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Mutual Information in Gaussian Channels

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In the Gaussian channel $Y(t) = \Phi(t) + X(t)$ = message + noise, where $\Phi(t)$ and $X(t)$ are mutually independent, the information $I(Y, \Phi)$ is evaluated. One of the results is that $I(Y, \Phi) < \infty$ if and only if $\Phi \in \mathscr{H}(X)$ = the reproducing kernel Hilbert space for $X(\cdot)$. And the causal formula of $I(Y, \Phi)$ is given.

1. INTRODUCTION

The model of Gaussian channels to be discussed here is

$$Y(t) = \Phi(t) + X(t) \qquad 0 \leqslant t \leqslant T\,(\leqslant \infty), \tag{1.1}$$

where $X(t)$ is a zero mean Gaussian noise and $\Phi(t)$ is a zero mean Gaussian message which is independent of $\{X(t); t \in [0, T]\}$, defined on a probability space $(\Omega, \mathscr{B}, P)$. The main interest of this paper is to evaluate the mutual information $I(Y, \Phi)$ between $\Phi(\cdot)$ and $Y(\cdot)$.

One of our results is that $I(Y, \Phi) < \infty$ if and only if $\Phi(\cdot) \in \mathscr{H}(X)$ with probability one, where $\mathscr{H}(X)$ is the reproducing kernel Hilbert space (RKHS) corresponding to the noise $X(\cdot)$ (Theorem 1).

Also, in case $I(Y, \Phi) < \infty$, the causal expression of the evaluation of $I(Y, \Phi)$ is given, by using a causal mean-square filtering error. In a special case where $X(t)$ is a Wiener process, the analogous result was given by Duncan [1] and Kadota-Zakai-Ziv [5].

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2. The Main Theorem

In this section, we concern the Gaussian channel which has been given by (1.1). The main theorem is stated as follows:

THEOREM 1. *The mutual information $I(Y, \Phi)$ is finite if and only if*

$$\Phi(\cdot, \omega) \in \mathscr{H}(X) \qquad (\textit{with probability one}),$$

where $\mathscr{H}(X)$ is the RKHS *corresponding to the Gaussian noise $X(\cdot)$.*

In proving Theorem 1, the following Lemma is used.

LEMMA 1. *The Gaussian process $Y(\cdot)$ given by* (1.1) *is equivalent to $X(\cdot)$ if and only if $\Phi(\cdot, \omega)$ belongs to $\mathscr{H}(X)$ with probability one.*

Proof. Parzen [6] showed that the Gaussian process

$$Y(\cdot) = X(\cdot) + a(\cdot), \; a(\cdot) \text{ is a deterministic function,}$$

is equivalent to the process $X(\cdot)$ if and only if $a(\cdot) \in \mathscr{H}(X)$. In our case, in which $X(\cdot)$ and $\Phi(\cdot)$ are mutually independent, $Y(\cdot)$ is not equivalent to $X(\cdot)$ if $P(\Phi(\cdot) \notin \mathscr{H}(X)) > 0$, by the use of Parzen's result; while if $P(\Phi(\cdot) \in \mathscr{H}(X)) = 1$, $Y(\cdot) = X(\cdot) + \Phi(\cdot)$ is equivalent to $X(\cdot)$ for almost all fixed $\Phi(\cdot)$. According to the Fubini's theorem, it follows that $Y(\cdot)$ is equivalent to $X(\cdot)$. Thus the Lemma is proved.

Proof of Theorem 1

"Only if" part. Define μ_Y, μ_Φ, and $\mu_{(Y,\Phi)}$'s as measures on function spaces induced by the processes $Y(\cdot)$, $\Phi(\cdot)$, and $(Y(\cdot), \Phi(\cdot))$, respectively. In order to realize the measure $\mu_Y \times \mu_\Phi$, we define three mutually independent processes $B_1(\cdot)$, $\Phi_1(\cdot)$, and $\Phi_2(\cdot)$ on a probability space $(\Omega_1, \mathscr{B}_1, P_1)$ in such a way that $X_1(\cdot)$ is a version of the process $X(\cdot)$ and that both $\Phi_1(\cdot)$ and $\Phi_2(\cdot)$ are versions of the process $\Phi(\cdot)$. In such a scheme, the vector-valued process

$$(Y_1(t), \Phi_2(t)) = (X_1(t) + \Phi_1(t), \Phi_2(t)), \qquad 0 \leqslant t \leqslant T, \tag{2.1}$$

induces the measure $\mu_Y \times \mu_\Phi$ on a certain function space W. Assume that $I(Y, \Phi) < \infty$. Then by the well-known result (Gelfand-Yaglom [2]), $\mu_{(Y,\Phi)}$ is absolutely continuous with respect to $\mu_Y \times \mu_\Phi$. Hence, there exists a density $M(\omega)$ such that on the probability space $(\Omega_1, \mathscr{B}_1, M(\omega) P_1)$, $\mu_{(Y,\Phi)}$ is the induced measure by $(Y_1(\cdot), \Phi_2(\cdot))$. Therefore, the process defined on $(\Omega_1, \mathscr{B}_1, M(\omega) P_1)$ by

$$X_2(t) = Y_1(t) - \Phi_2(t), \qquad 0 \leqslant t \leqslant T, \tag{2.2}$$

must be a version of the process $X(\cdot)$. On the other hand, the right-hand side of (2.2) is $X_1(t) + \Phi_1(t) - \Phi_2(t)$. Since $\Phi_1(t) - \Phi_2(t)$, $0 \leqslant t \leqslant T$, is independent of $X_1(t)$ in the space $(\Omega_1, \mathscr{B}_1, P_1)$, the process $\Psi_1(\cdot) = \Phi_1(\cdot) - \Phi_2(\cdot)$ belongs to $\mathscr{H}(X)$ with probability one by Lemma 1. As $\Phi_1(\cdot)$ and $\Phi_2(\cdot)$ are mutually independent Gaussian processes with the same covariance function, $\Psi_1(\cdot)/\sqrt{2}$ is a common version of $\Phi_1(\cdot)$, $\Phi_2(\cdot)$, and $\Phi(\cdot)$. Therefore, $\Phi(\cdot)$ belongs to the space $\mathscr{H}(X)$. Thus the proof for the "only if" part is completed.

In order to prove the "if" part, Lemma 1 is applied again. To begin with, we see that the measure $\mu_{(Y,\Phi)}$ induced by the process $(Y(\cdot), \Phi(\cdot))$ is equivalent to the measure $\mu_{(X,\Phi)} = \mu_X \times \mu_\Phi$ induced by the process $(X(\cdot), \Phi(\cdot))$, because for any fixed Φ, the conditional measure $\mu_{Y|\Phi}$ induced by $Y(\cdot) = X(\cdot) + \Phi(\cdot)$ is equivalent to μ_X induced by $X(\cdot)$ by Lemma 1. Also, the measure $\mu_Y \times \mu_\Phi$ is equivalent to $\mu_X \times \mu_\Phi$. Therefore, $\mu_{(Y,\Phi)}$ must be equivalent to $\mu_Y \times \mu_\Phi$. Noting that the both measures $\mu_{(Y,\Phi)}$ and $\mu_Y \times \mu_\Phi$ are Gaussian, we deduce that

$$E[\log(d\mu_{(Y,\Phi)}/d(\mu_Y \times \mu_\Phi))(Y(\cdot), \Phi(\cdot))] < \infty$$

by the use of the well-known Hajek–Feldman's theorem (see Rozanov [7]). On the other hand, since the mutual information is also given by

$$I(Y, \Phi) = E[\log(d\mu_{(Y,\Phi)}/d(\mu_Y \times \mu_\Phi))(Y(\cdot), \Phi(\cdot))]$$

(Gelfand–Yaglom [2]), the proof is completed.

Remark. It is obvious from the proof that the "only if" part is still valid, even if $\Phi(t)$, $0 \leqslant t \leqslant T$, is not a Gaussian process without atoms.

EXAMPLE. In the case where the process $X(\cdot)$ in (1.1) is a Wiener process, the RKHS is given by

$$\mathscr{H}(X) = \left\{ f(\cdot); f(t) = \int_0^t a(s)\, ds,\ a \in L^2(ds) \right\}.$$

Therefore, $I(Y, \Phi) < \infty$ if and only if

$$\Phi(t, \omega) = \int_0^t \varphi(s, \omega)\, ds, \qquad \varphi(\cdot, \omega) \in L^2(ds) \tag{2.3}$$

with probability one. As $\varphi(s, \omega)$ is Gaussian, (2.3) is equivalent to

$$\int_0^T E[\varphi^2(s, \omega)]\, ds < \infty.$$

3. The Causal Evaluation of the Mutual Information

In case $I(Y, \Phi) < \infty$, the causal evaluation of the mutual information is given in this section. The method used here is based upon the so-called Lévy–Hida canonical representation with respect to the vector-valued Gaussian processes with independent increments.

Lemma 2 (Hida [3, Theorem 1.5]). *A separable Gaussian process $X(\cdot)$ can be decomposed canonically in the form*

$$X(t) = \sum_{i=1}^{N} \int_0^t F_i(t, u)\, dB^{(i)}(u) + \sum_{\{t_j \leqslant t\}} \sum_{\ell=1}^{L_j} b_j^{\ell}(t)\, B_{t_j}^{\ell}, \qquad N, L_j \leqslant \infty, \tag{3.1}$$

where (i) *$B^{(i)}(u)$'s are Gaussian processes with independent increments such that $E[dB^{(i)}(u)^2] = m_i(du)$'s are continuous measures with the property $m_i \gg m_{i+1}$,*

(ii) *$B_{t_j}^{l}$'s are standard Gaussian variables,*

(iii) *all the $B^{(i)}(u)$'s and $B_{t_j}^{\ell}$'s are mutually independent, and*

(iv) *it holds that the σ-algebra $\mathfrak{X}(t)$ generated by $\{X(u);\, u \leqslant t\}$ is equal to the σ-algebra $\mathfrak{B}(t)$ generated by $\{B^{(i)}(u);\; u \leqslant t\}$, $\{B_{t_j}^{l};\, t_j < t,\; l = 1,..., L_j\}$ and $\{B_{t_j}^{\ell};\, t_j = t,\, b_j^{\ell}(t) \neq 0\}$.*

If we look at the representation (3.1), we can find an interesting fact that, for each t_j, there exists at most one integer l_0 such that $b_j^{\ell_0}(t) \neq 0$. For, if not, the representation can not satisfy condition (iv).

According to Lemma 3, the RKHS $\mathscr{H}(X)$ for $X(\cdot)$ can be written in the form

$$\mathscr{H}(X) = \Big\{ f(\cdot); f(t) = \sum_{i=1}^{N} a_i(t) + \sum_{j} \sum_{\ell=1}^{L_j} d_j^{\ell}(t), \sum_i \int_0^T \alpha_i(u)^2 m_i(du) + \sum_j \sum_{\ell} (c_j^{\ell})^2 < \infty \Big\}, \tag{3.2}$$

where $a_i(t) = \int_0^t F_i(t, u)\, \alpha_i(u)\, m_i(du)$, $\alpha_i \in L^2(m_i)$, and $d_j^{\ell}(t) = c_j^{\ell} b_j^{\ell}(t)$, c_j^{ℓ}'s are constants. (See Hida [3] or Hitsuda [4].) By (3.2), any given signal $\Phi(\cdot, \omega)$ can be represented as

$$\Phi(t, \omega) = \sum_{i=1}^{N} \int_0^t F_i(t, u)\, \varphi_i(u, \omega)\, m_i(du) + \sum_{\{j : t_j \leqslant t\}} \sum_{\ell} b_j^{\ell}(t)\, \varphi_j^{\ell}(\omega) \tag{3.3}$$

with probability one, where the system $\{\varphi_i(\cdot), \varphi_j^{\ell}\}$ is Gaussian and is independent of $\{B^{(i)}(u), B_{t_j}^{\ell}\}$.

We now evaluate the information $I(Y, \Phi)$ using the representations (3.1) and (3.3) of the noise and the signal.

THEOREM 2. *If $X(\cdot)$ and $\Phi(\cdot)$ are defined by (3.1) and (3.3) respectively, and if all the L_j's are finite, then the mutual information between $Y(\cdot) = X(\cdot) + \Phi(\cdot)$ and $\Phi(\cdot)$ is given by*

$$\begin{aligned} I(Y, \Phi) &= I_T(Y, \Phi) \\ &= \frac{1}{2}\Big\{\sum_{i=1}^{N} \int_0^T E[\varphi_i(u)^2 - \hat{\varphi}_i(u)^2]\, m_i(du) \\ &\quad + \sum_{j,\ell_0 : b_j^{\ell_0}(t_j) \neq 0} E[\log\{1 + E[(\varphi_j^{\ell_0})^2 - (\hat{\varphi}_j^{\ell_0})^2 \mid \mathfrak{Y}(t_j -)]\}] \\ &\quad + \sum_j E\{\log[\det(I + \mathbf{A}_j)]\}, \end{aligned} \tag{3.4}$$

where $\hat{\varphi}_i(u) = E[\varphi_i(u) \mid \mathfrak{Y}(u)]$, $\hat{\varphi}_j^{\ell_0} = E[\varphi_j^{\ell_0} \mid \mathfrak{Y}(t_j -)]$, for $\ell_0 = \ell_0(j)$ such that $b_j^{\ell_0}(t_j) \neq 0$, $\mathfrak{Y}(t_j -) = \bigvee_{t<t_j} \mathfrak{Y}(t)$, and $\mathbf{A}_j$ is the matrix whose entries $a_j^{\ell_1\ell_2}$'s are given by

$$a_j^{\ell_1\ell_2} = E\{\varphi_j^{\ell_1}\varphi_j^{\ell_2} - E[\varphi_j^{\ell_1} \mid \mathfrak{Y}(t_j)]\, E[\varphi_j^{\ell_2} \mid \mathfrak{Y}(t_j)] \mid \mathfrak{Y}(t_j)\} \qquad \ell_1, \ell_2 \neq \ell_0. \tag{3.5}$$

Proof. By (3.1) and (3.3), $Y(t)$ can be represented as

$$\begin{aligned} Y(t) &= \sum_{i=1}^{N} \int_0^t F_i(t, u)(dB^{(i)}(u) + \varphi_i(u, \omega)\, m_i(du)) + \sum_{\{j : t_j \leq t\}} \sum_{\ell=1}^{L_j} b_j^{\ell}(t)(B_{t_j}^{\ell} + \varphi_j^{\ell}) \\ &= \sum_{i=1}^{N} \int_0^t F_i(t, u)\, dZ^{(i)}(u) + \sum_{\{j : t_j \leq t\}} \sum_{\ell=1}^{L_j} b_j^{\ell}(t)\, Z_{t_j}^{\ell}, \end{aligned} \tag{3.6}$$

where $Z^{(i)}(t) = B^{(i)}(t) + \int_0^t \varphi_i(u)\, m_i(du)$, and $Z_{t_j}^{\ell} = B_{t_j}^{\ell} + \varphi_j^{\ell}$. We first note that the vector-valued process

$$Z(t) = (Z^{(1)}(t), \ldots, Z^{(i)}(t), \ldots, \chi_{(t_1, T]}(t)\, Z_{t_1}^1, \ldots, \chi_{[t_1, T]}(t)\, Z_{t_1}^{\ell_0},$$
$$\ldots, \chi_{(t_1, T]}(t)\, Z_{t_1}^{L_1}, \chi_{(t_2, T]}(t)\, Z_{t_2}^1, \ldots)$$

is equivalent to the process

$$B(t) = (B^{(1)}(t), \ldots, B^{(i)}(t), \ldots, \chi_{(t_1, T]}(t)\, B_{t_1}^1, \ldots, \chi_{[t_1, T]}(t)\, B_{t_1}^{\ell_0},$$
$$\ldots, \chi_{(t_1, T]}(t)\, B_{t_1}^{L_1}, \chi_{(t_2, T]}(t)\, B_{t_2}^1, \ldots),$$

because

$$E\Big[\sum_i \int_0^T \varphi_i^2(u)\, m_i(du) + \sum_j \sum_\ell (\varphi_j^{\ell})^2\Big] < \infty.$$

With this fact, we know that the σ-algebra $\mathfrak{Y}(t)$ generated by $\{Y(u);\ u \leqslant t\}$ coincides with the σ-algebra $\mathfrak{Z}(t)$ generated by $\{Z(u);\ u \leqslant t\}$. In other words, the process $Y(t)$ is canonically represented with respect to $Z(t)$. Furthermore, if we set

$$\Psi(t) = \Big(\int_0^t \varphi_1(u, \omega)\, m_1(du), \ldots, \int_0^t \varphi_i(u, \omega)\, m_i(du), \ldots, \chi_{(t_1, T]}(t)\, \varphi_1^{\,1}, \ldots, \chi_{(t_1, T]}(t) \times \varphi_1^{\ell_0}, \ldots, \chi_{[t_1, T]}(t)\, \varphi_1^{L_1}, \chi_{(t_2, T]}(t)\, \varphi_2^{\,1}, \ldots\Big),$$

then the σ-algebra $\mathbf{\Phi}(t)$ generated by $\{\Phi(u);\ u \leqslant t\}$ coincides with the σ-algebra generated by $\{\Psi(u);\ u \leqslant t\}$. Because the equation

$$\sum_{i=1}^{N} \int_0^t F_i(t, u)\, \alpha_i(u)\, m_i(du) + \sum_j \sum_\ell b_j^{\,\ell}(t)\, c_j^{\,\ell} = 0$$

implies that $(\alpha_1, \ldots, \alpha_i, \ldots, c_1^{\,1}, \ldots, c_1^{\ell_0}, \ldots, c_1^{L_1}, c_2^{\,1}, \ldots) = (0, \ldots, 0, \ldots, 0, \ldots, 0, \ldots, 0, 0, \ldots)$, we can conclude that

$$I(Y, \Phi) = I(Z, \Psi). \tag{3.7}$$

Now, let us evaluate the right-hand side of (3.7). We know that

$$\begin{aligned}(d\mu_{(Z,\Psi)}/d_{(\mu_Z \times \mu_\Psi)})(Z(\cdot), \Psi(\cdot)) &= \frac{(d\mu_{(Z,\Psi)}/d\mu_{(B,\Psi)})(Z(\cdot), \Psi(\cdot))}{(d(\mu_Z \times \mu_\Psi)/d\mu_{(B,\Psi)})(Z(\cdot), \Psi(\cdot))} \\ &= \frac{d\mu_{Z|\Psi(\cdot)}/d\mu_B(Z(\cdot))}{d\mu_Z/d\mu_B(Z(\cdot))}, \end{aligned} \tag{3.8}$$

where $\mu_{(Z,\Psi)}$, $\mu_{(B,\Psi)} = \mu_B \times \mu_\Psi$, μ_Z and μ_Ψ are the induced measures on function spaces corresponding to the respective processes. The numerator and the denominator of the last expression of (3.8) are calculated as follows:

$$\begin{aligned}\frac{d\mu_{Z|\Psi}}{d\mu_B}(Z(\cdot)) &= \exp\Big\{\sum_{i=1}^{N}\Big(\int_0^T \varphi_i(u)\, dZ^{(i)}(u) - \frac{1}{2}\int_0^T \varphi_i^{\,2}(u)\, m_i(du)\Big) \\ &\qquad + \sum_j \sum_\ell \big[\tfrac{1}{2}(Z_{t_j}^{\ell})^2 - \tfrac{1}{2}(Z_{t_j}^{\ell} - \varphi_j^{\,\ell})^2\big]\Big\} \\ &= \exp\Big\{\sum_{i=1}^{N}\Big(\int_0^T \varphi_i(u)\, dB^{(i)}(u) + \frac{1}{2}\int_0^T \varphi_i^{\,2}(u)\, m_i(du)\Big) \\ &\qquad + \sum_j \sum_\ell \big[\tfrac{1}{2}(B_{t_j}^{\ell} + \varphi_j^{\,\ell})^2 - \tfrac{1}{2}(B_{t_j}^{\ell})^2\big]\Big\}. \end{aligned} \tag{3.9}$$

Noting $\mathfrak{Y}(t) = \mathfrak{Z}(t)$, we have

$$\frac{d\mu_Z}{d\mu_B}(Z(\cdot)) = \exp\Big\{\sum_{i=1}^{N}\Big(\int_0^T \hat{\varphi}_i(u)\, dZ^{(i)}(u) - \frac{1}{2}\int_0^T \hat{\varphi}_i(u)^2\, m_i(du)\Big)$$

$$+ \frac{1}{2}\sum_{b_j(\ell_0)\neq 0}\Big[-\log(1 + E[(\varphi_j^{\ell_0})^2 - (\hat{\varphi}_j^{\ell_0})^2 \mid \mathfrak{Y}(t_j -)]$$

$$+ (Z_{t_j}^{\ell_0})^2 - \frac{(Z_{t_j}^{\ell_0} - \hat{\varphi}_j^{\ell_0})^2}{1 + E[(\varphi_j^{\ell_0})^2 - (\hat{\varphi}_j^{\ell_0})^2 \mid \mathfrak{Y}(t_j -)]}\Big]$$

$$+ \frac{1}{2}\sum_j [-\log(\det(I + \mathbf{A}_j)) + \tilde{\mathbf{Z}}_j \mathbf{Z}_j^* - \tilde{\mathbf{Z}}_j(I + \mathbf{A}_j)^{-1}\tilde{\mathbf{Z}}_j^*]\Big\},$$

where $\mathbf{A}_j = (a_j^{\ell_1\ell_2})$ is given by (3.5),

$$\mathbf{Z}_j = \{Z_{t_j}^1, \ldots, Z_{t_j}^{L_j}\}, \qquad \text{and} \qquad \tilde{\mathbf{Z}}_j = \{Z_{t_j}^1 - \hat{\varphi}_j^1, \ldots, Z_{t_j}^{L_j} - \hat{\varphi}_j^{L_j}\}.$$

(If there exists an integer $\ell_0 = \ell_0(j)$ such that $b_j^{\ell_0}(t_j) \neq 0$, then $Z_{t_j}^{\ell_0}$ and $Z_{t_j}^{\ell_0} - \varphi_j^{\ell_0}$ should be removed from $\mathbf{Z}_j$ and $\tilde{\mathbf{Z}}_j$, respectively.) Combining (3.7)–(3.10), the desired result follows:

Remark. In the case where some of the L_j's are infinite, we define

$$X_K(t) = \sum_{i=1}^{N}\int_0^t F_i(t, u)\, dB^{(i)}(u) + \sum_{\{j: t_j \leqslant t\}}\sum_{\ell=1}^{L_j \wedge K} b_j^{\ell}(t)\, B_{t_j}^{\ell}, \tag{3.11}$$

and define $\Phi_K(t)$ and $Y_K(t)$ analogously. The representation (3.11) is also canonical, and the σ-algebras $\mathfrak{Y}_K(t) = \sigma\{Y_K(u); u \leqslant t\}$ and $\mathbf{\Phi}_K(t) = \sigma\{\Phi_K(u); u \leqslant t\}$, $K \geqslant 1$, increase up to $\mathfrak{Y}(t)$ and $\mathbf{\Phi}(t)$, respectively, as K tends to ∞. We can therefore apply the standard limiting procedure to deduce

$$I(Y, \Phi) = \lim_{K\to\infty} I(Y_K, \Phi_K).$$

EXAMPLE. If there are no discrete parts in the representation (3.1) of $X(t)$:

$$X(t) = \sum_{i=1}^{N}\int_0^t F_i(t, u)\, dB^{(i)}(u),$$

then $I(Y, \Phi)$ is easily computed, and the result is

$$I(Y, \Phi) = \frac{1}{2}\sum_{i=1}^{N} E\Big[\int_0^T (\varphi_i(u)^2 - \hat{\varphi}_i(u)^2)\, m_i(du)\Big].$$

For a special case where $X(t)$ is the Brownian motion, this formula was given by Duncan [1] and Kadota-Zakai-Ziv [5].

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