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Parabolic Orbits in the Elliptic Restricted Three Body Problem*

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The main goal of this paper is to describe the parabolic orbits of the planar restricted elliptic three body problem. The method used is based on a standard blow up of the periodic orbits at the infinity and the perturbation of the integrable case corresponding to mass parameter equal to zero. We give an asymptotic formula for the distance between the stable and unstable manifolds of the infinity which allows us to describe how the heteroclinic orbits are created, changing the eccentricity of the primaries. Some conclusions of the quantitative study are given at the end.

1. INTRODUCTION

In many problems of celestial mechanics, the existence of oscillatory solutions such that either all the masses or some of them go far away and return infinitely many times has been shown. The basic ideas to prove the existence of that kind of motion were introduced by Alekseev [Al]. After his work, many authors contributed to related problems (Sitnikov [Si], Moser [Mo], Easton and McGehee [E.Mc], Llibre and Simó [Ll.S], Moeckel [Mk], Martínez and Simó [M.S], etc.). One important point in all that papers is the existence of some "homoclinic solution to the infinity," that is, a solution that belongs to the stable and unstable manifold of the infinity. In most of the studied problems, the infinity can be seen as a periodic orbit that, in general, corresponds to the motion of two of the bodies in a bounded orbit. So, the intersection of the invariant manifolds of the periodic orbit gives the necessary recurrence to prove the existence of oscillatory solutions. All such problems have two degrees of freedom and a first integral, so the phase space with a given energy is three-dimensional.

Easton and McGehee [E.Mc] showed that in the planar three body problem, the infinity can be seen as an invariant S^3 where all the orbits are periodic. In that case the phase space is five-dimensional. They pointed out

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that in a situation like that, the fact that the stable and unstable manifolds of S^3 intersect, can not ensure the existence of oscillatory solutions. This is due to the fact that the α- and ω-limit sets of an orbit in that intersection can be different.

In this paper we study the elliptic planar restricted three body problem. This is a nonautonomous, time periodic, two-degrees of freedom Hamiltonian problem. We consider two bodies of masses $m_1 = 1-\mu$ and $m_2 = \mu$, respectively, where $\mu \in (0, 1)$, describing an elliptic orbit with eccentricity $e \in [0, 1)$. A third body of zero mass is moving in the same plane under the gravitational effect of the positive masses. We remark that the phase space is five-dimensional as in the planar three body problem. Using a standard blow up, the periodic orbits at the infinity can be parametrized by two parameters $(\rho_\infty, \alpha_\infty)$, provided that some limit velocity is zero. Let $W^s(\rho_\infty, \alpha_\infty)$, $(W^u(\rho_\infty, \alpha_\infty))$ be the stable (unstable) manifold of the periodic orbit $(\rho_\infty, \alpha_\infty)$. Our purpose is to describe the intersections of the stable and unstable manifolds of these periodic orbits for small values of $\mu > 0$ and large values of ρ_∞. The orbits in $W^s(\rho_\infty, \alpha_\infty) \cap W^u(\hat{\rho}_\infty, \hat{\alpha}_\infty)$ for some parameters $\rho_\infty, \alpha_\infty, \hat{\rho}_\infty, \hat{\alpha}_\infty$ are heteroclinic orbits. We determine the heteroclinic orbits by the three parameters ρ_0, α_0, ω that we shall describe briefly. We consider the pericenter of the primaries on the x axes. Let us assume the third body crossing some direction $\alpha_0 \in [0, 2\pi)$ with zero radial velocity and an angular velocity related to ρ_0. The angle ω gives the position of m_1 on the ellipse.

In what follows we consider values of $\mu > 0$ and $e > 0$ small enough.

THEOREM A. *If $\rho_0 > 0$ is large enough, the parameters ρ_0, α_0 and ω corresponding to heteroclinic orbits are the zeros of some function $\mathscr{L}(\rho_0, \alpha_0, \omega, \mu) = D(\rho_0, \alpha_0, \omega) + O(\mu)$, where $D(\rho_0, \alpha_0, \omega)$ is exponentially small in ρ_0.*

The limit periodic orbits at the infinity for the heteroclinic orbit corresponding to the parameters $(\rho_0, \alpha_0, \omega)$ are

$$\rho_\infty = \rho_0 + \mu\left(-\frac{e}{\rho_0^3}\right)\left\{5e\cos 2\alpha_0 + \frac{1}{\rho_0^2}\left[\left(\frac{5}{2} + \frac{15}{8}e^2 + O(e^4)\right)\cos\alpha_0 \pm\left(\frac{15}{16} + \frac{45}{64}e^3 + O(e^5)\right)\sin\alpha_0\right] + O\left(\frac{1}{\rho_0^3}\right)\right\} + O(\mu^2),$$

$$\alpha_\infty = \alpha_0 \mp \pi + \mu\frac{1}{\rho_0^4}\left[\frac{3\pi}{4} + e^2\left(\frac{9\pi}{8} - \frac{45\pi}{16}\cos 2\alpha_0 \pm 80\sin 2\alpha_0\right) + O(e^3) + \frac{1}{\rho_0^2}\left(\frac{-15\pi e}{2}\cos\alpha_0 \pm 50e\sin\alpha_0 + O(e^3)\right) + O\left(\frac{1}{\rho_0^3}\right)\right] + O(\mu^2),$$

where the upper (lower) sign stands for the limit parameters when time tends to $+\infty$ $(-\infty)$.

Moreover, the following asymptotic expression holds for values of $r = 2e\rho_0$ *large enough*

$$D \simeq \exp\left(-\frac{\rho_0^3}{3}\right)\left(\frac{\pi\rho_0}{2}\right)^{1/2}$$
$$\times\left[\cos(\alpha_0 - \omega)\,\bar{e}\rho_0 a_1(\rho_0, \alpha_0) + \sin(\alpha_0 - \omega)\left(\frac{1}{8} + \bar{e}\rho_0 b_1(\rho_0, \alpha_0)\right)\right].$$

Where

$$a_1(\rho_0, \alpha_0) = \frac{4}{\sqrt{\pi}} r^{-5/2}\left\{3\sin\left(-\frac{5}{2}\alpha_0 \pm \frac{5\pi}{2}\right)\right.$$
$$\left. + 2r\exp(r\cos\alpha_0)\sin\left(r\sin\alpha_0 - \frac{3}{2}\alpha_0\right)\right\},$$
$$b_1(\rho_0, \alpha_0) = -1 + \frac{4}{\sqrt{\pi}} r^{-5/2}\left\{3\cos\left(-\frac{5}{2}\alpha_0 \pm \frac{5\pi}{2}\right)\right.$$
$$\left. + 2r\exp(r\cos\alpha_0)\cos\left(r\sin\alpha_0 - \frac{3}{2}\alpha_0\right)\right\},$$

where the sign $+(-)$ *is taken for* $-\pi/2 < \alpha_0 < \pi/2$ $(\pi/2 < \alpha_0 < 3\pi/2)$, *and* $\bar{e} = \exp(1)$.

THEOREM B. *Let us fix a small value of* $e > 0$ *and consider* $\rho_0 > 0$ *such that* $r = 2e\rho_0$ *is large enough. Then, except for a sequence* $\{\rho_0^{(k)}$, *k an even integer*$\}$ *of values of* ρ_0, *the function* $D(\rho_0, \alpha_0, \omega)$ *has two continuous curves of zeros* $\omega = \omega(\alpha_0)$ *and* $\omega = \omega(\alpha_0) + \pi$ *in the torus defined by* α_0 *and* ω.

Furthermore, $\{\rho_0^{(k)}\}$ *is an increasing sequence with* $\rho_0^{(k)} \to \infty$ *when* $k \to \infty$. *If k is an even integer such that* $|k|$ *is large enough, we have* $\rho_0^{(k)} \simeq (1/2e)(3\pi/4 + k\pi)$. *For these values of* ρ_0 *the winding number of the curve* $\omega = \omega(\alpha_0)$ *changes. These bifurcations take place when* $\omega = \omega(\alpha_0)$ *and* $\omega = \omega(\alpha_0) + \pi$ *have four contact points at*

$$\alpha_0 = \frac{\pi}{2} - \varepsilon, \qquad \alpha_0 = \frac{3\pi}{2} + \varepsilon, \qquad \omega \simeq \alpha_0, \qquad \omega \simeq \alpha_0 + \pi,$$

where

$$\varepsilon \simeq \arctan\left\{\frac{1}{k\pi}\ln\left[\frac{\sqrt{\pi}}{8}(k\pi)^{3/2}\right]\right\}.$$

The bifurcations imply that for a fixed value of the eccentricity $e>0$ small enough, if $\rho_0 \in (\rho_0^{(k-1)}, \rho_0^{(k)})$ for any $\omega \in [0, 2\pi)$, there exist $2k$ possible directions for the third body in order to have a heteroclinic orbit.

We remark that if $e=0$, the asymptotic formula of Theorem A does not apply. Nevertheless, from the expression of $D(\rho_0, \alpha_0, \omega)$ given in Section 7 we recover the results of [Ll.S]. In this case the initial conditions corresponding to a heteroclinic orbit show the three masses on a line. In fact, for a fixed direction, there are two heteroclinic orbits depending on the mass lying in the middle. In our notation, they correspond to $\alpha_0 = \omega$ and $\alpha_0 = \omega + \pi$ respectively. The winding number of the curve $\omega = \omega(\alpha_0)$ is now 1.

As a consequence of the theorems we have:

COROLLARY. *Let $\mu > 0$, $e > 0$ be small enough.*

For any $\alpha_\infty \in [0, 2\pi)$ and for almost all $\rho_\infty > 0$ large enough, there exists a transversal heteroclinic orbit, $\gamma \in W^s(\rho_\infty, \alpha_\infty) \cap W^u(\hat{\rho}_\infty, \hat{\alpha}_\infty)$ where

$$\hat{\alpha}_\infty = \alpha_\infty - \mu \frac{2}{\rho_\infty^4} \left[\frac{3\pi}{4} + e^2 \left(\frac{9\pi}{8} - \frac{45\pi}{16} \cos 2\alpha_\infty \right) + O(e^3) + F_1(\rho_\infty, \alpha_\infty, e) \right] + O(\mu^2),$$

$$\hat{\rho}_\infty = \rho_\infty - \mu \left\{ \frac{e}{\rho_\infty^5} \left[\left(\frac{15}{8} + e^3 \frac{45}{32} \right) \sin \alpha_\infty + O\left(\frac{1}{\rho_\infty^3} \right) \right] + F_2(\rho_\infty, \alpha_\infty, e) \right\} + O(\mu^2),$$

where F_1 is a function of order $1/\rho_\infty^2$ and F_2 contains only terms exponentially small of the type $\exp(-\rho_\infty^3/3)$.

We use the same technique as in [Ll.S] for the planar circular restricted three body problem. We perturb the integrable case corresponding to $\mu = 0$ and we carry the eccentricity, e, of the primaries as an additional parameter. This procedure allows us to obtain the asymptotic formula of Theorem A, which gives the distance between the stable and unstable manifolds of the infinity in terms of the initial conditions taken on a suitable section.

2. THE ELLIPTIC RESTRICTED THREE BODY PROBLEM

Let m_1, m_2 be the masses of the primaries normalized in such a way that $m_1 = 1 - \mu$, $m_2 = \mu$ for $\mu \in (0, 1)$. We consider primaries describing an elliptic orbit with eccentricity e and semimajor axis a. Introducing dimensionless variables, we can take $a=1$ and then the distance between the primaries can be written as

$$r=\frac{(1-e^2)}{1+e\cos f}, \tag{1}$$

where f is the true anomaly of m_1. The angular motion of the primaries is given by

$$\frac{df}{dt}=\frac{(1+e\cos f)^2}{(1-e^2)^{3/2}}, \tag{2}$$

where t is the dimensional time.

On the plane of motion we consider a third body with infinitesimal mass moving under the attraction of m_1 and m_2. In a fixed coordinate system X, Y we represent the position of the third body using complex notation $Z=X+iY$.

Let

$$z_1=\frac{\mu(1-e^2)}{1+e\cos f}$$

and

$$z_2=\frac{(1-\mu)(1-e^2)}{1+e\cos f}.$$

Then, $Z_1=z_1\exp(if)$ and $Z_2=-z_2\exp(if)$ locate the primaries.

The motion of the third body can be described by the equation [Sz]

$$\frac{d^2Z}{dt^2}=-(1-\mu)\frac{Z-Z_1}{R_1^3}-\mu\frac{Z-Z_2}{R_2^3}, \tag{3}$$

where $R_1=|Z-Z_1|$ and $R_2=|Z-Z_2|$ are the distances between the primaries and the third body.

In order to study the motion near the infinity we introduce McGehee coordinates [Mc] x, y, ρ, and α defined by

$$Z=\frac{2}{x^2}\exp(i\alpha), \qquad \frac{dZ}{dt}=\left(y+i\frac{x^2\rho}{2}\right)\exp(i\alpha), \tag{4}$$

where $i=\sqrt{-1}$.

This change brings the infinity to the origin $x=0$. Equations (3) are written as

$$\dot{x} = -\frac{1}{4}x^3 y,$$
$$\dot{y} = -\frac{1}{4}x^4 + \frac{1}{8}x^6\rho^2 + F_1,$$
$$\dot{\alpha} = \frac{x^4\rho}{4}, \tag{5}$$
$$\dot{\rho} = F_2,$$
$$\dot{f} = \frac{(1+e\cos f)^2}{(1-e^2)^{3/2}},$$

where

$$F_1 = \frac{x^4}{4}\left(1 - \left(\frac{1-\mu}{\sigma_1^3} + \frac{\mu}{\sigma_2^3}\right)\right) + \frac{x^6}{8}\cos(\alpha - f)\,\frac{(1-\mu)\,\mu(1-e^2)}{(1+e\cos f)}\left(\frac{1}{\sigma_1^3} - \frac{1}{\sigma_2^3}\right),$$
$$F_2 = \frac{\mu(1-\mu)(1-e^2)}{4(1+e\cos f)}x^4\sin(\alpha - f)\left(-\frac{1}{\sigma_1^3} + \frac{1}{\sigma_2^3}\right),$$
$$\sigma_1^2 = 1 - z_1 x^2\cos(\alpha - f) + z_1^2\frac{x^4}{4},$$
$$\sigma_2^2 = 1 + z_2 x^2\cos(\alpha - f) + z_2^2\frac{x^4}{4},$$

and the overdot stands for the derivative with respect to t.

The flow defined by (5) extends analytically to $x=0$. We define the set

$$I = \{(x, y, \alpha, \rho, f) \mid x=0, y\in\mathbb{R}, \alpha\in S^1, \rho\in\mathbb{R}, f\in S^1\},$$

which is invariant under the flow given by (5). In what follows we will call I the infinity manifold. We refer to $I_0 = I \cap \{y=0\}$, as the parabolic infinity.

We note that α and ρ remain constant on orbits of the flow restricted to I. Moreover, these orbits are periodic and can be characterized by the values of α and ρ, that we call α_∞ and ρ_∞, respectively.

The results on the planar three body problem obtained by Robinson [R] and Easton [E] can be applied to the restricted elliptic three body problem to obtain the existence of invariant manifolds for I_0. In fact, every periodic orbit given by $(\alpha_\infty, \rho_\infty)$ has two-dimensional stable $(W^s(\alpha_\infty, \rho_\infty))$

and unstable ($W^u(\alpha_\infty, \rho_\infty)$) manifolds. We define $W^s(I_0) = \bigcup (W^s(\alpha_\infty, \rho_\infty))$ where the union is taken for $\alpha_\infty \in S^1$ and $\rho_\infty \in \mathbb{R}$. In a similar way we define $W^u(I_0)$. We will refer to the orbits of $W^{s(u)}(I_0)$ as parabolic orbits for $t \to +\infty$ ($t \to -\infty$).

3. The Case $\mu = 0$

For $\mu = 0$, the elliptic restricted three body problem is equivalent to two copies of the two body problem. Equations (5) reduce to

$$\begin{aligned}
\dot{x} &= -\frac{1}{4} x^3 y, \\
\dot{y} &= -\frac{1}{4} x^4 + \frac{1}{8} x^6 \rho^2, \\
\dot{\alpha} &= \frac{x^4 \rho}{4}, \\
\dot{\rho} &= 0, \\
\dot{f} &= \frac{(1 + e\cos f)^2}{(1 - e^2)^{3/2}}.
\end{aligned} \tag{6}$$

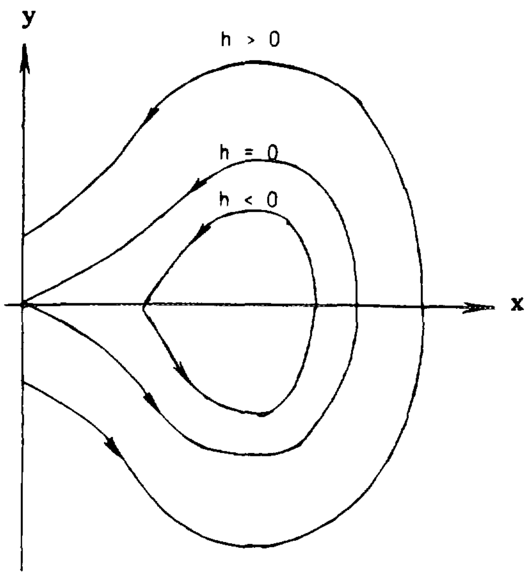

FIGURE 1

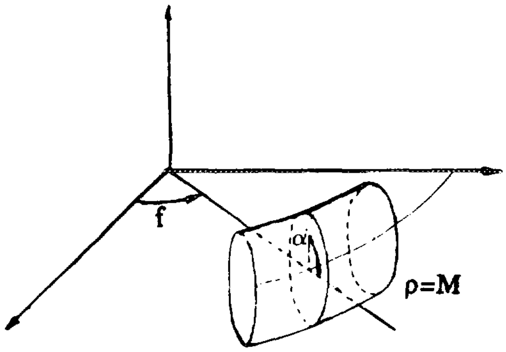

FIGURE 2

This is an integrable system with two independent integrals of motion

$$\rho = M, \qquad 2h = y^2 - x^2 + \tfrac{1}{4}x^4\rho^2, \tag{7}$$

which correspond to the angular momentum and to the energy of the third body with respect to m_1, respectively.

When $\mu = 0$, the manifold $W^s(I_0)$ coincides with $W^u(I_0)$. The projection of that manifold on the plane (x, y) can be seen as the curve $h = 0$ in Fig. 1.

The points of the curves $h > 0$ and $h < 0$ correspond to hyperbolic and elliptic orbits of the two body problem respectively.

Notice that the manifold of parabolic orbits intersects the hypersurface $y = 0$. Using the integrals M and h, the intersection of $W^s(I_0)$ (respectively $W^u(I_0)$) with $y = 0$ is given by set of points

$$T = \left\{ (x_0, y_0, \alpha_0, \rho_0, f_0) \mid x_0 = \frac{2}{|M|}, y_0 = 0, \alpha_0 \in S^1, \rho_0 = M \in \mathbb{R}, f_0 \in S^1 \right\}.$$

It is easy to see that the orbits with initial conditions $(x_0, y_0, \alpha_0, \rho_0, f_0) \in T$ tend to the periodic orbit of the parabolic infinity with $\alpha_\infty = \alpha_0 - \pi$ and $\rho_\infty = \rho_0$.

Topologically, T is a solid torus foliated by two-dimensional tori whose points correspond to orbits with a constant value of ρ (see Fig. 2).

4. THE CASE $\mu \neq 0$

If $\mu \neq 0$ is small enough, the manifolds $W^s(I_0)$ and $W^u(I_0)$ intersect the hypersurface $\{y = 0\}$ but they do not coincide.

Equations (5) are invariant under the symmetry

$$S:(x, y, \alpha, \rho, f, t) \to (x, -y, -\alpha, \rho, -f, -t).$$

Therefore $W^u(I_0) = S(W^s(I_0))$, and it is easy to see that for every periodic orbit at the parabolic infinity we have $W^u(\alpha_\infty, \rho_\infty) = S(W^s(-\alpha_\infty, \rho_\infty))$.

We are interested in computing $W^s(I_0) \cap \{y=0\}$ for μ small enough.

Introducing $z = (x, y, \alpha, \rho, f)$, system (5) can be written as

$$\dot{z} = F(z) + \mu \mathscr{T}(z) + R(z, \mu), \tag{8}$$

where $R(z, \mu)$ is a function of order two in μ.

We look for initial conditions of the type

$$z_i = (x_i, y_i, \alpha_i, \rho_i, f_i) = (x_0(\rho_0) + \mu L(\rho_0, \alpha_0, f_0, \mu), 0, \alpha_0, \rho_0, f_0),$$

where $z_0 = (x_0, y_0, \alpha_0, \rho_0, f_0) \in T$.

Let ω be the time of pericenter passage of the two primaries, then $f_0 = \omega + O(e)$ where $O(e)$ is an odd 2π-periodic function on ω of order 1 in the eccentricity e (see [B.C]). So we can consider

$$L(\rho_0, \alpha_0, f_0, \mu) = \Delta x_0(\rho_0, \alpha_0, \omega) + O(\mu).$$

Let $z(t, z_i, \mu)$ be the solution of (8) such that $z(0, z_i, \mu) = z_i$. We put $z(t, z_i, \mu) = z(t, z_0, 0) + \mu z_b(t, \Delta x_0) + O_2$, where $z_b(t, \Delta x_0)$ is the solution of the variational equations

$$\begin{aligned} &\dot{z}_b(t, \Delta x_0) = DF(z(t, z_0, 0))\, z_b(t, \Delta x_0) + \mathscr{T}(z(t, z_0, 0)), \\ &z_b(0, \Delta x_0) = (\Delta x_0, 0, 0, 0, 0)^T, \end{aligned} \tag{9}$$

and O_2, both here and in what follows, stands for $O(\mu^2)$.

If $z(t, z_i, \mu)$ tends to the parabolic infinity I_0 for $t \to \infty$, then the parameters of the limit periodic orbit are given by

$$\alpha_\infty = \alpha_0 - \pi + \mu \alpha_{b,\infty}(\rho_0, \alpha_0, \omega) + O_2,$$

$$\rho_\infty = \rho_0 + \mu \rho_{b,\infty}(\rho_0, \alpha_0, \omega) + O_2,$$

where $\alpha_{b,\infty} = \lim_{t\to\infty} \alpha_b(t, \Delta x_0)$ and $\rho_{b,\infty} = \lim_{t\to\infty} \rho_b(t, \Delta x_0)$.

The points $(x_i, y_i, \alpha_i, \rho_i, f_i) = (x_i, 0, \alpha_0, \rho_0, f_0) \in W^s(I_0) \cap W^u(I_0) \cap \{y=0\}$ are heteroclinic points. A necessary and sufficient condition to have a heteroclinic point is

$$L(\rho_0, \alpha_0, f_0, \mu) - L(\rho_0, 2\pi - \alpha_0, -f_0, \mu) = 0.$$

For small values of μ we can apply the implicit function theorem, provided some transversality condition holds (see later). So we can reduce the problem to solve

$$D(\rho_0, \alpha_0, \omega) := \Delta x_0(\rho_0, \alpha_0, \omega) - \Delta x_0(\rho_0, 2\pi - \alpha_0, -\omega) = 0.$$

If $(x_i, 0, \alpha_0, \rho_0, f_0)$ is a heteroclinic point, then the limit periodic orbit for $t \to -\infty$ is characterized by the parameters

$$\hat{\alpha}_\infty = \alpha_0 + \pi - \mu\alpha_{b,\infty}(\rho_0, 2\pi - \alpha_0, -\omega) + O_2,$$
$$\hat{\rho}_\infty = \rho_0 + \mu\rho_{b,\infty}(\rho_0, 2\pi - \alpha_0, -\omega) + O_2.$$

A heteroclinic point is called homoclinic point if $\alpha_\infty = \hat{\alpha}_\infty$ and $\rho_\infty = \hat{\rho}_\infty$. In order to have homoclinic points for small values of μ, we must be able to solve

$$\alpha_{b,\infty}(\rho_0, \alpha_0, \omega) + \alpha_{b,\infty}(\rho_0, 2\pi - \alpha_0, -\omega) = 0,$$
$$\rho_{b,\infty}(\rho_0, \alpha_0, \omega) - \rho_{b,\infty}(\rho_0, 2\pi - \alpha_0, -\omega) = 0,$$

for the zeros, $(\rho_0, \alpha_0, \omega)$, of the function $D(\rho_0, \alpha_0, \omega)$. We remark that the function D depends also on the eccentricity e. Therefore, in fact, the equations above involve four parameters. However, for any value of e, if we take $\alpha_0 = 0$ or π, $f_0 = 0$ or π and an arbitrary value of ρ_0, the initial conditions $z_i = (2/|\rho_0| + \mu L(\rho_0, \alpha_0, f_0, \mu), 0, \alpha_0, \rho_0, f_0)$ correspond to heteroclinic points. In order to study the existence of additional heteroclinic points we will determine the zeros of $D(\rho_0, \alpha_0, \omega)$. We are also interested in computing $\alpha_{b,\infty}$ and $\rho_{b,\infty}$.

5. Expansions of Δx_0, $\alpha_{b\infty}$, and $\rho_{b\infty}$

Let $(\hat{x}(t), \hat{y}(t), \hat{\alpha}(t), \rho_0, \hat{f}(t))$ be the solution $z(t, z_0, 0)$ of the two body problem. Then, Eq. (9) can be written as

$$\begin{aligned}
\dot{x}_b &= -\frac{3}{4}\hat{x}^2\hat{y}x_b - \frac{1}{4}\hat{x}^3 y_b,\\
\dot{y}_b &= \left(-\hat{x}^3 + \frac{3}{4}\hat{x}^5\rho_0^2\right)x_b + \frac{1}{4}\hat{x}^6\rho_0\rho_b + \left.\frac{\partial F_1}{\partial\mu}\right|_{\mu=0},\\
\dot{\alpha}_b &= \hat{x}^3\rho_0 x_b + \frac{1}{4}\hat{x}^4\rho_b,\\
\dot{\rho}_b &= \left.\frac{\partial F_2}{\partial\mu}\right|_{\mu=0}.
\end{aligned} \tag{10}$$

We remark that $f_b(t)=0$ if we take the initial conditions such that $f_b(0)=0$.

We introduce an independent variable τ through $dt/d\tau=(2/\hat{x}^2)\,\rho_0$. Then,

$$\hat{x}(\tau)=\frac{2}{\rho_0(1+\tau^2)^{1/2}},\qquad \hat{y}(\tau)=\frac{2\tau}{\rho_0(1+\tau^2)},\qquad \hat{\alpha}(\tau)=2\arctan\tau+\alpha_0. \tag{11}$$

Note that $\rho_0\tau$ is the eccentric anomaly of the motion of m_3 when $\mu=0$.

Let $'$ denote the derivative with respect to τ. System (10) becomes

$$\begin{aligned}
x_b' &= -\frac{3\tau}{(1+\tau^2)}x_b-\frac{1}{(1+\tau^2)^{1/2}}y_b,\\
y_b' &= \left(\frac{-4}{(1+\tau^2)^{1/2}}+\frac{12}{(1+\tau^2)^{3/2}}\right)x_b+\frac{8}{\rho_0^2(1+\tau^2)^2}\rho_b+\frac{\rho_0^2(1+\tau^2)}{2}\frac{\partial F_1}{\partial\mu}\bigg|_{\mu=0},\\
\alpha_b' &= \frac{4\rho_0}{(1+\tau^2)^{1/2}}x_b+\frac{2}{\rho_0(1+\tau^2)}\rho_b,\\
\rho_b' &= \frac{2(1-e^2)\sin(\hat{\alpha}-\hat{f})}{\rho_0(1+\tau^2)(1+e\cos\hat{f})}\left(-1+\frac{1}{\sigma_2^3}\right).
\end{aligned} \tag{12}$$

Integrating the equation for ρ_b and substituting in the equation of y_b', we can restrict ourselves to study the solutions of

$$\begin{aligned}
x_b' &= -\frac{3\tau}{(1+\tau^2)}x_b-\frac{1}{(1+\tau^2)^{1/2}}y_b,\\
y_b' &= \left(-\frac{4}{(1+\tau^2)^{1/2}}+\frac{12}{(1+\tau^2)^{3/2}}\right)x_b+b_1(\tau),
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
b_1(\tau) &= \frac{8}{\rho_0^2(1+\tau^2)^2}\rho_b(\tau)+\frac{2}{\rho_0(1+\tau^2)}\left(1-\frac{1}{\sigma_2^3}\right)\\
&\quad -\frac{4(1-e^2)\cos(\hat{\alpha}-\hat{f})}{\rho_0^3(1+e\cos\hat{f})(1+\tau^2)^2}\left(2+\frac{1}{\sigma_2^3}\right),
\end{aligned} \tag{14}$$

where, now,

$$\sigma_2^2=1+4\frac{1-e^2}{1+e\cos\hat{f}}\frac{\cos(\hat{\alpha}-\hat{f})}{\rho_0^2(1+\tau^2)}+4\frac{(1-e^2)^2}{(1+e\cos\hat{f})^2}\frac{1}{\rho_0^4(1+\tau^2)^2}.$$

We remark that $\alpha_b(\tau)$ can be obtained by integration from $x_b(\tau)$ and $\rho_b(\tau)$.

Let (v_1, w_1), (v_2, w_2) be a fundamental system of solutions of the homogeneous system associated to (13) such that $(v_1, w_1)(0) = (1, 0)$ and $(v_2, w_2)(0) = (0, 1)$. As in [L1.S] these solutions are given by

$$v_1(\tau) = -\frac{1}{5}\left[(1+\tau^2)^{1/2} + \frac{2}{(1+\tau^2)^{1/2}} + \frac{8}{(1+\tau^2)^{3/2}} - \frac{16}{(1+\tau^2)^{5/2}}\right],$$

$$w_1(\tau) = \frac{1}{5}\left[4\tau + \frac{4\tau}{(1+\tau^2)} + \frac{32\tau}{(1+\tau^2)^3}\right],$$

$$v_2(\tau) = -\frac{\tau}{(1+\tau^2)^{5/2}},$$

$$w_2(\tau) = \frac{1}{(1+\tau^2)^2}\left(\frac{2}{(1+\tau^2)} - 1\right).$$

Using the method of variation of the constants, the solutions of (13) can be written as

$$\begin{aligned} x_b(\tau) &= \beta(\tau)\, v_1(\tau) + \gamma(\tau)\, v_2(\tau), \\ y_b(\tau) &= \beta(\tau)\, w_1(\tau) + \gamma(\tau)\, w_2(\tau), \end{aligned} \tag{15}$$

where $\beta(\tau)$ and $\gamma(\tau)$ satisfy the equations

$$\begin{aligned} \beta' &= \frac{\tau}{(1+\tau^2)}\, b_1(\tau), \\ \gamma' &= -\frac{1}{5}\left[(1+\tau^2)^2 + 2(1+\tau^2) + 8 - \frac{16}{(1+\tau^2)}\right] b_1(\tau), \end{aligned} \tag{16}$$

with initial conditions $\beta(0) = \Delta x_0$, $\gamma(0) = 0$.

LEMMA 1. *If ρ_0 is large enough and $\Delta x_0 = -\int_0^\infty (s/(1+s^2))\, b_1(s)\, ds$, then the solution given by* (15) *verifies that $x_b(\tau) \to 0$ and $y_b(\tau) \to 0$ when $\tau \to \infty$.*

Proof. From (16) we get

$$\beta(\tau) = -\int_\tau^\infty \frac{s}{1+s^2}\, b_1(s)\, ds. \tag{17}$$

When μ is small enough and ρ_0 large enough, the third body does not approach the primaries, then σ_2 is bounded from below. It is also clear from (12) that $|\rho_b(\tau)|$ is bounded.

Using (14), it is easy to see that

$$|b_1(\tau)| \leqslant \frac{C}{(1+\tau^2)^2} + \frac{D}{(1+\tau^2)},$$

for some constants C and D. Therefore, (17) shows that $\tau\beta(\tau) \to 0$ when $\tau \to \infty$ and so, $\beta(\tau)\, v_1(\tau) \to 0$ and $\beta(\tau)\, w_1(\tau) \to 0$ when $\tau \to \infty$.

In a similar way we prove that $\gamma(\tau)/\tau^4 \to 0$, when $\tau \to \infty$, and then $\gamma(\tau)\, v_2(\tau)$ and $\gamma(\tau)\, w_2(\tau)$ tend to zero when $\tau \to \infty$. ∎

From (12) and (15) we get

$$\alpha_b' = \frac{4\rho_0}{(1+\tau^2)^{1/2}} [\beta(\tau)\, v_1(\tau) + \gamma(\tau)\, v_2(\tau)] + \frac{2}{\rho_0(1+\tau^2)} \rho_b(\tau).$$

By integrating by parts and taking the limit when $\tau \to \infty$, we obtain

$$\alpha_{b\infty} = \rho_0 \int_0^\infty \frac{-1+\tau^2}{(1+\tau^2)} b_1(\tau)\, d\tau + \frac{2}{\rho_0} \int_0^\infty \frac{\rho_b(\tau)}{(1+\tau^2)}\, d\tau,$$

which can be reduced to the following expression:

$$\begin{aligned} \alpha_{b\infty} = {} & -\frac{12(1-e^2)}{\rho_0^2} \int_0^\infty \frac{(-1+\tau^2)\cos(\hat{\alpha}-\hat{f})}{(1+e\cos\hat{f})(1+\tau^2)^3}\, d\tau \\ & + \int_0^\infty \left\{ \frac{2(-1+\tau^2)}{(1+\tau^2)^2} + \frac{4(1-e^2)}{\rho_0^2(1+e\cos\hat{f})} \cdot \right. \\ & \cdot \left[\frac{(-1+\tau^2)\cos(\hat{\alpha}-\hat{f})}{(1+\tau^2)^3} \right. \\ & \left.\left. - \left(\frac{\tau}{(1+\tau^2)^2} + \frac{2\tau}{(1+\tau^2)^3} \right) \sin(\hat{\alpha}-\hat{f}) \right] \left[1 - \frac{1}{\sigma_2^3} \right] \right\} d\tau. \end{aligned} \tag{18}$$

Our purpose is to compute asymptotic expansions for the integrals which appear in the expressions of Δx_0, $\alpha_{b\infty}$, and $\rho_{b\infty}$. Of course, for fixed values of e, ρ_0, α_0, and ω, we can compute numerically Δx_0, $\rho_{b,\infty}$, and $\alpha_{b,\infty}$ by integrating the corresponding functions on the real line; but, to obtain a good approximation of the behaviour of those functions for ρ_0 large becomes a hopeless numerical task, due to the exponentially small character that they display.

Following [L1.S] and [Be], we use the Gegenbauer polynomials $C_n^{(\nu)}(x)$ given by

$$(1-2xw+w^2)^{-\nu} = \sum_{n \geq 0} C_n^{(\nu)}(x)\, w^n,$$

in order to develop

$$\sigma_2^{-3} = \left(1 + \frac{4(1-e^2)}{\rho_0^2(1+e\cos\hat{f})} \frac{1}{(1+\tau^2)} \cos\hat{\theta} + \frac{4(1-e^2)^2}{\rho_0^4(1+e\cos\hat{f})^2 (1+\tau^2)^2} \right)^{-3/2},$$

where $\hat{\theta} = \hat{\alpha} - \hat{f}$. We denote by $\beta_k^{(n)}$ the coefficients of $C_n^{(3/2)}(-\cos\hat{\theta})$, that is,

$$C_n^{(3/2)}(-\cos\hat{\theta}) = \sum_{k=n(-2)0} \beta_k^{(n)} \cos(k\hat{\theta}).$$

We define

$$\mathscr{R}_{n,k}^{(m)} = \left[\frac{2(1-e^2)}{\rho_0^2}\right]^n \int_0^\infty \frac{\exp(im\hat{\theta})}{(1+e\cos\hat{f})^n\,(1+\tau^2)^{n+k}}\,d\tau := \mathscr{C}_{n,k}^{(m)} + i\mathscr{S}_{n,k}^{(m)},$$

$$\mathscr{T}_{n,k}^{(m)} = \left[\frac{2(1-e^2)}{\rho_0^2}\right]^n \int_0^\infty \frac{\tau\exp(im\hat{\theta})}{(1+e\cos\hat{f})^n\,(1+\tau^2)^{n+k}}\,d\tau := \hat{\mathscr{C}}_{n,k}^{(m)} + i\hat{\mathscr{S}}_{n,k}^{(m)}. \tag{19}$$

Let $n_1 = 1$ for $m = 1$ and $n_1 = 0$ for $m \geqslant 2$. Then, an elementary but long computation shows that

$$\Delta x_0 = \frac{1}{\rho_0}\left\{\sum_{n\geqslant 1} \zeta_n^{(0)}\hat{\mathscr{C}}_{2n,2}^{(0)} + \sum_{m\geqslant 1}\sum_{n\geqslant n_1}\left[\zeta_n^{(m)}\hat{\mathscr{C}}_{m+2n,2}^{(m)} - \gamma_n^{(m)}\mathscr{S}_{m+2n,2}^{(m)}\right]\right\}, \tag{20a}$$

$$\begin{aligned}\alpha_{b,\infty} = &-\sum_{n\geqslant 1}\zeta_n^{(0)}\left[\mathscr{C}_{2n,1}^{(0)} - 2\mathscr{C}_{2n,2}^{(0)}\right]\\ &+\sum_{m\geqslant 2}\sum_{n\geqslant n_1} -\zeta_n^{(m)}\left[\mathscr{C}_{m+2n,1}^{(1)} - 2\mathscr{C}_{m+2n,2}^{(m)}\right]\\ &+\gamma_n^{(m)}\left[\hat{\mathscr{S}}_{m+2n,1}^{(m)} + 2\hat{\mathscr{S}}_{m+2n,2}^{(m)}\right],\end{aligned} \tag{20b}$$

$$\rho_{b,\infty} = \frac{\rho_0}{2}\sum_{m\geqslant 2}\sum_{n\geqslant n_1}\gamma_n^{(m)}\mathscr{S}_{m+2n,0}^{(m)}, \tag{20c}$$

where

$$\begin{aligned}\zeta_n^{(0)} &= 2\beta_0^{(2n)} + \beta_1^{(2n-1)} = 2(2n+1)\,\Delta_0^{(2n)},\\ \zeta_n^{(1)} &= 2\beta_0^{(2n)} + \beta_2^{(2n)} + 2\beta_1^{(2n+1)}\\ \zeta_n^{(m)} &= 2\beta_m^{(m+2n)} + \beta_{m-1}^{(m+2n-1)} + \beta_{m+1}^{(m+2n-1)}\\ &= (-1)^m\,4(m+2n+1)\,\Delta_m^{(2n)}, \qquad \text{if} \quad m\geqslant 1, n\geqslant 0,\\ \gamma_n^{(1)} &= 2\beta_0^{(2n)} - \beta_2^{(2n)},\\ \gamma_n^{(m)} &= \beta_{m-1}^{(m+2n-1)} - \beta_{m+1}^{(m+2n-1)}\\ &= (-1)^{m-1}\,4m\,\Delta_m^{(2n)}, \qquad \text{if} \quad m\geqslant 1, n\geqslant 0,\end{aligned}$$

with

$$\Delta_m^{(0)} = \frac{(2m-1)!!}{(2m)!!}, \qquad \text{for} \quad m\geqslant 0,$$

$$\Delta_m^{(2n)} = \frac{(2n-1)!!}{(2n)!!}\,\frac{(2m+2n-1)!!}{(2m+2n)!!}, \qquad \text{for} \quad m\geqslant 0, n\geqslant 1.$$

Using (11) we have $\hat{\theta} = \alpha_0 + 2 \arctan \tau - \hat{f}$ and

$$\exp[im\hat{\theta}] = \exp[im\alpha_0] \exp[i2m \arctan \tau] \exp[-im\hat{f}].$$

Let $T_{2m}(x)$ and $U_{2m-1}(x)$ be the Chebyshev polynomials of first and second class with degree $2m$ and $2m-1$, respectively. We define $\bar{T}_m(x^2) = T_{2m}(x)$ and $\bar{U}_m(x^2) = xU_{2m-1}(x)$. The coefficients $\bar{t}_j^{(m)}$ and $\bar{u}_j^{(m)}$ of x^j in $\bar{T}_m(x)$ and $\bar{U}_m(x)$, respectively, are given by

$$\bar{t}_j^{(m)} = (-1)^{m-j} m 4^j \frac{(m+j-1)!}{(2j)!\,(m-j)!}, \qquad \bar{u}_j^{(m)} = \frac{j}{m} \bar{t}_j^{(m)}.$$

Using these polynomials, we have [Be]

$$\exp[i2m \arctan \tau] = \sum_{j=0}^{m} \left[\frac{\bar{t}_j^{(m)}}{(1+\tau^2)^j} + i \frac{\bar{u}_j^{(m)} \tau}{(1+\tau^2)^j} \right]. \tag{21}$$

Let l be the mean anomaly of the motion of m_1 and m_2. Then

$$l = \frac{\rho_0^3}{2} \left(\frac{\tau^3}{3} + \tau \right) + \omega. \tag{22}$$

We develop, as Fourier series in the mean anomaly l, the functions of the type $r^n \exp[im\hat{f}]$ where r is given in (1)

$$r^n \exp[im\hat{f}] = \sum_{k \in \mathbf{Z}} c_k^{n,m} \exp[ikl]. \tag{23a}$$

The coefficients $c_k^{n,m}$ depend on the eccentricity e and can be written as

$$c_k^{n,m} = \left(\frac{1 + (1-e^2)^{1/2}}{2} \right)^m \sum_{j=0}^{n-m+1} \binom{n-m+1}{j} \left(-\frac{e}{2} \right)^j \sum_{l=0}^{j} \binom{j}{l}$$
$$\cdot \sum_{s=0}^{2m} \binom{2m}{s} \left(-\frac{1-(1-e^2)^{1/2}}{e} \right)^s J_{k+2l+s-j-m}(ke), \tag{23b}$$

where $J_q(x)$ is the Bessel function of order q.

LEMMA 2. (a) *If* $e = 0$, *then*

$$c_m^{n,m} = 1, \quad \textit{for} \quad n \in \mathbb{N},$$
$$c_k^{n,m} = 0, \quad \textit{for} \quad n \in \mathbb{N} \quad \textit{and} \quad k \neq m.$$

(b) *If* $e \in (0, 1)$,

$$c_k^{n,m} = e^{|k-m|}\left(a_0 + \sum_{j \geqslant 1} a_j e^{2j}\right),$$

where $a_j = a_j(n, m, k)$ *are constants depending on the indexes* n, m, *and* k.

(c) *The following equalities hold*:

$$a_0(m, m, 1) = \frac{(-1)^{m-1}(2m+1)!}{2^{m-1}(m+2)!\,(m-1)!}\,\bar{e}\left(1 - \frac{5}{m} + O(m^{-2})\right),$$

$$a_0(m, m, -1) = \frac{(-1)^m (2m+1)!}{2^{m+1} m!\,(m+1)!}\,\bar{e}\left(1 - \frac{13}{8m} + O(m^{-2})\right).$$

Where $\bar{e}$ *is the basis of natural logarithms.*

Proof. Part (a) is obvious from (23).

The expression in (b) follows from the fact that the powers of the eccentricity e which appear in $c_k^{n,m}$ are $j + s + |k + 2l + s - j - m| + 2u$ for $u \geqslant 0$.

Some tedious but not difficult computations show that

$$a_0(m, m, 1) = \frac{(-1)^{m-1}(2m+1)!}{2^{m-1}(m+2)!\,(m-1)!} \cdot \left(1 + \frac{m-1}{m+3} + \frac{(m-1)(m-2)}{2!\,(m+3)(m+4)} + \frac{(m-1)(m-2)(m-3)}{3!\,(m+3)(m+4)(m+5)} + \cdots\right),$$

$$a_0(m, m, -1) = \frac{(-1)^m (2m+1)!}{2^{m+1} m!(m+1)!} \cdot \left(\frac{m}{2!\,(m+2)} - \frac{m(m-1)}{3!\,(m+2)(m+3)} + \frac{m(m-1)(m-2)}{4!\,(m+2)(m+3)(m+4)} + \cdots\right).$$

Taking the dominant terms of the sumes inside parenthesis we get (c). ∎

Using (21) and (23), the integrals defined in (19) can be written in terms of

$$\mathcal{I}_k(\delta) = I_k(\delta) + iJ_k(\delta) := \int_0^\infty \frac{\exp[i\delta(\tau^3/3 + \tau)]}{(1+\tau^2)^k}\,d\tau,$$
$$\mathcal{M}_k(\delta) = N_k(\delta) + iM_k(\delta) := \int_0^\infty \frac{\tau \exp[i\delta(\tau^3/3 + \tau)]}{(1+\tau^2)^k}\,d\tau, \qquad (24)$$

where $\delta = \rho_0^3/2$ and $k \geqslant 1$.

By integrating by parts, it is easy to see that the following recurrent relations are satified for $k \geqslant 1$ (see [L1.S] and [Be])

$$\begin{aligned} M_{k+2}(\delta) &= \frac{\delta}{2(k+1)} I_k(\delta), \\ I_{k+4}(\delta) &= \frac{\delta^2}{4(k+1)(k+3)} I_k(\delta) + \left(1 - \frac{1}{2(k+3)}\right) I_{k+3}(\delta), \\ N_{k+2}(\delta) &= \frac{1}{2(k+1)} (1 - \delta J_k(\delta)), \\ J_{k+4}(\delta) &= \frac{\delta^2 J_k(\delta)}{4(k+1)(k+3)} + \left(1 - \frac{1}{2(k+3)}\right) \frac{\delta J_{k+3}(\delta)}{4(k+1)(k+3)}. \end{aligned} \tag{25}$$

Then

$$\begin{aligned} \mathscr{R}^{(m)}_{n,k} &= \left(\frac{2}{\rho_0^2}\right)^n \left\{ \sum_{s \geqslant 0} c_s^{n,m} \exp[i(m\alpha_0 - s\omega)] \sum_{j=0}^{m} (\bar{t}_j^{(m)} \bar{\mathscr{I}}_{n+k+j}(s\delta) \right. \\ &\quad + i\bar{u}_j^{(m)} \bar{\mathscr{M}}_{n+k+j}(s\delta)) + \sum_{s>0} c_{-s}^{n,m} \exp[i(m\alpha_0 + s\omega)] \\ &\quad \left. \times \sum_{j=0}^{m} (\bar{t}_j^{(m)} \mathscr{I}_{n+k+j}(s\delta) + i\bar{u}_j^{(m)} \mathscr{M}_{n+k+j}(s\delta)) \right\}, \\ \mathscr{T}^{(m)}_{n,k} &= \left(\frac{2}{\rho_0^2}\right)^n \cdot \left\{ \sum_{s \geqslant 0} c_s^{n,m} \exp[i(m\alpha_0 - s\omega)] \sum_{j=0}^{m} (\bar{t}_j^{(m)} \bar{\mathscr{M}}_{n+k+j}(s\delta) \right. \\ &\quad + i\bar{u}_j^{(m)} (\bar{\mathscr{I}}_{n+k+j-1}(s\delta) - \bar{\mathscr{I}}_{n+k+j}(s\delta))) + \sum_{s>0} c_{-s}^{n,m} \exp[i(m\alpha_0 + s\omega)] \\ &\quad \left. \times \sum_{j=0}^{m} (\bar{t}_j^{(m)} \mathscr{M}_{n+k+j}(s\delta) + i\bar{u}_j^{(m)} (\mathscr{I}_{n+k+j-1}(s\delta) - \mathscr{I}_{n+k+j}(s\delta))) \right\}, \end{aligned} \tag{26}$$

where $\bar{\mathscr{M}}_n$ and $\bar{\mathscr{I}}_n$ stand for complex conjugates of $\mathscr{M}_n$ and $\mathscr{I}_n$, respectively.

From (20a) we can write Δx_0 in terms of the integrals defined in (24).

As we are interested in the difference $D(\rho_0, \alpha_0, \omega)$, it is sufficient to consider the terms in $\sin(m\alpha_0 - s\omega)$.

If $s = 0$, we get terms which are independent of ω. They can be written as

$$\frac{2}{\rho_0} \sum_{m \geqslant 1} \sum_{n \geqslant n_1} \left(\frac{2}{\rho_0^2}\right)^{m+2n} c_0^{m+2n,m} \sin m\alpha_0$$
$$\cdot \sum_{j=0}^{m} \left\{ -\zeta_n^{(m)} \bar{u}_j^{(m)} (I_{m+2n+j+1}(0) - I_{m+2n+j+2}(0)) - \gamma_m^{(m)} \bar{t}_j^{(m)} I_{m+2n+j+2}(0) \right\}.$$

We claim that for any value of m and n, the summation on j is zero. To see that, we write the sum as

$$(-1)^m 4\Delta_m^{(2n)} \left\{(m+2n+1) \sum_{j=0}^{m} \bar{u}_j^{(m)}(I_{m+2n+j+1}(0) - I_{m+2n+j+2}(0)) - m \sum_{j=0}^{m} \bar{t}_j^{(m)} I_{m+2n+j+2}(0)\right\}$$

$$= (-1)^m 4\Delta_m^{(2n)} \left\{(m+2n+1) \int_0^\infty \frac{\tau \sin(2m \arctan \tau)}{(1+\tau^2)^{m+2n+2}} d\tau - m \int_0^\infty \frac{\cos(2m \arctan \tau)}{(1+\tau^2)^{m+2n+2}} d\tau\right\} = 0.$$

The last equality is obtained integrating by parts.

From (24) and (26) we see that the terms which appear in Δx_0 coming from integrals of the type $J_k(\delta)$ and $N_k(\delta)$ are multiplied by $\cos(m\alpha_0 - s\omega)$ and so they do not contribute to $D(\rho_0, \alpha_0, \omega)$. Then we only need to consider the terms which contain the integrals $I_k(\delta)$ and $M_k(\delta)$. Using (25), we get

$$D(\rho_0, \alpha_0, \omega) = 2 \sum_{m \geq 0} T^{(m)},$$

with

$$T^{(0)} = -\sum_{s>0} \sin s\omega \sum_{n \geq 1} (c_s^{2n,0} + c_{-s}^{2n,0})\, s\, 2^{2n-1} \rho_0^{-4n+2} \Delta_0^{(2n)} I_{2n}(s\delta),$$

$$T^{(m)} = \sum_{s>0} \left\{\sin(m\alpha_0 - s\omega) \sum_{n \geq n_1} c_s^{m+2n,m} f_{m,s,n}^{-} + \sin(m\alpha_0 + s\omega) \sum_{n \geq n_1} c_{-s}^{m+2n,m} f_{m,s,n}^{+}\right\}, \tag{27}$$

where $n_1 = 1$ if $m = 1$ and $n_1 = 0$ if $m \geq 2$,

$$f_{m,s,n}^{-(+)} = (-1)^m \frac{2^{m+2n+2}}{\rho_0^{2m+4n+1}} \Delta_m^{(2n)} \sum_{j=0}^{m} \bar{t}_j^{(m)} \left\{+(-)\frac{s\delta}{2} I_{m+2n+j}(s\delta) - \frac{(m+2n+1)}{m} j I_{m+2n+j+1}(s\delta) + \frac{j(m+2n+1)+m^2}{m} I_{m+2n+j+2}(s\delta)\right\}.$$

6. The Basic Integrals

In order to get asymptotic developments for $I_k(\delta)$ and $J_k(\delta)$ for large values of $\delta > 0$ we follow the same ideas of [Ll.S] and [Be] (see also [Er]). In fact, the integrals were computed in these references but some terms were omitted. Consequently, the formula (5.23) of [Ll.S] should be corrected to $I = (1/16)(\pi C)^{1/2} \exp(-C^3/24)(1 + o(1))$, by using the right estimates of the I_k integrals, to be obtained later on in Lemma 4 of this section.

Let us define $h_k(\tau) = \exp(i\delta(\tau + \tau^3/3))/(1+\tau^2)^k$ for $k \geqslant 1$. If there is not a possibility of confusion we simply put $h(\tau)$ instead $h_k(\tau)$. Then

$$I_k(\delta) + iJ_k(\delta) = \lim_{R \to \infty} \int_0^R h(\tau)\, d\tau.$$

We consider the curve Γ of Fig. 3, for R large enough, where points C and D belong to the circle $\tau = i + \varepsilon \exp(i\phi)$ for ε small. Let $\tau = \xi + i\eta$. The curve CB is a branch of the hyperbola $3 + \xi^2 - 3\eta^2 = 0$. Over the curve the argument of exp is real and negative.

The integration of $h(\tau)$ along Γ gives

$$\int_0^\infty h(\tau)\, d\tau = -\lim_{R \to \infty} \int_R^{B(R)} h(\tau)\, d\tau + \lim_{R \to \infty} \int_C^{B(R)} h(\tau)\, d\tau$$
$$+ \int_D^C h(\tau)\, d\tau + \int_0^D h(\tau)\, d\tau.$$

It is shown in [Ll.S] that $\lim_{R \to \infty} \int_R^{B(R)} h(\tau)\, d\tau = 0$.

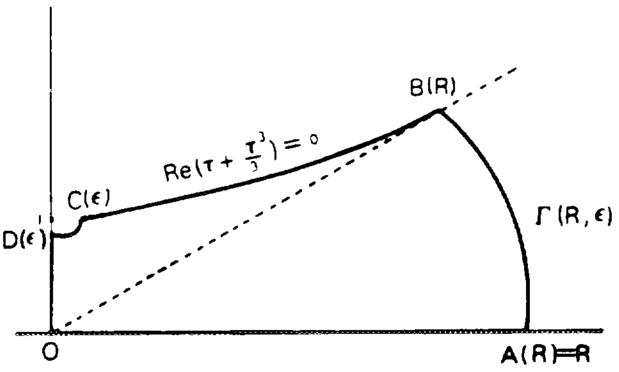

Figure 3

We note that the integral on the imaginary axis is imaginary and so, it only contributes to $J_k(\delta)$. Using the parameter $u(t) = t - (t^3/3)$ where $\tau = it$, the following development is computed in [Be]:

$$\int_0^D h(\tau)\,d\tau = i(\delta^{-1} + 2(k+1)\,\delta^{-3} + 4(k+1)(3k+10)\,\delta^{-5} + \cdots). \qquad (28)$$

In fact, the terms above give the main contribution to $J_k(\delta)$ because we will show that the integrals of $h(\tau)$ on the curves CB and DC give exponentially small terms.

For the computation of $I_k(\delta) = \mathrm{Re}\{\int_D^C h(\tau)\,d\tau + \int_C^B h(\tau)\,d\tau\}$ we will use the recurrence given in (25). If we define $I_0(\delta) = \int_0^\infty \cos(\delta(\tau + \tau^3/3))\,d\tau$, the recurrent formula can be applied for $k \geqslant 0$. Therefore, we only need to compute $I_k(\delta)$ for $k = 0, 1, 2$, and 3.

An asymptotic development of $I_0(\delta)$ for δ large is given in [Er] as

$$I_0(\delta) = \exp\left(\frac{-2\delta}{3}\right)\left\{\frac{1}{2}\left(\frac{\pi}{\delta}\right)^{1/2} + \frac{1}{2\sqrt{\delta}}\sum_{m\geqslant 1}\frac{(-1)^m\,\Gamma(3m+1/2)}{(2m)!\,(9\delta)^m}\right\}.$$

We define $B_k(\varepsilon) := \int_D^C h_k(\tau)\,d\tau$, for $k = 1, 2, 3$. For $\varepsilon > 0$ small enough we introduce a new parameter θ as $\tau = i + \varepsilon e^{i\theta}$. Then

$$B_k(\varepsilon) = \exp\left(\frac{-2\delta}{3}\right)\int_{-\pi/2}^{\theta(\varepsilon)} \frac{\exp(-\delta\varepsilon^2 e^{2i\theta} + (i\delta/3)\,\varepsilon^3 e^{3i\theta})}{(2i)^k\,\varepsilon^k e^{ik\theta}(1 + \varepsilon e^{i\theta}/2i)^k}\,\varepsilon i e^{i\theta}\,d\theta,$$

where $\theta(\varepsilon)$ is implicitely defined by $\varepsilon = 6\sin\theta(\varepsilon)/(1 - 4\sin^2\theta(\varepsilon))$, and so $\theta(\varepsilon) = \varepsilon/6 + O(\varepsilon^3)$.

After some computations we get

$$\mathrm{Re}(B_1(\varepsilon)) = \exp\left(\frac{-2\delta}{3}\right)\left[\frac{\pi}{4} + O(\varepsilon)\right],$$

$$\mathrm{Re}(B_2(\varepsilon)) = \exp\left(\frac{-2\delta}{3}\right)\left[\frac{\pi}{8} + \frac{\cos\theta(\varepsilon)}{4\varepsilon} + O(\varepsilon)\right],$$

$$\mathrm{Re}(B_3(\varepsilon)) = \exp\left(\frac{-2\delta}{3}\right)\left[\frac{\pi}{16}\left(\delta + \frac{3}{2}\right) - \frac{\sin 2\theta(\varepsilon)}{16\varepsilon^2} + \frac{3\cos\theta(\varepsilon)}{16\varepsilon} + O(\varepsilon)\right].$$

Let us define $A_k(\varepsilon) := \int_C^\infty h(\tau)\,d\tau$.

If $\tau = \xi + i\eta$, we can write

$$A_k(\varepsilon) = \exp\left(\frac{-2\delta}{3}\right)\int_{\hat\xi}^\infty \frac{\exp[-\delta\{\eta((2/3) + (8/9)\,\xi^2) - (2/3)\}]}{4^k\xi^{2k}(1 + 4\xi^2/9)^k} \times\left(\frac{2}{3}\,\xi^2 - 2i\xi\eta\right)^k\left(1 + \frac{\xi}{3\eta}\,i\right)d\xi,$$

where $\eta = (1 + \xi^2/3)^{1/2}$, and $\hat\xi = \varepsilon\cos\theta(\varepsilon)$.

Some computations show that

$$\begin{aligned}
\mathrm{Re}(A_1(\varepsilon)) &= \frac{1}{3}\exp\left(\frac{-2\delta}{3}\right)\int_{\xi}^{\infty}\frac{\exp[-\delta u]}{1+4\xi^2/9}\,d\xi,\\
\mathrm{Re}(A_2(\varepsilon)) &= -\frac{1}{4}\exp\left(\frac{-2\delta}{3}\right)\int_{\xi}^{\infty}\frac{\exp[-\delta u]}{\xi^2(1+4\xi^2/9)^2}\,d\xi,\\
\mathrm{Re}(A_3(\varepsilon)) &= -\frac{1}{8}\exp\left(\frac{-2\delta}{3}\right)\int_{\xi}^{\infty}\frac{\exp[-\delta u]}{\xi^2(1+4\xi^2/9)^2}\left(\frac{4}{3}+\frac{8}{27}\xi^2\right)d\xi,
\end{aligned} \tag{29}$$

where

$$u=\eta\left(\frac{2}{3}+\frac{8}{9}\xi^2\right)-\frac{2}{3}=(\tau-i)^2-i\frac{(\tau-i)^3}{3}.$$

In order to compute the integrals of (29) we use u as a new variable (see [Er]). The inversion theorem of Lagrange can be applied in a neighbourhood of the point $\tau-i$ to get

$$\tau-i=y\sum_{p\geqslant 0}\alpha_p(iy)^p, \tag{30}$$

where $y=u^{1/2}$. The series above is convergent if $|u^{1/2}|<2/\sqrt{3}$.

From (30), we have

$$\xi=y\sum_{n\geqslant 0}(-1)^n\alpha_{2n}y^{2n}, \tag{31}$$

$$d\xi=\frac{1}{y}\left(\sum_{n\geqslant 0}(-1)^n\frac{(2n+1)}{2}\alpha_{2n}y^{2n}\right)du,$$

where

$$\alpha_p=\frac{\Gamma((3/2)(p+1)-1)}{(p+1)!\,3^p\Gamma((p+1)/2)},\qquad \text{if}\quad p\geqslant 0.$$

Let u_0 be a real number such that $|u_0|<4/3$, and $\xi_0>0$ defined by $u(\xi_0)=u_0$. It is easy to check that if $0<\xi\leqslant\xi_0$ then $4\xi^2/9<1$ and so $(1+4\xi^2/9)^{-k}$ can be developped as a power series in ξ^2. Let $f(\xi)$ be a bounded function for $\xi>\xi_0$. Then

$$\int_{\xi_0}^{\infty}\exp[-\delta u(\xi)]\,f(\xi)\,d\xi=\exp(-\delta u_0)\int_{\xi_0}^{\infty}\exp[-\delta(u-u_0)]\,f(\xi)\,d\xi$$

If δ is large enough we can neglect terms of order $\exp(-\delta u_0)$ in front of the terms $\exp(-2\delta/3)$. Therefore we can use formal developments of the functions to be integrated in (29). The neglected terms are of order $\exp(-4\delta/3)$.

We define

$$g_r = \int_{\hat{u}}^{\infty} \exp(-\delta u)\, u^r\, du,$$

where

$$\hat{u} = -\frac{2}{3} + \left(1 + \frac{\hat{\xi}^2}{3}\right)^{1/2} \left(\frac{2}{3} + \frac{8\hat{\xi}^2}{9}\right) = O(\varepsilon^2). \tag{32}$$

The following formula is obtained integrating by parts:

$$g_{((-2q-1)/2)} = \frac{2}{2q-1}\left(\exp(-\delta\hat{u})\,\hat{u}^{(-2q+1)/2}\right) + \frac{2\delta}{-2q+1}\, g_{((-2q+1)/2)}. \tag{33}$$

It is clear that for $r \geqslant -1$, $\lim_{\varepsilon \to 0} g_r = (1/\delta^{r+1})\,\Gamma(r+1)$.

We return to the computation of the integrals of (29). Using (31), we can write

$$\left(1 + \frac{4\xi^2}{9}\right)^{-1} d\xi = \left(\frac{\alpha_0}{2} u^{-1/2} + \sum_{j \geqslant 1} \lambda_j u^{(2j-1)/2}\right) du,$$

for some coefficients λ_j.

Then

$$\begin{aligned} \operatorname{Re}(A_1(\varepsilon)) &= \frac{1}{3}\exp\left(-\frac{2\delta}{3}\right)\left[\frac{\alpha_0}{2} g_{-1/2} + \sum_{j \geqslant 1} \lambda_j g_{(2j-1)/2}\right] \\ &\xrightarrow[\varepsilon \to 0]{} \frac{1}{3}\exp\left(-\frac{2\delta}{3}\right)\left[\frac{1}{2}\left(\frac{\pi}{\delta}\right)^{1/2} + \sum_{j \geqslant 1} \frac{\lambda_j}{\delta^{(2j+1)/2}}\,\Gamma\left(\frac{2j+1}{2}\right)\right]. \end{aligned} \tag{34}$$

In a similar way, we get

$$\begin{aligned} \operatorname{Re}(A_2(\varepsilon)) = -\frac{1}{4}\exp\left(-\frac{2\delta}{3}\right)\Big[&\exp(-\delta\hat{u})\,\hat{u}^{-1/2} \\ &-\left(\delta + \frac{23}{48}\right) g_{-1/2} + \sum_{j \geqslant 1} \bar{\lambda}_j\, g_{(2j-1)/2}\Big], \end{aligned} \tag{35}$$

where (33) has been used for $q=1$, and

$$\operatorname{Re}(A_3(\varepsilon)) = -\frac{1}{8}\exp\left(-\frac{2\delta}{3}\right)\left[\frac{4}{3}\exp(-\delta\hat{u})\,\hat{u}^{-1/2} - \left(\frac{4}{3}\delta+\frac{85}{108}\right)g_{-1/2} + \sum_{j\geq 1}\bar{\bar{\lambda}}_j g_{(2j-1)/2}\right], \tag{36}$$

In (35) and (36), $\bar{\lambda}_j$, $\bar{\bar{\lambda}}_j$ are suitable coefficients.

Using (32) it is easy to see that the singular terms for $\varepsilon=0$ in (35) and (36) are cancelled by the singular terms appearing in $\operatorname{Re}(B_2(\varepsilon))$ and $\operatorname{Re}(B_3(\varepsilon))$.

Adding $\operatorname{Re}(A_k(\varepsilon))$ and $\operatorname{Re}(B_k(\varepsilon))$, we get the following lemma.

LEMMA 3. *If $\delta>0$ is sufficiently large the following expressions hold:*

$$I_1(\delta) = \exp\left(-\frac{2\delta}{3}\right)\left[\frac{\pi}{4}+\frac{1}{6}\left(\frac{\pi}{\delta}\right)^{1/2} + O(\delta^{-3/2})\right],$$

$$I_2(\delta) = \exp\left(-\frac{2\delta}{3}\right)\left[\frac{\pi}{8}+\left(\frac{\delta}{4}+\frac{23}{192}\right)\sqrt{\frac{\pi}{\delta}} + O(\delta^{-3/2})\right],$$

$$I_3(\delta) = \exp\left(-\frac{2\delta}{3}\right)\left[\frac{\pi}{16}\left(\delta+\frac{3}{2}\right)+\frac{1}{8}\left(\frac{4\delta}{3}+\frac{85}{108}\right)\left(\frac{\pi}{\delta}\right)^{1/2} + O(\delta^{-3/2})\right].$$

LEMMA 4. *For $k\geq 0$ and $\delta>0$ large enough*

$$I_k(\delta) = \exp\left(-\frac{2\delta}{3}\right)\left[\pi\delta^{[(k-1)/2]}\hat{I}_k(1+O(\delta^{-1})) + \sqrt{\pi}\,\delta^{q}\bar{I}_k(1+O(\delta^{-1}))\right], \tag{37}$$

where

$$q = \left[\frac{k}{2}\right]-\frac{1}{2},$$

$$\hat{I}_k = \begin{cases} \dfrac{\gamma_k}{\sqrt{2}} & \text{if } k \text{ is even,} \\[2ex] \dfrac{\bar{\gamma}_k}{\sqrt{2}} & \text{if } k \text{ is odd,} \end{cases} \qquad \bar{I}_k = \begin{cases} \bar{\gamma}_k & \text{if } k \text{ is even} \\ \gamma_k & \text{if } k \text{ is odd,} \end{cases} \tag{38}$$

with

$$\gamma_k = \frac{k+1}{6.2^{(k+1)/2}(k-2)!!}, \qquad \bar{\gamma}_k = \frac{1}{2^{(k+2)/2}(k-1)!!}, \qquad \text{for} \quad k\geq 1,$$

and $\hat{I}_0=0$, $\bar{I}_0=1/2$.

In (37), [ ] stands for the integer part.

Proof. The recurrence formula (25) shows that the numbers $\hat{I}_k$ satisfy

$$\hat{I}_k = \frac{1}{4(k-3)(k-1)} \hat{I}_{k-4} + \frac{2k-3}{2k-2} \hat{I}_{k-1}, \qquad \text{if } k \text{ is even,} \tag{39}$$

and

$$\hat{I}_k = \frac{1}{4(k-3)(k-1)} \hat{I}_{k-4}, \qquad \text{if } k \text{ is odd.} \tag{40}$$

The initial values $\hat{I}_1 = 1/4$, $\hat{I}_2 = 1/8$, $\hat{I}_3 = 1/16$ are given by Lemma 3.
The numbers $\tilde{I}_k$ verify (39) if k is odd, and (40) if k is even.
The relations given in (38) are proved easily by induction. ∎

LEMMA 5. *If $\delta > 0$ is large enough, for any $k > 0$ we have*

$$J_k(\delta) = \delta^{-1} + 2(k+1)\,\delta^{-3} + 4(k+1)(3k+10)\,\delta^{-5} + O(\delta^{-2}).$$

Proof. We have seen that the integrals of $h(\tau)$ on the curves DC and CB give exponentially small contributions. Therefore, the dominant terms of $\int_0^\infty h(\tau)\,d\tau$ are given in (28). ∎

7. AN ASYMPTOTIC FORMULA FOR THE FUNCTION $D(\rho_0, \alpha_0, \omega)$

Using Lemma 4, we get the following expression for the dominant terms of $T^{(m)}$, $m \geqslant 0$;

$$T^{(0)} = -\exp\left(-\frac{\rho_0^3}{3}\right) \sin\omega \sum_{n \geqslant 1} (c_1^{2n,0} + c_{-1}^{2n,0}) \left(\frac{\pi}{2}\right)^{1/2} \frac{1}{2\rho_0^{n-1/2}} \frac{(2n-1)!!}{[(2n)!!]^2}, \tag{41}$$

$$\begin{aligned} T^{(m)} = (-1)^m \sqrt{\pi} \exp\left(-\frac{\rho_0^3}{3}\right) \sum_{n \geqslant n_1} \Delta_m^{(2n)} \frac{\rho_0^{m-n+1/2}}{2^{m+1/2}} \\ \cdot \left\{ \sin(m\alpha_0 - \omega)\, c_1^{m+2n,m} \sum_{k=0}^{m} \frac{\xi_k^{(1)}(m,n)}{\rho_0^{3k/2}} \right. \\ \left. + \sin(m\alpha_0 + \omega)\, c_{-1}^{m+2n,m} \sum_{k=0}^{m} \frac{\xi_k^{(2)}(m,n)}{\rho_0^{3k/2}} \right\}, \end{aligned} \tag{42}$$

where $n_1 = 1$ if $m = 1$, and $n_1 = 0$ if $m \geqslant 2$, and the ξ coefficients are defined by

$$\xi_0^{(1,2)}(m, n) = \frac{\bar{t}_m^m(\pm 1 + 1)}{(2m + 2n - 1)!!}, \tag{43}$$

while for $1 \leqslant k \leqslant m$ we have

$$\xi_k^{(1,2)}(m, n) = \frac{\bar{t}_{m-k}^{(m)} 2^k L_k}{(2m + 2n - k + 1)!!} \left\{ \pm (2m + 2n - k + 1) + \frac{m^2(2m - 2k + 1) + (m + 2n + 1)(2m^2 + m - k)}{m(2m - 2k + 1)} \right\}, \tag{44}$$

$$L_k = \begin{cases} 1, & \text{if } k \text{ is even,} \\ \left(\dfrac{\pi}{2}\right)^{1/2}, & \text{if } k \text{ is odd.} \end{cases}$$

We note that in (41) and (42) terms of the order $\exp(-s\rho_0^3/3)$ with $s > 1$ have been neglected.

In what follows we develop an asymptotic formula which is a good approximation of $D(\rho_0, \alpha_0, \omega)$ for some values of the parameters e and ρ_0. We consider $\rho_0 > 0$ sufficiently large. Then, the dominant part of $T^{(m)}$, for $m \geqslant 2$, is given by the terms corresponding to $n = 0$. The rest can be bounded by a geometric series in ρ_0^{-1}. Due to Lemma 2, if $e = 0$, then $D(\rho_0, \alpha_0, \omega) \simeq T^{(1)}$, so the terms corresponding to $m = 1$ become important when the eccentricity tends to zero.

We are interested in determining values of ρ_0 and e such that $T^{(m)}$, for $m \geqslant 1$ is well approximated neglecting the terms with $k \geqslant 1$ in the finite sums of (42). Note that $|\xi_k^{(2)}(m, 0)| \leqslant |\xi_k^{(1)}(m, 0)|$ for all $m \geqslant 0$. Moreover $\xi_0^{(2)}(m, 0) = 0$ if $m \geqslant 0$. Therefore for our purpose we can restrict us to consider the terms with $\sin(m\alpha_0 - \omega)$, that is,

$$T^{(1)} \simeq -\exp\left(-\frac{\rho_0^3}{3}\right) \sin(\alpha_0 - \omega) \frac{(2\pi)^{1/2}}{16} \rho_0^{1/2}, \tag{45a}$$

$$\sum_{m \geqslant 2} T^{(m)} \simeq -\frac{8\bar{e}}{e} \left(\frac{\rho_0}{2}\right)^{1/2} \exp\left(-\frac{\rho_0^3}{3}\right) \cdot \sum_{m \geqslant 2} \sin(m\alpha_0 - \omega)(2e\rho_0)^m \frac{1}{2^{m+1}} \frac{(2m+1)!!\,(2m-1)!!}{(2m+4)!!\,(2m-2)!!} \cdot \sum_{k=0}^{m} \frac{\xi_k^{(1)}(m, 0)}{\rho_0^{3k/2}}, \tag{45b}$$

where the dominant terms of $c_1^{m,m}$ given in Lemma 2 have been used.

LEMMA 6. *Let δ be a positive number sufficiently small. For any integer $m_0 \geqslant 1$, if $\rho_0 \geqslant m_0 \varepsilon^{-1}$ where $\varepsilon = \varepsilon_\delta := \{(1/2)[\log(1+\delta/3)]^2\}^{1/3}$, one has*

$$\left| \sum_{k=1}^{m} \frac{\xi_k^{(1)}(m,0)}{\rho_0^{3k/2} \xi_0^{(1)}(m,0)} \right| < \delta, \qquad \text{for} \quad m \leqslant m_0. \tag{46}$$

Proof. For $k \geqslant 1$, from (44), we have

$$\xi_k^{(1)}(m,0) = \frac{(-1)^k 4^{m-k} 2^k L_k (2m-k-1)!}{k!\,(2m-2k+1)!\,(2m-k+1)!!} B_{m,k},$$

where

$$\begin{aligned} |B_{m,k}| &= |m(2m-k+1)(2m-2k+1) \\ &\quad + m^2(2m-2k+1) + (m+1)(2m^2+m-k)| \\ &\leqslant 8m^3 + 3m^2 + m. \end{aligned} \tag{47}$$

If k is even then $(2m-1)!!/(2m-k+1)!! \leqslant (2m)^{k/2-1}$, and for k odd, $(2m-1)!!/(2m-k+1)!! \leqslant ((2m)^{(k-1)}/((\pi/2)m))^{1/2}$. Using $(2m-k-1)!/(2m-2k+1)! \leqslant (2m)^{k-1}/(2m-k)$, we have

$$\left| \frac{\xi_k^{(1)}(m,0)}{\rho_0^{3k/2} \xi_0^{(1)}(m,0)} \right| \leqslant \frac{B_{m,k} 2^{k/2-2} m^{3k/2}}{(2m-k)\, m^2 \rho_0^{3k/2} k!}.$$

From (47), we get $|B_{m,k}/(m^2(2m-k))| \leqslant 12$, for all $m \geqslant 1$. Then

$$\begin{aligned} \left| \sum_{k=1}^{m} \frac{\xi_k^{(1)}(m,0)}{\rho_0^{3k/2} \xi_0^{(1)}(m,0)} \right| &\leqslant 3 \sum_{k=1}^{\infty} \left(\frac{2m^3}{\rho_0^3} \right)^{k/2} \frac{1}{k!} \\ &= 3 \left[\exp\left(\left(\frac{2m^3}{\rho_0^3} \right)^{1/2} \right) - 1 \right] \\ &\leqslant 3[\exp((2\varepsilon^3)^{1/2}) - 1], \end{aligned}$$

where the last equality holds if $\rho_0 \geqslant m\varepsilon^{-1}$. ∎

LEMMA 7. *Let ρ_0 be a positive number large enough. Then*

$$\left| \sum_{k=1}^{m} \frac{\xi_k^{(1)}(m,0)}{\rho_0^{3k/2}} \right| \leqslant 6 \left(\frac{\pi}{2} \right)^{1/2} \frac{2^{2m} m}{(m-2)!!}. \tag{48}$$

Proof. Using $(2m-k+1)!! \geqslant m^{(m-k+2)/2}(m-2)!!$, for $1 \leqslant k \leqslant m$, and (47) we get from (45) that

$$\left| \sum_{k=1}^{m} \frac{\xi_k^{(1)}(m,0)}{\rho_0^{3k/2}} \right| \leqslant 6 \left(\frac{\pi}{2} \right)^{1/2} \frac{2^{2m}}{m^{m/2}(m-2)!!} \sum_{k=1}^{m} \frac{m^{3k/2}}{\rho_0^{3k/2} k!}. \tag{49}$$

If $m \leqslant \rho_0$, we have

$$\sum_{k=1}^{m} \frac{m^{3k/2}}{\rho_0^{3k/2} k!} \leqslant m \left(\frac{m}{\rho_0}\right)^{3/2} < m$$

and (48) holds in that case.

We consider now $m > \rho_0$.

We define $\gamma_k := (m/\rho_0)^{3k/2}(1/k!)$. Let η be a natural number $1 \leqslant \eta \leqslant m$ such that $\gamma_\eta \geqslant \gamma_k$, for all $1 \leqslant k \leqslant m$. Then

$$\sum_{k=1}^{m} \gamma_k \leqslant m \left(\frac{m}{\rho_0}\right)^{3\eta/2} \frac{1}{\eta!}. \tag{50}$$

It is easy to see that $\eta \geqslant (m/\rho_0)^{3/2}$.

Let us assume $\eta \leqslant m$. Using $m^{(\eta - m)/2} < 1$ and the Stirling formula, we get from (49) and (50)

$$\left| \sum_{k=1}^{m} \frac{\xi_k^{(1)}(m, 0)}{\rho_0^{3k/2}} \right| \leqslant 3 \frac{2^{2m}}{(m-2)!!} m \left(\frac{\bar{e}}{\sqrt{\rho_0}}\right)^{\eta} < 3.2^{2m} \frac{m}{(m-2)!!},$$

where the last inequality holds for $\rho_0 > \bar{e}^2$. We recall that here $\bar{e} = \exp(1)$. Then, (48) follows because $1 < (2\pi)^{1/2}$.

In the case that $\eta > m$, it is obvious that

$$\sum_{k=1}^{m} \gamma_k \leqslant m \left(\frac{m}{\rho_0}\right)^{3m/2} \frac{1}{m!}.$$

The Stirling formula gives

$$\left| \sum_{k=1}^{m} \frac{\xi_k^{(1)}(m, 0)}{\rho_0^{3k/2}} \right| \leqslant 3 \frac{m^{1/2}}{(m-2)!!} \left(\frac{4\bar{e}}{\rho_0^{3/2}}\right)^{m},$$

and we get (48) by taking ρ_0 sufficiently large. ▮

LEMMA 8. *Let δ be a positive number small enough and $\varepsilon < \varepsilon_\delta$ as defined in Lemma 6. For a fixed value $r > 0$, let $m_0(r) > 0$ be an integer number such that*

$$6 \left(\frac{\pi}{2}\right)^{1/2} \sum_{m \geqslant m_0(r)} r^m \beta_m \frac{2^{2m} m}{(m-2)!!} < \varepsilon,$$

where

$$\beta_m = \frac{1}{2^{m+1}} \frac{(2m+1)!!\,(2m-1)!!}{(2m+4)!!\,(2m-2)!!}. \tag{51}$$

Then, for any real value $\rho_0 > m_0(r)/\varepsilon$, we have

$$\sum_{m\geq 2} r^m\beta_m \sin(m\alpha_0-\omega)\sum_{k=0}^{m}\frac{\xi_k^{(1)}(m,0)}{\rho_0^{3k/2}} = \sum_{m\geq 2} r^m\beta_m \sin(m\alpha_0-\omega)\,\xi_0^{(1)}(m,0)(1+\eta_m)+\gamma, \tag{52}$$

where $|\eta_m|<\delta$ for all $m\geq 2$, and $|\gamma|\leqslant\varepsilon$.

Proof. We define

$$\eta_m = \begin{cases} 0, & \text{if } m\geq m_0(r),\\ \displaystyle\sum_{k=1}^{m}\frac{\xi_k^{(1)}(m,0)}{\rho_0^{3k/2}\xi_0^{(1)}(m,0)}, & \text{if } m<m_0(r). \end{cases}$$

The left-hand side of (52) can be written as

$$\sum_{m\geq 2}\sin(m\alpha_0-\omega)\,r^m\beta_m\xi_0^{(1)}(1+\eta_m)+\sum_{m\geq m_0(r)}\sin(m\alpha_0-\omega)\,r^m\beta_m\sum_{k=1}^{m}\frac{\xi_k^{(1)}}{\rho_0^{3k/2}}.$$

Using Lemma 7, the second summation is less than ε. Otherwise, for a fixed value of r, $m_0(r)$ is a fixed integer and then Lemma 6 can be applied to obtain $|\eta_m|<\delta$. ▮

It is easy to see that the maximum term of the series $\sum_{m\geq 2} r^m\beta_m\xi_0^{(1)}$ is obtained for some m in the interval determined by $[r-1]$ and $[r-1]+1$, where [ ] means the integer part.

We note that, if r and ρ_0 are in the hypotheses of Lemma 8, for some δ and ε sufficiently small, and if the eccentricity is defined by $r/(2\rho_0)$, then we can approximate $\sum_{m\geq 1} T^{(m)}$ by taking the terms with $k=0$ in (45b); that is,

$$\begin{aligned}
&-\exp\left(-\frac{\rho_0^3}{3}\right)(2\pi\rho_0)^{1/2}\left\{\frac{1}{16}\sin(\alpha_0-\omega)+8\bar{e}\rho_0\right.\\
&\qquad\left.\times\sum_{m\geq 1}\sin((m+1)\,\alpha_0-\omega)\frac{r^m}{m!}\frac{(2m+3)!!}{(2m+6)!!}\right\}\\
&\quad=-\exp\left(-\frac{\rho_0^3}{3}\right)(\pi\rho_0)^{1/2}\,\mathrm{Im}\left\{\exp(i(\alpha_0-\omega)\right.\\
&\qquad\left.\times\left[\frac{\sqrt{2}}{16}+\bar{e}\rho_0\frac{16}{\sqrt{2}}\sum_{m\geq 1}\frac{z^m}{m!}\frac{(2m+3)!!}{(2m+6)!!}\right]\right\}\\
&\quad=-\exp\left(-\frac{\rho_0^3}{3}\right)\left(\frac{\pi\rho_0}{2}\right)^{1/2}\mathrm{Im}\{\tilde{D}(e,\rho_0,\alpha_0,\omega)\},
\end{aligned}$$

where $z=r\exp(i\alpha_0)$.

Let $M(5/2, 4; z)$ be the confluent hypergeometric function defined by (see [A.S])

$$M\left(\frac{5}{2}, 4; z\right) = 16 \sum_{m \geq 0} \frac{z^m}{m!} \frac{(2m+3)!!}{(2m+6)!!}.$$

It is easy to see that

$$\operatorname{Im}\{\tilde{D}(e, \rho_0, \alpha_0, \omega)\} = \operatorname{Im}\left\{\exp(i(\alpha_0 - \omega))\left[\frac{1}{8} + \bar{e}\rho_0\left(M\left(\frac{5}{2}, 4; z\right) - 1\right)\right]\right\}.$$

We recall that if the eccentricity e is zero, then $M(5/2, 4; z) = 1$, and then the zeros of $\operatorname{Im}\{\tilde{D}(0, \rho_0, \alpha_0, \omega)\}$ are given by $\omega = \alpha_0 + k\pi$, $k \in \mathbb{Z}$. Furthermore, by using $C = 2\rho_0$, the result coincides with [L1.S] after the correction made at the beginning of Section 6 (and it agrees quite well with the numerical results given in the same reference for values of C larger than 5). However, if we fix e, and we take ρ_0 sufficiently large, then the term corresponding to $M(5/2, 4; z)$ becomes important.

The following asymptotic formula for $|z|$ large is well known (see [A.S])

$$M\left(\frac{5}{2}, 4; z\right) = \frac{12}{\sqrt{\pi}} \exp\left(\pm\frac{5}{2}\pi i\right) z^{-5/2}(1 + O(|z|^{-1})) + \frac{8}{\sqrt{\pi}} \exp(z)\, z^{-3/2}(1 + O(|z|^{-1})),$$

where the sign $+(-)$ is taken for $-\pi/2 < \alpha_0 < \pi/2$ $(\pi/2 < \alpha_0 < 3\pi/2)$.

The expression above allows us to get an asymptotic expression for $\operatorname{Im}\{\tilde{D}(e, \rho_0, \alpha_0, \omega)\}$ which can be written as

$$\operatorname{Im}\{\tilde{D}(e, \rho_0, \alpha_0, \omega)\} = \cos(\alpha_0 - \omega)\left(\frac{1}{8} + \bar{e}\rho_0 b_1\right) - \sin(\alpha_0 - \omega)\, \bar{e}\rho_0 a_1, \quad (53)$$

where

$$a_1(\rho_0, \alpha_0) = \frac{12}{\sqrt{\pi}} r^{-5/2} \sin\left(-\frac{5}{2}\alpha_0 \pm \frac{5\pi}{2}\right) + \frac{8}{\sqrt{\pi}} r^{-3/2} \exp(r \cos\alpha_0) \sin\left(r \sin\alpha_0 - \frac{3}{2}\alpha_0\right),$$

$$b_1(\rho_0, \alpha_0) = -1 + \frac{12}{\sqrt{\pi}} r^{-5/2} \cos\left(-\frac{5}{2}\alpha_0 \pm \frac{5\pi}{2}\right) + \frac{8}{\sqrt{\pi}} r^{-3/2} \exp(r \cos\alpha_0) \cos\left(r \sin\alpha_0 - \frac{3}{2}\alpha_0\right), \quad (54)$$

and we recall that here $r = 2e\rho_0$ should be large enough.

8. NUMERICAL RESULTS CONCERNING HETEROCLINIC ORBITS

The zeros of $D(\rho_0, \alpha_0, \omega)$ can be computed numerically using the expression of the function $D(\rho_0, \alpha_0, \omega)$ given in (27).

Given a value of ρ_0, we take $\delta = \rho_0^3/2$. The integrals $I_i(\delta)$, $i \geqslant 4$ are obtained from the recurrent formula (25) once $I_0(\delta)$, $I_1(\delta)$, $I_2(\delta)$, and $I_3(\delta)$ are known. Note that for $i = 0, 1, 2, 3$, $I_i(\delta)$ can be computed by numerical integration on the real axis only for small values of δ, due to the factor $\exp(-2\delta/3)$ given by Lemma 3. The method we have used to compute them is to perform a numerical integration along the path defined in section 6. More precisely, we compute numerically the integrals appearing in (29) and we add to them the dominant terms of $\exp(2\delta/3)\,\mathrm{Re}(B_i)$, $i = 1, 2, 3$. In that way we skip the exponencial factor. A good agreement between the results given by the two methods is obtained for small values of δ (which allows to compute the integrals on the real axis without numerical problems).

For fixed values of e and ρ_0, two curves of zeros, $\omega = \omega(\alpha_0)$ and $\omega = \omega(\alpha_0) + \pi$ are obtained. We represent them on a torus parametrized by α_0 and ω. Due to the symmetry of the function $D(\rho_0, \alpha_0, \omega)$ it is sufficient to compute $\omega = \omega(\alpha_0)$ for $\alpha_0 \in [0, \pi]$. Figures 4, 5, and 6 show that curves for some values of e and ρ_0, for $\alpha_0 \in [0, \pi]$ and $\omega_0 \in [0, 2\pi]$.

We remark that, if $e = 0$, the dominant term of $D(\rho_0, \alpha_0, \omega)$ is given by (45a), so, for large values of ρ_0, the zero curves are well approximated by $\omega = \alpha_0$ and $\omega = \alpha_0 + \pi$. In that case, for a fixed value of ρ_0, only two heteroclinic orbits exist. In these way we recover the results obtained for the circular case in [L1.S].

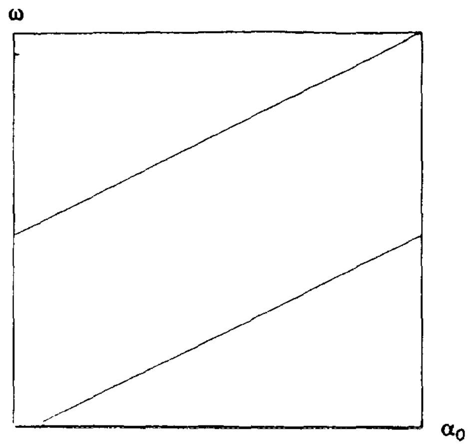

FIG. 4. $\rho_0 = 20.23$, $e = 0$.

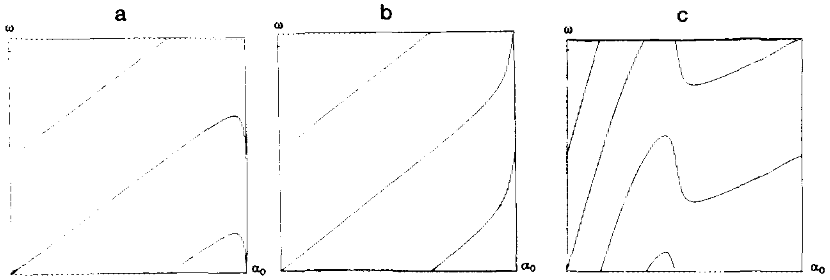

FIG. 5. (a) $\rho_0 = 20.23$, $e = 0.2D\text{-}3$; (b) $\rho_0 = 20.23$, $e = 0.23D\text{-}3$; (c) $\rho_0 = 20.23$, $e = 0.18$.

We observe that if we fix a value of ρ_0 and increase e, some bifurcations appear which change the winding number of the zero curves. That occurs for values of ρ_0 and e such that the curves $\omega = \omega(\alpha_0)$ and $\omega = \omega(\alpha_0) + \pi$ coincide at some point (α_0^*, ω^*) where the slope of curves becomes unbounded.

In order to compute these bifurcations we must solve the system of nonlinear equations

$$\begin{aligned} D &= 0, \\ D_\alpha &= 0, \\ D_\omega &= 0, \end{aligned} \tag{55}$$

where D_α and D_ω mean the partial derivatives with respect to α_0 and ω, respectively.

We recall that D is a function of the variables $(\rho_0, \alpha_0, \omega)$ once the eccentricity e has been fixed. So, if we take e as an additional parameter, we get some curves in the plane (ρ_0, e) as solutions of (55). Using a continuation method, we have computed these bifurcation curves. Three of them are shown in Figs. 7, 8, and 9. We not that the scale in Fig. 7 is different from the ones in Figs. 8 and 9.

As an example points (ρ_0, e) = (10.031842, 0.000887), (ρ_0, e) = (10.058603), 0.396653) and $(\rho_0, e) =$ (10.047798, 0.749382) belong to the curves of Figs. 7, 8, and 9, respectively.

Finally, we present the results obtained using the approximation given by (53). Figures 10 and 11 shows the curves of zeros of function $\text{Im}\{\tilde{D}(e, \rho_0, \alpha_0, \omega)\}$ for some values of e and ρ_0 that can be compared with Figs. 6b and 6c. Figure 12 displays the bifurcation curves corresponding to $\text{Im}\{\tilde{D}\}$.

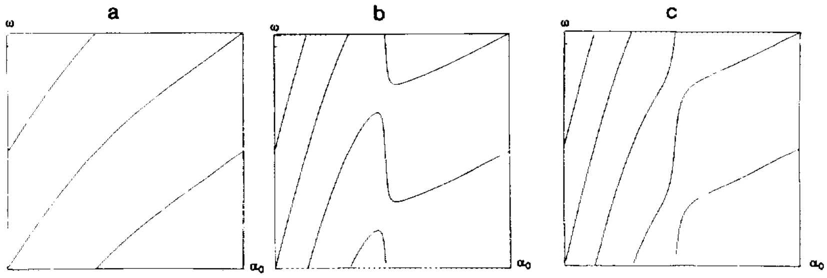

FIG. 6. (a) $\rho_0 = 10$, $e = 0.1$; (b) $\rho_0 = 10$, $e = 0.39$; (c) $\rho_0 = 10$, $e = 0.42$.

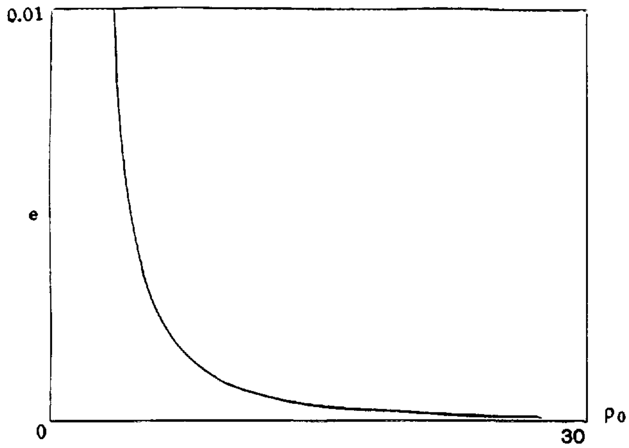

FIGURE 7

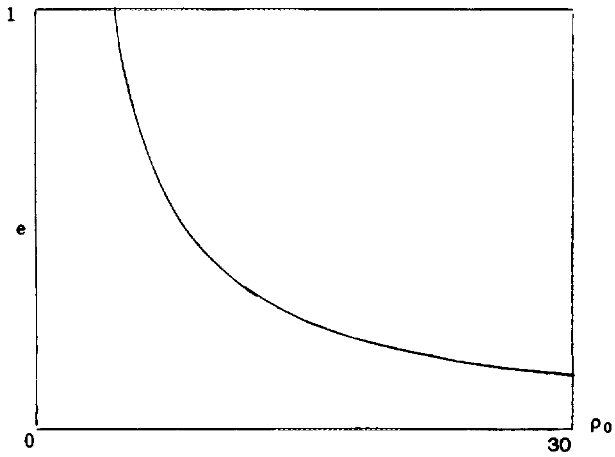

FIGURE 8

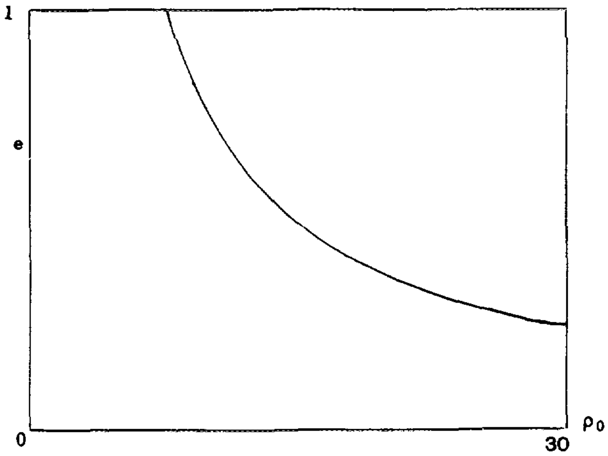

FIGURE 9

We recall that the asymptotic formula has only meaning for large values of $r = 2e\rho_0$. However, if r is large enough, we can approximate

$$\mathrm{Im}\{\tilde{D}\} \simeq \bar{e}\rho_0\{\cos\omega(b_1 \sin\alpha_0 + a_1\cos\alpha_0) + \sin\omega(-b_1\cos\alpha_0 + a_1\sin\alpha_0)\},$$

where a_1, and b_1 are given in (54), and so the bifurcation curves in the plane (ρ_0, e) are hyperbolas.

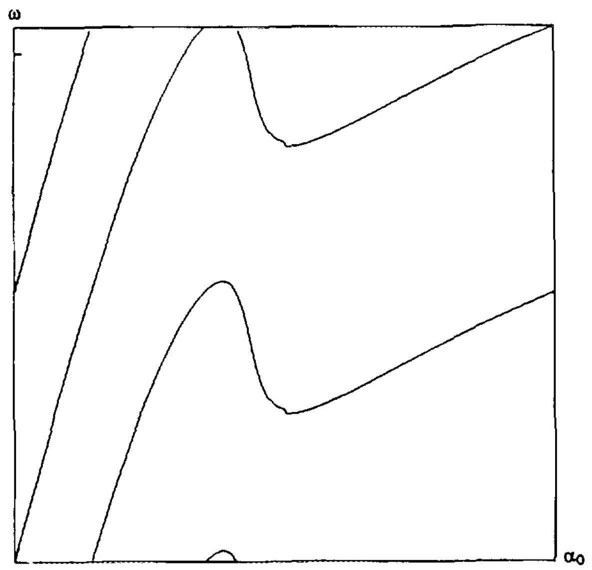

FIG. 10. $\rho_0 = 10$, $e = 0.39$.

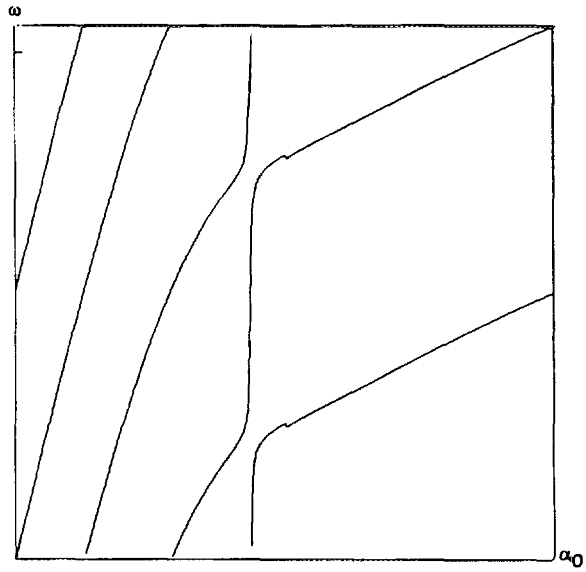

FIG. 11. $\rho_0 = 10$, $e = 0.43$.

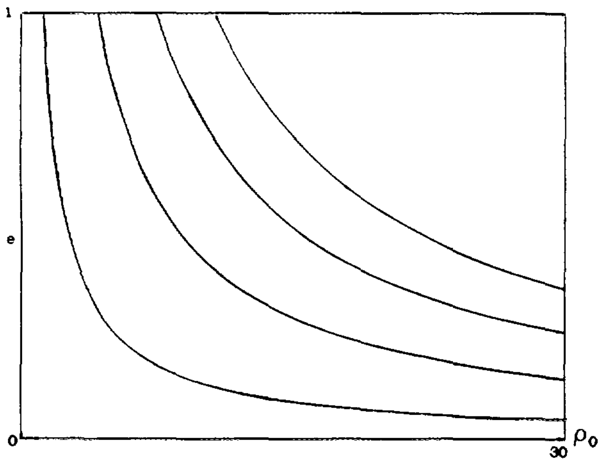

FIGURE 12

9. On the Homoclinic Orbits

In a similar way as that for Δx_0, from (20b) and (20c) if we neglect the exponential terms we can write $\alpha_{b\infty}$ and $\rho_{b\infty}$ as

$$\alpha_{b\infty} = A^0 - \sum_{m\geq 1}\sum_{n\geq n_1}\left(\frac{2}{\rho_0^2}\right)^{m+2n}\sum_{j=0}^{m} t_j^{(m)}\left[\bar{B}_0 + \sum_{s\geq 0}(\bar{B}_s + \bar{B}_{-s})\right],$$

where

$$A^0 = -\sum_{n\geq 1}\xi_n^{(0)}\left(\frac{2}{\rho_0^2}\right)^{2n}\left[\frac{-2nc_0^{2n,0}}{2n+1} I_{2n+1}(0)\right.$$
$$\left.\times\sum_{s>0}(c_s^{2n,0}+c_{-s}^{2n,0})\sin s\omega\frac{2}{s\rho_0^3}\left(1+O\left(\frac{2}{s\rho_0^3}\right)^2\right)\right],$$

$$\bar{B}_0 = c_0^{m+2n,m}\left\{\cos m\alpha_0\left[\frac{3j}{2m(m+2n+j+1)}\gamma_n^{(m)}\right.\right.$$
$$\left.-\frac{2(m+2n+j)-1}{m+2n+j+1}\xi_n^{(m)}\right]I_{m+2n+j}(0)$$
$$\left.+\sin m\alpha_0\left[\gamma_n^{(m)}\frac{m+2n+j+2}{2(m+2n+j)}+\frac{j}{2m}\xi_n^{(m)}\right]\right\},$$

and

$$\bar{B}_{\pm s} = c_{\pm s}^{m+2n,m}\left(\frac{2}{\rho_0^3}\right)\sin(m\alpha_0 \mp s\omega)$$
$$\times\left\{\pm\xi_n^{(m)}\left(1-\frac{j}{m}\left(\frac{2}{\rho_0^3}\right)\right)-3\gamma_n^{(m)}\left(\frac{2}{\rho_0^3}\right)+O\left(\left(\frac{2}{\rho_0^3}\right)^3\right)\right\}.$$

In a similar way,

$$\rho_{b\infty} = \frac{\rho_0}{2}\left[\sum_{m\geq 1}\sum_{n\geq n_1}\gamma_n^{(m)}\left(\frac{2}{\rho_0^2}\right)^{m+2n}\sum_{j=0}^{m} t_j^{(m)}\left[B_0 - \sum_{s\geq 0}(B_s + B_{-s})\right]\right],$$

where

$$B_0 = c_0^{m+2n,m}\left[\cos m\alpha_0\frac{j}{2m(m+2n+j-1)} + \sin m\alpha_0 I_{m+2n+j}(0)\right],$$

$$B_\pm = \left(\frac{2}{\rho_0^3}\right)^2 c_{\pm s}^{m+2n,m}\cos(m\alpha_0 \mp s\omega)$$
$$\times\left[\frac{j}{m}\pm 2(m+2n+j+1)\frac{2}{s\rho_0^3}+O\left(\frac{2}{s\rho_0^3}\right)^2\right],$$

and $n_1 = 1$ if $m = 1$, and $n_1 = 0$ for $m \geq 2$.

Therefore, for a fixed value e, if ρ_0 is large enough we have that

$$\alpha_{b,\infty} = \frac{1}{\rho_0^4}\left[\frac{3\pi}{4} + e^2\left(\frac{9\pi}{8} - \frac{45\pi}{16}\cos 2\alpha_0 + 80\sin 2\alpha_0\right) + O(e^3)\right)$$

$$+\frac{1}{\rho_0^2}\left(\frac{-15\pi e}{2}\cos\alpha_0 + 50e\sin\alpha_0 + O(e^3)\right) + O\left(\frac{1}{\rho_0^3}\right)\Bigg], \quad (56)$$

$$\rho_{b,\infty} = -\frac{e}{\rho_0^3}\left\{5e\cos 2\alpha_0 + \frac{1}{\rho_0^2}\left[\left(\frac{5}{2} + \frac{15}{8}e^2 + O(e^4)\right)\cos\alpha_0\right.\right.$$

$$\left.\left.+\left(\frac{15}{16} + \frac{45}{64}e^3 + O(e^5)\right)\sin\alpha_0\right] + O\left(\frac{1}{\rho_0^3}\right)\right\},$$

where $O(e^n)$ are functions depending on α_0 and ω.

In order to have homoclinic points for small values of μ we must solve $D_2(\rho_0, \alpha_0, \omega) := \alpha_{b,\infty}(\rho_0, \alpha_0, \omega) + \alpha_{b,\infty}(\rho_0, 2\pi - \alpha_0, -\omega) = 0$. Taking into account (56), $D_2(\rho_0, \alpha_0, \omega) \simeq \rho_0^{-4}(3\pi/2) \neq 0$. Therefore, for a fixed value e sufficiently small, if μ is small enough and ρ_0 large enough, there are not homoclinic points *at the first intersection*, in the sense that the third body escapes to infinity for positive and negative time without crossing the section $\{y=0\}$.

10. Proof of the Main Results

Proof of Theorem A. We proved in Section 5 that the only integrals that contribute to function $D(\rho_0, \alpha_0, \omega)$ are $I_k(s\delta)$ and $M_k(s\delta)$ for $k \geqslant 2$ and $s > 0$. In Section 6, it is shown that these integrals are exponentially small in ρ_0. This proves the first part of the theorem.

The asymptotic expression for $D(\rho_0, \alpha_0, \omega)$ is given in Section 7 where the formula (53) is obtained.

The last part of the theorem follows from the expression of the limit parameters $(\rho_\infty, \alpha_\infty)$ and $(\hat{\rho}_\infty, \hat{\alpha}_\infty)$ given in Section 4 and (56). ∎

Proof of Theorem B. In order to determine when the implicit function theorem does not apply, we must look for the points $(\rho_0, \alpha_0, \omega)$ such that

$$D(\rho_0, \alpha_0, \omega) = 0,$$

$$\frac{\partial D(\rho_0, \alpha_0, \omega)}{\partial \omega} = 0.$$

We have seen that for a fixed value of $e > 0$ sufficiently small and ρ_0 large enough, the zeros of $D(\rho_0, \alpha_0, \omega)$ are well approximated by the ones of $\operatorname{Im}\{\tilde{D}(e, \rho_0, \alpha_0, \omega)\}$. In the same way, we can approximate the zeros of $\partial D/\partial\omega$ by the ones of $\operatorname{Re}\{\tilde{D}(e, \rho_0, \alpha_0, \omega)\}$.

We consider the system of equations

$$\begin{aligned} \operatorname{Im} \tilde{D} &= \cos(\alpha_0 - \omega)\, \bar{e}\rho_0 a_1 + \sin(\alpha_0 - \omega)(\tfrac{1}{8} + \bar{e}\rho_0 b_1) = 0, \\ \operatorname{Re} \tilde{D} &= \cos(\alpha_0 - \omega)(\tfrac{1}{8} + \bar{e}\rho_0 b_1) - \sin(\alpha_0 - \omega)\, \bar{e}\rho_0 a_1 = 0, \end{aligned} \tag{57}$$

where $a_1(\rho_0, \alpha_0)$, $b_1(\rho_0, \alpha_0)$ are given in (54). System (57) has a solution for $\cos(\alpha_0 - \omega)$, $\sin(\alpha_0 - \omega)$ if and only if

$$\begin{aligned} a_1(\rho_0, \alpha_0) &= 0, \\ b_1(\rho_0, \alpha_0) + \frac{1}{8\bar{e}\rho_0} &= 0. \end{aligned} \tag{58}$$

Let (ρ_0, α_0) be such that $a_1(\rho_0, \alpha_0) = 0$ and

$$\sin\left(r \sin \alpha_0 - \frac{3}{2}\alpha_0\right) = 0, \tag{59}$$

where we recall that $r = 2e\rho_0$. Then

$$\sin\left(-\frac{5}{2}\alpha_0 \pm \frac{5\pi}{2}\right) = 0. \tag{60}$$

There is a discret set $\mathscr{P}$ of values (ρ_0, α_0) which satisfy (59) and (60) that can be written as

$$\mathscr{P} = \left\{(\rho_0, \alpha_0) \mid \alpha_0 \in \{\pi/5, 3\pi/5, 7\pi/5, 9\pi/5\}, \right.$$
$$\left. \rho_0 = \frac{1}{2e \sin \alpha_0}\left(\frac{3}{2}\alpha_0 + k\pi\right), k \in \mathbb{Z}\right\}.$$

It is easy to check that if $(\rho_0, \alpha_0) \in \mathscr{P}$, then $b_1(\rho_0, \alpha_0) + (1/8\bar{e}\rho_0) \neq 0$. Therefore, for any (ρ_0, α_0) solution of (58) we have

$$r \exp(r \cos \alpha_0) = -\frac{3 \sin(-(5/2)\,\alpha_0 \pm (5\pi/2))}{2 \sin(r \sin \alpha_0 - (3/2)\,\alpha_0)}. \tag{61}$$

By substituting the left-hand side of the equation above in the second equation of (58), we have

$$\frac{12}{\sqrt{\pi}} \sin\left(r \sin \alpha_0 + \alpha_0 \pm \frac{5\pi}{2}\right) - \left(1 - \frac{1}{8\bar{e}\rho_0}\right) r^{5/2} \sin\left(r \sin \alpha_0 - \frac{3}{2}\alpha_0\right) = 0. \tag{62}$$

Note that the first term above is bounded, so Eq. (62) holds for large values of r if and only if

$$r \sin \alpha_0 - \frac{3}{2}\alpha_0 = \nu + k\pi, \qquad k \in \mathbb{Z} \tag{63}$$

for some small value ν. However $\sin \alpha_0 \neq 0$ for the solutions of (58), because if $\sin \alpha_0 = 0$ then (59) or (60) holds, but there is not any value of ρ_0 such that $a_1(\rho_0, \alpha_0) = 0$. So we can assume that $\sin \alpha_0$ is different from zero and then from (63) we have for $k \in \mathbb{Z}$, $|k|$ large enough

$$r = \frac{k\pi}{\sin \alpha_0}\left(1 + \frac{\nu + (3/2)\,\alpha_0}{k\pi}\right) = \frac{k\pi}{\sin \alpha_0}(1 + O_1), \tag{64}$$

where $O_1(O_n)$ stands for terms of order $1/|k\pi|$ $(1/|k\pi|^n)$.

Introducing the parameter ν in (62), we have

$$\tan \nu = \mp \frac{12}{\sqrt{\pi}} r^{-5/2} \cos \frac{5}{2}\alpha_0 \frac{1}{(1 - (1/(8\bar{e}\rho_0)) \mp (12/\sqrt{\pi})\, r^{-5/2} \sin (5/2)\,\alpha_0)}.$$

Using (64), we get

$$\nu = \mp \frac{12}{\sqrt{\pi}} \cos \frac{5}{2}\alpha_0 \left(\frac{\sin \alpha_0}{k\pi}\right)^{5/2} (1 + O_1). \tag{65}$$

Then $\sin(r \sin \alpha_0 - (3/2)\,\alpha_0) = (-1)^k \sin \nu = (-1)^k\,(\mp(12/\sqrt{\pi}))\cos((5/2)\,\alpha_0)((\sin \alpha_0)/(k\pi))^{5/2}\,(1 + O_1)$ and using (64) we get from (61)

$$\exp\left[\frac{k\pi}{\tan \alpha_0}(1 + O_1)\right] = \frac{\sqrt{\pi}}{8}(-1)^k \left(\frac{k\pi}{\sin \alpha_0}\right)^{3/2} (1 + O_1). \tag{66}$$

Therefore, k must be an even integer, positive for $\alpha_0 \in (0, \pi)$ and negative for $\alpha_0 \in (\pi, 2\pi)$. Let us consider values of k as before with $|k|$ large enough. It is easy to see that for any k, there exist two solutions of (66), corresponding to $\alpha_0 = \pi/2 - \varepsilon$ and $\alpha_0 = 3\pi/2 + \varepsilon$ where $\varepsilon > 0$ is a small quantity that can be computed from (66). As a first approximation we get

$$\varepsilon \simeq \arctan\left\{\frac{1}{k\pi} \ln\left(\frac{\sqrt{\pi}}{8}(k\pi)^{3/2}\right)\right\},$$

and using (64), $r \simeq 3\pi/4 + k\pi$ or equivalently $\rho_0 \simeq (1/2e)(3\pi/4 + k\pi)$. However, a bifurcation point $(\rho_0, \alpha_0, \omega)$ satisfies also that $\partial\tilde{D}/\partial\alpha_0 = 0$, that is,

$$\cos(\alpha_0 - \omega)\frac{\partial a_1}{\partial \alpha_0} + \sin(\alpha_0 - \omega)\frac{\partial b_1}{\partial \alpha_0} = 0.$$

Some computations show that

$$\tan(\alpha_0-\omega) = -\frac{-(15/2)\cos\gamma_1\sin\gamma_2-\sin\gamma_1[3r\cos(\gamma_2+\alpha)-(9/2)\cos\gamma_2]}{\sin\gamma_1[3\sin\gamma_2+3r\sin(\gamma_2+\alpha)]}, \tag{67}$$

where $\gamma_1=-(5/2)\,\alpha_0\pm(5\pi/2)$ and $\gamma_2=r\sin\alpha_0-(3/2)\,\alpha_0$.

Using the following approximation, $\cos\gamma_2\simeq 1$, $\cos(\gamma_2+\alpha)\simeq\varepsilon$ and neglecting the terms in $\sin\gamma_2=\nu=O(1/|k\pi|^{5/2})$, we have

$$\tan(\alpha_0-\omega)\simeq\frac{-(9/2)+3r\sin\varepsilon}{3\sin\gamma_2+3r\sin(\gamma_2+\alpha)}\underset{r\to\infty}{\longrightarrow}0.$$

Then, for any value, $\alpha_0=\pi/2-\varepsilon$ or $\alpha_0=3\pi/2+\varepsilon$, taking $r(\alpha_0)$ given in (64), there exist two values of ω which satisfy (67). So if we fix $\rho_0=\rho_0^{(k)}$ for some k, with $|k|$ large enough, there are four bifurcation points located near the following ones

$$(\alpha_0,\omega)=\left(\frac{\pi}{2},\frac{\pi}{2}\right),\quad\left(\frac{\pi}{2},\frac{3\pi}{2}\right),\quad\left(\frac{3\pi}{2},\frac{\pi}{2}\right),\quad\left(\frac{3\pi}{2},\frac{3\pi}{2}\right). \quad ∎$$

We remark that for the bifurcation curves computed in Fig. 12 the values of r are 8.5405, 14.8915, and 21.2120, respectively. The corresponding values using the expression $(3\pi/4)+k\pi$ given in the proof of Theorem B gives the values 8.6394, 14.9225, and 21.2018 for $k=2$, 4 and 6, respectively.

Proof of the Corollary. Given some values $\alpha_\infty\in[0,2\pi)$ and $\rho_\infty>0$ large enough, we must show that there exist values of ρ_0, α_0, and ω such that

$$\begin{aligned} D(\rho_0,\alpha_0,\omega)+O_1&=0,\\ -\alpha_\infty+\alpha_0-\pi+\mu\alpha_{b\infty}(\rho_0,\alpha_0,\omega)+O_2&=0,\\ -\rho_\infty+\rho_0+\mu\rho_{b\infty}(\rho_0,\alpha_0,\omega)+O_2&=0. \end{aligned} \tag{68}$$

We recall that the first equation ensures that the initial conditions given by ρ_0, α_0 and ω correspond to a heteroclinic orbit. The second and third equations are setting to demand that the heteroclinic orbit belongs to $W^s(\rho_\infty,\alpha_\infty)$ (see Section 4).

We have seen in the proof of Theorem B that on the points (ρ_0,α_0,ω) such that $D(\rho_0,\alpha_0,\omega)=0$, $\partial D/\partial\omega\neq 0$ except for a discrete set of points. Then, for almost all values of ρ_0, the implicit function theorem can be applied to (68). The transversality of the invariant manifolds is also a consequence of that.

The relations between the limiting parameters $(\rho_\infty, \alpha_\infty)$ and $(\hat{\rho}_\infty, \hat{\alpha}_\infty)$ for positive and negative time follows from the estimates given by Theorem B. ▌

11. Conclusions

We have seen that if $e=0$ in order to have a heteroclinic orbit, the three bodies must be on a line.

Let e_0 a fixed value of eccentricity sufficiently small. If we increase ρ_0, we see from the bifurcation diagram of Fig. 12 that the line $e=e_0$ intersects the bifurcation curves in a sequence $\{\rho_0^{(2i)}\}_{i\geq 1}$ such that $\lim_{i\to\infty} \rho_0^{(2i)}=\infty$. Going from $\rho_0^{(2i)}$ to $\rho_0^{(2i+2)}$ for some $i\geq 1$, some quantitative change appears. More precisely, if we fix the pericenter passage of the primaries, ω, two new directions, α_0, for a heteroclinic orbit appear as can be seen in Fig. 6. Theorem B predicts this behaviour for ρ_0 sufficiently large. In this case we have an asymptotic formula to compute $\rho_0^{(2i)}$. These values show a good agreement with the ones computed numerically using the complete series even for values of ρ_0 not very large.

In the circular case, numerical computations show in [Ll.S] that increasing μ, four heteroclinic orbits appear. Of course, in the elliptic case, for large values of μ, the behaviour can change from the one given by the asymptotic formula; that is, either new heteroclinic orbit can appear or some of them can be destroyed. We remark that to see which is the situation in this case, we must compute numerically the intersection of the stable manifold of the infinity, which has dimension 4, with the section $\{y=0\}$. This gives a three-dimensional object. To obtain the intersection of this object with its symmetrical, that is, the unstable manifold, could be a difficult task.

We define the map

$$(\alpha_\infty, \rho_\infty) \xrightarrow{T} (\hat{\alpha}_\infty, \hat{\rho}_\infty),$$

where $(\hat{\alpha}_\infty, \hat{\rho}_\infty)$ are given in the Corollary.

We remark that from any fixed point of T we can ensure the existence of a homoclinic orbit. For small fixes values of μ and e, and large values of ρ_∞, the map T does not have fixed points. However, the periodic points of T will give a chain of heteroclinic orbits which begins and ends at the same periodic orbit of the infinity. A more detailed study of this map will give this kind of information.

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REFERENCES

[A.S] M. ABRAMOWITZ AND I. STEGUN, "Handbook of Mathematical Functions," Dover, New York.

[Al] V. M. ALEKSEEV, Quasirandom dynamical system I, II, III, *Math. USSR* **5**, **6**, **7** (1968–1969).

[B.C] D. BROUWER AND G. M. CLEMENCE, "Methods of Celestial Mechanics," Academic Press, New York, 1961.

[Be] A. BENSENY, "Contribuciò a l'estudi del problema restringit de 3 cossos per a valors petits del paràmetre de masses," Tesis Doctoral, Universitat de Barcelona, Barcelona, 1984.

[E] R. EASTON, Parabolic orbits in the planar three-body problem, *J. Differential Equations* **52** (1984), 116–134.

[E.Mc] R. EASTON AND R. MCGEHEE, Homoclinic phenomena for orbits doubly asymptotic to an invariant three-sphere, *Indiana Univ. Math. J.* **28** (1979), 211–240.

[Er] A. ERDÉLYI, "Asymptotic Expansions," Dover, New York, 1956.

[Ll.S] J. LLIBRE AND C. SIMÓ, Oscillatory solutions in the planar restricted three-body problem, *Math. Ann.* **248** (1980), 153–184.

[M.S] R. MARTÍNEZ AND C. SIMÓ, Qualitative study of the planar isosceles three-body problem, *Celestial Mech.* **41** (1988), 179–251.

[Mc] R. MCGEHEE, A stable manifold for degenerate fixed points with applications to celestial mechanics, *J. Differential Equations* **14** (1973), 70–88.

[Mk] R. MOECKEL, Heteroclinic phenomena in the isosceles three-body problem, *SIAM J. Math. Anal.* **15** (1984), 857–876.

[Mo] J. MOSER, "Stable and Random Motions in Dynamical Systems," Annals of Mathematics Studies No. 77, Princeton Univ. Press, Princeton, NJ, 1973.

[R] C. ROBINSON, Homoclinic orbits and oscillation for the planar three-body problem, *J. Differential Equations* **52** (1984), 356–377.

[Si] K. SITNIKOV, The existence of oscillatory motion in the restricted three-body problem, *Dokl. Akad. Nauk USSR* **133** (1960), 303–306.

[Sz] V. SZEBEHELY, "Theory of Orbits," Academic Press, New York, 1967.