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C^k Surface Approximation from Surface Patches

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Abstract—We study the problem of constructing a smooth approximant from a finite set of patches given on a surface defined by an equation $x_3 = f(x_1, x_2)$. As an approximant of f, a discrete smoothing spline belonging to a suitable piecewise polynomial space is proposed. Error results and numerical results are given. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Spline approximation, Numerical integration, Convergence.

1. INTRODUCTION

The problem of constructing a surface from given patches on this surface appears, for instance, in geophysics or geology processes like migration of time-maps or depth-maps as shown in [1]. Classical algorithms used to solve this class of problems usually select points on the patches to define a Lagrange dataset, and subsequently make use of classical spline functions (e.g., [2–4]), bivariate spline [5–7], or spline functions in Hilbert spaces (see, for instance, [8,9]). In the available literature, to our knowledge there are no classical methods that explicitly take into account the continuous aspect of the data (surface patches). Meanwhile, an alternative approach has been developed by Le Méhauté [10] using surface's extensions. Le Méhauté proposes a rational scheme interpolating a Taylor field of order 1 (or a function and its first partial derivatives) given on the edge of a triangle T with acute angles. This approach is very interesting but difficult to use on real datasets, and it does not perform well to get a C^1 (or higher) approximant. That is why we propose to use a penalized least square approach, as done for instance in [7,11,12].

In this paper, our purpose is to devise an approximation method which takes into account the original aspect of the data. To do that, we use a fidelity criterion to the data of integral type related to the L^2-norm of these data. It leads to the following abstract formulation: from a finite set of open subsets $\omega_j, j = 1, \dots, N$, in the closure of a bounded nonempty open set $\Omega \subset \mathbb{R}^n$ (see

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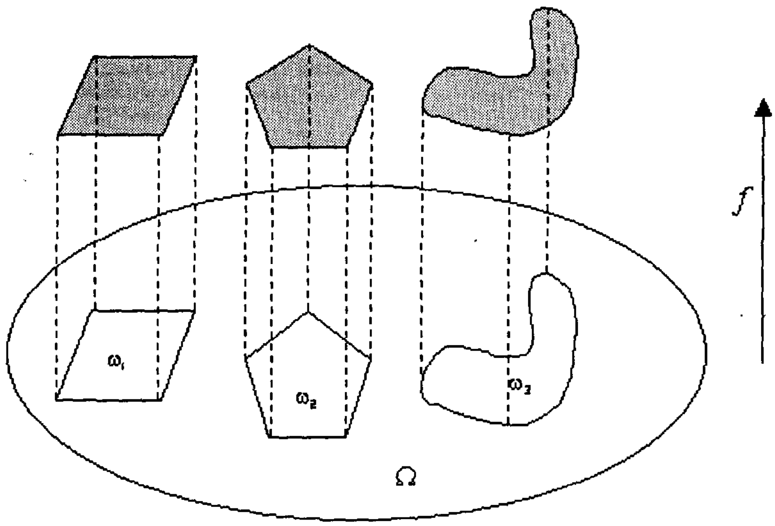

Figure 1. Example of an open set Ω with three subsets ω_1, ω_2, and ω_3. From these three surface patches, the goal is to propose a method to reconstruct the surface on Ω.

Figure 1) and from a function f defined on $\omega = \bigcup_{j=1}^{N} \omega_j$, construct a regular function Φ on Ω, approximating f on ω.

In practice, $n = 2$ and ω is the finite union of pairwise disjoint open subsets ω_j, $j = 1, \dots, N$ of Ω, which are called patches (see again Figure 1).

REMARK 1. As a fit criterion, we have chosen the L^2-norms (which define a distance). Of course, it would have been possible to use the L^1-norms (the fidelity criterion to the data being a minimization of the volume located between the patches and the approximant), but, in order to obtain a functional with "good properties", we have chosen an L^2-norm approach. This choice permits us to study the problem in Hilbert spaces (L^1-norm only defines reflexive Banach spaces).

The paper is organized as follows. We first introduce some notations and state an existence-uniqueness result. In Section 2, we focus on the quadrature formula used to approximate the fidelity criterion to the data. Section 3 is devoted to the proof of the convergence theorem which is the main result of the paper. We give numerical results in Section 4.

We assume the following.

- Ω is a nonempty, connected, bounded open set in $\mathbb{R}^n$, with Lipschitz-continuous boundary (in the sense of Nečas [13]).
- For an integer $j, j = 1, \dots, N$, let ω_j be a nonempty connected subset in $\mathbb{R}^n$.
- $k \in \mathbb{N}$ and $m > n/2$.
- For simplicity, f is the restriction on ω of a function f defined on Ω, given on $\omega (= \bigcup_{j=1}^{N} \omega_j)$ and unknown on $\Omega \setminus \omega$, which belongs to the Sobolev space $H^m(\Omega)$. Finally, we impose the condition that the approximant Φ belongs to $H^m(\Omega) \cap C^k(\bar{\Omega})$, where $\bar{\Omega}$ is the closure of Ω. The main interest of such a regularity for Φ is that it allows one to obtain a final surface that can later be used directly as an input model in a different application, such as ray tracing, image synthesis, or numerical simulation (e.g., [14]).

The problem of the approximation of f on ω is a fitting problem on each surface patch $\{(x, f(x)) : x \in \omega\}$.

When $m > k + n/2$, the corresponding interpolation problem $\Phi_{|\omega} = f_{|\omega}$ has an infinity of solutions because, in this case, $H^m(\Omega) \hookrightarrow C^k(\bar{\Omega})$. We can obtain a solution using Duchon's theory (see [8]). Unfortunately, Duchon's theory leads to linear systems whose order increases rapidly with the number of data points, which makes the method inefficient in the case of large datasets. We can obtain another solution using the following (interpolation) method. We introduce the convex set

$$K = \{v \in H^m(\Omega),\ v_{|\omega} = f_{|\omega}\}.$$

Then we consider the following interpolation problem on the data patches: find $\sigma \in K$ such that

$$\forall v \in K, \qquad |\sigma|_{m,\Omega} \leq |v|_{m,\Omega}, \tag{1}$$

where

$$|v|_{m,\Omega} = \left(\sum_{|\alpha|=m} \int_\Omega (\partial^\alpha v(x))^2 \, dx \right)^{1/2},$$

with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \sum_{i=1}^n \alpha_i$, $x = (x_i)_{i=1}^n$, and $\partial^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$.

Let $L^2(\omega)$ be equipped with the usual norm

$$\|v\|_{0,\omega} = \left(\sum_{j=1}^N \int_{\omega_j} v^2(x) \, dx \right)^{1/2}.$$

Then, Gout [15] has established the existence and uniqueness of the problem of equation (1) using compactness argument (see [13]) to establish the equivalence of the usual norm of $H^m(\Omega)$ denoted by

$$\|u\|_{m,\Omega} = \left(\sum |\alpha| \leq m \int_\Omega (\partial^\alpha v)^2 \, dx \right)^{1/2}$$

and the norm

$$|||u||| = \left(\|u\|_{0,\omega}^2 + |u|_{m,\Omega}^2 \right)^{1/2}. \tag{2}$$

Then, as K is a convex, closed, and nonempty set in $H^m(\Omega)$, the solution σ of (1) is nothing but the unique element of minimal norm $|||.|||$.

Hence, we could take the solution $\Phi = \sigma$ when $m > k + n/2$. Unfortunately, it is often impossible to compute σ using a discretization of problem (1), because in a finite-dimensional space, it is not possible to satisfy an infinity of interpolation conditions. Therefore, to take into account the continuous aspect of the data $(f_{|\omega})$, we instead choose to define the approximant Φ as a fitting surface.

We propose to construct a "*smoothing D^m-spline*", as defined by Arcangéli [9], that will be discretized in a suitable piecewise polynomial space. In order to do that, we introduce the functional J_ε defined on $H^m(\Omega)$ by

$$J_\varepsilon(v) = \|v - f\|_{0,\omega}^2 + \varepsilon \, |v|_{m,\Omega}^2, \tag{3}$$

where $\varepsilon > 0$ is a classical smoothing parameter. The key idea here is that the fidelity criterion to the data $\|v - f\|_{0,\omega}^2$ honors their continuous aspect.

We now need to numerically estimate this term, which is done using a quadrature formula. In this regard, the approach is quite different from more classical techniques that usually simply make use of a large number of data points on ω in order to solve the fitting problem.

In this paper, C denotes a generic positive constant and may take different values at different occurrences.

2. APPROXIMATION OF $\|.\|_{0,\omega}$

In this section, we propose a quadrature formula to approximate $\|.\|_{0,\omega}$ with a certain order of approximation.

We introduce a bounded subset E in $\mathbb{R}_+^*$ for which 0 is an accumulation point, and for any $\eta \in E$ and for an integer $j, j = 1, \dots, N$, a set $\{\zeta_i\}_{1\le i\le L}$ of $L = L(\eta, j)$ distinct points $\zeta_i = \zeta_i(\eta, j)$ of $\bar{\omega}_j$ such that

$$\left(\max_{i,k=1,\dots,L} \delta(\zeta_i, \zeta_k)\right), \qquad \text{depends on } \eta,$$

where δ is the Euclidean distance in $\mathbb{R}^n$ and the ζ_i are the nodes of a numerical integration formulae. We also introduce a set $\{\lambda_i\}_{1\le i\le L}$ of real numbers $\lambda_i = \lambda_i(\eta, j) > 0$.

Then we define, for any $\eta \in E$ and any $v \in C^0(\bar{\omega}_j)$,

$$\ell_j^\eta(v) = \sum_{i=1}^{L} \lambda_i v(\zeta_i), \qquad j = 1, \dots, N,$$

and for any $v \in C^0(\bar{\omega})$,

$$\ell^\eta(v) = \sum_{j=1}^{N} \ell_j^\eta(v), \qquad j = 1, \dots, N.$$

In all that follows, we assume that there exists C, $t > 0$, such that, for any $\eta \in E$, and for any $v \in H^m(\Omega)$,

$$\left|\ell_j^\eta\left(v^2\right) - \int_{\omega_j} v^2\, dx\right| \le C\eta^t \|v\|_{m,\Omega}^2, \qquad j = 1, \dots, N. \tag{4}$$

For simplicity, we shall write v instead of $v_{|\omega_j}$ or $v_{|\omega}$ and we shall consider ℓ^η as a linear continuous form defined in $C^0(\bar{\Omega})$, or in $C^0(\bar{\Omega}')$ for all open sets Ω' such that $\Omega \subset \Omega'$.

REMARK 2. When the hypothesis (4) is satisfied, the relation $\|v\|_{0,\omega}^2 \sim \ell^\eta(v^2)$ gives us an abstract numerical integration formula for $\|.\|_{0,\omega}^2$. In order to obtain the convergence of this integration formula when η tends to 0, the nodes ζ_i must satisfy

$$\max_{1\le i\le L} \min_{1\le k\le L, k\ne i} \delta\left(\zeta_i, \zeta_k\right) \le \eta,$$

where $\delta(.,.)$ represents the Euclidean distance in $\mathbb{R}^n$.

Let us assume, for example, that ω is a polyhedron. One can introduce a suitable triangulation $\mathcal{T}_\eta$ on $\bar{\omega}$ by means of n-simplexes T of diameter $\le \eta$, and a P_{m-1} integration formula on each T,

$$\int_T v(x)\, dx \simeq \text{meas } (T) \sum_{i=1}^{N} \gamma_i v(\zeta_{iT}),$$

where the $(\lambda_i)_{i=1,\dots,N}$ and the $(\gamma_i)_{i=1,\dots,N}$ are classically the weights and the nodes of the integration formula. In that context, Arcangéli *et al.* [16] showed that when $m \ge n$, relation (4) is satisfied with $t = m$ and

$$\ell^\eta(v) = \sum_{T\in\mathcal{T}_\eta} (\text{meas } T) \sum_{i=1}^{N} \gamma_i v(\zeta_{iT}).$$

In practice, when $n = 2$, one can use the P_3-exact formula on a triangle T with vertices $(a_i)_{i=1,2,3}$, with midpoints of the sides $(b_i)_{i=1,2,3}$, and with barycenter c

$$\int_T v\, dx \simeq \text{meas } (T) \left[\frac{1}{20} \sum_{i=1}^{3} v(a_i) + \frac{2}{15} \sum_{i=1}^{3} v(b_i) + \frac{9}{20} v(c)\right].$$

Thus, for any $\eta \in E$ and for any $v \in C^0(\bar{\omega})$, we obtain

$$\ell^\eta(v) = \sum_{T \in \mathcal{T}_\eta} \text{meas}\,(T) \left[\frac{1}{20} \sum_{i=1}^{3} v(a_i) + \frac{2}{15} \sum_{i=1}^{3} v(b_i) + \frac{9}{20} v(c) \right].$$

When ω is not a polyhedron, one can use quadrature formulas adapted to the geometry of ω (see, for instance, [17,18]). ∎

3. DISCRETE SMOOTHING D^m-SPLINE

In order to compute a discrete approximant Φ, we could use Bézier-polynomials space or any other finite-dimensional space. We select a finite element representation of Φ, which allows us to obtain a very small sparse linear system (see [19,20] for more details) and makes the study of the approximation error easier.

Let H be a bounded subset in $\mathbb{R}_+^*$ for which 0 is an accumulation point, let $\tilde{\Omega}$ be a bounded polygonal open set in $\mathbb{R}^n$ such that $\Omega \subset \tilde{\Omega}$, and, for any $h \in H$, let $\tilde{\mathcal{T}}_h$ be a triangulation on $\bar{\tilde{\Omega}}$ by means of elements K whose diameter h_K are $\leq h$ and let $\tilde{V}_h$ be a finite-element space constructed on $\tilde{\mathcal{T}}_h$ such that

$$\tilde{V}_h \text{ is a finite-dimensional subspace of } H^m\left(\tilde{\Omega}\right) \cap C^k\left(\bar{\tilde{\Omega}}\right). \tag{5}$$

Furthermore, to study the convergence of the approximation, we assume that there exists a family of operators $(\tilde{\Pi}_h)_{h\in H} \subset \mathcal{L}(H^m(\tilde{\Omega}), \tilde{V}_h)$ satisfying

$$\begin{aligned}
&\text{(i)} \quad && \exists C > 0 \text{ such that for any } h \in H, \text{ for any } l = 0, \dots, m-1, \\
& && \text{and for any } v \in H^m\left(\tilde{\Omega}\right), \left|v - \tilde{\Pi}_h v\right|_{l,\tilde{\Omega}} \leq C h^{m-l} |v|_{m,\tilde{\Omega}}\,; \\
&\text{(ii)} \quad && \text{for any } v \in H^m\left(\tilde{\Omega}\right), \lim_{h \to 0} \left|v - \tilde{\Pi}_h v\right|_{m,\tilde{\Omega}} = 0.
\end{aligned} \tag{6}$$

Condition (6) does not need the classical hypothesis of regularity of the finite element method $H^m(\tilde{\Omega}) \hookrightarrow C^s(\bar{\tilde{\Omega}})$, where s is the maximal order of the derivatives appearing in the definition of the degrees of freedom of the generic finite element of $(\tilde{V}_h)_{h\in H}$, but it is assumed that

$$\text{the family } \left(\tilde{\mathcal{T}}_h\right)_{h\in H} \text{ is regular (see [21]).} \tag{7}$$

Moreover, condition (6) needs the following hypothesis: the generic finite element (K, P_K, Σ_K) of the family $(\tilde{V}_h)_{h\in H}$ satisfies the inclusion $P_K \supset P_m(K)$.

In fact, (i) only uses the inclusion $P_{m-1}(K) \subset P_K$, and a property of uniformity of the generic finite element of $(\tilde{V}_h)_{h\in H}$, which is satisfied in the usual cases (see [22,23]).

REMARK 3. In most problems, one would want to solve in practice, the value of m would be either 2 or 3, allowing one to get either a C^1 or a C^2 approximant. When $m = 2$, the finite elements used to solve the problem could typically be classical elements of class C^1 or C^2, such as the Argyris or the Bell triangle (see [12,24]) or the Bogner-Fox-Schmit quadrangle (e.g., [19,25]). When $m = 3$, one could use the same finite element of class C^2 as for $m = 2$. When $m > 3$, one could generalize the Bogner-Fox-Schmit quadrangle into a finite element of class C^{m-1}. Other elements, such as isoparametric finite elements [19] or rational finite elements [26] could also be used. Isoparametric finite elements are useful to impose boundary conditions, but this is not usually a critical problem in the context of surface approximation. On the other hand, the use of rational finite elements is expensive, therefore, we choose to use the Bogner-Fox-Schmit quadrangle of class C^1, which allows us to obtain a C^1-approximant. Note that in certain classes of interpolation problems, each data point must also be a node of the finite element grid, in which

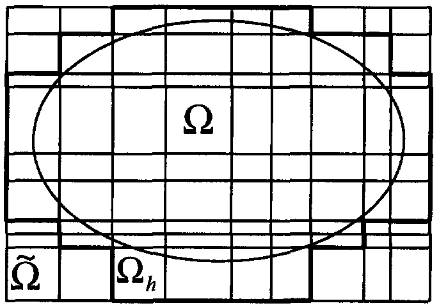

Figure 2. Definition of the sets Ω, Ω_h, and $\tilde{\Omega}$.

case the use of triangles, as opposed to quadrangles, greatly facilitates the creation of a suitable finite element mesh to numerically solve the problem. This is not the case in a surface fitting problem (data points do not need to be linked with the geometry of the finite element grid), and as rectangles are less expensive than triangles, we will use the Bogner-Fox-Schmit rectangles. In all cases, we could verify that conditions (5) and (6) are satisfied under (7). ∎

Now, for any $h \in H$, we consider the subset Ω_h (see Figure 2) defined by

$$\Omega_h \text{ is the inside of the union of the rectangles } K \text{ of } \mathcal{T}_h \text{ such that } K \cap \Omega \neq \emptyset. \tag{8}$$

It is clear that the family $(\Omega_h)_{h \in H}$ satisfies the following relations:

$$\forall\, h \in H, \qquad \Omega \subset \Omega_h \subset \tilde{\Omega}, \tag{9}$$

$$\lim_{h \longrightarrow 0} \text{meas}\ \left(\Omega_h \setminus \bar{\Omega}\right) = 0. \tag{10}$$

For any $h \in H$, we define V_h as follows:

$$V_h \text{ is the vector space of the restrictions to } \Omega_h \text{ of the functions of } \tilde{V}_h. \tag{11}$$

For any $\varepsilon > 0$, any $h \in H$, and any $\eta \in E$, we consider the minimization problem of finding $\sigma^{\eta}_{\varepsilon,h} \in V_h$ satisfying, for any $v_h \in V_h$,

$$J^{\eta}_{\varepsilon,h}\left(\sigma^{\eta}_{\varepsilon,h}\right) \leq J^{\eta}_{\varepsilon,h}\left(v_h\right), \tag{12}$$

where $J^{\eta}_{\varepsilon,h}$ is the functional defined by

$$J^{\eta}_{\varepsilon,h}\left(v_h\right) = \ell^{\eta}\left[\left(v_h - f\right)^2\right] + \varepsilon \left|v_h\right|^2_{m,\Omega_h}.$$

Then, we consider the variational problem of finding $\sigma^{\eta}_{\varepsilon,h} \in V_h$ satisfying

$$\forall\, v_h \in V_h, \qquad \ell^{\eta}\left(\sigma^{\eta}_{\varepsilon,h} v_h\right) + \varepsilon \left(\sigma^{\eta}_{\varepsilon,h}, v_h\right)_{m,\Omega_h} = \ell^{\eta}\left(f v_h\right), \tag{13}$$

where $(u,v)_{m,\Omega_h} = \sum_{|\alpha|=m} \int_{\Omega_h} \partial^{\alpha} u(x) \partial^{\alpha} v(x)\, dx$. Then, we have the following theorem.

THEOREM 3.1. *We assume that Ω, ω, m, and f are defined as in Section 1 and that hypotheses (4), (5), (8), and (11) are satisfied. Then, for any $\varepsilon > 0$, any $h \in H$, there exists $\eta_0 > 0$ such that for any $\eta \in E$, $\eta \leq \eta_0$, problems (12) and (13) have the same unique solution $\sigma^{\eta}_{\varepsilon,h}$.*

PROOF. This proof is divided into two parts.

PART 1. Using compactness arguments (see [13]), we show, under the relation

$$\text{for any } p \in P_{m-1}\left(\bar{\Omega}_h\right), \qquad p_{|\omega} = 0 \Longrightarrow p \equiv 0,$$

that the function $[|.|]_h$ defined on $H^m(\Omega_h)$ by

$$[|v|]_h = \left(\|v\|_{0,\omega}^2 + |v|_{m,\Omega_h}^2\right)^{1/2}$$

is a norm on $H^m(\Omega_h)$ which is equivalent to the usual norm

$$\|v\|_{m,\Omega_h} = \left(\sum_{|\alpha|\le m} \int_{\Omega_h} (\partial^\alpha v)^2 \, dx\right)^{1/2}.$$

PART 2. As a consequence of the definition of ℓ^η, the symmetric bilinear form

$$(u_h, v_h) \longmapsto \ell^\eta(u_h v_h) + \varepsilon(u_h, v_h)_{m,\Omega_h}$$

is continuous on $V_h \times V_h$. Likewise, this form is V_h-elliptic for η small enough because using (4), we have

$$\begin{aligned} \ell^\eta\left(v_h^2\right) + \varepsilon\,|v_h|_{m,\Omega_h}^2 &\ge \|v_h\|_{0,\omega}^2 - C\eta^t\,\|v_h\|_{m,\Omega}^2 + \varepsilon\,|v_h|_{m,\Omega_h}^2 \\ &\ge \min(1,\varepsilon)\,[|v_h|]_h^2 - C\eta^t\,\|v_h\|_{m,\Omega}^2 \\ &\ge \left(C' \min(1,\varepsilon) - C\eta^t\right)\|v_h\|_{m,\Omega_h}^2, \end{aligned} \tag{14}$$

where C' is a constant related to the equivalence of norms ($[|.|]_h$ and $\|.\|_{m,\Omega_h}$).

Let us assume (see also [16] for more details) that there exists $\beta > 0$, such that $\forall\,\varepsilon > 0$, $\forall\,\eta \in E$, $\eta^t/\min(1,\varepsilon) < \beta$. Taking $\beta = C'/C$, there exists $C'' > 0$ such that $\ell^\eta(v_h^2) + \varepsilon|v_h|_{m,\Omega_h}^2 \ge C''\,\|v_h\|_{m,\Omega_h}^2$. Then, the Lax-Milgram lemma gives the result. ∎

The function $\sigma_{\varepsilon,h}^\eta$ is called the V_h-discrete smoothing D^m-spline of f relative to ω, η, and ε.

REMARK 4. Denoting by $M = M(h)$ the dimension of V_h and by $(\varphi_j)_{1\le j\le M}$ a basis of V_h, we set

$$\sigma_{\varepsilon,h}^\eta = \sum_{j=1}^{M} \alpha_j \varphi_j,$$

with $\alpha_j \in \mathbb{R}$, $1 \le j \le M$. Introducing the matrices

$$\mathcal{A} = \left(\ell^\eta\left(\varphi_i \varphi_j\right)\right)_{1\le i,j\le M},$$

$$\mathcal{R} = \left(\left(\varphi_i, \varphi_j\right)_{m,\Omega_h}\right)_{1\le i,j\le M},$$

and

$$\mathcal{F} = \left(\ell^\eta\left(f\varphi_i\right)\right)_{1\le i\le M},$$

we see that (13) is equivalent to the problem

$$\text{find } \alpha = (\alpha_1, \dots, \alpha_i, \dots, \alpha_M) \in \mathbb{R}^M \text{ solution of } (\mathcal{A}+\varepsilon\mathcal{R})\,\alpha = \mathcal{F}.$$

Then, we can take as an approximation of f the function $\Phi = \sigma_{\varepsilon,h|\Omega}^\eta$ which, using the hypotheses (5), (8), and (11), belongs to $H^m(\Omega) \cap C^k(\bar{\Omega})$. We have to know in which sense Φ is an approximation of f. The next result proves the convergence of $\sigma_{\varepsilon,h}^\eta$ to the solution σ of (1) and gives an error result between $\sigma_{\varepsilon,h}^\eta$ and f on ω. Other convergence results are given in [15,27] when the number of data patches increases to infinity.

THEOREM 3.2. *Under the hypotheses of Theorem 3.1, if moreover we assume that (6) is satisfied, then the solution $\sigma_{\varepsilon,h}^\eta$ of (12) and (13) satisfies the following.*

(i)
$$\lim_{\varepsilon\to 0,\, h^{2m}/\varepsilon \to 0,\, \eta^t/\varepsilon < \beta} \left\|\sigma_{\varepsilon,h}^\eta - \sigma\right\|_{m,\Omega} = 0,$$

where σ is the solution of (1), and where β is introduced in Theorem 3.1.

(ii) *There exists a positive constant C such that*

$$\left\|\sigma_{\varepsilon,h}^\eta - f\right\|_{0,\omega}^2 \le C\left(h^{2m} + \eta^t o(1) + \varepsilon\right), \qquad \text{when } \varepsilon \to 0, \quad \frac{h^{2m}}{\varepsilon} \to 0, \quad \frac{\eta^t}{\varepsilon} < \beta.$$

PROOF. The proof of Point (i) will be split into four steps.

STEP 1. Let σ be the unique solution of (1). We have $\sigma_{|\omega} = f_{|\omega}$ and

$$\left\|\sigma^{\eta}_{\varepsilon,h} - f\right\|^2_{0,\omega} = \left\|\sigma^{\eta}_{\varepsilon,h} - \sigma\right\|^2_{0,\omega}.$$

We obtain, using (4),

$$\left\|\sigma^{\eta}_{\varepsilon,h} - f\right\|^2_{0,\omega} \leq \ell^{\eta}\left(\left(\sigma^{\eta}_{\varepsilon,h} - \sigma\right)^2\right) + C\eta^t \left\|\sigma^{\eta}_{\varepsilon,h} - \sigma\right\|^2_{m,\Omega}. \tag{15}$$

Then, from (12), we have

$$\forall v_h \in V_h, \ell^{\eta}\left(\left(\sigma^{\eta}_{\varepsilon,h} - \sigma\right)^2\right) + \varepsilon\left|\sigma^{\eta}_{\varepsilon,h}\right|^2_{m,\Omega h} \leq \ell^{\eta}\left((v_h - \sigma)^2\right) + \varepsilon\left|v_h\right|^2_{m,\Omega_h}. \tag{16}$$

Let $\tilde{\sigma}$ be an m-extension of σ on $\tilde{\Omega}$, taking (with (6)) $v_h = \tilde{\Pi}_h\tilde{\sigma} \in H^m(\tilde{\Omega})$, we obtain

$$\ell^{\eta}\left(\left(\sigma^{\eta}_{\varepsilon,h} - \sigma\right)^2\right) \leq \ell^{\eta}\left(\left(\tilde{\Pi}_h\tilde{\sigma} - \sigma\right)^2\right) + \varepsilon\left|\tilde{\Pi}_h\tilde{\sigma}\right|^2_{m,\Omega_h}, \tag{17}$$

but from (6)(ii), there exists $h_0 \in H$ such that

$$\forall h \leq h_0, \qquad \left|\tilde{\Pi}_h\tilde{\sigma}\right|^2_{m,\Omega h} \leq \left|\tilde{\Pi}_h\tilde{\sigma}\right|^2_{m,\tilde{\Omega}} \leq C\left|\tilde{\sigma}\right|^2_{m,\tilde{\Omega}} \leq C\left\|\sigma\right\|^2_{m,\Omega}. \tag{18}$$

Moreover, using (4), we have

$$\ell^{\eta}\left(\left(\tilde{\Pi}_h\tilde{\sigma} - \sigma\right)^2\right) \leq \left\|\tilde{\Pi}_h\tilde{\sigma} - \sigma\right\|^2_{0,\omega} + C\eta^t\left\|\tilde{\Pi}_h\tilde{\sigma} - \sigma\right\|^2_{m,\Omega},$$

and finally, with (6)(i),

$$\left\|\tilde{\Pi}_h\tilde{\sigma} - \sigma\right\|_{0,\omega} \leq \left\|\tilde{\Pi}_h\tilde{\sigma} - \tilde{\sigma}\right\|_{0,\tilde{\Omega}} \leq Ch^m\left|\tilde{\sigma}\right|_{m,\tilde{\Omega}}.$$

Thus, using (6), we get

$$\ell^{\eta}\left(\left(\tilde{\Pi}_h\tilde{\sigma} - \sigma\right)^2\right) \leq Ch^{2m}\left|\tilde{\sigma}\right|^2_{m,\tilde{\Omega}} + C\eta^t o(1), \qquad h \to 0. \tag{19}$$

From (17) and (19), we deduce that

$$\ell^{\eta}\left(\left(\sigma^{\eta}_{\varepsilon,h} - \sigma\right)^2\right) \leq Ch^{2m}\left|\tilde{\sigma}\right|_{m,\tilde{\Omega}} + C\eta^t o(1) + \varepsilon\left|\tilde{\Pi}_h\tilde{\sigma}\right|^2_{m,\Omega_h}, \qquad h \to 0, \tag{20}$$

and therefore, using (15), we have

$$\begin{aligned}\left\|\sigma^{\eta}_{\varepsilon,h} - \sigma\right\|^2_{0,\omega} &\leq Ch^{2m}\left|\tilde{\sigma}\right|_{m,\tilde{\Omega}} + C\eta^t o(1) + \varepsilon\left|\tilde{\Pi}_h\tilde{\sigma}\right|^2_{m,\Omega h} + C\eta^t\left\|\sigma^{\eta}_{\varepsilon,h} - \sigma\right\|^2_{m,\Omega} \\ &\leq C\left(h^{2m} + \eta^t o(1)\right) + \varepsilon\left|\tilde{\Pi}_h\tilde{\sigma}\right|^2_{m,\Omega h} + C\eta^t\left\|\sigma^{\eta}_{\varepsilon,h} - \sigma\right\|^2_{m,\Omega}, \qquad h \to 0.\end{aligned} \tag{21}$$

Likewise, the relations (16) with $v_h = \tilde{\Pi}_h\tilde{\sigma}$, (18) and (19) involve that

$$\left|\sigma^{\eta}_{\varepsilon,h} - \sigma\right|^2_{m,\Omega} \leq C\left(\frac{h^{2m}}{\varepsilon} + \frac{\eta^t}{\varepsilon}o(1) + \left\|\sigma\right\|^2_{m,\Omega}\right), \qquad h \to 0. \tag{22}$$

We finally obtain, using (21) and (22),

$$\left|\left|\left|\sigma^{\eta}_{\varepsilon,h}-\sigma\right|\right|\right|^2 \le C(\varepsilon+1)\left(\frac{h^{2m}}{\varepsilon}+\frac{\eta^t}{\varepsilon}o(1)+\|\sigma\|^2_{m,\Omega}\right)+C\eta^t\left\|\sigma^{\eta}_{\varepsilon,h}-\sigma\right\|^2_{m,\Omega}, \qquad h\to 0, \quad (23)$$

where $|||.|||$ is defined in (2). Because the norm $|||.|||$ is equivalent to the usual norm $\|.\|_{m,\Omega}$ in $H^m(\Omega)$, we deduce from (23) that

$$\left(1-C\eta^t\right)\left\|\sigma^{\eta}_{\varepsilon,h}-\sigma\right\|^2_{m,\Omega} \le C(\varepsilon+1)\left(\frac{h^{2m}}{\varepsilon}+\frac{\eta^t}{\varepsilon}o(1)+\|\sigma\|^2_{m,\Omega}\right), \qquad h\to 0.$$

Let η' be a positive constant less than $(1/C)^{1/t}$. Therefore, the family $(\sigma^{\eta}_{\varepsilon,h})_{\eta,\varepsilon,h}$ is bounded in $H^m(\Omega)$ if

$$\varepsilon\to 0, \quad \frac{h^{2m}}{\varepsilon}\to 0, \qquad \eta\le\eta', \quad \frac{\eta^t}{\varepsilon}\le\beta. \quad (24)$$

The previous result involves that $H^m(\Omega)$ contains a sequence $(\sigma^{\eta_n}_{\varepsilon_n,h_n})_{n\in\mathbb{N}^*}$ extracted from the family $(\sigma^{\eta}_{\varepsilon,h})$ and a function $\sigma^*\in H^m(\Omega)$ such that

$$\begin{gathered}\sigma^* = \text{weak}\lim_{n\to+\infty}\sigma^{\eta_n}_{\varepsilon_n,h_n}, \qquad \text{in } H^m(\Omega),\\ \text{with} \lim_{n\to+\infty}\varepsilon_n=\lim_{n\to+\infty}\frac{h_n^{2m}}{\varepsilon_n}=0,\ (\eta_n)\subset(0,\eta') \text{ and } \left(\frac{\eta_n^t}{\varepsilon_n}\right)\subset(0,\beta).\end{gathered} \quad (25)$$

STEP 2. Now, we have to show that $\sigma^*=\sigma$. Let us consider the relations (20) and (22) applied to the sequence $(\sigma^{\eta_n}_{\varepsilon_n,h_n})_n$

$$\begin{gathered}\exists C>0,\\ \left|\sigma^{\eta_n}_{\varepsilon_n,h_n}\right|^2_{m,\Omega}\le C\left(\frac{h_n^{2m}}{\varepsilon_n}+\frac{\eta_n^t}{\varepsilon_n}o(1)\right)+\left|\tilde{\Pi}_{h_n}\tilde{\sigma}\right|^2_{m,\Omega_{h_n}}, \qquad n\to+\infty,\\ \ell^{\eta_n}\left(\left(\sigma^{\eta_n}_{\varepsilon_n,h_n}-\sigma\right)^2\right)\le Ch_n^{2m}|\tilde{\sigma}|_{m,\tilde{\Omega}}+C\eta_n^t o(1)+\varepsilon_n\left|\tilde{\Pi}_{h_n}\tilde{\sigma}\right|^2_{m,\Omega_{h_n}}, \qquad n\to+\infty.\end{gathered} \quad (26)$$

But, for n large enough, we have, knowing that $\lim_{n\to+\infty}|\tilde{\Pi}_{h_n}\tilde{\sigma}|_{m,\Omega_{h_n}}=|\sigma|_{m,\Omega}$,

$$\left|\tilde{\Pi}_{h_n}\tilde{\sigma}\right|^2_{m,\Omega_{h_n}}\le|\sigma|^2_{m,\Omega}+\mu^2, \qquad \mu>0 \text{ arbitrarily small},$$

and then it comes that

$$\left|\sigma^{\eta_n}_{\varepsilon_n,h_n}\right|^2_{m,\Omega}\le C\left(\frac{h_n^{2m}}{\varepsilon_n}+\frac{\eta_n^t}{\varepsilon_n}o(1)\right)+|\sigma|^2_{m,\Omega}+\mu^2, \qquad n\to+\infty. \quad (27)$$

From (4), we have

$$\left\|\sigma^{\eta_n}_{\varepsilon_n,h_n}-\tilde{\sigma}\right\|^2_{0,\omega}\le\ell^{\eta_n}\left(\left(\sigma^{\eta_n}_{\varepsilon_n,h_n}-\tilde{\sigma}\right)^2\right)+C\eta_n^t\left\|\sigma^{\eta_n}_{\varepsilon_n,h_n}-\tilde{\sigma}\right\|^2_{m,\Omega},$$

thus, using (26), and because the sequence $(\sigma^{\eta_n}_{\varepsilon_n,h_n})$ is bounded in $H^m(\Omega)$, we obtain

$$\left\|\sigma^{\eta_n}_{\varepsilon_n,h_n}-\tilde{\sigma}\right\|^2_{0,\omega}\le C\varepsilon_n\left(\frac{h_n^{2m}}{\varepsilon_n}+\frac{\eta_n^t}{\varepsilon_n}o(1)+|\sigma|^2_{m,\Omega}+\mu^2+\frac{\eta_n^t}{\varepsilon_n}\right), \qquad n\to+\infty. \quad (28)$$

Therefore, from (27) and (28), we finally have

$$\begin{gathered}|\sigma^*|^2_{m,\Omega}\le\lim_{n\to+\infty}\inf\left|\sigma^{\eta_n}_{\varepsilon_n,h_n}\right|^2_{m,\Omega}\le|\sigma|^2_{m,\Omega}+\mu^2,\\ \|\sigma-\sigma^*\|^2_{0,\omega}\le\lim_{n\to+\infty}\inf\left\|\sigma^{\eta_n}_{\varepsilon_n,h_n}-\tilde{\sigma}\right\|^2_{0,\omega}\le 0.\end{gathered} \quad (29)$$

It comes from (29), as μ is arbitrarily small,

$$\sigma^* \in H^m(\Omega),$$
$$|\sigma^*|_{m,\Omega} \le |\sigma|_{m,\Omega},$$
$$\sigma = \sigma^*, \qquad \text{on } \omega.$$

And thus, $\sigma = \sigma^*$ in $H^m(\Omega)$ by considering the existence and uniqueness of the solution of the problem of equation (1) (see Section 1).

STEP 3. We now have to show that $\lim_{n\to+\infty} \sigma^{\eta_n}_{\varepsilon_n,h_n} = \sigma$ in $H^m(\Omega)$. Using the compact embedding of $H^m(\Omega)$ in $H^{m-1}(\Omega)$ and knowing that $\sigma = \sigma^*$ in $H^m(\Omega)$, we have $\sigma^{\eta_n}_{\varepsilon_n,h_n} \to_{n\to+\infty} \sigma$ in $H^{m-1}(\Omega)$. Furthermore, for any $n \in \mathbb{N}$,

$$\left|\sigma^{\eta_n}_{\varepsilon_n,h_n} - \sigma\right|^2_{m,\Omega} \le \left|\sigma^{\eta_n}_{\varepsilon_n,h_n}\right|^2_{m,\Omega} + |\sigma|^2_{m,\Omega} - 2\left(\sigma^{\eta_n}_{\varepsilon_n,h_n}, \sigma\right)_{m,\Omega}.$$

Therefore, using (27), we obtain

$$\left|\sigma^{\eta_n}_{\varepsilon_n,h_n} - \sigma\right|^2_{m,\Omega} \le C\left(\frac{h_n^{2m}}{\varepsilon_n} + \frac{\eta_n^t}{\varepsilon_n} o(1)\right) + 2|\sigma|^2_{m,\Omega} + \mu^2 - 2\left(\sigma^{\eta_n}_{\varepsilon_n,h_n}, \sigma\right)_{m,\Omega}, \qquad n \to +\infty,$$

and when $n \to +\infty$, it comes because $\sigma^{\eta_n}_{\varepsilon_n,h_n} \rightharpoonup \sigma$ in $H^m(\Omega)$ that

$$\lim_{n\to+\infty} \left|\sigma^{\eta_n}_{\varepsilon_n,h_n} - \sigma\right|^2_{m,\Omega} \le 2|\sigma|^2_{m,\Omega} + \mu^2 - 2(\sigma,\sigma)_{m,\Omega} = \mu^2,$$

and the result follows because μ is arbitrarily small.

STEP 4. To achieve this proof, we now assume that

$$\lim \left\|\sigma^{\eta}_{\varepsilon,h} - \sigma\right\|^2_{m,\Omega} \ne 0,$$

with $\varepsilon \to 0$, $h^{2m}/\varepsilon \to 0$, $\eta^t/\varepsilon \le \beta$. It means there exists a sequence $(d'_n, \eta'_n, \varepsilon'_n, h'_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} \varepsilon'_n = \lim_{n\to+\infty} h'^{2m}_n/\varepsilon'_n = 0$, $\eta'^t_n/\varepsilon'_n \le \beta$ satisfying for any $n \in \mathbb{N}^*$,

$$\left\|\sigma^{d'_n,\eta'_n}_{\varepsilon'_n,h'_n} - f\right\|_{m,\Omega} > \alpha.$$

But such a sequence is bounded in $H^m(\Omega)$ and using the previous argument, we reach a contradiction.

Point (ii) is an immediate consequence of equation (21) and Point (i), taking into account that $\sigma_{|\omega} = f_{|\omega}$. ∎

4. NUMERICAL RESULTS

The CPU time for each of the following examples was less than 15 seconds (on a PC Intel Pentium III 500 Mhz, 256 Megabytes).

We have chosen two functions f and g defined by

$$f(x,y) = e^{-(3x-1)^2-(3y-1)^2}$$

and

$$g(x,y) = \frac{3}{4} e^{\left(-(1/4)(9x-2)^2-(1/4)(9y-2)^2\right)} + \frac{3}{4} e^{\left(-(1/49)(9x+1)^2-(1/10)(9y+1)^2\right)} + \frac{1}{2} e^{\left(-(1/4)(9x-7)^2-(1/4)(9y-3)^2\right)} - \frac{1}{5} e^{\left(-(9x-4)^2-(9y-7)^2\right)}.$$

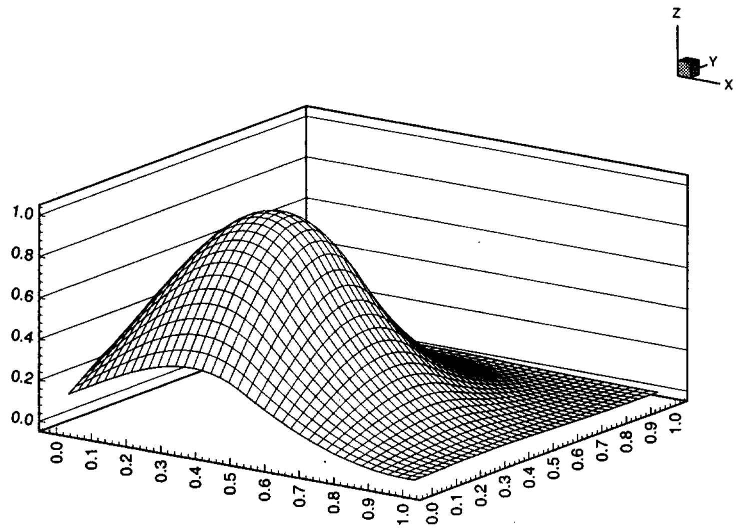

Figure 3. Function f.

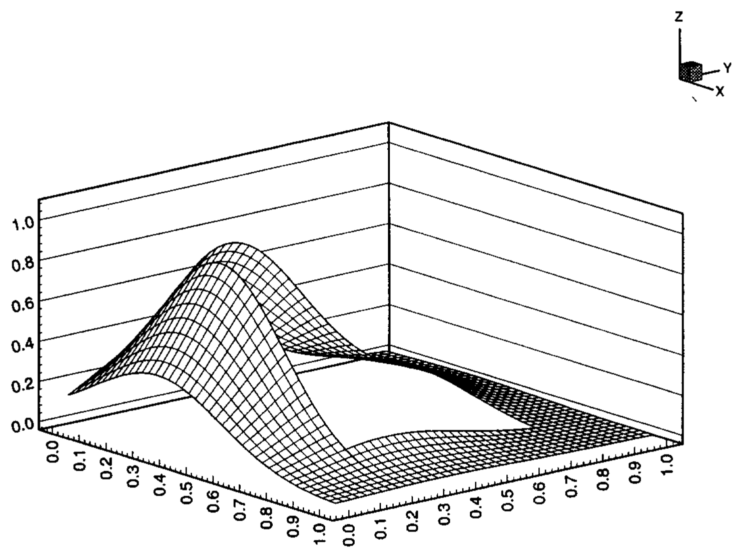

Figure 4. Data corresponding to $f_{|\omega_1}$.

The open Ω is $]0,1[\times]0,1[$. The chosen generic finite element is the Bogner-Fox-Schmit rectangle of class C^1. For the numerical integration on the given subsets, we use the P_3 formula on triangles:

$$\int_T u\,dx \simeq \text{meas}\,(T)\left(\frac{1}{20}\sum_{i=1}^{3}u(a_i)+\frac{2}{15}\sum_{i=1}^{3}u(b_i)+\frac{9}{20}u(c)\right),$$

as introduced in Remark 2.

We take $\varepsilon = 10^{-6}$. To our knowledge, there is no mathematical method to optimize this choice. In Deshpande and Girard [28,29], cross validation methods are studied for a similar kind of problems.

For the function f (see Figure 3), the data corresponding to the first example are reproduced in Figure 4. Only one surface patch ω_1 is used and the triangulation is made using 288 triangles (a total of 288 distinct vertices for ω). We use a finite element grid which divides Ω into 81 ($= 9\times 9$) rectangles. Therefore, the dimension of the associated finite element space V_h is equal to 400.

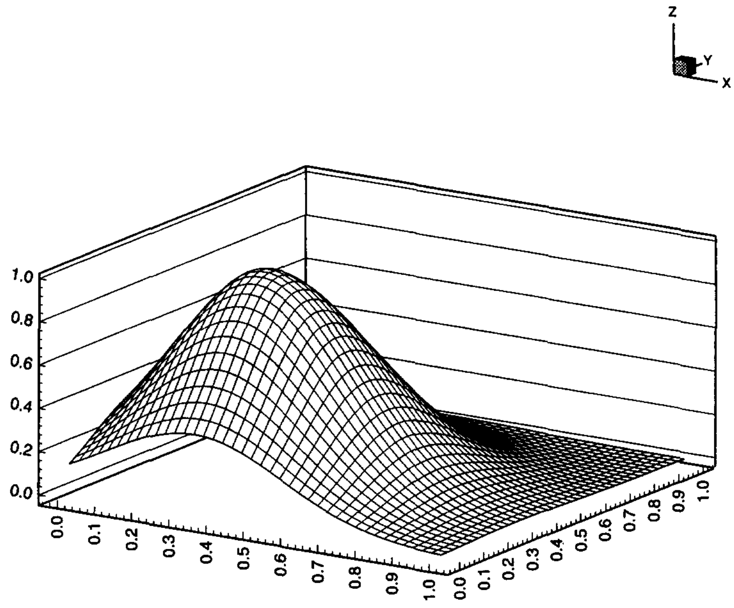

Figure 5. C^1 approximant.

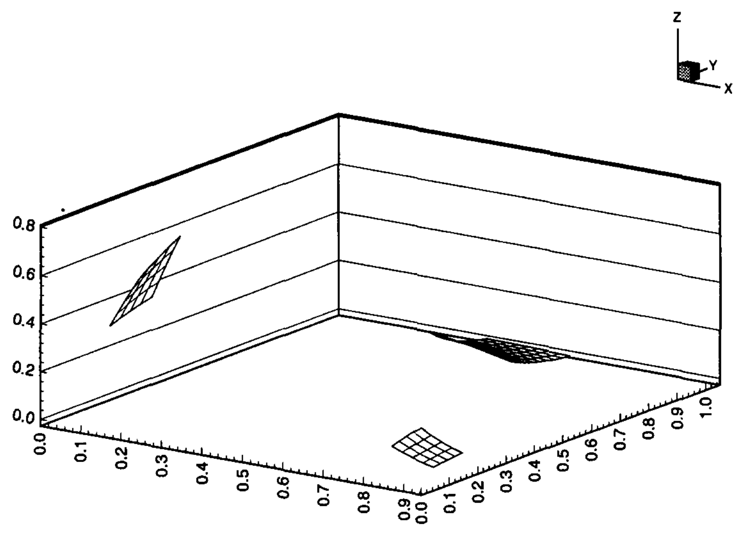

Figure 6. Data corresponding to $f_{|\omega_{1,2,3}}$.

For the second example, data are reproduced in Figure 6. In this case, ω has three connected components $(\omega_1, \omega_2, \omega_3)$ and the triangulation is made using two triangles with a common side per $\omega_{i:i=1,2,3}$ (a total of 12 distinct vertices for ω). We use a finite element grid which divides Ω into four rectangles (dim $V_h = 36$). We show the three surface patches. The corresponding approximant surface are given, for the first example, in Figure 5, and for the second one in Figure 7.

For the function g (see Figure 8), we have used the same kind of data (and the same finite element grids) as for the function f, one surface patch (see Figure 9) or three surface patches (see Figure 11). The corresponding approximating surfaces are displayed in Figures 10 and 12, respectively.

In all cases, we have evaluated the approximant on a regular 40×40 grid of points. To estimate the accuracy of the method, we evaluate the quadratic error (see Tables 1 and 2) on Ω (for the

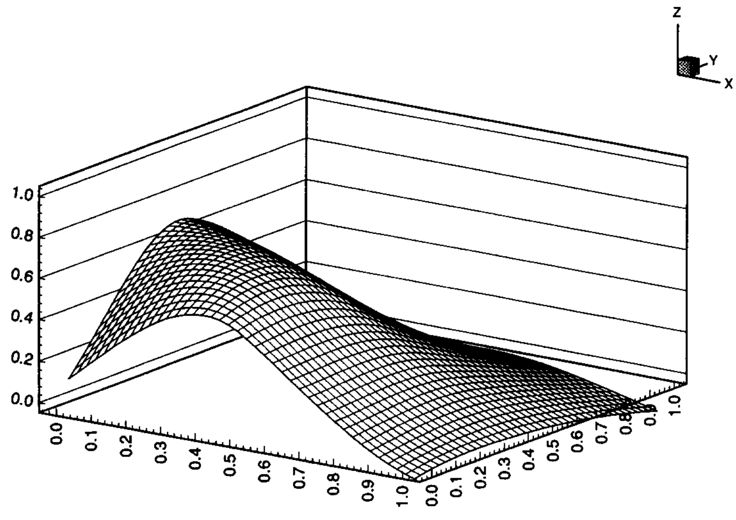

Figure 7. C^1 approximant.

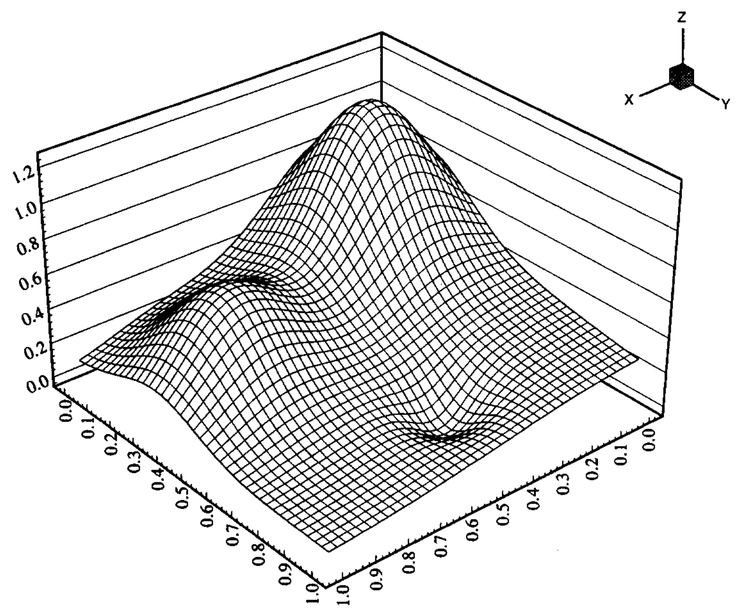

Figure 8. Function g.

approximant and a usual D^m-spline (see, for instance, [9]) based upon the classical formula

$$Q_{\text{error}}\left(\bigcup_i (x_i, y_i, z_i)\right) = \left(\frac{\sum\limits_{i=1}^{1600} (\tilde{z}_i - z_i)^2}{\sum\limits_{i=1}^{1600} z_i^2}\right),$$

where z_i represents the z-data value, and where $\tilde{z}_i$ is the z-approximant for the same (x_i, y_i).

For the usual D^m-spline, the Lagrange data are the nodes of the triangulation, and of course, we have chosen exactly the same finite element grid (and the Bogner-Fox-Schmit finite element of class C^1).

Considering the dataset corresponding to the first triangulation (288 triangles), such values are considered very good in the context of surface approximation. The second triangulation (six triangles) illustrates the method in the case of fewer data points.

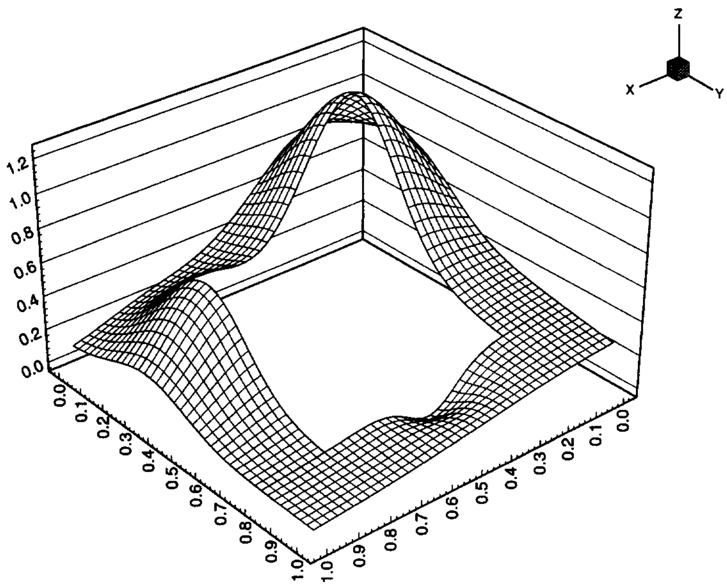

Figure 9. Data corresponding to $g_{|\omega_1}$.

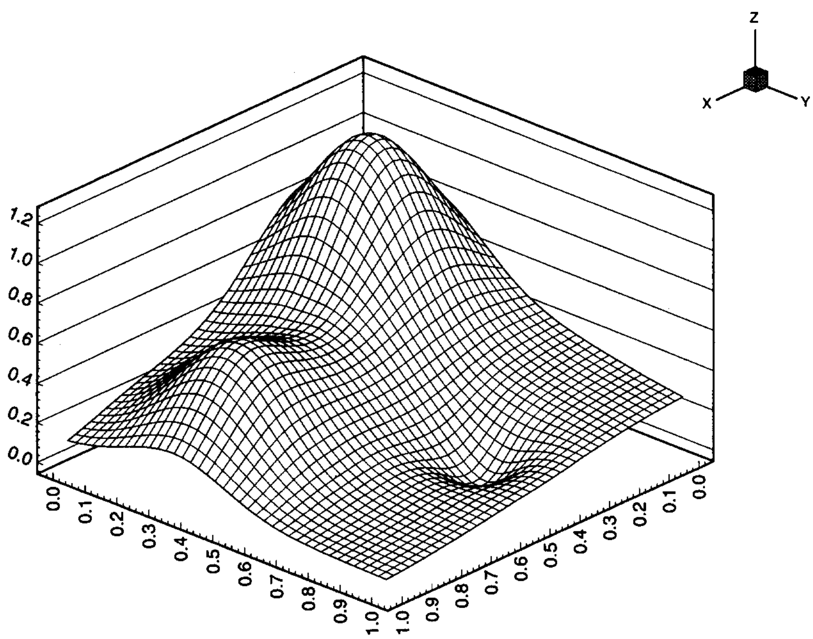

Figure 10. C^1 approximant.

We now give a numerical example from a set of 7049 data points (Figure 13) in the Big Island area in Hawaii. The maximum height of the big island is 4.7 km, and the depth of the seafloor reaches more than 4 km in several places. The topogaphy and bathymetry of the Hawaiian Islands in the Pacific Ocean result from the activity of a huge hot spot combined with the effect of erosion. Being able to describe the topography of such regions exhibiting rapid local variations with at least C^0 regularity, or even C^1 regularity, is important in many fields in geophysics. For example, this description of the topography can be an input to numerical modeling codes that study the propagation of pyroclastic flows or lava flows, and related hazard; other examples are seismic site effects and ground motion amplification due to topographic features. The data points

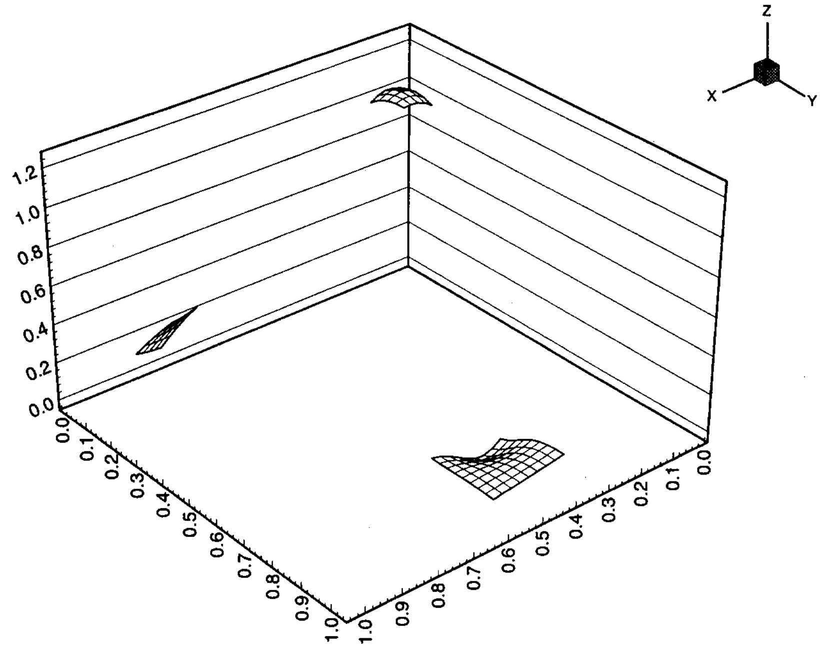

Figure 11. Data corresponding to $g_{|\omega_{1,2,3}}$.

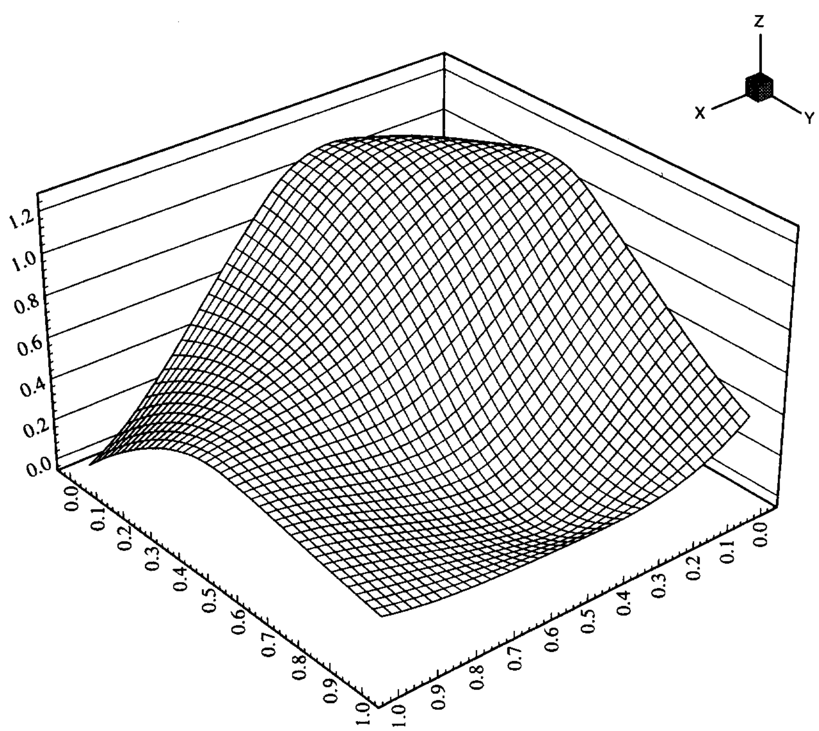

Figure 12. C^1 approximant.

in the digital elevation model (DEM) have been obtained by digitizing a not complete map of the seafloor. A zone without data is located from the northwest corner to the southeast one (see Figure 13). This zone divides the domain into two regions. We have made a triangulation on each region using the software Mefisto (see [30]). We use a 23×23 rectangular C^1-BFS finite element

Table 1.

Function f	$Q_{\text{error}}\left(\bigcup_i (x_i, y_i, z_i)\right)$	Data
Approximant	$9.5\ 10^{-4}$	ω_1: 1152 data points
D^m-Spline	$1.17\ 10^{-3}$	ω_1: 1152 data points
Approximant	$8.91\ 10^{-3}$	$\omega_{i:i=1,2,3}$: 33 data points
D^m-Spline	$1.07\ 10^{-2}$	$\omega_{i:i=1,2,3}$: 33 data points

Table 2.

Function g	$Q_{\text{error}}\left(\bigcup_i (x_i, y_i, z_i)\right)$	Data
Approximant	$1.34\ 10^{-3}$	ω_1: 1152 data points
D^m-Spline	$1.30\ 10^{-3}$	ω_1: 1152 data points
Approximant	0.18	$\omega_{i:i=1,2,3}$: 33 data points
D^m-Spline	0.24	$\omega_{i:i=1,2,3}$: 33 data points

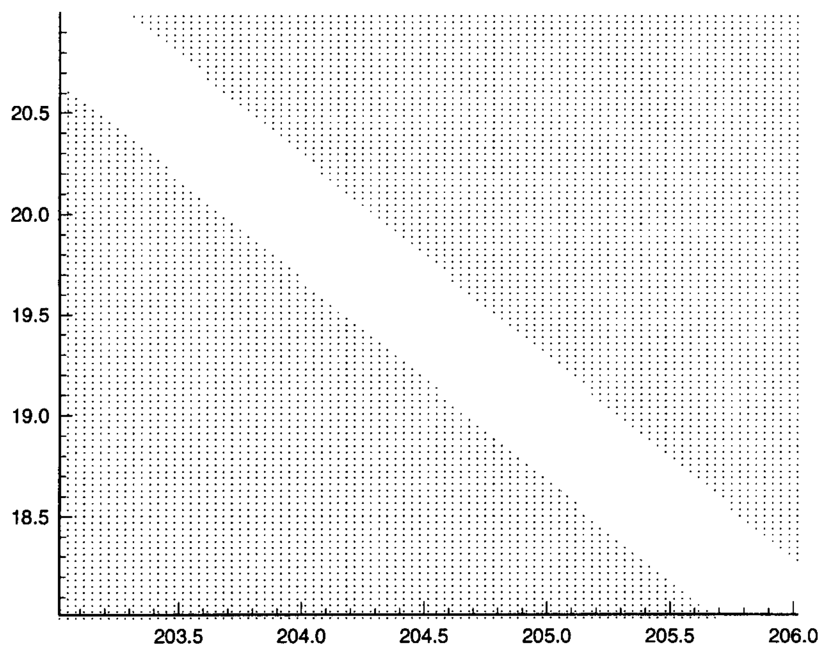

Figure 13. Dataset of 7049 points, Big Island, Hawaii.

grid. We give the C^1 approximant in Figure 14. We also evaluate the approximant obtained at the 7049 data points of the dataset. To estimate the error quantitatively, we then evaluate the quadratic error on the dataset: we obtain a value of $1.02\ 10^{-4}$, which is a satisfactory result.

For the reader interested in learning more about possible applications of this technique to realistic cases (and more complicated datasets), numerical examples to real geophysical data (large datasets up to several hundreds of thousands data points coming from an old glacial valley located in the Vallée d'Ossau, Pyrénées Mountains, France) are given in [1]. The regularity obtained, which can be C^0, C^1, or higher, allows us to describe the topography of real geophysical surfaces accurately. Future work will focus on investigating automatic methods to choose the finite element grid and, when necessary, the local refinement, as well as on the choice of quadrature formula with better order of approximation.

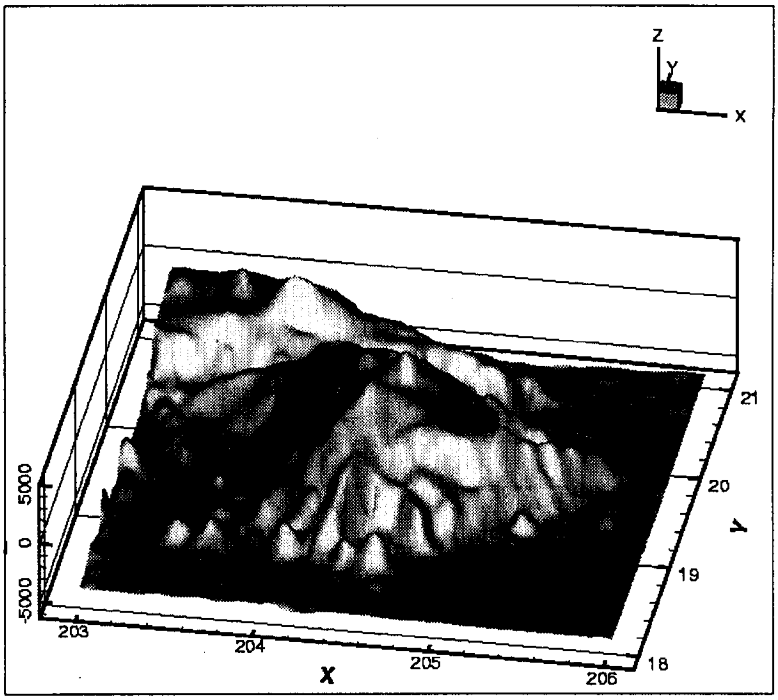

Figure 14. C^1 approximant (evaluated on a regular 250×250 grid of points).

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