β is surjective, so the bottom long exact sequence splits and the map α is injective. We deduce by the Five Lemma that the middle vertical map is also an isomorphism, as desired.

Finally, if the filtration is infinite, we have

$$
\widetilde{E}_G^*(X) = \varprojlim \widetilde{E}_G^*(X_i) = \varprojlim \left(E_G^*(X_i) \otimes_{E_G^*} \widetilde{E}_G^* \right)
$$

$$
= \left(\varprojlim E_G^*(X_i) \right) \widehat{\otimes}_{E_G^*} \widetilde{E}_G^* = E_G^*(X) \widehat{\otimes}_{E_G^*} \widetilde{E}_G^*. \qquad \square
$$

Assume now that X is a CW complex with G-invariant cells, [2] that the filtration (2.1) is the usual filtration by skeleta (indexed by $\mathbb{N}$), and that

$$
E_G^*(X) = H_G^*(X) := H^*(X \times_G EG)
$$

is ordinary equivariant cohomology. In this case, we can give a canonical set of free generators for $H_G^*(X)$. As before, we let $F = \cup F_i$, where F_i is now the set of the centers of the i-dimensional cells. We write $|v| = i$ whenever $v \in F_i$ and recall the notation $x_v(w)$ for $x_v|_w$.

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[2] Careful: we do not mean that X is a G-CW complex.

Proposition 4.3. *Let X be a CW complex as above. Then there is a unique set $\{x_v\}_{v \in F}$ of free generators for the H_G^*-module $H_G^*(X)$ satisfying the conditions*

1. *each x_v is homogeneous of degree $|v|$;*
2. *if $|w| \leqslant |v|$, $w \neq v$, then $x_v(w) = 0 \in H_G^*$; and*
3. *the element $x_v(v)$ is the equivariant Euler class $e(V_v) := e(V_v \times_G EG \to BG)$ in H_G^*, where V_v is the cell of X with center v.*

Proof. We first construct the classes x_v. Assume by induction that we have classes x_w' in $H_G^*(X_{i-1})$ for $|w| < i$. To extend these to $H_G^*(X_i)$, consider the short exact sequence

$$0 \longrightarrow H_G^*(X_i, X_{i-1}) \longrightarrow H_G^*(X_i) \longrightarrow H_G^*(X_{i-1}) \longrightarrow 0$$

and note that

$$H_G^* (X_i, X_{i-1}) \cong H_G^* \left(\bigvee_{|v|=i} Th(V_v) \right) \cong \prod_{|v|=i} H_G^* (Th(V_v)).$$

The spaces $Th(V_i)$ are G-spheres, so each $H_G^*(Th(V_v))$ has a canonical generator u_v. The restriction of u_v to the center v of V_v is the equivariant Euler class $e(V_v)$. The classes x_w' of $H_G^*(X_{i-1})$ have a unique lift x_w to $H_G^*(X_i)$ because $H_G^k(X_i, X_{i-1})$ is zero for all $k < i$. It is straightforward to check that these lifts, along with the images x_v of the chosen generators u_v of $H_G^*(X_i, X_{i-1})$, satisfy the above conditions and generate $H_T^*(X_i)$. We take a limit over i to obtain the generators $x_v \in H_T^*(X)$.

We show that Conditions 1–3 characterize the generators x_v. Let $\{\widetilde{x}_v\}$ be another set of generators satisfying the same conditions. Write them as $\widetilde{x}_v = \sum_w b_{vw} x_w$. By Condition 2, we have $b_{vw} = 0$ whenever $|w| \leqslant |v|$ and $w \neq v$. By Condition 3, $b_{vv} = 1$. Finally, $b_{vw} = 0$ when $|w| > |v|$, because otherwise $\widetilde{x}_v$ would not be homogeneous. $\quad \square$

Remark 4.4. Suppose X is a manifold with a G-invariant Morse function f and a CW decomposition constructed from the Morse flow. Then the above construction is the same as the following: given a fixed point v, consider the flow-up manifold Σ_v of codimension $|v|$. By Poincaré duality, it represents a cohomology class x_v. It is straightforward to see that the x_v satisfy Conditions 1–3 of Proposition 4.3.

Remark 4.5. There are other situations when it is possible to find canonical module generators. For example, such generators exist when X is a complex algebraic variety or a symplectic manifold, and E_G^* is equivariant K-theory. The algebraic construction involves resolving the structure sheaf of the "flow-up" varieties Σ_v. See [BFM79] for details. The symplectic construction can be found in [GK03].

We illustrate these generators for some examples in Section 6.

5. Kac–Moody flag varieties

We now turn our attention to the main examples that motivate the results in this paper. These are homogeneous spaces $\mathcal{G}/\mathcal{P}$ for a (not necessarily symmetrizable) Kac–Moody group $\mathcal{G}$, defined over $\mathbb{C}$, with $\mathcal{P}$ a parabolic subgroup. Specific examples of such homogeneous spaces include finite-dimensional Grassmannians, flag manifolds, and based loop spaces ΩK of compact simply connected Lie groups K.

We first take a moment to explicitly describe ΩK as a homogeneous space $\mathcal{G}/\mathcal{P}$. Let LK be the group of polynomial loops

$$LK := \{\gamma : S^1 \to K\},$$

where the group structure is given by pointwise multiplication. By polynomial, we mean that the loop is the restriction $S^1 = \{z \in \mathbb{C} : |z| = 1\} \to K$ of an algebraic map $\mathbb{C}^* \to K_{\mathbb{C}}$. The space of based polynomial loops is defined by

$$\Omega K := \{\sigma \in LK \mid \sigma(1) = 1 \in K\}.$$

The group LK acts transitively on ΩK by

$$(\gamma \cdot \sigma)(z) = \gamma(z)\sigma(z)\gamma(1)^{-1}. \tag{5.1}$$

The stabilizer of the constant identity loop is exactly K, the subgroup of constant loops. Thus $\Omega K \cong LK/K$.

Now let $\mathcal{G}$ be the affine Kac–Moody group $\mathcal{G} = \widehat{LK_{\mathbb{C}}} \rtimes \mathbb{C}^*$. Here, $LK_{\mathbb{C}}$ is the group of algebraic maps $\mathbb{C}^* \to K_{\mathbb{C}}$, $\widehat{LK_{\mathbb{C}}}$ is the universal central extension of $LK_{\mathbb{C}}$, and the $\mathbb{C}^*$ acts on $LK_{\mathbb{C}}$ by rotating the loop. The parabolic $\mathcal{P}$ is $\widehat{L^+K_{\mathbb{C}}} \rtimes \mathbb{C}^*$, where $L^+K_{\mathbb{C}}$ is the subgroup of $LK_{\mathbb{C}}$ consisting of maps $\mathbb{C}^* \to K_{\mathbb{C}}$ that extend to maps $\mathbb{C} \to K_{\mathbb{C}}$. It is shown in [PS86, 8.3] that ΩK can be identified as a homogeneous space $\mathcal{G}/\mathcal{P}$. We briefly sketch this argument. The group LK acts on $\mathcal{G}/\mathcal{P}$ by left multiplication, and the stabilizer of the identity is $\mathcal{P} \cap LK$. This intersection is the set of polynomial maps $\mathbb{C}^* \to K_{\mathbb{C}}$ which extend over 0, and which send S^1 to K. Thus, a loop γ in $\mathcal{P} \cap LK$ satisfies the condition $\gamma(z) = \theta(\gamma(1/\bar{z}))$, where θ is the Cartan involution on $K_{\mathbb{C}}$. Therefore, since γ extends over zero, by setting $\gamma(\infty) = \theta(\gamma(0))$, it also extends over ∞. But then γ is an algebraic map from $\mathbb{P}^1$ to $K_{\mathbb{C}}$, and is therefore constant, since $K_{\mathbb{C}}$ is affine. Hence $\mathcal{P} \cap LK = K$.

Remark 5.1. We have only considered the space of polynomial loops in K. However, our results still apply to other spaces of loops, such as smooth loops, $\frac{1}{2}$-Sobolev loops, etc. Indeed, the polynomial loops are dense in these other spaces of loops [PS86, 3.5.3, Mit87]. By Palais' theorem [Pal66, Theorem 12], these dense inclusions are weak homotopy equivalences. The inclusions of T'-fixed point sets for T' a closed subgroup of T are also equivalences. So the various forms of ΩK are actually equivariantly weakly homotopy equivalent.

Let us return to the general case. Let $T_{\mathcal{G}}$ be the maximal torus of $\mathcal{G}$. The center $Z(\mathcal{G})$ acts trivially on $X = \mathcal{G}/\mathcal{P}$, so the quotient group $T := T_{\mathcal{G}}/Z(\mathcal{G})$ acts on X. We need to check that this space X with this T-action satisfy Assumptions 1–4 that are the hypotheses of Theorem 3.1. It is known (see for example [BD94,KP83,KK87,Mit87]) that $\mathcal{G}/\mathcal{P}$ admits a T-invariant CW decomposition

$$\mathcal{G}/\mathcal{P} = \coprod_{[w] \in W_{\mathcal{G}}/W_{\mathcal{P}}} \mathcal{B}\tilde{w}\mathcal{P}/\mathcal{P}, \tag{5.2}$$

where $W_{\mathcal{G}}$ and $W_{\mathcal{P}}$ are the Weyl groups of $\mathcal{G}$ and of (the semisimple part of) $\mathcal{P}$, respectively, and $\tilde{w}$ is a representative of w in $\mathcal{G}$. This is the filtration of Assumption 1. Each cell is homeomorphic to a T-representation and has a single T-fixed point $\bar{w} := \tilde{w}\mathcal{P}/\mathcal{P}$ at its center. These cells are the V_i and the fixed points are the F_i. The T-representation V_i is isomorphic to the tangent space

$$\mathrm{T}_{\bar{w}}\mathcal{B}\bar{w} = \mathrm{T}_{\bar{w}}\mathcal{B}\tilde{w}\mathcal{P}/\mathcal{P} = \mathfrak{b}/\mathfrak{b} \cap \tilde{w}\mathfrak{p}\tilde{w}^{-1} = \mathfrak{b}/\mathfrak{b} \cap w \cdot \mathfrak{p}.$$

This tangent space decomposes into 1-dimensional representations, corresponding to the roots contained in $\mathfrak{b}$ but not in $w \cdot \mathfrak{p}$. These subspaces are the V_{ij} of Assumption 2.

We now check Assumption 3. Since the F_i are points, we only need to show that the attaching map $\varphi_i : S(V_i) \to X_{i-1}$ maps each $S(V_{ij})$ onto the point F_j. In other words, we need to show that the closure of V_{ij} is a 2-sphere with north and south poles F_i and F_j. Pick a root α in $\mathfrak{b}$ but not in $w \cdot \mathfrak{p}$. Let $e_\alpha, e_{-\alpha}$ be the standard root vectors for $\alpha, -\alpha$. Let $SL(2, \mathbb{C})_\alpha$ be the subgroup of $\mathcal{G}$ with Lie algebra spanned by $e_\alpha, e_{-\alpha}$ and $[e_\alpha, e_{-\alpha}]$, and let $\mathcal{B}_\alpha$ be the Borel of $SL(2, \mathbb{C})_\alpha$ with Lie algebra spanned by e_α and $[e_\alpha, e_{-\alpha}]$. Let $\tilde{r}_\alpha := exp(\pi(e_\alpha - e_{-\alpha})/2)$ represent the element r_α of the Weyl group which is reflection along α. Let F_i be the point $\bar{w}$ and F_j the point $r_\alpha\bar{w}$. The α-eigenspace in the cell $\mathcal{B}\bar{w}$ is $\mathcal{B}_\alpha\bar{w} = V_{ij} \cong \mathbb{C}$. Its closure is $SL(2, \mathbb{C})_\alpha\bar{w} \cong \mathbb{P}^1$, and the point at infinity is given by $\tilde{r}_\alpha w\mathcal{P}/\mathcal{P} = r_\alpha\bar{w} = F_j$, as desired.

Finally, we need to check Assumption 4. To do this, we must show that for the roots contained in $\mathfrak{b}$ but not in $w \cdot \mathfrak{p}$, the corresponding Euler classes are pairwise relatively prime. This is true for a large class of T-equivariant complex oriented cohomology theories including $H_T^*(-; \mathbb{Z})$, K_T^* and MU_T^*.

Lemma 5.2. *Let E_T^* be $H_T^*(-; \mathbb{Z})$, K_T^* or MU_T^*. Let α_i be any finite set of non-zero characters such that no two are collinear. Moreover, if $E_T^* = H_T^*(-; \mathbb{Z})$, assume that no prime p divides two of the α_i. Then the corresponding Euler classes $e(\alpha_i)$ are pairwise relatively prime in E_T^*.*

Proof. The equivariant cohomology ring H_T^* is the symmetric algebra [3] $Sym^*(\Lambda)$ on the weight lattice of T. This is a unique factorization domain, and the Euler classes

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[3] This is true if one restricts the $RO(T)$-grading of [May96] to the more familiar $\mathbb{Z}$-grading. Otherwise, one has various periodicities with respect to all zero-dimensional virtual T-representations.

$e(\alpha_i) = \alpha_i$ decompose into an integer times a primitive character. The result follows immediately in this case.

The equivariant K-theory ring K_T^0 is the group ring $\mathbb{Z}[\Lambda]$ generated by symbols e^α. For each α in our set of characters, let $\bar\alpha$ be the primitive character in that direction, so $\alpha = n\bar\alpha$. The Euler classes $e(\alpha_i) = 1 - e^{\alpha_i}$ factorize as a product of cyclotomic polynomials

$$1 - e^{\alpha_i} = \prod_{d|n_i} \Phi_d(e^{\bar\alpha_i}).$$

The factors $\Phi_d(e^{\bar\alpha_i})$ are all distinct, so the result follows.

To prove the result about complex cobordism, we argue by induction on the number of characters in our set. The base case is trivial. Assume by induction that the result holds for n characters and that we are given a set $\alpha, \beta_1, \ldots, \beta_n$ of $n+1$ characters satisfying the hypotheses of the lemma. Let x be a class in MU_T^* which is divisible by each of the Euler classes of the above characters. By induction, x is divisible by the product $\prod_i e(\beta_i)$, so there exists a class b such that $b \cdot \prod_i e(\beta_i) = x$. We now consider the short exact sequence [Sin01, Theorem 1.2]

$$0 \longrightarrow MU_T^* \xrightarrow{\cdot e(\alpha)} MU_T^* \xrightarrow{res} MU_{Ker(\alpha)}^* \longrightarrow 0.$$

Since x is divisible by $e(\alpha)$,

$$res(b) \cdot \prod_i res(\beta_i) = res(x) = 0.$$

By assumption, the restrictions $\beta_i|_{Ker(\alpha)}$ are non-torsion in the group of characters of $Ker(\alpha)$. So by a result of Sinha [Sin01, Theorem 5.1] their Euler classes $e(\beta_i|_{Ker(\alpha)}) = res(e(\beta_i))$ are not zero divisors. We conclude that $res(b) = 0$. Hence b is a multiple of $e(\alpha)$, completing the proof. $\square$

Remark 5.3. It is shown in [CGK02] that any complex oriented T-equivariant co-homology theory E_T^* is an algebra over MU_T^*. Combining this with Proposition 4.2 and Lemma 5.2, we may use our main Theorem 3.1 to compute $E_T^*(\mathcal{G}/\mathcal{P}) = MU_T^*(\mathcal{G}/\mathcal{P})\widehat{\otimes}_{MU_T^*} E_T^*$.

We conclude this section with an explanation of how to obtain the pictures that we draw in Section 6. The GKM graph associated to $\mathcal{G}/\mathcal{P}$ has vertices $W_\mathcal{G}/W_\mathcal{P}$, with an edge connecting $[w]$ and $[r_\alpha w]$ for all reflections r_α in $W_\mathcal{G}$. The weight label on such an edge is α. It turns out that it is possible to embed this GKM graph in $\mathfrak{t}^*$, the dual of the Lie algebra of T. Under this embedding, the weight α_{ij} is then the primitive element of $\Lambda \subset \mathfrak{t}^*$ in the direction of the corresponding edge. To produce
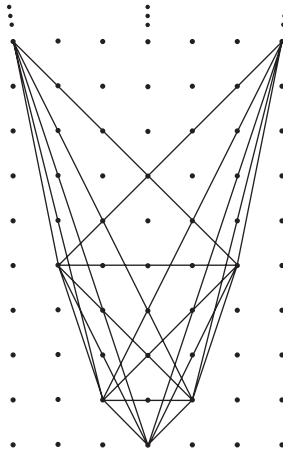
Fig. 2. This is the GKM graph embedded in $\mathfrak{t}^*$ for $\Omega SU(2)$, a homogeneous space for the loop group $LSL(2, \mathbb{C})$.

this embedding, we pick a point in $\mathfrak{t}_{\mathcal{G}}^*$ whose $W_{\mathcal{G}}$-stabilizer is exactly $W_{\mathcal{P}}$, take its $W_{\mathcal{G}}$-orbit, and draw an edge connecting any two vertices related by a reflection in $W_{\mathcal{G}}$. This graph sits in a fixed level of $\mathfrak{t}_{\mathcal{G}}^*$ (this is only relevant when $\mathcal{G}$ is of affine type) and can therefore be thought of as sitting in $\mathfrak{t}^*$.

These ideas are borrowed from the theory of moment maps in symplectic geometry. In that context, X is a symplectic manifold with T-action and admits a moment map $\mu : X \to \mathfrak{t}^*$. Consider the set $X^{(1)}$ of points with stabilizer of codimension at most 1. The GKM graph is the image of $X^{(1)}$ under the moment map μ. In our situation, $X^{(1)}$ corresponds exactly to the union of the V_{ij}. Fig. 2 shows the image of the moment map for the example $\Omega SU(2)$.

6. Examples

6.1. A homogeneous space for G_2

The complex Lie group G_2 contains two conjugacy classes of maximal parabolic subgroups. They correspond to the two simple roots of G_2. We consider the case $X = G_2/\mathcal{P}$ and its natural torus action, where $\mathcal{P} = \mathcal{P}_{\text{long}}$ is the parabolic generated by the Borel subgroup and the exponential of the negative long simple root. Equivalently, X is the quotient of the compact group G_2 by a subgroup isomorphic to $U(2)$. The GKM graph is a complete graph on 6 vertices and is embedded in $\mathfrak{t}^* \cong \mathbb{R}^2$ as a regular hexagon.

We now compute explicitly module generators x_v of $E_T^*(X)$ for a large class of cohomology theories E_T^*, following Section 4. We will represent them by their restrictions $x_v(w) := x_v|_w$ to the various T-fixed points $w \in F$. In this example, all the $x_v(w)$ happen to be Euler classes of complex T-representations. This allows us to use the following convenient notation to represent the classes x_v. On every vertex w of Γ we

draw a bouquet of arrows $\beta_j \in \Lambda$. By this, we mean that the class $x_v(w) \in E_T^*(\{w\})$ is the Euler class

$$x_v(w) = e\left(\bigoplus_j \beta_j\right) = \prod_j e(\beta_j).$$

The vertices with no arrows coming out of them carry the class 0. Using these conventions, we draw the six module generators $1, x, y, z, s, t$ of $E_T^*(G_2/\mathcal{P})$ in Fig. 3.

Recall that Assumptions 1–4 are satisfied for the cohomology theories $H_T^*(-;\mathbb{Z})$, K_T^* and MU_T^*, as shown in Section 5. To check that the elements shown in Fig. 3 are module generators, we need to check two things. First, we notice that the conditions (4.1) are satisfied. Second, we need to verify that the elements x, y, z, s, t satisfy the criteria (3.1) for being elements of $E_T^*(X)$.

To check (3.1), note that $e(\alpha) \in E_T^*$ divides $e(\beta) - e(\gamma)$ whenever $\beta - \gamma$ is a multiple of α in Λ. This is a trivial fact when E_T^* is ordinary T-equivariant cohomology or T-equivariant K-theory, and is a consequence of the theory of equivariant formal group laws when E_T^* is an arbitrary T-equivariant complex oriented cohomology theory [CGK00, p. 374]. Similarly $e(\alpha)$ divides a difference of products $\prod e(\beta_j) - \prod e(\gamma_j)$ if the $\beta_j - \gamma_j$ are all multiples of α. Now, for each of the classes in Fig. 3, and for each edge (v, w) of Γ with direction α, we note that the two bouquets of arrows $\{\beta_j\}$ at v and $\{\gamma_j\}$ at w can be ordered in such a way that the differences $\beta_j - \gamma_j$ are each in the direction of α. So, we have checked (3.1) and hence by Theorem 3.1, the classes in Fig. 3 are elements of $E_T^*(X)$. Thus, by Proposition 4.1, they are free module generators.
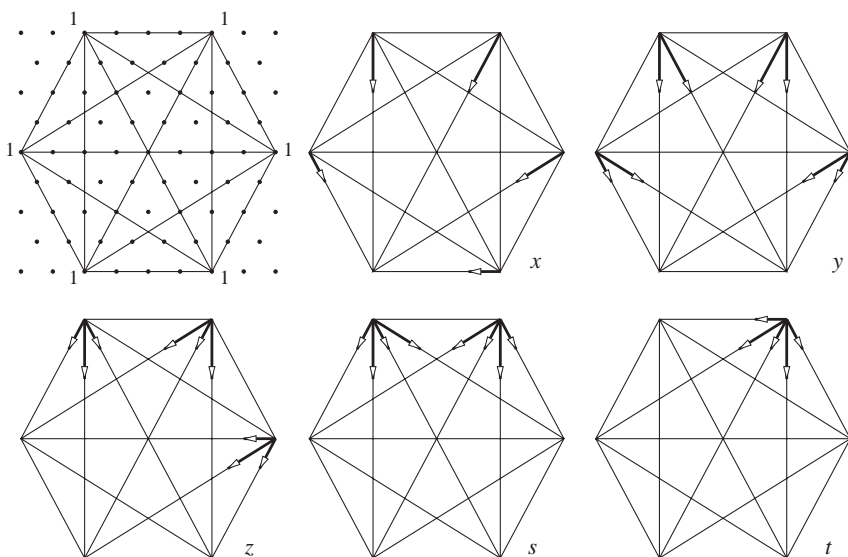

Fig. 3. The module generators for $E_T^*(G_2/\mathcal{P})$. We include the lattice Λ in the first diagram.

Even though the module generators look very similar in all cohomology theories, the ring structures are different. We compute the ordinary T-equivariant cohomology and K-theory of $X = G_2/\mathcal{P}$ to exhibit this phenomenon.

For cohomology theories $\widetilde{E}_T^*$ such as $H_T^*(-; \mathbb{Z}/2)$, $H^*(-; \mathbb{Z})$, K^*, or MU^* for which Assumption 4 fails, we still have a good understanding of $\widetilde{E}_T^*(X)$ by Proposition 4.2. We exploit this to compute $H^*(X; \mathbb{Z})$ from $H_T^*(X; \mathbb{Z})$ and $K^*(X)$ from $K_T^*(X)$ below.

For the computation of $H_T^*(X; \mathbb{Z})$, it is convenient to let $a := e(\rightarrow), b := e(\searrow)$ be the Euler classes in H_T^2 of the characters $\rightarrow, \searrow \in \Lambda$. One then has $H_T^* = \mathbb{Z}[a, b]$. Using the embedding (2.2) $H_T^*(X; \mathbb{Z}) \hookrightarrow \prod_F H_T^*$, we compute:

$$x(x + a) = y,$$

$$x(x + a)(x + b) = 2z,$$

$$x(x + a)(x + b)(x + 2a + b) = 2s,$$

and

$$x(x + a)(x + b)(x + 2a + b)(x + 2b + a) = 2t.$$

To get the non-equivariant cohomology $H^*(X; \mathbb{Z})$, it suffices by Proposition 4.2 to set $a = b = 0$:

$$x^2 = y, \quad x^3 = 2z, \quad x^4 = 2s, \quad x^5 = 2t, \quad x^6 = 0. \tag{6.1}$$

In K-theory, it is more convenient to let $a, b \in K_T^0$ be the characters $\rightarrow$ and $\searrow \in \Lambda$ themselves (not their Euler classes). We then have $K_T^0 = \mathbb{Z}[a, a^{-1}, b, b^{-1}]$, and all other K-groups are either zero or isomorphic to K^0. We use the convention that the Euler class of a line bundle L is $1 - L$. We can now compute:

$$x(ax + 1 - a) = y,$$

$$x(ax + 1 - a)(bx + 1 - b) = (1 + a^{-1})z - a^{-1}s,$$

$$x(ax + 1 - a)(bx + 1 - b)(a^2bx + 1 - a^2b) = (1 + b^{-1})s - b^{-1}t,$$

and

$$x(ax + 1 - a)(bx + 1 - b)(a^2bx + 1 - a^2b)(ab^2x + 1 - ab^2) = (1 + a^{-1}b^{-1})t.$$

To get the non-equivariant K-theory, we set $a = b = 1$ according to Proposition 4.2:

$$x^2 = y, \quad x^3 = 2z - s, \quad x^4 = 2s - t, \quad x^5 = 2t, \quad x^6 = 0. \tag{6.2}$$

We note that, as expected, the cohomology ring (6.1) of $G_2/\mathcal{P}$ is the associated graded of the K-theory ring (6.2).

6.2. Loops in $SU(2)$

We now compute explicitly the ring structure of $H_T^*(\Omega SU(2); \mathbb{Z})$ using the GKM graph $\Gamma \subset \mathfrak{t}^*$ and the module generators x_v as constructed in Section 4. In this example, as in the previous one, all the restrictions $x_v(w)$ at fixed points are elementary tensors in $H_T^*(\{w\}) \cong Sym^*(\Lambda)$. So as before, we will represent the classes x_v by drawing on every vertex w a bouquet of arrows $\beta_j \in \Lambda$ such that $x_v(w) = \prod \beta_j$. The vertices with no arrows coming out of them carry the class 0.

The first few module generators are illustrated in Fig. 4. We call x the generator of degree 2, and express the others in terms of it. The arrows in the expressions denote elements in $H_T^2 = \Lambda$.

The map $H_T^*(\Omega SU(2); \mathbb{Z}) \to H^*(\Omega SU(2); \mathbb{Z})$ is simply the map that sends the arrows to zero. So we recover the well-known fact that the ordinary cohomology $H^*(\Omega SU(2); \mathbb{Z})$ is a divided powers algebra on a class in degree 2.

Note that the classes in Fig. 4 are *not* generators for K-theory. Indeed, the conditions (3.1) are only satisfied when the classes in Fig. 4 are interpreted in cohomology, but not when they are interpreted in K-theory.

To compute the generators of $K_T^*(\Omega SU(2))$, we introduce the following notation. Let

$$p_k(\lambda_1, \ldots, \lambda_n) := (1 - \lambda_1) \cdots (1 - \lambda_n) \sum_{0 \leqslant |\alpha| < k} \lambda^\alpha,$$

where $\lambda^\alpha = \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The first such polynomial $p_1(\lambda_1, \ldots, \lambda_n)$ is exactly the Euler class $e(\bigoplus \lambda_i)$ that appeared in Section 6.1. The other ones are slightly more complicated. To best draw our K-theory classes, we introduce a pictorial
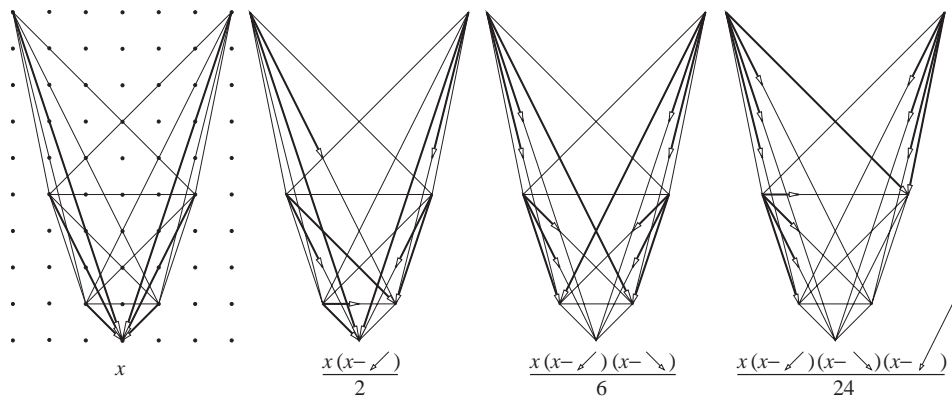

Fig. 4. The degree $2, 4, 6$, and 8 generators for $H_T^*(\Omega SU(2); \mathbb{Z})$.
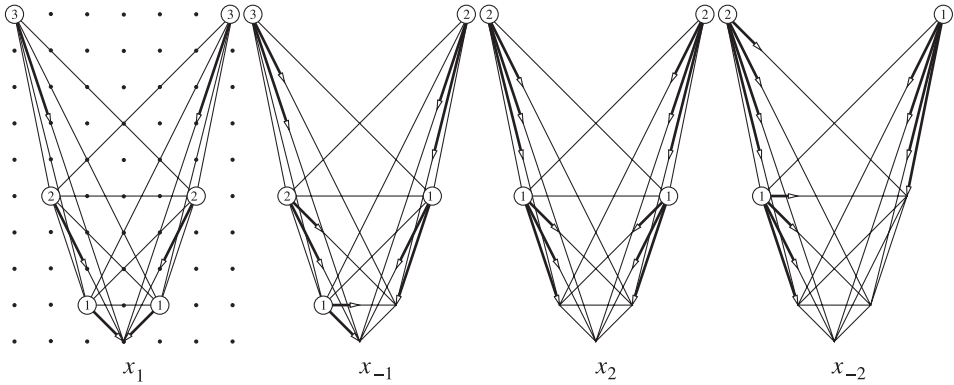
Fig. 5. The first few module-generators of $K_T^*(\Omega SU(2))$. (The class $x_0 = 1$ is omitted.)

notation for $p_k(\lambda_1, \ldots, \lambda_n)$, for $\lambda_i \in \Lambda$. We will represent them by a bouquet $\{\lambda_i\}$ of arrows, and a small number k at the vertex. We illustrate our generators using this notation in Fig. 5. To check that these elements are indeed the generators of $K_T^*(\Omega SU(2))$, we need to check (4.1), which is immediate, and that they satisfy the GKM conditions (3.1). These latter turn out to be quite hard to check.

Let $a := \longrightarrow$ and $q := \uparrow \in K_T^0$. Let us also identify the vertex set F of Γ with $\mathbb{Z}$ by taking the horizontal coordinate. The class x_i drawn in Fig. 5 is given by

$$x_i(m) = \begin{cases} p_{m-k}\left(a^{-1}q^{-m-k}, a^{-1}q^{-m-k+1}, \ldots, a^{-1}q^{-m+\ell}\right) & \text{if } m > k, \\ 0 & \text{if } -\ell \leqslant m \leqslant k, \\ p_{-m-\ell}(aq^{m-\ell}, aq^{m-\ell+1}, \ldots, aq^{m+k}) & \text{if } m < -\ell, \end{cases}$$

where $\ell = |i| - 1$ and $k = |i - \frac{1}{2}| - \frac{1}{2}$. Given an edge $(m, n) \in \Gamma$, we must check the condition given in (3.1), namely that the Euler class $1 - aq^{m+n}$ divides the difference

$$x_i(m) - x_i(n).$$

This involves several different cases. However, the problem has a few symmetries that allow us to reduce the cases to the following three.

If m is between $-\ell$ and k then $x_i(n)$ has either $(1 - aq^{m+n})$ or $(1 - a^{-1}q^{-m-n})$ as a factor and we are done.

If both m and n are bigger than k, then we must check that $1 - aq^{m+n}$ divides

$$p_{m-k}\left(a^{-1}q^{-m-k}, \ldots, a^{-1}q^{-m+\ell}\right) - p_{n-k}\left(a^{-1}q^{-n-k}, \ldots, a^{-1}q^{-n+\ell}\right). \quad (6.3)$$

This is equivalent to checking that (6.3) evaluates to 0 after setting $a^{-1} = q^{m+n}$. So we are reduced to checking that

$$p_{m-k}(q^{n-k}, \ldots, q^{n+\ell}) = p_{n-k}(q^{m-k}, \ldots, q^{m+\ell}).$$

The above formula is invariant under adding the same constant to the indices m, n and k, and subtracting it from ℓ. So by letting $k = 0$, we must prove the equivalent formula

$$p_m(q^n, \ldots, q^{n+\ell}) = p_n(q^m, \ldots, q^{m+\ell}). \tag{6.4}$$

This is the content of Lemma 6.1.

Finally, if $m > k$ and $n < -\ell$ then we are reduced to checking that

$$p_{m-k}(q^{n-k}, \ldots, q^{n+\ell}) = p_{-n-\ell}(q^{-m-\ell}, \ldots, q^{-m+k}).$$

By replacing q with q^{-1}, reversing the order of the arguments in the polynomial p, and a couple changes of indices, this also reduces to Lemma 6.1.

Lemma 6.1. *The expression*

$$a_{mn\ell} := p_m(q^n, q^{n+1}, \ldots, q^{n+\ell})$$

is symmetric in m and n.

Proof. Let $\left(\; \right)_q$ denote the quantum binomial coefficient

$$\binom{a}{b}_q = \frac{a!_q}{b!_q (a-b)!_q},$$

where $a!_q$ is the q-factorial [4] $a!_q = (1-q)(1-q^2)\ldots(1-q^a)$. We can then rewrite the expression $a_{mn\ell}$ as

$$a_{mn\ell} = (1-q^n)\cdots(1-q^{n+\ell}) \sum_{i=0}^{m-1} q^{in} \binom{\ell+i}{\ell}_q. \tag{6.5}$$

See for example [And76, Section 3.3] for more detail. In particular, (6.5) is a truncated version of Eq. (3.3.7) in [And76].

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[4] Some authors define the quantum factorial $a!_q$ to be $1(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^a)$. This agrees with our expression up to a power of $1-q$.

Now recall from [Zei93] that a "difference form"

$$\omega = f(i, j)\delta i + g(i, j)\delta j$$

has "exterior difference"

$$d\omega = [f(i, j + 1) - f(i, j)]\, \delta j\ \delta i + [g(i + 1, j) - g(i, j)]\, \delta i\ \delta j,$$

where δi and δj are anti-commuting symbols. Such a difference form can be viewed as a cellular 1-cochain on the standard square tiling of $\mathbb{R}^2$, the exterior difference being the usual cellular coboundary operator. Consider the difference form

$$\omega = q^{ij}\, \frac{(i + \ell)!_q (j + \ell)!_q}{i!_q\, j!_q\, \ell!_q} \left[(1 - q^j)\delta i + (1 - q^i)\delta j\right].$$

It is an easy exercise to verify that ω is closed. Therefore, by the discrete Stokes' theorem [Zei93],

$$\int_{\partial L} \omega = 0,$$

where L is the rectangle $[0, m] \times [0, n]$. One now checks that the above integral is zero on the sides $\{0\} \times [0, n]$ and $[0, m] \times \{0\}$, and equals $a_{nm\ell}$ and $-a_{mn\ell}$ on the remaining two sides. $\square$

Remark 6.2. We do not know whether the generators illustrated in Fig. 5 are the same as those mentioned in Remark 4.5.

6.3. A homogeneous space of type $A_1^{(4)}$

For our last example, we let $\mathcal{G}$ be the affine group associated to the Cartan matrix

$$\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}.$$

This group is $\widehat{LSL(3, \mathbb{C})}^{\mathbb{Z}/2\mathbb{Z}} \rtimes \mathbb{C}^*$, where the $\mathbb{Z}/2\mathbb{Z}$-action on $LSL(3, \mathbb{C})$ is given by precomposition with the antipodal map $z \mapsto -z$ on $\mathbb{C}^*$ and composition with the outer automorphism $A \mapsto (A^t)^{-1}$ of $SL(3, \mathbb{C})$.

We consider the homogeneous space $\mathcal{G}/\mathcal{P}$ where the parabolic $\mathcal{P}$ has Lie algebra generated by b and the negative of the simple short root. The degree 2, 4, 6, and 8 module generators for $H_T^*(\mathcal{G}/\mathcal{P}; \mathbb{Z})$ are illustrated in Fig. 6. The denominator in the degree nth module generator is given by $n!2^{\lfloor n/2 \rfloor}$.
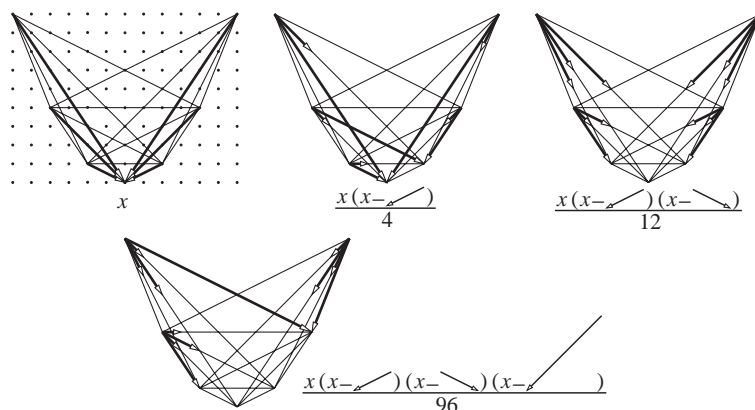
Fig. 6. The degree 2, 4, 6, and 8 generators for $H_T^*(\mathcal{G}/\mathcal{P}; \mathbb{Z})$.

Acknowledgments

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