Strong proximinality of closed convex sets

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Received 23 February 2010; received in revised form 27 December 2010; accepted 17 January 2011
Available online 23 January 2011

Abstract

We show that in a Banach space $X$, every closed convex subset is strongly proximinal if and only if the dual norm is strongly subdifferentiable and for each norm 1 functional $f$ in the dual space $X^*$, $J_X(f)$ - the set of norm 1 elements in $X$ where $f$ attains its norm - is compact. As a consequence, it is observed that if the dual norm is strongly subdifferentiable, then every closed convex subset of $X$ is strongly proximinal if and only if the metric projection onto every closed convex subset of $X$ is upper semi-continuous.

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Keywords: Strong proximinality; SSD points; Metric projections; Upper (Hausdorff) semi-continuous

1. Introduction

Let $X$ be a Banach space and $C$ a closed subset of $X$. The metric projection of $X$ onto $C$ is the set valued map defined by $P_C(x) = \{y \in C : ||x - y|| = d(x, C)\}$ for $x \in X$, where $d(x, C)$ denotes the distance of $x$ from $C$. If for every $x \in X$, $P_C(x) \neq \emptyset$, we say that $C$ is a proximinal subset of $X$.

For a Banach space $X$, we denote the closed unit ball and the unit sphere by $B_X$ and $S_X$ respectively. If $f \in S_{X^*}$ is a norm attaining functional, we define $J_X(f) = \{x \in S_X : f(x) = 1\}$.

For $x \in X \setminus C$ and given any $t > 0$, there exists $y \in C$ such that $||x - y|| < d(x, C) + t$. If we call such a $y$ a nearly best approximation to $x$ in $C$, a natural question is whether $y$ is close to an actual best approximation of $x$ in $C$. Clearly we are demanding more than proximinality of $C$ in $X$ and in [4] the authors called such a subset a strongly proximinal subset.

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doi:10.1016/j.jat.2011.01.001
Definition 1.1. Let $C$ be a closed subset in a Banach space $X$ and $x \in X$. For $t > 0$, consider the following set:

$$P_C(x, t) = \{ y \in C : \| x - y \| < d(x, C) + t \}.$$ 

A proximinal set $C$ is said to be strongly proximinal at $x \in X$ if for given $\epsilon > 0$ there exists a $t > 0$ such that

$$P_C(x, t) \subseteq P_C(x) + \epsilon B_X.$$ 

If $C$ is strongly proximinal at all points of $X$ we say that $C$ is strongly proximinal.

Some sufficient (and necessary) conditions for strong proximality of certain subspaces of some classical Banach spaces are studied in the literature [2,3,6,5,4].

In this paper we are motivated by the following question:

**Question 1.2.** Under what condition (necessary or sufficient) is every closed convex subset of $X$ strongly proximinal?

A known necessary condition for every closed convex subset of $X$ to be strongly proximinal is that the norm of $X^*$ is strongly subdifferentiable (see below for the definition). This follows from the fact that every closed hyperplane in $X$ is strongly proximinal if and only if the norm of $X^*$ is strongly subdifferentiable (this was noted as a corollary to a main theorem in [4]; however we will present a direct proof of this fact in Section 2 for completeness).

Our aim is to find an additional condition such that this necessary condition and the additional one become necessary and sufficient.

**Definition 1.3.** Let $X$ be a Banach space. The norm $\| \cdot \|$ is said to be strongly subdifferentiable (for short SSD) at $x \in X$ if the one-sided limit

$$\lim_{t \to 0^+} \frac{\| x + th \| - \| x \|}{t}$$

exists uniformly for $h \in S_X$. If the norm $\| \cdot \|$ of $X$ is SSD at all points of $S_X$, we say that $\| \cdot \|$ is SSD or the space $X$ is SSD.

Recall that (see [7]) a closed set $C \subseteq X$ is said to be approximatively compact if every minimizing sequence in $C$ has a convergent subsequence. It is easy to see that if $C$ is approximatively compact then $C$ is strongly proximinal. Also every closed convex subset of $X$ is approximatively compact if and only if $X$ is reflexive and (KK), where (KK) means that the relative weak and norm topologies coincide on the unit sphere $S_X$ of $X$. This is implicit in [9] (see also [7]).

Therefore a sufficient condition for every closed convex subset of $X$ to be strongly proximinal is that $X$ is reflexive and (KK). Interestingly, this condition also turns out to be necessary.

Our main result in this paper is the following:

**Theorem 1.4.** Let $X$ be a Banach space. Then the following statements are equivalent.

(a) $X^*$ is SSD and $J_X(f)$ is compact for every $f \in S_{X^*}$.
(b) $X$ is reflexive and (KK).
(c) Every closed convex subset of $X$ is approximatively compact.
(d) Every closed convex subset of $X$ is strongly proximinal.

Compare this with the well known result that every closed convex subset in $X$ is proximinal if and only if $X$ is reflexive.
We relate our main result to the continuity of metric projection as follows.

**Definition 1.5.** Let $C \subseteq X$ and $x \in X$. $P_C$ is said to be

(a) upper semi-continuous (for short usc) at $x$ if for every open set $U \subseteq X$ such that $P_C(x) \subseteq U$, there exists $\delta > 0$ such that $P_C(z) \subseteq U$ for every $z$ satisfying $\|z - x\| < \delta$,

(b) upper Hausdorff semi-continuous (for short uHsc) at $x$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $P_C(z) \subseteq P_C(x) + \varepsilon B_X$ for every $z$ satisfying $\|z - x\| < \delta$.

It is well known [1] that if $C$ is a subspace of $X$ then $P_C$ is usc at $x$ if and only if $P_C$ is uHsc at $x$ and $P_C(x)$ is compact. Also it is straightforward to see that if $C$ is a strongly proximinal subset then $P_C$ is uHsc. We will show that if $X^*$ is SSD then every closed convex subset of $X$ is strongly proximinal if and only if $P_C$ is usc for every closed convex subset of $X$. In this case we also get that $P_C$ is uHsc for every closed convex subset $C$ of $X$ and $P_C(x)$ is compact for every $x \in X$.

**2. Main results**

We first give a straightforward proof of the fact that the condition that $X^*$ is SSD is necessary for every closed convex subset of $X$ to be strongly proximinal. We will be using [4, Lemma 1.1] which we state as a fact.

**Fact 1.** Let $X$ be a Banach space and $f \in S_{X^*}$. The following assertions are equivalent.

(a) The dual norm on $X^*$ is SSD at $f$.

(b) $f$ is norm attaining and for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(x) > 1 - \delta \Rightarrow d(x, J_X(f)) < \varepsilon.$$ 

**Proposition 2.1.** Let $X$ be a Banach space and $f \in S_{X^*}$. Then the following statements are equivalent.

(a) For every $c \in \mathbb{R}$, the hyperplane $H_c = \{x \in X : f(x) = c\}$ is strongly proximinal.

(b) The norm of $X^*$ is SSD at $f$.

**Proof.** (a) $\Rightarrow$ (b): Since the hyperplane $H = \{x \in X : f(x) = 1\}$ is proximinal, $J_X(f) \neq \emptyset$. Let $(x_n)$ be a sequence in $B_X$ such that $f(x_n) \to 1$. We will show that $d(x_n, J_X(f)) \to 0$. By Fact 1 it follows that the norm of $X^*$ is SSD at $f$. Note that $d(0, H) = 1$ and $P_H(0) = J_X(f)$. We put $y_n = \frac{x_n}{f(x_n)}$. Then $y_n \in H$ and $\|y_n\| \to 1$. Therefore $y_n \in P_H(0, \delta_n)$ for some $\delta_n \to 0$ and since $H$ is strongly proximinal $d(y_n, P_H(0)) \to 0$. Since $\|x_n - y_n\| \to 0$, $d(x_n, P_H(0)) = d(x_n, J_X(f)) \to 0$.

(b) $\Rightarrow$ (a): Since the norm of $X^*$ is SSD at $f$, by Fact 1, $J_X(f) \neq \emptyset$. Therefore for every $c \in \mathbb{R}$, the hyperplane $H_c = \{x \in X : f(x) = c\}$ is proximinal. To show that $H_c$ is strongly proximinal, let $x_0 \in X \setminus H_c$. Without loss of generality we assume $f(x_0) > c$. For $n \geq 1$ let $x_n \in P_{H_c}(x_0, \frac{1}{n})$. We show that $d(x_n, P_{H_c}(x_0)) \to 0$ which completes the proof. We first note that $H_c = \{x \in X : f(x) = f(x_0) - d\}$ where $d = d(x_0, H_c)$. Let $y_n = \frac{x_0 - x_n}{d + \frac{2}{n}}$. Then $y_n \in B_X$ and $f(y_n) \to 1$. Therefore, by the assumption and Fact 1 there exists a sequence $(z_n)$ from $J_X(f)$ such that $\|y_n - z_n\| \to 0$. Now the sequence $(x_0 - dz_n)$ is in $P_{H_c}(x_0)$ and $\|x_n - x_0 + dz_n\| = \|x_0 - y_n(d + \frac{2}{n}) - x_0 + dz_n\| \to 0$. □
Proposition 2.1. We say that a Banach space $S_{X^*}$ is strongly proximinal if every convex subset of $S_{X^*}$ is proximinal. We show that the norm of $X^*$ is SSD at all norm attaining functionals of $S_{X^*}$ then all proximinal hyperplanes of $X$ are strongly proximinal. However, the condition that the norm of $X^*$ is SSD at all norm attaining functionals of $S_{X^*}$ is not sufficient for every proximinal convex subset to be strongly proximinal. To show this we construct an example of a proximinal convex subset of $c_0$ which is not strongly proximinal. Note that the norm of $\ell_1$ is SSD at every norm 1 norm attaining functional on $c_0$.

Lemma 2.2. Let $H = \{x : f(x) = c\}$ be a closed hyperplane in $X$ and $(x_n)$ be a sequence in $X$ such that $x_n \to x_0$ weakly for some $x_0$. Suppose $f(x_n) > c$ for all $n$. Then $(\overline{co}\{x_n : n \in \mathbb{N}\}) \cap H = \emptyset$. Suppose $f(x_0) \in H$, and in this case $(\overline{co}\{x_n : n \in \mathbb{N}\}) \cap H = \{x_0\}$.

Example 2.3. Consider the sequence $(x_n)$ in $c_0$ where $x_n = \left(-\frac{1}{n}, 0, \ldots, \frac{1}{2}, 0, \ldots\right)$, where $\frac{1}{2}$ occurs at the $n$th place. It is easy to see that $x_n \to 0$ weakly. Let $X = \overline{co}\{x_n : n \in \mathbb{N}\}$. Then $X$ is weakly compact and hence proximinal. We show that $X$ is not strongly proximinal.

Lemma 2.4. We say that a Banach space $X$ has the property strong (HR) (for short SHR) if for any $f \in S_{X^*}$ such that $J_X(f) \neq \emptyset$, and any sequence $(y_n)$ in $X$ such that $f(y_n) \geq 1$ and $d(y_n, J_X(f)) \to 0$ we have

$$d(y_n, (\overline{co}\{y_n : n \in \mathbb{N}\}) \cap S_X) \to 0.$$ 

The property (SHR) is a stronger version of the property (HR) defined in [8] as follows: $X$ has the property (HR) if for any $f \in S_{X^*}$ such that $J_X(f) \neq \emptyset$, and any sequence $(y_n)$ in $X$ such that $f(y_n) = 1$ and $d(y_n, J_X(f)) \to 0$, we have $d(y_n, (\overline{co}\{y_n : n \in \mathbb{N}\}) \cap S_X) \to 0$.

Proposition 2.5. Let $X$ be such that $J_X(f)$ is compact for every $f \in S_{X^*}$. Then $X$ has the property (SHR). Conversely, if $X$ is reflexive, then having the property (SHR) implies that $J_X(f)$ is compact for every $f \in S_{X^*}$.

Lemma 2.6. Let $C = \overline{co}\{x_n : n \in \mathbb{N}\}$ for all $n$. Then $X$ is compact for some $x \in X$.

Proposition 2.7. Let $X$ be such that $J_X(f)$ is compact for every $f \in S_{X^*}$. Then $X$ has the property (SHR). Conversely, if $X$ is reflexive, then having the property (SHR) implies that $J_X(f)$ is compact for every $f \in S_{X^*}$.

Proof. Let $f \in S_{X^*}$ be such that $J_X(f) \neq \emptyset$ and $J_X(f)$ be compact. Let $(y_n)$ be such that $f(y_n) \geq 1$, $d(y_n, J_X(f)) \to 0$ and $d(y_n, (\overline{co}\{y_n : n \in \mathbb{N}\}) \cap S_X) \geq \epsilon$ for some subsequence $(y_{n_k})$ and for some $\epsilon > 0$. Then by the compactness of $J_X(f)$ there is a subsequence of $(y_{n_k})$ which converges to some $x \in J_X(f)$. Note that $x \in (\overline{co}\{y_n : n \in \mathbb{N}\}) \cap S_X$ and this contradicts the assumption.

Now let $X$ be reflexive. Then for every $f \in S_{X^*}$, $J_X(f)$ is nonempty and weakly compact. Suppose $X$ has the property (SHR). Let $x_n$ be a sequence in $J_X(f)$. Define $y_n = \left(1 + \frac{1}{n}\right)x_n$ for every $n$. Then $f(y_n) > 1$ for every $n$ and $d(y_n, J_X(f)) \to 0$. Since $J_X(f)$ is weakly compact, we may choose a subsequence $(y_{n_k})$ of $(y_n)$ such that $y_{n_k} \to y_0$ weakly for some $y_0 \in J_X(f)$. By Lemma 2.2,

$$(\overline{co}\{y_{n_k} : k \in \mathbb{N}\}) \cap H = \{y_0\}$$
where $H = \{ x \in X : f(x) = 1 \}$. Since $y_0 \in S_X$ and $\overline{co}\{y_{n_k} : k \in \mathbb{N}\} \cap S_X \subseteq (\overline{co}\{y_{n_k} : k \in \mathbb{N}\}) \cap H$, we have $(\overline{co}\{y_{n_k} : k \in \mathbb{N}\}) \cap S_X = \{ y_0 \}$. By the property (SHR) we get that $d(y_{n_k}, (\overline{co}\{y_{n_k} : k \in \mathbb{N}\}) \cap S_X) \to 0$. That is $y_{n_k} \to y_0 \in J_X(f)$. This implies that $x_{n_k} \to y_0$ and the proof is complete. \qed

Before we prove our main result, we give an example to show that the property (SHR) is strictly stronger than the property (HR) considered in [8].

**Example 2.6.** Consider the following set in $\ell_2$: $$B' = \left\{ x = (x(1), x(2), \ldots) \in l_2 : \| x \|_2 \leq 1, |x(1)| \leq \frac{1}{2} \right\}.$$ Let $\| \cdot \|$ be the Minkowski functional of $B'$. Then $\| \cdot \|$ is an equivalent norm on $\ell_2$ and let $X = (\ell_2, \| \cdot \|)$. It is shown by Osman [8] that the space $X$ has the property (HR). We will show that the space does not have the property (SHR).

Consider $f \in X^*$ defined by $f((x(1), x(2), \ldots)) = 2x(1)$. It is clear that $f \in S_{X^*}$. Let $H = \{ x = (x(1), x(2), \ldots) \in X : f(x) = 1 \} = \left\{ x \in X : x(1) = \frac{1}{2} \right\}$. Then the closed hyperplane $H$ supports the unit ball $B'$ and $J_X(f) = H \cap B'$ is not compact. Hence by Proposition 2.5, the space cannot have the property (SHR).

**Proof of Theorem 1.4.** (a) $\Rightarrow$ (b): If the norm of $X^*$ is SSD at some $f \in S_{X^*}$ then $f$ is norm attaining on $S_X$ [4, Lemma 1.1]. Hence if $X^*$ is SSD then $X$ is reflexive.

To show being (KK), let $(x_n)$ be a sequence in $S_X$ such that $x_n \to x$ weakly for some $x \in S_X$. Suppose $f(x) = 1$ for some $f \in S_{X^*}$. Then $f(x_n) \to 1$. Thus by [4, Lemma 1.1], $d(x_n, J_X(f)) \to 0$. By the compactness of $J_X(f)$, there exists a norm convergent subsequence $(x_{n_k})$. Since $x_{n_k} \to x$ weakly, $x_{n_k} \to x$ in norm. Starting with any subsequence of $(x_n)$ we can produce, by the above argument, a further subsequence which is norm convergent to $x$. Hence $(x_n)$ converges to $x$ in norm.

(b) $\Rightarrow$ (c): This is essentially proved in [9]. See also [7].

(c) $\Rightarrow$ (d) is easy.

(d) $\Rightarrow$ (a): Since every closed convex subset of $X$ is strongly proximinal, by Proposition 2.1, $X^*$ is SSD. We will show that $X$ has the property (SHR). This, with Proposition 2.5, will show that $J_X(f)$ is compact for every $f \in S_{X^*}$.

Let $(y_{n})$ be any sequence in $X$ such that $f(y_{n}) \geq 1$ for all $n$ and $d(y_{n}, J_X(f)) \to 0$ for some $f \in S_{X^*}$. Take $C = \overline{co}\{y_{n} : n \in \mathbb{N}\}$. Since $\| y_{n} \| \to 1, d(0, C) = 1$ and $y_{n} \in P_{C}(0, \delta_{n})$ for some $\delta_{n} \to 0$. By the strong proximinality of $C$, we have $d(y_{n}, P_{C}(0)) \to 0$. Since

$$P_{C}(0) = (\overline{co}\{y_{n} : n \in \mathbb{N}\}) \cap S_X,$$

we have $d(y_{n}, \overline{co}\{y_{n} : n \in \mathbb{N}\} \cap S_X) \to 0$. \qed

**Remark 2.7.** In Question 1.2 we demand that every closed convex set is strongly proximinal in $X$. This automatically forces the space $X$ to be reflexive. However, in a non-reflexive set up one may ask when every proximinal convex subset is strongly proximinal. From Proposition 2.1, we observe that the condition that the norm of $X^*$ is SSD at all norm attaining functionals of $S_{X^*}$ is necessary for every proximinal convex subset to be strongly proximinal. In the following proposition we show that the conditions that the norm of $X^*$ is SSD at all norm attaining functionals of $S_{X^*}$ and $X$ has the property (SHR) are sufficient for this.
Proposition 2.8. Let $X$ be a Banach space such that the norm of $X^*$ is SSD at every norm attaining functional of $S_{X^*}$ and $X$ has the property (SHR). Then every proximinal convex subset in $X$ is strongly proximinal.

Proof. Suppose $C$ is a proximinal convex subset such that $d(0, C) = 1$ and $y_n \in P_C(0, \frac{1}{n})$ for every $n$. Let $H = \{x \in X : f(x) = 1\}$, $f \in S_{X^*}$. Then $H$ separates $C$ and $B_X$. Choose $y'_n \in [0, y_n] \cap H$. Then

$$1 = d(0, H) \leq \|y'_n\| \leq 1 + \frac{1}{n} \to 1.$$  

This implies that $\|y'_n\| \to 1$. Since $f$ is norm attaining on $S_X$, the norm of $X^*$ is SSD at $f$. By Proposition 2.1, $H$ is strongly proximinal and therefore $d(y'_n, P_H(0)) \to 0$. Note that $\|y_n - y'_n\| \to 0$ and $P_H(0) = J_X(f)$. Therefore $d(y_n, J_X(f)) \to 0$. Since $f(y_n) \geq 1$, by the property of (SHR), $d(y_n, (\overline{\text{co}}\{y_n : n \in \mathbb{N}\}) \cap S_X) \to 0$. This implies that $d(y_n, P_C(0)) \to 0$ because $(\overline{\text{co}}\{y_n : n \in \mathbb{N}\}) \cap S_X \subseteq P_C(0)$. This proves that $C$ is strongly proximinal. \[\square\]

Remark 2.9. Note that in the case $X = c_0$, the dual norm is SSD at all norm attaining functionals of $S_{X^*}$. Proposition 2.8 and Example 2.3 together show that $c_0$ does not have the property (SHR). It is also evident that for every norm attaining $f \in S_{\ell_1}$, $J_{c_0}(f)$ is never compact. We do not know of an example of a non-reflexive Banach space $X$ satisfying the condition of Proposition 2.8 but $J_X(f)$ is not compact for a norm attaining functional $f \in S_{X^*}$.

We now relate our main result to the continuity property of the metric projection.

It is easy to see that if $C$ is a strongly proximinal subset then $P_C$ is uHsc. In the next result we see that if every closed convex subset of $X$ is strongly proximinal then the metric projection becomes uHsc for every closed convex subset of $X$.

It is well known [1] that if $C$ is a subspace of $X$ then $P_C$ is uHsc at $x$ if and only if $P_C$ is uHsc at $x$ and $P_C(x)$ is compact. The following result characterizes the upper semi-continuity of $P_C(\cdot)$ for every closed convex subset $C$ of $X$.

Proposition 2.10. Let $X$ be a Banach space such that $X^*$ is SSD. Then the following statements are equivalent.

(a) For every $f \in S_{X^*}$, $J_X(f)$ is compact.
(b) Every closed convex subset of $X$ is strongly proximinal.
(c) For every closed convex subset $C$ of $X$, the metric projection $P_C(\cdot)$ is uHsc and $P_C(x)$ is compact for every $x \in C$.
(d) For every closed convex subset $C$ of $X$, the metric projection $P_C(\cdot)$ is uHsc.

Proof. (a) $\Rightarrow$ (b): This follows from Theorem 1.4.
(b) $\Rightarrow$ (c): By Theorem 1.4, every closed convex subset of $X$ is approximatively compact. The compactness of $P_C(x)$ follows from the approximative compactness of $C$.
(c) $\Rightarrow$ (d): This is known [1].
(d) $\Rightarrow$ (a): For given $f \in S_{X^*}$ consider the hyperspace $G = \{x \in X : f(x) = 0\}$. Since $G$ is a subspace, by (d) and a result of [1], $P_G(x)$ is compact for every $x \in X$. This implies that $J_X(f)$ is compact because $P_G(x) = \{x - f(x)z : z \in J_X(f)\}$.

\[\square\]
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