JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 76, 174–182 (1980)

Borsuk Presentations and the Fixed-Point Index*

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Submitted by Ky Fan

DEDICATED TO PROFESSOR DR.-ING. HEINZ UNGER ON THE OCCASION OF HIS 65TH BIRTHDAY

We develop a theory of fixed-point index for maps $f: X \to X$ such that the fixed-point set of f is contained in a compact invariant set A, and A has a Borsuk presentation as an intersection of a decreasing sequence of ANR $Z_n \subset X$ and Z_{n+1} is a retract of Z_n.

1. INTRODUCTION

When one tries to develop a theory of fixed-point index for a class of spaces and a class of maps, the least thing one has to assume is that the fixed-point sets behave decently. To be somewhat more specific, let $f: X \to X$ be a map, then it will be necessary to assume that $\text{Fix}\, f := \{x \in X \mid fx = x\}$ be compact. It seems, however, inconvenient to have an assumption on the topological structure of Fix f itself, since this set will be "unknown" in general. So we consider the case where there is a compact invariant set $A \supset \text{Fix}\, f$, and we state some conditions on A. A traditional assumption would require A to be a compact ANR; in [6, 7] the author and H.-O. Peitgen considered the case where X is an ANR (not assumed to be compact), f is locally compact, and A has arbitrarily small invariant neighbourhoods. Now open subsets of an ANR are again ANR, so we are immediately reduced to a classical situation. Here, we consider a much more general situation—we make no assumptions on X except that for convenience we assume X to be metric. As to A, we assume that A has a Borsuk presentation as intersection of a descending sequence $Z_1 \supset \cdots \supset Z_n \supset Z_{n+1} \supset \cdots$ of compact ANR. We owe this idea to Dugundji's interesting article [4], which in turn extends relevant work of Borsuk [3]. Of course, each compactum is representable as an intersection of a descending sequence of compact ANR, so in view of Borsuk's [1] and Kinoshita's [8] examples this condition

* The author gratefully acknowledges support by the Deutsche Forschungsgemeinschaft—SFB 72 an der Universitat Bonn.

0022-247X/80/070174-09$02.00/0

cannot be adequate for fixed-point theory. Dugundji [4], however, has already pointed out how to circumvent this difficulty: it is sufficient to assume that all Z_n be subspaces of X, which is quite a natural condition, of course. Dugundji assumed f to be a Borsuk map, i.e., $f: A \to A$ can be approximated arbitrarily closely by a map $g: Z_n \to A$. We avoid this condition by posing stronger conditions on the Borsuk presentation, but our conditions will only involve A, and we make no assumptions on f. This allows us to generalize our results immediately to more complicated types of mappings, i.e., multivalued mappings. Since, however, most readers will not be interested in multivalued mappings, we will not burden the main body of the presentation with such complications but defer this to the final section. So we will now desribe the situation we shall investigate, and try to be somewhat more precise.

2. Notation and Terminology

We will use Čech homology, throughout, and we will use rational coefficients; all homological terms are to be understood in the Čech theory. Now let (X, d) be a metric space and $f: X \to X$ a continuous mapping. Consider the following assumptions:

(2.1) There is a compact invariant A containing $\text{Fix } f := \{x \in X \mid fx = x\}$ and satisfying the following conditions:

(2.2) There is a descending sequence of compact ANR, $(Z_n)_{n \in \mathbb{N}}$, contained in X such that $A = \bigcap_{n=1}^{\infty} Z_n$.

(2.3) For each n there is a retraction $\rho_{n+1}^n: Z_n \to Z_{n+1}$. For $m > n$ write $\rho_m{}^n := \rho_m^{m-1} \cdots \rho_{n+1}^n: Z_n \to Z_m$.

(2.4) For each $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $m > n \geqslant n_0$ and all $x \in Z_n$ we have $d(x, \rho_m{}^n x) < \epsilon$.

Definition 2.1. Let (X, d) be a metric space and $f: X \to X$ a continuous mapping. We say that the fixed-point set of f is *Borsuk-approachable* if there is a compact invariant set A satisfying conditions (2.1)–(2.4) above.

We retain the notation from above and denote by $i_n^{n+1}: Z_{n+1} \to Z_n$ the inclusion. For $m > n$ we again let $i_n{}^m := i_n^{n+1} \cdots i_{m-1}^m: Z_m \to Z_n$. For $n \in \mathbb{N}$ we denote by $i_n: A \to Z_n$ the inclusion. Since Z_n is an ANR and X is metric, there is an open set $\Omega_n \subset X$ and a retraction $r_n: \Omega_n \to Z_n$. Especially, we may assume, and we will always do so, that $\Omega_{n+1} \subset \Omega_n$ and $r_{n+1} = \rho_{n+1}^n r_n \mid \Omega_{n+1}$, hence $r_{n+1} = \rho_{n+1}^1 r_1 \mid \Omega_{n+1}$. In view of (2.4), it is obvious that we may choose Ω_n so small that we have

(2.5) For each $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $m > n \geqslant n_0$ and all $x \in \Omega_n$ we have $d(x, \rho_m{}^n r_n x) < \epsilon$.

In the sequel we shall always assume that Ω_n is chosen in such a way that (2.5) is satisfied. Moreover, whenever we have a map f such that Fix f is Borsuk-approachable, we will assume that we have chosen a set A according to (2.1) and we will use all of the notation above without further explanation.

Remark 2.2. Condition (2.3) does not seem to be very realistic, for it implies that H_*Z_{n+1} is an epimorphic image of H_*Z_n. So the homology of Z_n gets simpler the more one approaches A. In fact, it forces us to choose A in such a way that the homology of A is of finite type (cf. infra, 4.1). Note that, if the answer to Borsuk's Problem, (11.5) in [2], is affirmative, this would mean that we could assume that all ρ_{n+1}^n are deformations, hence A would be a FANR [2, 11.1].

3. The Index

We now prepare the definition of the index.

PROPOSITION 3.1. *Let (X, d) be a metric space, $f: X \to X$ a continuous map, and U open in X. Assume that $fx \neq x$ whenever $x \in \partial U$, and that* Fix f *is Borsuk-approachable. Then for $n \in \mathbb{N}$ sufficiently large, there exists $m_n \in \mathbb{N}$ such that* $\operatorname{ind}(Z_m, \rho_m{}^n r_n f, Z_m \cap U)$ *is defined whenever $m \geqslant m_n$.*

Here, ind is the classical fixed-point index for compact ANR, of course.

Proof. Choose $\mu > 0$ such that $d(x, fx) > \mu$ for all $x \in Z_1 \cap \partial U$. According to (2.5), we find n_0 such that $d(x, \rho_m{}^n r_n x) < (\mu/2)$ whenever $m > n \geqslant n_0$ and $x \in \Omega_n$. So let $n \geqslant n_0$. Since by (2.1) $f(A) \subset A$, we may choose m_n such that $f(Z_{m_n}) \subset \Omega_n$. Now if $x \in Z_m \cap \partial U$ we have that

$$d(x, \rho_m{}^n r_n fx) \geqslant d(x, fx) - d\left(fx, \rho_m{}^n r_n fx\right) > \frac{\mu}{2}.$$

But this means that $\operatorname{ind}(Z_m, \rho_m{}^n r_n f, U \cap Z_m)$ is defined. Q.E.D.

We have not been able to decide whether actually

$$\operatorname{ind}(Z_m, \rho_m{}^n r_n f, Z_m \cap U) = \operatorname{ind}(Z_{m+1}, \rho_{m+1}^n r_n f, Z_{m+1} \cap U).$$ [1]

It is not difficult to see that equality would hold if the retractions ρ_{n+1}^n were deformations. As to the dependence on n, by our choice of the retractions r_n it is obvious that

$$\operatorname{ind}(Z_m, \rho_m{}^n r_n f, Z_m \cap U) = \operatorname{ind}(Z_m, \rho_m^{n+1} r_{n+1} f, Z_m \cap U)$$

provided both sides are defined. Thus, the choice of n is rather irrelevant, for

[1] *Note added in proof.* In a subsequent paper we shall show that the index in fact takes values in $\mathbb{Z} \subset \mathbb{Z}^*$ and does not depend on the choice of Borsuk presentation.

replacing n by $n + 1$ will just result in increasing m_n. So if we agree to identify two sequences of integers, if they differ only in finitely many terms, we could define $\operatorname{ind}(X, f, U)$ to be the sequence

$$(\operatorname{ind}(Z_m, \rho_m{}^n r_n f, Z_m \cap U))_{m \geqslant m_n}.$$

Formally:

DEFINITION 3.2. Let $\mathbb{Z}^*$ denote the quotient ring of $\mathbb{Z}^{\mathbb{N}}$ (a countable product of the integers) by $\bigoplus_{\mathbb{N}} \mathbb{Z}$ (a countable sum of the integers). If $x \in \mathbb{Z}^{\mathbb{N}}$, denote the class of x in $\mathbb{Z}^*$ by $[x]$. We embed $\mathbb{Z}$ into $\mathbb{Z}^*$ by associating with $n \in \mathbb{Z}$ the class of the sequence with constant term n.

The idea of using $\mathbb{Z}^*$ rather than $\mathbb{Z}$ is due to Wong [11] (cf. also Weinberg [10]). Note that, if we have a sequence $x = (x_n)_{n > n_0}$ of integers, then x determines a unique element in $\mathbb{Z}^*$.

DEFINITION 3.3. Let (X, d) be a metric space, $f: X \to X$ a continuous map, and U open in X. Assume that $fx \neq x$ for $x \in \partial U$, and that $\operatorname{Fix} f$ is Borsuk-approchable. Then $\operatorname{ind}_A(X, f, U)$ is defined to be the unique element in $\mathbb{Z}^*$ determined by the sequence $(\operatorname{ind}(Z_m, \rho_m{}^n r_n f, Z_m \cap U))_{m > m_n}$.

In fact, $\operatorname{ind}_A(X, f, U)$ will depend not only on the choice of A but also on the Borsuk presentation of A and the retractions. We do not indicate this dependence, since we want to avoid too cumbersome a notation. The dependence on A, however, should be indicated, since it is easy to see that different choices of $A \supset \operatorname{Fix} f$ will result in different values of the index. Finally, a $\mathbb{Z}^*$-valued index is, of course, not as satisfactory as an integer-valued one. We think, however, that this is a tolerable inconvenience, since we shall provide a decent normalization theorem in the next section.

Remark 3.4. We assumed that f is defined on the whole space X. If we just had a mapping $f: \bar{U} \to X$, we would have to assume in Condition (2.1) that there is a compact invariant set A with $\operatorname{Fix} f \subset A \subset U$, the ANR Z_n will then eventually lie in U, and we could proceed exactly as above.

We now list some properties of the index. We do not formulate the commutativity property, since it would require some rather awkward hypotheses on the Borsuk presentations, and the interested reader should be able to find these out for himself.

PROPOSITION 3.5. *Let (X, d) be a metric space and U open in X.*

(1) *Let $f: X \to X$ be a continuous map with Borsuk-approachable fixed-point set. Let U_1, U_2 be disjoint open subsets of U, and assume that $fx \neq x$ for $x \in \bar{U} \setminus U_1 \cup U_2$. Then*

$$\operatorname{ind}_A(X, f, U) = \operatorname{ind}_A(X, f, U_1) + \operatorname{ind}_A(X, f, U_2).$$

In particular, $\operatorname{ind}_A(X, f, U) \neq 0$ implies the existence of a fixed point in U.

(2) *Let* $h: X \times [0, 1] \to X$ *be a continuous map such that* $h(x, t) \neq x$ *for all* $x \in \partial U$ *and* $t \in [0, 1]$. *Assume that each* $h(\cdot, t)$ *has a Borsuk-approachable fixed-point set, and that we may choose the same set* A *and the same retractions for all* $h(\cdot, t)$. *Then* $\mathrm{ind}_A(X, h(\cdot, 0), U) = \mathrm{ind}_A(X, h(\cdot, 1), U)$.

(3) *Let* $x_0 \in X$, *and let* $c: X \to X$ *be the constant map with* $c(x) = x_0$. *If* U *is a neighborhood of* x_0, *then*

$$\mathrm{ind}_{\{x_0\}}(X, c, U) = 1.$$

(4) *Let* (Y, δ) *be another metric space, and let* $f: X \to X$, $g: Y \to Y$ *be continuous maps with Borsuk-approachable fixed-point sets. Let* V *be open in* Y, *and assume that* $fx \neq x$ *for* $x \in \partial U$ *and* $gy \neq y$ *for* $y \in \partial V$. *Assume that* A' *satisfies* (2.1)–(2.4) *for* g. *Then*

$$\mathrm{ind}_{A \times A'}(X \times Y, f \times g, U \times V) = \mathrm{ind}_A(X, f, U) \cdot \mathrm{ind}_{A'}(Y, g, V).$$

Proof. (1)

$$\begin{aligned}\mathrm{ind}_A(X, f, U) &= [(\mathrm{ind}(Z_m, \rho_m{}^n r_n f, Z_m \cap U))_{m \geqslant m_n}] \\ &= [(\mathrm{ind}(Z_m, \rho_m{}^n r_n f, Z_m \cap U_1))_{m \geqslant m_n}] \\ &\quad + [(\mathrm{ind}(Z_m, \rho_m{}^n r_n f, Z_m \cap U_2))_{m \geqslant m_n}] \\ &= \mathrm{ind}_A(X, f, U_1) + \mathrm{ind}_A(X, f, U_2).\end{aligned}$$

Now assume that $\mathrm{ind}_A(X, f, U) \neq 0$. Choose n_0 and m_{n_0} such that for $m \geqslant m_{n_0}$ $\mathrm{ind}(Z_m, \rho_m{}^n r_n f, Z_m \cap U)$ is defined. Then there is m_n' such that $\mathrm{ind}(Z_m, \rho_m{}^n r_n f, Z_m \cap U) \neq 0$ for $m \geqslant m_n'$. Then for each $m \geqslant m_{n_0}'$ choose n_m such that $n_m \to_{m \to \infty} \infty$ and $\mathrm{ind}(Z_m, \rho_m^{n_m} r_{n_0} f, Z_m \cap U)$ is defined and equals $\mathrm{ind}(Z_m, \rho_m^{n_0} r_{n_0} f, Z_m \cap U) \neq 0$. So for each $m \geqslant m_{n_0}'$ there is $x_m \in Z_m$ such that $\rho_m^{n_m} r_{n_m} f x_m = x_m$. Without loss of generality we may assume that $x_m \to x_0 \in A$. But then (2.5) will imply

$$d(x_0, fx_0) \leqslant d(x_0, x_m) + d(\rho_m^{n_m} r_{n_m} f x_m, f x_m) + d(f x_m, f x_0) \xrightarrow[m \to \infty]{} 0.$$

(2) is trivial.

(3) Fix $c = \{x_0\}$ is, of course, Borsuk-approachable, since we may choose $A := Z_n := \{x_0\}$ for all n. Hence

$$\mathrm{ind}_{\{x_0\}}(X, c, U) = \mathrm{ind}(\{x_0\}, c, \{x_0\}) = 1.$$

(4) Choose Z_m', r_n', and $\rho_m'^n$ for Y and g. Now it is easy to see that we may choose the same n and m_n for f and g. But then

$$\begin{aligned}
&\operatorname{ind}_{A\times A'}(X \times Y, f \times g, U \times V) \\
&\quad = [(\operatorname{ind}(Z_m \times Z'_m, \rho_m{}^n r_n f \times \rho'^n_m r'_n g, (Z_m \cap U) \times (Z'_m \cap V)))_{m \geqslant m_n}] \\
&\quad = [(\operatorname{ind}(Z_m, \rho_m{}^n r_n f, Z_m \cap U))_{m \geqslant m_n}] \cdot [(\operatorname{ind}(Z'_m, \rho'^n_m r'_n g, Z'_m \cap V))_{m \geqslant m_n}] \\
&\quad = \operatorname{ind}_A(X, f, U) \cdot \operatorname{ind}_{A'}(Y, g, V). \qquad \text{Q.E.D.}
\end{aligned}$$

4. The Normalization Property

As we announced already, the index for maps with a Borsuk-approachable approachable fixed-point set satisfies a normalization property. Of course, one cannot expect a formula like $\operatorname{ind}_A(X, f, X) = \Lambda(f)$ (we denote the Lefschetz number of f by $\Lambda(f)$), since we provided no proviso guaranteeing that $\Lambda(f)$ actually exists, whereas it is easily seen that $\Lambda(f\,|_A)$ exists. Moreover, the choice of the ambient space X is rather arbitrary (consider, e.g., Proposition 3.5, 3))—what we are really concerned with is the invariant set A. So what we should expect is a formula like $\operatorname{ind}_A(X, f, X) = \Lambda(f\,|_A)$. We prepare the proof of this property by establishing the existence of $\Lambda(f\,|_A)$.

Proposition 4.1. *Let $(Z_n)_{n\in\mathbb{N}}$ be a descending sequence of compact* ANR *such that for each n there is a retraction $\rho_{n+1}\colon Z_n \to Z_{n+1}$. Then*

(a) *The homology of $A := \bigcap_{n=1}^{\infty} Z_n$ is of finite type, i.e., $\dim_{\mathbb{Q}} H_*A < \infty$.*

(b) *For k sufficiently large, the inclusion $i_k\colon A \to Z_k$ induces isomorphisms $i_{k*}\colon H_*A \to H_*Z_k$.*

Proof. We continue to use the notation of Section 2.

(a) Since for all m we have a retraction $\rho_m{}^1\colon Z_1 \to Z_m$, we see that $\dim_{\mathbb{Q}} H_*Z_m \leqslant \dim_{\mathbb{Q}} H_*Z_1$. Let $N := \dim_{\mathbb{Q}} H_*Z_1$. Then we claim that $\dim_{\mathbb{Q}} H_*A \leqslant N$. For, let us assume that there are $N+1$ linearly independent elements in H_*A, say $a_1, \ldots, a_{N+1}$. Since the Čech theory is continuous on compact spaces, we may choose m so large that i_{m*} is injective on the subspace of H_*A generated by $a_1, \ldots, a_{N+1}$ (cf., (1.2) in [4]). But this would mean $\dim_{\mathbb{Q}} H_*Z_m \geqslant N+1$.

(b) By (a) and continuity of the Čech theory we may now choose m_0 such that $i_{m*}\colon H_*A \to H_*Z_m$ is injective for $m \geqslant m_0$ (here, we use again (1.2) in [4]). According to Borsuk's homology embedding theorem [4, Theorem 2.3] for each $m \geqslant m_0$ there is $k(m)$ such that $\operatorname{Image} i_{m*} = \operatorname{Image} i^k_{m*}$ for all $k \geqslant k(m)$. So fix $m \geqslant m_0$ and let $k \geqslant k(m)$. Since Z_k is a retract of Z_m, we see that i^k_{m*} is injective, and i_{m*} is injective by our choice of m_0. Since $k \geqslant k(m)$ the composition $(i_m{}^k)_*^{-1} i_{m*}\colon H_*A \to H_*Z_k$ is an isomorphism. Now ρ^m_{k*} is a left inverse for i^k_{m*}, so $(i_m{}^k)_*^{-1} i_{m*} = \rho_{k*} i_{m*}$. But it is obvious that $\rho_k{}^m i_m = i_k$.

Q.E.D.

Now the proof of the Normalization Theorem is easy.

THEOREM 4.2. *Let* (X, d) *be a metric space and* $f: X \to X$ *a continuous map such that* Fix f *is Borsuk-approachable. Then*

$$\operatorname{ind}_A(X, f, X) = \Lambda(f|_A).$$

In fact, there is n_0 *such that for all* $n \geqslant n_0$

$$\operatorname{ind}_A(X, f, X) = \operatorname{ind}(Z_m, \rho_m{}^n r_n f, Z_m) = \Lambda(\rho_m{}^n r_n f|_{Z_m}) = \Lambda(f|_A)$$

for infinitely many m.

Proof. Choose n_0 so large that for $n \geqslant n_0$ there is m_n such that $\operatorname{ind}(Z_m, \rho_m{}^n r_n f, Z_m)$ is defined for $m \geqslant m_n$. Fix $n \geqslant n_0$. According to Proposition 4.1 we may then choose $m_n' \geqslant m_n$ such that $i_{m*}: H_*A \to H_*Z_m$ is isomorphic for $m \geqslant m_n'$. Fix $m \geqslant m_n'$. The following diagram is obviously commutative:

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & A \\ {\scriptstyle i_m}\downarrow & & \downarrow{\scriptstyle i_m} \\ Z_m & \xrightarrow[\rho_m{}^n r_n f]{} & Z_m \end{array}.$$

The vertical arrows induce isomorphisms in homology, hence

$$\Lambda(f|_A) = \Lambda(\rho_m{}^n r_n f|_{Z_m}) = \operatorname{ind}(Z_m, \rho_m{}^n r_n f, Z_m).$$

By the definition of ind and $\mathbb{Z}^*$ this means $\Lambda(f|_A) = \operatorname{ind}_A(X, f, X)$. Q.E.D.

Remark 4.3. Note that the Normalization Theorem (or rather Proposition 4.1) is the first place where we made full use of the fact that we have retractions $\rho_{m+1}^m: Z_m \to Z_{m+1}$; for in the proof we need the fact that $(i_m{}^k)_*^{-1}: i_{m*}(H_*A) \to H_*Z_k$ is actually the restriction to $i_{m*}(H_*A)$ of a homomorphism, which is induced by a map, viz., $\rho_k{}^m$.

5. MULTIVALUED MAPPINGS

If X is a topological space denote the set of nonvoid compact subsets of X by $K(X)$. If Y is another topological space, a map $f: X \to K(Y)$ is u.s.c. (*upper semicontinuous*) if for each $x \in X$ and neighbourhood U of fx there is a neighbourhood V of x such that $f(V) \subset U$. Define $\Gamma(f) := \{(x, y) \in X \times Y \mid y \in fx\}$, and denote by p_X, q_Y the projections $p_X: \Gamma(f) \to X$, $q_Y: \Gamma(f) \to Y$ with $p_X(x, y) = x$, $q_Y(x, y) = y$. For technical reasons we will now use Čech homology with compact supports (and rational coefficients). By $A(X)$ we will then denote the set of compact acyclic subsets of a space X. Now let (X, d) be again a metric space and call a mapping $f: X \to K(X)$ *admissible* if there are metric spaces $X = X_0, \ldots, X_{n+1} = X$ and u.s.c. maps $f_i: X_i \to A(X_{i+1})$ for $i \in \{0, \ldots, n\}$ such that $f = f_n \cdots f_0$. By the Vietoris–Begle theorem for Čech homology with compact supports [9], $p_{X_i}: \Gamma(f_i) \to X_i$ induces isomorphisms in homology, and one puts $f_{i*} := q_{X_{i+1}*}(p_{X_i}^*)^{-1}$, and then $f_* := f_{n_*} \cdots f_{0_*}$. Note that the representation of an admissible mapping as a composition of acyclic-valued mappings is by no means unique, and so one cannot expect that commutative diagrams for admissible mappings induce commutative diagrams in homology.

If $f: X \to K(X)$ is u.s.c., we put $\operatorname{Fix} f := \{x \in X \mid x \in fx\}$. As in Section 2, we say that an admissible map $f: X \to K(X)$ has a Borsuk-approachable fixed-point set, if there is a compact invariant set A satisfying (2.1)–(2.4). There is a fixed-point index for admissible maps of compact ANR (cf. [5]). (As a matter of fact, there is only an index $\operatorname{ind}_{\mathscr{U}}$ for sufficiently fine open covers $\mathscr{U}$ of X depending on the choice of the covering, but we ignore this complication since it will cause no problems in our context, and the Lefschetz number is uniquely defined, anyway.) Sections 2 and 3 then carry over almost verbatim; especially, there is again a fixed point index—denoted in the same way as before, and we will only explain the Normalization Theorem. (The point is that we have to make sure that the diagram displayed in the proof induces a commutative diagram in homology.)

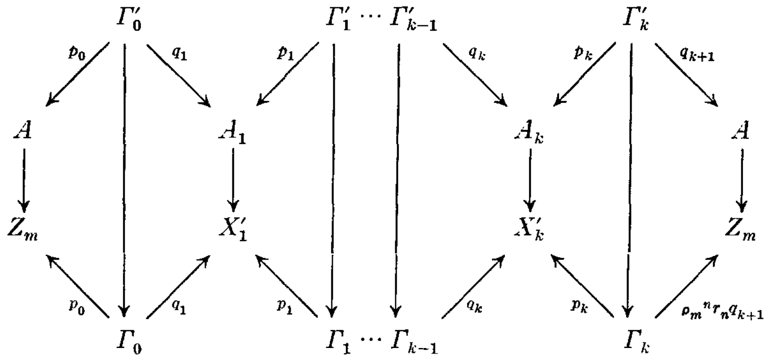
THEOREM 5.1. *Let (X, d) be a metric space and $f: X \to K(X)$ an admissible map such that* Fix f *is Borsuk-approachable. Then*

$$\operatorname{ind}_A(X, f, X) = \Lambda(f\,|_A).$$

Proof. Choose n_0, $n \geqslant n_0$, m_n', and $m \geqslant m_n'$ as in the proof of Theorem 4.2. We have to check that

$$\begin{array}{ccc} H_*A & \xrightarrow{\quad f_* \quad} & H_*A \\ {\scriptstyle i_{m*}}\downarrow & & \downarrow{\scriptstyle i_{m*}} \\ H_*Z_m & \xrightarrow{\quad (\rho_m{}^n r_n f)_* \quad} & H_*Z_m \end{array}$$

is commutative. Since f is admissible, there are metric spaces $X = X_0, \ldots, X_{k+1} = X$ and u.s.c. maps $f_i: X_i \to A(X_{i+1})$ such that $f = f_k \cdots f_0$. Write $p_i := p_{X_i}$, $q_i := q_{X_i}$, and let $\Gamma_i := \Gamma(f_i \cdots f_0 \mid Z_m)$, $\Gamma_i' := \Gamma(f_i \cdots f_0 \mid A)$, $A_i := f_{i-1} \cdots f_0(A)$, $X_i' := f_{i-1} \cdots f_0(Z_m)$. Then there is a commutative diagram of single-valued maps, where all vertical arrows are inclusions:



Applying H_* we see that in fact $i_{m*}f_* \mid H_*A = \rho_{m*}^n r_{n*} f_* i_{m*}$, which proves the theorem. Q.E.D.

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