THE RATIONAL HERMITE INTERPOLATION PROBLEM
AND SOME RELATED RECURRENCE FORMULAS

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Abstract—Some recurrence relations between adjacent elements in the rational Hermite interpolation table
are proved. This enables us to derive two methods for calculating the coefficients of the rational Hermite
interpolants. These methods are generalizations of known algorithms for classical Padé approximations.

1. THE RATIONAL HERMITE INTERPOLATION PROBLEM

Consider a real- or complex-valued function $f$. Suppose functional values are given in the points $\{z_i\}_{i=0}^m$, with the usual convention for multiple points [5, p. 53]. Then we can construct in a formal
manner the Newton interpolation series

$$f(z) = f_{00} + f_{01} \cdot (z - z_0) + f_{02} \cdot (z - z_0)(z - z_1) + \ldots$$

$$+ f_{0i} \cdot (z - z_0)(z - z_1) \ldots (z - z_{i-1}) + \ldots .$$

For abbreviation we put $w_{00}(z) = 1$ and $w_{0i}(z) = (z - z_{i-1}) \cdot w_{0i-1}(z)$ for $i \geq 0$, consequently

$$f(z) = \sum_{i=0}^\infty f_{0i} \cdot w_{0i}(z).$$

(1)

The coefficients of the $w_{0i}(z)$ are the divided differences (with possible confluent arguments) of
$f_{0i} = f[z_0, z_1, \ldots, z_i]$ for $i \geq 0$. By $f_0$ we denote the divided difference of order $j - i$ (if $j \geq i$),
determined in the interpolation points $z_0, z_1, \ldots, z_j$.

If $j < i$ then $f_j = 0$ by convention. For divided differences the following product-formula holds [6, p. 18]:

$$(fg)_i = \sum_{k=0}^i f_k g_{i-k},$$

(2)

where

$$g(z) = \sum_{i=0}^\infty g_{0i} w_{0i}(z).$$

Let's consider a rational function $r$

$$r(z) = \frac{p(z)}{q(z)} = \sum_{i=0}^m a_{0i} w_{0i}(z) \sum_{i=0}^n b_{0i} w_{0i}(z)$$

with degree of numerator (resp. denominator) less than or equal to $m$ (resp. $n$), which satisfies

$$q(z) \cdot f(z) - p(z) = 0(w_{0,i+n+1}(z)).$$

(3)

The expression (3) means that the coefficients of $w_{0i}$ in the Newton interpolation series of $qf - p$
are zero for $i = 0, 1, \ldots, m + n$, under the condition that we interpolate $qf - p$ in the same
sequence of points \( \{ z_i \}_{i=0}^\infty \) as \( f \). Consequently (3) is equivalent to saying that

\[
(qf - p)_i = 0
\]

for \( i = 0, 1, \ldots, m + n \).

By making use of (2) condition (3) is then equivalent to the following set of equations:

\[
\sum_{i=0}^I b_{i0}f_i = \begin{cases} a_{0i}, & i = 0, 1, \ldots, m \\ 0, & i = m + 1, m + 2, \ldots, m + n \end{cases}
\]

Hence the determination of \( p \) and \( q \) such that (3) is satisfied is equivalent to a solution \( a_{0i}, a_{1i}, \ldots, a_{mi}, b_{0i}, b_{1i}, \ldots, b_{ni} \) of (4). This is a homogeneous set of \( m + n + 1 \) linear equations in the \( m + n + 2 \) unknowns \( a_{0i}, a_{1i}, \ldots, a_{mi} \) and \( b_{0i}, b_{1i}, \ldots, b_{ni} \). Consequently (4) always has a nontrivial solution with \( q \neq 0 \), for \( q = 0 \) would imply \( p = 0 \). We now prove the following result.

**Theorem 1.** The rational forms \( p/q \), where the polynomials \( p = \sum_{i=0}^\infty a_{0i}w_0 \) and \( q = \sum_{i=0}^\infty b_{0i}w_0 \) are constructed by using a nontrivial solution \( a_{0i}, a_{1i}, \ldots, a_{mi}, b_{0i}, b_{1i}, \ldots, b_{ni} \) of (4), have the same irreducible form \( p_m/q_m \) with \( q_m \neq 0 \), which equals 1 is always possible.

**Proof.** Suppose that the coefficients of \( p_1, q_1 \) and \( p_2, q_2 \) satisfy (4). Then,

\[
q_1f - p_1 = 0(w_{0,m+n+1}(z)) \quad \text{for} \quad i = 1, 2,
\]

and

\[
p_1q_2 - p_2q_1 = -q_1(q_1f - p_1) + q_1(q_2f - p_2) = 0(w_{0,m+n+1}(z)).
\]

The left-hand side is a polynomial of degree at most \( m + n \), which means that \( p_1q_2 - p_2q_1 = 0 \).

Consequently \( p/q_1 \) and \( p/q_2 \) have the same irreducible form \( p_m/q_m \) with \( q_m \neq 0 \), which proves the first part of the theorem. Consider a rational form \( p/q \) with \( p = \sum_{i=0}^\infty a_{0i}w_0 \) and \( q = \sum_{i=0}^\infty b_{0i}w_0 \) such that \( p \) and \( q \) satisfy (3), then \( p_m/q_m \) is the irreducible form of \( p/q \). Now suppose \( q_m = 0(w_{0\lambda}(z)) \), with \( \lambda > 0 \) and \( b_{0\lambda} \neq 0 \). Then \( \lambda = n \), because otherwise we should have \( q = 0 \). From (4a) we then conclude that the coefficients \( a_{0i} \) of \( p \) satisfy \( a_{0i} = 0 \) for \( i = 0, 1, \ldots, \lambda - 1 \) and \( a_{0\lambda} = f_{\lambda,b}b_{0\lambda} \).

Consequently, \( p_m(z) = 0(w_{0\lambda}(z)) \). Hence, the normalization \( b_{0\lambda} = 1 \), if \( b_{0i} = 0 \) for \( i = 0, 1, \ldots, \lambda - 1 \) is always possible. \( \square \) This theorem is the analogue of a theorem due to Frobenius [3, p. 10] for Padé approximants.

The uniquely determined rational function \( r_m \) defined by

\[
r_m(z) = \frac{p_m(z)}{q_m(z)}
\]

for every \( z \) with \( q_m(z) \neq 0 \) is called the rational Hermite interpolant of order \([m, n]\). This rational Hermite interpolant does not necessarily satisfy (3). Necessary and sufficient conditions, in order that \( p_m \) and \( q_m \) should satisfy (3), can be found in [7]. These results were proved in the case of real-valued functions, but are easily extended to the complex case.

For a given Newton series (1) we can arrange all these interpolants in a two-dimensional array, called the rational Hermite interpolation table (see Table 1).

Note that the elements appearing in the first column, are just the partial sums of the Newton series for \( f \).
The rational Hermite interpolation problem

Table 1. The rational Hermite interpolation table

<table>
<thead>
<tr>
<th>r_{m,0}</th>
<th>r_{0,0}</th>
<th>r_{0,1}</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>r_{0,0}</td>
<td>r_{0,1}</td>
<td>r_{0,2}</td>
<td>\ldots</td>
</tr>
<tr>
<td>r_{m,0}</td>
<td>r_{m,1}</td>
<td>r_{m,2}</td>
<td>\ldots</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
</tbody>
</table>

In the following sections we will prove certain recurrence relations, which exist between the coefficients of the rational Hermite interpolants. Therefore we will assume that the degree of \( p_m \) (resp. \( q_m \)) is exactly \( m \) (resp. \( n \)). This is e.g. just what is defined as the normality of rational Hermite interpolation table [6, p. 39].

2. SOME IDENTITIES IN THE RATIONAL HERMITE INTERPOLATION TABLE

In the sequel we will denote the coefficients of \( w_{0i}(z) \) in \( p_m \) (resp. \( q_m \)) by \( a_{m,i} \) (resp. \( b_{m,i} \)).

First we prove two theorems, which relate certain triples of elements in the rational Hermite interpolation table. A first theorem concerns the elements \( r_{m,n} = P_i / Q_i \), \( r_{m,n-1} = P_3 / Q_3 \) and \( r_{m+1,n-1} = P_3 / Q_3 \).

**Theorem 2.**

\[
\frac{P_1}{Q_1} = a_{m,n-1}P_3 - a_{m+1,n-1}(z - z_{m+1})P_3
\]

**Proof.** Since \( a_{m+1,n-1} \neq 0 \), it is clear that the denominator of the right side of (5) has exactly degree \( n \).

On the other hand, since

\[
a_{m,n-1} - a_{m+1,n-1} = 0
\]

the numerator has at most degree \( m \).

Now let

\[
U(z) = \left[ a_{m,n-1}Q_3(z) - a_{m+1,n-1}(z - z_{m+1})Q_3(z) \right] f(z)
\]

\[
- \left[ a_{m,n-1}P_3(z) - a_{m+1,n-1}(z - z_{m+1})P_3(z) \right],
\]

or

\[
U(z) = a_{m,n-1} \left[ (Q_3(z)f(z) - P_3(z)) \right]
\]

\[
- a_{m+1,n-1}(z - z_{m+1})[Q_3(z)f(z) - P_3(z)].
\]

We will show that

\[
U(z) = 0(w_{0,m+n+1}(z)).
\]

Suppose in the set \( \{z_i\} \), \( i = 0, 1, \ldots, m + n \) there are \( l \) distinct points \( z_{m,i} \) \( i = 1, 2, \ldots, l \) with resp. multiplicity \( m_i \). Then \( \sum_{i=1}^{l} m_i = m + n + 1 \). Consider also the formal \( \ell \)-th derivative of \( U(z) \),

\[
U^{(\ell)}(z) = a_{m,n-1} \left[ (Q_3(z)f(z) - P_3(z))^{(\ell)} \right]
\]

\[
- a_{m+1,n-1}(z - z_{m+1})[Q_3(z)f(z) - P_3(z)]^{(\ell)},
\]

with \( i \geq 1 \).

Then it is easy to conclude, using the definition of \( P_3/Q_3 \) and \( P_3/Q_3 \), that
for $j = 0, 1, \ldots, m_i - 1$ and $i = 1, 2, \ldots, l$, which implies (6).

Because of the supposed normality of the rational Hermite interpolation table and because of
the unicity of the rational Hermite interpolant, the function associated with the right side of (5)
must be equal to the rational Hermite interpolant of order $[m, n]$ (and hence the numerator has
exactly degree $m$).

This concludes the proof. □

Considering the elements $r_{m-1,n} = (P_1/Q_1), r_{m,n} = (P_2/Q_2)$ and $r_{m,n-1} = (P_3/Q_3), we can prove in
the same way the following result.

**Theorem 3.**

\[
\frac{P_1}{Q_1} = \frac{a_{m,n}^{(m)} - a_{m-1,n}^{(m)} a_{m,n}^{(m)} P_3}{a_{m,n-1}^{(m)} Q_2 - a_{m,n}^{(m)} Q_3}.
\]  

(7)

Note that in (5) because of the appearance of the factor $z - z_{n+1}$, the representation as a
Newton series for the numerator and the denominator of the rational Hermite interpolant of
order $[m, n]$ has been lost. We will show in the next theorem how to recover this representation.
We also remark that the denominator in the right side of (5) and (7) does not in general satisfy the
normalizing condition.

The relations (5) and (7) can be used for calculating the rational Hermite interpolants. Indeed,
note that relation (5) enables us to go to the right in the rational Hermite interpolation table, while
relation (7) allows us to move upwards.

3. A FIRST METHOD

Consider the elements in the rational Hermite interpolation table lying on an ascending
staircase,

\[ T_k = \{r_{k,0}, r_{k-1,0}, r_{k-1,1}, \ldots, r_{0,k}\}, \]  

(8)

with $k \geq 1$.

**Theorem 4.** To compute the coefficients of the numerator and denominator in the sequence $T_k$, the
following recurrence formulas exist:

\[
a_{k-j}^{(i)} = \frac{a_{k-j}^{(i)} - a_{k-j+1}^{(i)} [a_{k-j+1}^{(i)} (z_k - z_i) a_{k-j}^{(i)}] - a_{k-j}^{(i)} (z_k - z_i) a_{k-j+1}^{(i)}}{a_{k-j}^{(i)} + (z_k - z_i) a_{k-j+1}^{(i)}},
\]  

(9)

for $j = 1, 2, \ldots, k$ and

\[
a_{k-j}^{(i)} = \frac{a_{k-j}^{(i)} - a_{k-j+1}^{(i)} a_{k-j}^{(i)}}{a_{k-j}^{(i)} + (z_k - z_i) a_{k-j+1}^{(i)}},
\]  

for $j = 0, 1, \ldots, k - 1$.

\[
b_{k-j}^{(i)} = \frac{b_{k-j}^{(i)} - b_{k-j+1}^{(i)} [b_{k-j+1}^{(i)} (z_k - z_i) b_{k-j}^{(i)}] - b_{k-j}^{(i)} (z_k - z_i) b_{k-j+1}^{(i)}}{b_{k-j}^{(i)} + (z_k - z_i) b_{k-j+1}^{(i)}},
\]  

(10)

for $j = 1, 2, \ldots, k$ and

\[
b_{k-j}^{(i)} = \frac{b_{k-j}^{(i)} - b_{k-j+1}^{(i)} b_{k-j}^{(i)}}{b_{k-j}^{(i)} + (z_k - z_i) b_{k-j+1}^{(i)}},
\]  

for $j = 0, 1, \ldots, k - 1$. 
Proof. First we rewrite (5) and (7) in the respective forms
\[
\begin{align*}
\frac{p_{k-j}}{q_{k-j}} &= \frac{a_{k-j-1}^{(k-j)}}{a_{k-j-1}^{(k-j-1)}} p_{k-j+1}(z) - a_{k-j-1+1}^{(k-j)} (z - z_0) p_{k-j-1} \\
\frac{p_{k-j-1}}{q_{k-j-1}} &= \frac{a_{k-j-1}^{(k-j-1)}}{a_{k-j-1}^{(k-j-1-1)}} q_{k-j}(z) - a_{k-j+1}^{(k-j)} (z - z_0) q_{k-j-1}
\end{align*}
\]

and
\[
\begin{align*}
\frac{p_{k-j-1}}{q_{k-j-1}} &= \frac{a_{k-j}^{(k-j-1)}}{a_{k-j-1}^{(k-j-1)}} p_{k-j+1}(z) - a_{k-j}^{(k-j)} p_{k-j-1} \\
\frac{p_{k-j-2}}{q_{k-j-2}} &= \frac{a_{k-j}^{(k-j-2)}}{a_{k-j-1}^{(k-j-2-1)}} q_{k-j}(z) - a_{k-j}^{(k-j-1)} q_{k-j-1}
\end{align*}
\]

To determine the Newton coefficients of the numerator and denominator of \( r_{k-j} \), we proceed as follows.

The numerator \( N \) of the right side of (5') can be written as
\[
N = a_{k-j-1}^{(k-j)} \sum_{i=0}^{k-j+1} a_{k-j+1+1}^{(i)} w_{i}(z) - a_{k-j-1+1}^{(k-j)} \sum_{i=0}^{k-j} a_{k-j-1+1}^{(i)} w_{i}(z)
\]

\[
\times [(z_0 - z_k) w_{k-1}(z) + w_{k-1}(z)].
\]

By using (2) the second term on the right side becomes
\[
N = a_{k-j-1}^{(k-j)} \sum_{i=0}^{k-j+1} a_{k-j+1+1}^{(i)} w_{i}(z) - a_{k-j-1+1}^{(k-j)} \sum_{i=0}^{k-j} a_{k-j+1+1}^{(i)} + (z_k - z_{k-j}) a_{k-j-1+1}^{(i)} w_{i}(z)
\]

with the convention that \( a_{k-j-1+1}^{(i)} = 0 \), if \( i < 0 \) or \( i > k - j \) or,
\[
N = \sum_{i=0}^{k-j+1} (a_{k-j-1}^{(k-j)} a_{k-j+1+1}^{(i)} - a_{k-j-1+1}^{(k-j)} a_{k-j+1+1}^{(i+1)}) + (z_k - z_i) a_{k-j-1+1}^{(i)} w_{i}(z).
\]

Analogously the denominator \( D \) of (5') can be expressed as
\[
D = \sum_{i=0}^{k-j+1} (a_{k-j}^{(k-j)} b_{k-j+1+1}^{(i)} - a_{k-j-1+1}^{(k-j+1)} b_{k-j+1+1}^{(i+1)}) + (z_k - z_i) a_{k-j-1+1}^{(i)} b_{k-j+1+1}^{(i)} w_{i}(z),
\]

with \( b_{k-j+1+1}^{(i)} = 0 \) if \( i < 0 \) or \( i > j - 1 \).

Normalizing so that the denominator takes on the value 1 for \( z = z_0 \) we finally get the first set of recurrence formula (9). Note that the denominator in (9) can not vanish, since otherwise the numerator and denominator in (5) would have a common factor \( z - z_0 \), which contradicts the supposed normality of the rational Hermite interpolants. From (7') we immediately derive (10) by taking into account the normalizing condition. Again the denominator in (10) can not vanish for an analogous reason.

Making alternately use of (9) and (10) it is possible to construct the Newton coefficients of the elements of (8), since we know the first two elements as partial sums of the given Newton series.

It is remarkable that the a’s in (9) and (10) can be computed independently of the b’s, which means that the numerators can be computed independently of the denominators. On the basis of (4b) one should expect the opposite. It is however also possible to compute the denominators independently of the numerators.

Theorem 5. For the coefficients of the denominators of the rational Hermite interpolants we have the recurrence relation
\[
b_{k-j+1+1}^{(i)} = b_{k-j}^{(i)} (1 + (z_k - z_0) B_{k-j+1+1}^{(i)}) + B_{k-j+1+1}^{(i)} (b_{k-j+1+1}^{(i+1)} - (z_k - z_i) b_{k-j+1+1}^{(i)}),
\]

for \( i = 0, 1, \ldots, j - 1 \), where
\[
B_{k-j+1+1}^{(i)} = \frac{b_{k-j}^{(i)}}{b_{k-j}^{(i)} (z_k - z_0) + b_{k-j}^{(i+1)}},
\]
Proof. The second relation of (9) is our starting point. Taking into account that for $i = j$

$$b_{k-j}^{(j)} = \frac{a_{k-j}^{(j+1)} b_{k-j}^{(j-1)}}{a_{k-j}^{(j+1)}} - \frac{b_{k-j}^{(j)}}{a_{k-j}^{(j+1)}} (z_k - z_0) a_{k-j}^{(j+1)}$$

we find that

$$\frac{a_{k-j}^{(j+1)}}{a_{k-j}^{(j+1)}} = - \frac{b_{k-j}^{(j)}}{b_{k-j}^{(j)} (z_k - z_0) + b_{k-j}^{(j-1)}} = B_{k-j}^{(j)}.$$ 

Consequently the second relation of (9) becomes, after a suitable reordering,

$$b_{k-j+i+1}^{(j+1)} = b_{k-j+i}^{(j+1)} [1 + (z_k - z_0) B_{k-j+i+1}^{(j)} + B_{k-j+i}^{(j)} [b_{k-j}^{(j-1)} - (z_k - z_0) b_{k-j}^{(j-1)}]$$

for $i = 0, 1, \ldots, j - 1$. □

Relation (11) enables us to compute the denominators of the Hermite interpolants row by row, provided we know the first row. The method for getting the first row is based on the following property that if $f(z_i) \neq 0$ for $i = 0, 1, \ldots$, then $p/q$ is the rational Hermite interpolant of order $[m, n]$ of $f$ if and only if $q/p$ is the rational Hermite interpolant of order $[n, m]$ of $1/f$. Hence the first row can be computed e.g. by constructing a table of divided differences.

4. A SECOND METHOD

Suppose we are interested in the element of order $[m, n]$ in Table 1, then we could proceed as follows.

Calculate $p_{mn}$ by forming Table 2 column by column, using the first relation of (9).

<table>
<thead>
<tr>
<th>$p_{m,0}$</th>
<th>$p_{m,1}$</th>
<th>$p_{m+1,1}$</th>
<th>$p_{m,n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_{m+1,0}$</td>
<td>$p_{m+1,1}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_{m+n-1,0}$</td>
<td>$p_{m+n-1,1}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_{m+n,0}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Construct Table 3 row by row, using relation (11), to get $q_{mn}$.

<table>
<thead>
<tr>
<th>$q_{0,n}$</th>
<th>$q_{1,n}$</th>
<th>$q_{2,n}$</th>
<th>$q_{3,n}$</th>
<th>$q_{m,n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_{0,n+1}$</td>
<td>$q_{1,n+1}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$q_{0,n+2}$</td>
<td>$q_{1,n+2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$q_{0,n+3}$</td>
<td>$q_{1,n+3}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$q_{0,n+4}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This amounts to the following theorem.

**Theorem 6.** For the computation of an arbitrary element $r_{m,n}$ of Table 1, use can be made of the following recurrence relations

$$a_{m+n-j}^{(m+n-j)} = a_{m+n-j+1}^{(m+n-j+1)} [a_{m+n-j}^{(m+n-j+1)} - (z_{m+n} - z_i) a_{m+n-j}^{(m+n-j+1)}]$$

for $i = 0, 1, \ldots, m + n - j$, and

$$b_{m+n-j+1}^{(m+n-j+1)} = b_{m+n-j+1}^{(m+n-j+1)} [1 + (z_{m+n} - z_i) B_{m+n-j+1}^{(m+n-j+1)}]$$

for $i = 0, 1, \ldots, m + n - j$. □

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for \( i = 0, 1, \ldots, j - 1 \), where

\[
B_{m+n-j+1,i-j-1} = -\frac{b^{(i)}_{m+n-j}}{b^{(i-1)}_{m+n-j}(2^{(m+n)} - 2\alpha) + b^{(i-1)}_{m+n-j-1}}.
\]

Looking at the triangular structure, it is clear that this method can be of interest, if we have to know the following triangular array of rational Hermite interpolants:

\[
\begin{array}{cccc}
R_{0,0} & R_{0,1} & \cdots & R_{0,n-1} & R_{0,n} \\
R_{1,0} & R_{1,1} & \cdots & R_{1,n-1} \\
\vdots & & & & \\
R_{n-1,0} & R_{n-1,1} & \cdots & R_{n,n} \\
R_{n,0} & \\
\end{array}
\]

5. SOME REMARKS

In the case that \( z_i = 0 \) for \( i = 0, 1, \ldots \) the relations (9) and (10) become the recurrence relations for the algorithm of Baker[1, 2] for Padé approximation.

In the same way the relations (12) and (13) reduce to the formulas for the algorithm of Longman[4, 2].

Other extensions of methods for ordinary rational interpolation or for the Padé approximation to rational Hermite interpolation can be found in [6].

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REFERENCES