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Note

Kneser's Conjecture, Chromatic Number, and Homotopy

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If the simplicial complex formed by the neighborhoods of points of a graph is $(k - 2)$-connected then the graph is not k-colorable. As a corollary Kneser's conjecture is proved, asserting that if all n-subsets of a $(2n - k)$-element set are divided into $k + 1$ classes, one of the classes contains two disjoint n-subsets.

1. Introduction

Kneser [6] formulated the following conjecture in 1955, whose proof is the main objective of this note.

Theorem 1. *If we split the n-subsets of a $(2n + k)$-element set into $k + 1$ classes, one of the classes will contain two disjoint n-subsets.*

It is easy to split the n-subsets into $k + 2$ classes so that the assertion does not remain valid. For let $1, \ldots, 2n + k$ be the given elements and let K_i contain those subsets whose first element is i. Then $K_1, K_2, \ldots, K_{k+1}$, $K_{k+2} \cup \cdots \cup K_{k+n+1}$ is a partition of the n-subsets into $k + 2$ classes such that any two n-subsets in the same class intersect.

Let us construct a graph $KG_{n,k}$ as follows. The vertices of $KG_{n,k}$ are the n-subsets of $\{1, \ldots, 2n + k\}$ and two of them are joined by an edge iff they are disjoint. These graphs are often called Kneser's graphs. Note that $KG_{2,1}$ is the well-known Petersen graph. Now Theorem 1 can be rephrased as follows:

Theorem 1′. *The chromatic number of Kneser's graph $KG_{n,k}$ is $k + 2$.*

This conjecture, or special cases of it, have turned out to play many roles in various fields of graph theory. In particular, the case $n \leqslant 3$ has been proved and applied by Garey and Johnson [4] and Stahl [5]. Here we mention the following: Kneser's graph has the property that each odd circuit of it has

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lengths at least $2n/k + 1$. So if we know that it has high chromatic number, we see that Kneser's graph is an example of triangle-free high-chromatic graphs (and of even more).

Erdös and Hajnal [2, 3] have constructed several other classes of graphs with similar properties, among others the following graph, which is often called Borsuk's graph.[1] Let the vertices of graph B_k be the points of the k-sphere S^k, two of them being adjacent iff their distance is at least $2 - \epsilon$ for some $\epsilon > 0$ (i.e., iff they are almost antipodal). It is easy to see that if ϵ is small, this graph contains no short odd circuits. The fact that its chromatic number is $k + 2$ is equivalent to the following well-known theorem of Borsuk:

BORSUK'S THEOREM. *If $S^k = F_1 \cup \cdots \cup F_{k+1}$, where $F_1, \ldots, F_{k+1}$ are closed subsets of S^k, then one of the sets F_i contains two antipodal points.*

We shall prove Theorem 1 by using some techniques of algebraic topology and Borsuk's theorem. In fact we shall derive a more general lower bound for the chromatic number of certain graphs. To formulate this result we need some preparation.

Let G be a graph. Define the *neighborhood complex* $\mathcal{N}(G)$ as the simplicial complex whose vertices are the vertices of G and whose simplices are those subsets of $V(G)$ which have a common neighbor. For any complex $\mathcal{K}$, let $\tilde{\mathcal{K}}$ denote the polyhedron determined by $\mathcal{K}$. A topological space T is called *n-connected* if each continuous mapping of the surface S^r of the $(r + 1)$-dimensional ball into T extends continuously to the whole ball, for $r = 0, 1, \ldots, n$.

THEOREM 2. *If $\tilde{\mathcal{N}}(G)$ is $(k + 2)$-connected then G is not k-colorable.*

COROLLARY. *$\tilde{\mathcal{N}}(G)$ is never homotopically trivial.*

In the case $k = 2$ we obtain: If the neighborhood complex of G is connected then G is not bipartite. This is trivial since the color-classes of any 2-coloration of G are components of $\mathcal{N}(G)$. For connected graphs the converse is also true if $k = 2$: If G is not bipartite then any two vertices x, y can be connected by a walk $x = x_0, x_1, \ldots, x_{2p} = y$ of even length and then $x_0, x_2, \ldots, x_{2p-2}, x_{2p}$ is a walk in $\mathcal{N}(G)$ connecting x to y, thus $\mathcal{N}(G)$ is connected.

For $k \geqslant 3$ the condition of Theorem 2 is not necessary, which is shown by any graph with large chromatic number and girth. It seems to be an interesting question whether any topological property of $\tilde{\mathcal{N}}(G)$ is equivalent to the k-colorability of G. On the other hand, could Theorem 2 be strengthened

[1] I am indepted to Miklós Simonovits for pointing out the analogy between Kneser's and Borsuk's graphs, which is the underlying idea of this paper.

by considering homology instead of homotopy, or as follows? If the $(k-2)$-dimensional homotopy group of $\mathcal{N}(G)$ is trivial, then the chromatic number of G differs from k.

The fact that Kneser's graph satisfies the conditions of Theorem 2 is not quite obvious; we shall prove the following, slightly more general result:

THEOREM 3. *Let S be a finite set and n, k natural numbers. Consider the simplicial complex $\mathscr{K}$ whose vertices are the n-subsets of S and whose simplices are those sets $A_0, ..., A_m$ of n-subsets for which*

$$\left| \bigcup_{i=0}^{m} A_i \right| \leqslant n + k.$$

Then $\mathscr{K}$ is $(k-1)$-connected.

Since for $|S| = 2n + k$ the complex $\mathscr{K}$ above is the neighborhood complex of $KG_{n,k}$, Theorems 2 and 3 together imply Theorem 1.

2. PROOF OF THEOREM 2

Let $\mathcal{N}_1(G)$ denote the barycentric subdivision of $\mathcal{N}(G)$. The vertices of $\mathcal{N}_1(G)$ are those sets $X \subset V(G)$ whose elements have a common neighbor, and some of them span a simplex iff they form a chain with respect to inclusion. It is trivial that $\tilde{\mathcal{N}}(G)$ and $\tilde{\mathcal{N}}_1(G)$ are homeomorphic.

Let $X \subset V(G)$ and denote by $\nu(X)$ the set of common neighbors of X. Then ν maps the set of vertices of $\mathcal{N}_1(G)$ into itself, and since $X \subseteq Y$ implies $\nu(X) \supseteq \nu(Y)$, it is simplicial, i.e., maps the vertices of any simplex onto vertices of a simplex of $\mathcal{N}_1(G)$. Let us extend it simplicially to a continuous mapping of $\tilde{\mathcal{N}}_1(G)$ into itself. We denote this extended mapping by $\tilde{\nu}$.

Note that

$$\nu^3 = \nu \qquad \text{and} \qquad \tilde{\nu}^3 = \tilde{\nu}. \tag{1}$$

We define mappings

$$\varphi_r: S^r \to \tilde{\mathcal{N}}_1(G) \qquad (r = 0, 1, ..., k-1)$$

by induction on r such that

$$\varphi_r(-x) = \tilde{\nu}(\varphi_r(x)) \tag{2}$$

for all $x \in S^r$ (here $-x$ is the point antipodal to x).

First let $r = 0$ and v an arbitrary point of $\tilde{\mathcal{N}}_1(G)$. Set $\varphi_0(1) = \tilde{\nu}(v)$, $\varphi_0(-1) = \tilde{\nu}^2(v)$, then we have the desired mapping of S^0 into $\tilde{\mathcal{N}}_1(G)$.

Second, let $r \geqslant 1$ and assume that $\varphi_{r-1}: S^{r-1} \to \tilde{\mathcal{N}}_1(G)$ is defined so that (2) holds. Denote by S^+ and S^- the upper and lower hemisphere of S^r, so that $S^+ \cap S^- = S^{r-1}$. Let us extend φ_{r-1} to a continuous mapping $\psi: S^+ \to \tilde{\mathcal{N}}_1(G)$. This is possible by the assumption that $\tilde{\mathcal{N}}_1(G)$ is k-connected. Define now

$$\varphi_r(x) = \begin{cases} \tilde{\nu}^2(\psi(x)) & \text{if } x \in S^+, \\ \tilde{\nu}(\psi(-x)) & \text{if } x \in S^-. \end{cases} \tag{3}$$

On $S^{r-1} = S^+ \cap S^-$ the two definitions coincide, and in fact both yield φ_{r-1}:

$$\tilde{\nu}^2(\psi(x)) = \tilde{\nu}^2(\varphi_{r-1}(x)) = \tilde{\nu}(\varphi_{r-1}(-x)) = \varphi_{r-1}(x),$$

since (2) is valid for $r - 1$. Thus (3) defines a continuous mapping of S^r into $\tilde{\mathcal{N}}_1(G)$. Moreover, if $x \in S^+$ then

$$\varphi_r(-x) = \tilde{\nu}(\psi(x)) = \tilde{\nu}^3(\psi(x)) = \tilde{\nu}(\varphi_r(x))$$

by (1), and if $x \in S^-$ then

$$\varphi_r(-x) = \tilde{\nu}^2(\psi(-x)) = \tilde{\nu}(\varphi_r(x)).$$

So (2) is inherited and the definition of φ_r is complete for all $r \leqslant k - 1$.

Suppose now that G admits a k-coloration. Let $\mathcal{N}_i$ denote the subcomplex of $\mathcal{N}(G)$ formed by those simplices whose vertices have a common neighbor of color i ($1 \leqslant i \leqslant k$). Then trivially

$$\tilde{\mathcal{N}}(G) = \tilde{\mathcal{N}}_1 \cup \cdots \cup \tilde{\mathcal{N}}_k .$$

Moreover,

$$\tilde{\mathcal{N}}_i \cap \tilde{\nu}(\tilde{\mathcal{N}}_i) = \varnothing .$$

Assume indirectly that $x \in \tilde{\mathcal{N}}_i$ and $\nu(x) \in \tilde{\mathcal{N}}_i$. Then x belongs to the simplex of $\mathcal{N}(G)$ spanned by the neighborhood of a vertex $v \in V(G)$ of color i. In the barycentric subdivision $\mathcal{N}_1(G)$, x belongs to the interior of a unique simplex spanned by vertices of $\mathcal{N}_1(G)$, i.e., subsets of $V(G)$, say $X_1, \ldots, X_m$; and we have $X_1, X_2, \ldots, X_m \subseteq \nu(v)$. Then $Y_i = \nu(X_i) \ni v$ and $\tilde{\nu}(x)$ is contained in the interior of the simplex of $\tilde{\mathcal{N}}_1(G)$ spanned by some of $Y_1, Y_2, \ldots, Y_m$. Since $\tilde{\nu}(x) \in \tilde{\mathcal{N}}_i$, it follows similarly that there is a vertex $u \in V(G)$ of color i such that some $Y_j \subseteq \nu(u)$. But then u, v are adjacent vertices of G of color i, which is a contradiction.

Now let $F_i = \varphi_{k-1}(\tilde{\mathcal{N}}_i)$. Then F_i is a closed subset of S^{k-1} and clearly $F_1 \cup \cdots \cup F_k = S^{k-1}$. Also F_i contains no two antipodal points; in fact, if $x \in F_i$ and $-x \in F_i$ then

$$\varphi_{k-1}(x) \in \tilde{\mathcal{N}}_i$$

and

$$\varphi_{k-1}(-x) = \tilde{\nu}(\varphi_{k-1}(x)) \in \tilde{\mathscr{N}}_i ,$$

which is impossible. But the existence of such sets F_i contradicts Borsuk's theorem. This completes the proof of Theorem 2.

3. Proof of Theorem 3

Let $A = \{A_0 ,..., A_m\}$ be a simplex in $\mathscr{K}$. Put

$$U(A) = \bigcup_{i=0}^{m} A_i ,$$

and denote by $M(A)$ the simplex spanned by all n-subsets of $U(A)$. We call A *crowded* if $|U(A)| < n + k$.

The proof goes by induction on $|S|$. For $|S| \leqslant n + k$ the assertion is obvious, since $\mathscr{K}$ is a simplex. So we may assume that $|S| > n + k$.

Let $\mathscr{K}'$ denote the closed subcomplex of $\mathscr{K}$ whose simplices are the crowded simplices of $\mathscr{K}$, and let $\mathscr{K}_0$ be the subcomplex whose simplices are the simplices of dimension $\leqslant k - 1$ (the $(k - 1)$-dimensional skeleton of $\mathscr{K}$).

First we show that $\tilde{\mathscr{K}}_0$ can be deformed into $\tilde{\mathscr{K}}'$ in $\tilde{\mathscr{K}}$. We do so by defining a continuous mapping ψ: $\tilde{\mathscr{K}}_0 \to \tilde{\mathscr{K}}'$ such that

(*) for each simplex A of $\mathscr{K}_0$, $\psi(\tilde{A})$ lies in $\tilde{M}(A)$.

This condition clearly implies that ψ is homotopic in $\tilde{\mathscr{K}}$ to the injection of $\tilde{\mathscr{K}}_0$ into $\tilde{\mathscr{K}}$.

We define $\psi(\tilde{A})$ by induction on the dimension of A. If dim $A = 0$ then A is automatically crowded and we may set $\psi(A) = A$.

Assume now that dim $A > 0$ and that ψ is defined on the boundary $\dot{A}$ of $\tilde{A}$ such that (*) is fulfilled. Consider the subcomplex $\tilde{\mathscr{K}}'_A$ of $\mathscr{K}'$ induced by the vertices of $M(A)$. By the induction hypothesis, $\psi(\dot{A})$ lies in $\tilde{\mathscr{K}}'_A$ and hence, by the other induction hypothesis on $|S|$, from which we know that $\tilde{\mathscr{K}}'_A$ is $(r - 1)$-connected, we know that ψ can be extended over the interior of $\tilde{A}$. This completes the definition of ψ.

Next we take two elements $u, v \in S$ and define a continuous mapping φ_{uv}: $\tilde{\mathscr{K}}' \to \tilde{\mathscr{K}}'$ as follows. For each n-subset $X \subseteq S$, let

$$\varphi_{uv}(X) = \begin{cases} X - \{u\} \cup \{v\}, & \text{if } u \in X \text{ but } v \notin X, \\ X & \text{otherwise.} \end{cases}$$

Then φ_{uv} is simplicial, i.e., if $A = \{A_0 ,..., A_m\}$ is a simplex in $\mathscr{K}'$ then so is $\varphi_{uv}(A) = \{\varphi_{uv}(A_0),..., \varphi_{uv}(A_m)\}$. In fact, if $u \notin A_0 \cup \cdots \cup A_m$ or $v \in A_0 \cup \cdots \cup$

A_m then $U(\varphi_{uv}(A)) \subseteq U(A)$; if $u \in A_0 \cup \cdots \cup A_m$ and $v \notin A_0 \cup \cdots \cup A_m$ then $U(\varphi_{uv}(A)) = U(A) - \{u\} \cup \{v\}$. In both cases

$$|U(\varphi_{uv}(A))| \leqslant |U(A)| \leqslant n + k - 1.$$

Thus φ_{uv} can be considered as a continuous mapping of $\tilde{\mathscr{K}}'$ into itself. Also observe that

(**) $\varphi_{uv}(\tilde{A}) \cup \tilde{A}$ is contained in the simplex of $\tilde{\mathscr{K}}$ spanned by the n-subsets of $U(A) \cup \{v\}$.

Therefore φ_{uv} is homotopic in $\tilde{\mathscr{K}}$ to the injection of $\tilde{\mathscr{K}}'$ into $\tilde{\mathscr{K}}$.

Consider now the mapping

$$\varphi_{u_p u_1} \varphi_{u_p u_2} \cdots \varphi_{u_p u_{p-1}} \varphi_{u_{p-1} u_1} \cdots \varphi_{u_{p-1} u_{p-2}} \cdots \varphi_{u_2 u_1}.$$

This maps each n-subset of S on $\{u_1, ..., u_n\}$. Since it is, by the remark above, homotopic to the injection of $\mathscr{K}'$ into $\mathscr{K}$, it follows that $\mathscr{K}'$ can be contracted in $\mathscr{K}$ to a single point. Since we have shown that $\mathscr{K}_0$ can be deformed into $\mathscr{K}'$, it follows that $\mathscr{K}_0$ can be contracted in $\mathscr{K}$ to a single point. This completes the proof.

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