Stochastic Processes and their Applications 24 (1987) 151-155
North-Holland

A NOTE ON THE PROPERTIES OF SOME NONSTATIONARY ARMA PROCESSES

N. SINGH and M. Shelton PEIRIS

Department of Mathematics, Monash University, Clayton, VIC 3168, Australia

Received 17 February 1986
Revised 8 October 1986

The aim of this note is to study the properties of some nonstationary autoregressive-moving average (ARMA) processes that are considered important in real world situations. In particular, the covariance structure and linear predictors are obtained.

Hilbert space * linear prediction * time-dependent coefficients

1. Introduction

Many time series exhibit behaviour which cannot be explained using models for stationary processes. For instance, the economies of many developing countries show signs of steady growth. This indicates that there is a typical economic upward trend through time. The nonstationarity may be due to trends mostly found in the mean or the instability in the variance or both.

The usual approach to nonstationary problems in time series analysis is to assume some suitable difference of the stochastic process under study to be stationary. However, a suitable transformation may not always exist as its application depends on the constitution of the series. Hence, another way of studying the second-order nonstationary processes is the use of ARMA models with time varying coefficients.

Whittle [6], Abdrabbo and Priestley [1] have considered ARMA models with time-dependent coefficients and additive stationary white noise. In their monograph Granger and Newbold [3] have outlined briefly the forecasting of nonstationary ARMA processes. Recently Niemi [5] examined the effect of nonstationary noise on ARMA processes with constant AR and MA parameters and found the effect insignificant on the estimation of parameters.

In this note we consider a nonstationary ARMA process with time-dependent coefficients defined by

$$\Phi_t(B)X_t = \Theta_t(B)e_t, \tag{1.1}$$

where

$$\Phi_t(B) = 1 + \phi_{1t}B + \cdots + \phi_{pt}B^p, \qquad \Theta_t(B) = 1 + \theta_{1t}B + \cdots + \theta_{qt}B^q$$

are the autoregressive (AR) and moving average (MA) operators respectively satisfying AR and MA regularity conditions, namely, the zeros of the polynomials $\Phi_t(B)$ and $\Theta_t(B)$ lie in the region $|z| > \lambda$, where $\lambda > 1$; B is a lag operator ($B^s X_t = X_{t-s}$) and $\{e_t\}$ is a white noise process with mean zero and finite variance $\sigma_t^2 < M < \infty$, $\forall t \in Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of all integers.

The following sections are devoted to the study of some properties of such nonstationary ARMA processes:

2. A unique solution of a nonstationary ARMA model

Let the triplet (Ω, β, P) denote the probability space and let $\mathscr{L}^2(\Omega, \beta, P)$ denote the space of all real valued random variables on (Ω, β, ρ) with finite first and second order moments. Then $\mathscr{L}^2(\cdot)$ is a Hilbert space if the inner product and the norm are defined by

$$\langle U, V\rangle = E(U, V), \quad \|U\|^2 = E|U|^2, \quad U, V \in \mathscr{L}^2(\cdot)$$

respectively.

Assume that X_t^* is a solution to the stochastic difference equation (1.1). Let $M_t(X^*)$ (resp. $M_t(e)$) denote the subspace of $\mathscr{L}^2(\cdot)$ spanned by the random variables X_s^* (resp. e_s), $s \in Z$, $s \leq t$. Similarly let $M(X^*)$ (resp. $M(e)$) denote the subspace of $\mathscr{L}(\cdot)$ spanned by all random variables X_s^* (resp. e_s), $s \in Z$. From the assumption made on the zeros of $\Phi_t(z)$ and $\Theta_t(z)$ it follows that

$$M_t(X^*) \equiv M_t(e); \tag{2.1}$$

see also [5].

A stochastic difference equation of the form (1.1) represents an ARMA (p, q) process iff it admits a solution $\{X_t^*: t \in Z\}$ such that $X_t^* \in M_t(e)$.

Remark 2.1. It may be shown that (1.1) admits at most one solution in $M_t(e)$. Let the solution to equation (1.1) be

$$X_t^* = \sum_{s=-\infty}^{t} G(t, s) e_s, \tag{2.2}$$

where

$$G(t, s) = \begin{cases} \sum_{k=0}^{t-s} g(t, t-k)\theta_{t-s-k,t-k}, & t-q \leq s \leq t, \\ \sum_{k=0}^{q} g(t, k+s)\theta_{k,k+s}, & s < t-q. \end{cases} \tag{2.3}$$

The $g(t, s)$ are the Green functions associated with $\Phi_t(z)$ (see Miller [4]).

That is, $g(t,s)$ is a function of t in the solution of the homogeneous difference equation $\Phi_t(B)\psi_t = 0$ on Z with initial conditions

$$g(t,s) = \begin{cases} 1, & s = t, \\ 0, & s = t+1, \ldots, t+p-1. \end{cases} \tag{2.4}$$

Consider the following theorems:

Theorem 2.1. *If* (i) *the zeros of* $\Phi_t(z)$ *and* $\Theta_t(z)$ *lie in the region* $|z| > \lambda$, *where* $\lambda > 1$, *and*

(ii) $\sigma_t^2 < M < \infty$, $\forall t \in Z$, *then the* MA *process defined by* (2.2) *is a solution to* (1.1).

Proof. Let $u_t = \Theta_t(B)e_t$, then conditions (i) and (ii) imply that $E(u_t)^2$ is uniformly bounded on Z. Following Miller [4, p. 90], there exists a solution X_t^* for $\Phi_t(B)X_t = u_t$ given by

$$X_t^* = \sum_{s=-\infty}^{t} g(t,s)u_s. \tag{2.5}$$

Substituting for u_s in (2.5) and rearranging we get (2.2).

Theorem 2.2. *If* X_t^*, $t \in Z$ *is a valid stochastic process (i.e.* $X_t^* \in M_t(e)$) *then its covariance function* $\gamma_{t,\tau}$ *is given by*

$$\gamma_{t,\tau} = \mathrm{Cov}(X_t^*, X_\tau^*) = \sum_{s=-\infty}^{\min(t,\tau)} G(t,s)G(\tau,s)\sigma_s^2. \tag{2.6}$$

Proof. Straightforward, hence omitted.

Theorem 2.3. *Let* X_t^* *be the unique solution to* (1.1) *satisfying the conditions of Theorem* 2.1. *Then, for any fixed* t,

$$\lim_{\tau\to\infty} \gamma_{t,\tau} \to 0. \tag{2.7}$$

Proof. From conditions (i) and (ii) of Theorem 2.1, it follows that

$$|\gamma_{t,\tau}| < M \left| \sum_{s=-\infty}^{\min(t,\tau)} G(t,s)G(\tau,s) \right|.$$

Thus

$$\lim_{\tau\to\infty} |\gamma_{t,\tau}| < M \lim_{\tau\to\infty} \left| \sum_{s=-\infty}^{t} G(t,s)G(\tau,s) \right|$$

which tends to 0 by condition (i) of Theorem 2.1 as

$$|G(\tau,s)| \sim 0(\rho^{-\tau}) \quad \text{for } s \leqslant t, \text{ where } \rho > 1.$$

3. Linear prediction

Let X_t^* be a unique solution to (1.1) under conditions of Theorem 2.1. Then an optimal linear predictor $\hat{X}_t^*(h)$ of X_{t+h}^*, $h>0$, in terms of X_s^*, $s \leq t$, is given by

$$\hat{X}_t^*(h) = P_{M_t(X^*)} X_{t+h}^*, \tag{3.1}$$

where $P_{M_t(X^*)}$ is the orthogonal projection of $M(X^*)$ onto $M_t(X^*)$.

Theorem 3.1. *Let X_t^* be the unique solution to* (1.1) *under conditions* (i) *and* (ii) *of Theorem* 2.1, *then*

(i) *for any $h>0$,*

$$X_t^*(h) = \sum_{s=-\infty}^{t} G(t+h, s)e_s;$$

(ii) *the prediction error is*

$$X_{t+h}^* - \hat{X}_t^*(h) = \sum_{j=0}^{h-1} G(t+h, t+h-j)e_{t+h-j}$$

and its variance is

$$E|X_{t+h}^* - \hat{X}_t^*(h)|^2 = \sum_{j=0}^{h-1} |G(t+h, t+h-j|^2 \sigma_{t+h-j}^2;$$

(iii) *$\hat{X}_t^*(h)$ may be recursively obtained from*

$$\sum_{j=0}^{p} \phi_{j,t+h} \hat{X}_t^*(h-j) = \sum_{j=0}^{q} \theta_{j,t+h} e_{t+h-j},$$

where

$$e_s = X_{s+1}^* - \hat{X}_s^*(1), \qquad P_{M_t(X^*)} X_{t+l}^* = \begin{cases} X_{t+l}^*, & l \leq 0, \\ X_t^*(l), & l > 0. \end{cases}$$

Notice that $P_{M_t(X^*)}(\cdot) = P_{M_t(e)}(\cdot)$ since $M_t(X^*) = M_t(e)$ under the hypothesis of $X_t^* \in M_t(e)$ and $e_t \in M_t(X^*)$.

Proof. Straightforward, hence omitted.

Acknowledgements

The authors gratefully express their thanks to the Editor and the referee for their valuable comments and suggestions.

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