Note

Resolvable packings $\tilde{\text{R}}\text{MP}(3, 2; n, n - 3)$ and coverings
$\tilde{\text{R}}\text{MC}(3, 2; n, n - 2)$

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Abstract

Let $n \equiv k - 1, 0$ or $1 \pmod{k}$. An $\tilde{\text{R}}\text{MP}(k, \lambda; n, m)$ (resp. $\tilde{\text{R}}\text{MC}(k, \lambda; n, m)$) is a resolvable packing (resp. covering) with maximum (resp. minimum) possible number $m$ of parallel classes which are mutually distinct, each parallel class consists of $\lfloor (n-k+1)/k \rfloor$ blocks of size $k$ and one block of size $n-k\lfloor (n-k+1)/k \rfloor$, and its leave (resp. excess) is a simple graph. Such designs can be used to construct certain uniform designs which have been widely applied in industry, system engineering, pharmaceutics, and natural sciences. In this paper, direct and recursive constructions are discussed for such designs. The existence of an $\tilde{\text{R}}\text{MP}(3, 2; n, n - 3)$ and an $\tilde{\text{R}}\text{MC}(3, 2; n, n - 2)$ for $n \equiv 1 \pmod{3}$ is established with $n \geq 16$.

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1. Introduction

Let $n$ and $\lambda$ be positive integers. A packing (resp. covering) of $X$ is a collection of subsets of $X$ (called blocks) such that any pair of distinct points from $X$ occur together in at most (resp. at least) $\lambda$ blocks in the collection. Denote by $P(K, \lambda; n)$ (resp. $C(K, \lambda; n)$) a packing (resp. covering) on $n$ points with block sizes from $K$.

For any pair $e = \{x, y\}$ of distinct points, let $m(e)$ be the number of blocks containing $e$. The leave (resp. excess) of a packing (resp. covering) $P(K, \lambda; n)$ (resp. $C(K, \lambda; n)$) is the multigraph spanned by all pairs $e$ of distinct points with multiplicity $\lambda - m(e)$ (resp. $m(e) - \lambda$).

A packing or a covering is called resolvable if its block set admits a partition into parallel classes, each parallel class being a partition of the point set $X$. Denote by $\text{RP}(K, \lambda; n, m)$ (resp. $\text{RC}(K, \lambda; n, m)$) a resolvable $P(K, \lambda; n)$ (resp. $C(K, \lambda; n)$) with $m$ parallel classes. We write $\text{RP}(k, \lambda; n, m)$ (resp. $\text{RC}(k, \lambda; n, m)$) instead of $\text{RP}(K, \lambda; n, m)$ (resp. $\text{RC}(K, \lambda; n, m)$) when $K = \{k\}$.

Let $n \equiv k - 1, 0$ or $1 \pmod{k}$. An $\tilde{\text{R}}\text{MP}(k, \lambda; n, m)$ (resp. $\tilde{\text{R}}\text{MC}(k, \lambda; n, m)$) is a resolvable packing (resp. covering) $\text{RP}(\lfloor k-1, k, k+1 \rfloor, \lambda; n, m)$ (resp. $\text{RC}(\lfloor k-1, k, k+1 \rfloor, \lambda; n, m)$) which satisfies the following properties:

1. it contains the maximum (resp. minimum) possible number $m$ of parallel classes which are mutually distinct;
2. each parallel class consists of $\lfloor (n-k+1)/k \rfloor$ blocks of size $k$ and one block of size $n-k\lfloor (n-k+1)/k \rfloor$; 
3. its leave (resp. excess) is a simple graph, that is, $\lambda - m(e) \leq 1$ (resp. $m(e) - \lambda \leq 1$) for any pair $e$ of distinct points.

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Both $\tilde{R}MP(k, \lambda; n, m)$s and $\tilde{R}MC(k, \lambda; n, m)$s were first introduced by Fang et al. [7]. They can be used in the construction of uniform designs in statistics, which have been widely applied in industry, system engineering, pharmaceutics, and natural sciences [5,6,19].

**Theorem 1.1 (Fang et al. [7], Theorem 3.1).** Let $n, k, \lambda$, and $m$ be positive integers, and $n \equiv r \pmod{k}$ where $r \in \{0, 1, k - 1\}$. If there exists an $\tilde{R}MP(k, \lambda; n, m)$ (or $\tilde{R}MC(k, \lambda; n, m)$), then there exists a uniform design $U_n(q^m)$, where $q = [(n - k + 1)/k] + 1$.

In the particular case where the $\tilde{R}MP$ and $\tilde{R}MC$ are exact, that is, the leave and excess are empty, we simply write $\tilde{R}B(k, \lambda; n, m)$ for both $\tilde{R}MP(k, \lambda; n, m)$ and $\tilde{R}MC(k, \lambda; n, m)$. In the literature, an $\tilde{R}MP(k, 1; n, m)$ is also called a Kirkman packing design or a Kirkman school project design when $k = 3$, an $\tilde{R}MC(k, 1; n, m)$ is called a Kirkman covering design. When $k = 3$ and $\lambda \in \{1, 2\}$, the existence of an $\tilde{R}MP(3, 1; n, m)$ or $\tilde{R}MC(3, 1; n, m)$ has been solved for every positive integer $n$ with few possible exceptions [15,16,12,4,14,1]; the existence of an $\tilde{R}MP(3, 2; n, m)$ or $\tilde{R}MC(3, 2; n, m), n \equiv 0, 2, 3, 4, 5 \pmod{6}$, has been solved with some possible exceptions [17,7]. There are also some results on $\tilde{R}MP(4, \lambda; n, m)$ and $\tilde{R}MC(4, \lambda; n, m)$ for $\lambda \in \{1, 2\}$ [11,13,9,10,7]. The main known results concerning $\tilde{R}MP(3, \lambda; n, m)$ and $\tilde{R}MC(3, \lambda; n, m)$ are as follows.

**Theorem 1.2.** There exist

1. an $\tilde{R}MP(3, 1; n, [n - 1/2])$ and an $\tilde{R}MC(3, 1; n/2)$ when $n \equiv 0 \pmod{3}$ and $n \not\in \{6, 12\}$;
2. an $\tilde{R}MP(3, 1; n, [n - 3/2])$ when $n \equiv 1 \pmod{3}$ and except for $n \in \{1, 4, 7, 10, 13\}$ and possibly except for $n = 19$;
3. an $\tilde{R}MC(3, 1; n, [n - 1/2])$ when $n \equiv 1 \pmod{3}$ and $n \not\in \{16, 67\}$;
4. an $\tilde{R}MP(3, 1; n, [n/2])$ when $n \equiv 2 \pmod{3}$ and $n \not\in \{5, 11\}$;
5. an $\tilde{R}MC(3, 1; n, [n + 1/2])$ when $n \equiv 2 \pmod{3}$ and $n \not\in \{5, 11\}$;
6. an $\tilde{R}B(3, 2; n, n)$ when $n \equiv 0 \pmod{3}$ and $n \geq 9$;
7. an $\tilde{R}B(3, 2; n, n)$ when $n \equiv 2 \pmod{3}$ and $n \geq 8$;
8. an $\tilde{R}MP(3, 2; n, n - 3)$ and an $\tilde{R}MC(3, 2; n, n - 2)$ when $n \equiv 4 \pmod{6}$, $n \geq 16$ and $n \not\in \{28, 34, 40, 46, 58, 70, 82, 94, 142\}$.

In this paper, we mainly deal with the existence of an $\tilde{R}MP(3, 2; n, n - 3)$ and an $\tilde{R}MC(3, 2; n, n - 2)$ for $n \equiv 1 \pmod{3}$, we also solve some left possible exceptions in other infinity classes. Direct and recursive constructions are discussed for an $\tilde{R}MP$ and an $\tilde{R}MC$. The existence of an $\tilde{R}MP(3, 2; n, n - 3)$ and an $\tilde{R}MC(3, 2; n, n - 2)$ for $n \equiv 1 \pmod{3}$ is established with $n \geq 16$.

**Theorem 1.3.** There exist an $\tilde{R}MP(3, 2; n, n - 3)$ and an $\tilde{R}MC(3, 2; n, n - 2)$ for every $n \equiv 1 \pmod{3}$ and $n \geq 16$.

**2. Direct construction using $\ast DF(n, K, 2)$**

In this section, we construct some small $\tilde{R}MP(3, 2; n, n - 3)$s some of which will be used as input designs in recursive constructions of the next section. These $\tilde{R}MP(3, 2; n, n - 3)$s can be obtained from a special difference family $\ast DF(n - 4, K, 2)$.

**Definition 2.1.** Suppose $n \equiv 0 \pmod{3}$. Let $G$ be the additive group of $\mathbb{Z}_n$. Let $S = (B_1, B_2, \ldots, B_s)$ be a partition of $G$ with $|B_i| \in K, 1 \leq i \leq s$. $B_i$ are called base blocks. Then $S$ is called a special difference family $\ast DF(n, K, 2)$ if each non-zero element $g \neq n/3$ of $G$ occurs equally twice in the sum $\Delta S$ of the difference lists

$$\triangle B_j := (b - b': b, b' \in B_j, b \neq b')$$

of the base blocks $B_j$, and $g = n/3$ occurs exactly once in $\Delta S$.

The type of a $\ast DF(n, K, 2)$ is the multiset $\{|B_j| : B_j \in S\}$. We denote the type by $2^{u_2}3^{u_3}, \ldots$, where there are precisely $u_i$ occurrences of $i, i \geq 2$. 


Lemma 2.1. Let \( n \equiv 0 \pmod{3} \), \( n \geq 12 \) and \( u = (n - 12)/3 \). If there exists a \(*DF(n, K, 2)\) with type \( 2^43^44^1 \), then there exist an \( \bar{R}\text{MP}(3, 2; n + 4, n + 1) \) and an \( \text{RMC}(3, 2; n + 4, n + 2) \).

**Proof.** Take the point set \( V = Z_n \cup \{\infty_1, \infty_2, \infty_3, \infty_4\} \). Suppose \( S = (B_1, B_2, \ldots, B_4) \) is a \(*DF(n, K, 2)\) on \( Z_n \) with type \( 2^43^44^1 \). Let \( |B_i| = 2 \) for \( 1 \leq i \leq 4 \). Add \( \infty_i \) to the base block \( B_i \), we may obtain a new block \( B'_i \). It is obvious that \( (B'_1, B'_2, B'_3, B'_4, B_5, \ldots, B_8) \) is a partition of \( V \), that is, they form a parallel class \( P_1 \). Thus \( n \) parallel classes can be generated from \( P_1 \) by \( +1 \mod n \), where \( \infty_i + x = \infty_j \). These \( n \) parallel classes form an \( \text{RP}((3, 4), 2; n + 4, n) \), the leave contains \((n/3)K_3\) and two \( K_4 \). By the definition of a \(*DF(n, K, 2)\), these \((n/3)K_3\) is a partition of \( Z_n \). So, we can obtain a new parallel class \( P_{n+1} \) on \( V \) which contains \((n/3)K_3\) and one \( K_4 \). It is easy to check that the \( n + 1 \) parallel classes form an \( \bar{R}\text{MP}(3, 2; n + 4, n + 1) \) on \( V \) whose leave is a \( K_4 \) on \{\( \infty_1, \infty_2, \infty_3, \infty_4\} \). Furthermore, an \( \text{RMC}(3, 2; n + 4, n + 2) \) can be obtained by adding one more parallel class \( P_{n+2} \) to the \( \bar{R}\text{MP} \) constructed above. The blocks of \( P_{n+2} \) are as follows: \( \{3i, 3i + 1, 3i + 2\} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, i \in Z_n/3 \). □

**Remark.** Note that no pair of infinity points appears in the parallel classes \( P_1, \ldots, P_n \). This property is important for the later recursive constructions.

Lemma 2.2. There exists a \(*DF(n, K, 2)\) with type \( 2^43^44^1 \) for each \( n \in \{12, 24, 30, 36, 42, 54, 66, 78\} \), where \( u = (n - 12)/3 \).

**Proof.** The base blocks for each \( n \) are listed as follows:

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Lemma 2.3. There exists a \(*DF(n, K, 2)\) with type \( 2^43^44^1 \) for each \( n \in \{15, 21, 27, 33, 39, 45, 51, 57, 69, 81\} \), where \( u = (n - 12)/3 \).

**Proof.** The base blocks for each \( n \) are listed as follows:

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Combining Lemma 2.1 with Lemmas 2.2 and 2.3, we get the following two results.

**Lemma 2.4.** There exist an \( \tilde{R} \text{MP}(3, 2; n, n - 3) \) and an \( \tilde{R} \text{MC}(3, 2; n, n - 2) \) for each \( n \in \{16, 28, 34, 40, 46, 58, 70, 82\} \).

**Lemma 2.5.** There exist an \( \tilde{R} \text{MP}(3, 2; n, n - 3) \) and an \( \tilde{R} \text{MC}(3, 2; n, n - 2) \) for each \( n \equiv 1 \pmod{6}, 19 \leq n \leq 61 \) and \( n \in \{73, 85\} \).

3. Main results

A group-divisible design of index \( \lambda \), denoted by \((K, \lambda)\)-GDD, is a triple \((X, \mathcal{G}, \mathcal{B})\) which satisfies the following properties: (i) \( X \) is a finite set of points, (ii) \( \mathcal{G} \) is a partition of \( X \) into subsets called groups, (iii) \( \mathcal{B} \) is a set of subsets of \( X \) with sizes from \( K \) (called blocks), such that a group and a block contain at most one common point, and every pair of points from distinct groups occurs in exactly \( \lambda \) blocks.

The type of a GDD is the multiset \([|G| : G \in \mathcal{G}]\). We denote the type by \( 1^{u_1}2^{u_2}\ldots \), where there are precisely \( u_i \) occurrences of \( i, i \geq 1 \). A \((k, \lambda)\)-GDD is a GDD in which all the blocks have size \( k \). A transversal design \( \text{TD}(k, \lambda, n) \) is a \((k, \lambda)\)-GDD of type \( n^k \). It is idempotent if it contains a parallel class of blocks.

A \( GDD(X, \mathcal{G}, \mathcal{B}) \) is called frame resolvable if its block set \( \mathcal{B} \) can be partitioned into frame parallel classes, each class being a partition of \( X - G_j \) for some \( G_j \in \mathcal{G} \). A \((k, \lambda)\)-frame is a frame resolvable \((k, \lambda)\)-GDD in which all the blocks have size \( k \). It is well known that to each \( G_j \) there are exactly \( \lambda/G_j/(k - 1) \) frame parallel classes of triples that partition \( X - G_j \). The groups in a \((k, \lambda)\)-frame are often considered as holes. The existence of \((3, \lambda)\)-frame has been solved completely.

**Theorem 3.1** (Stinson [18]). There exists a \((3, \lambda)\)-frame of type \( g^u \) if and only if \( u \geq 4, \lambda g \equiv 0 \pmod{2} \) and \( g(u - 1) \equiv 0 \pmod{3} \).

For later use, we need the following weighting construction for frames.

**Lemma 3.2** (Furino et al. [8]). Suppose that there is a \((K, 1)\)-GDD of type \( g_1^{l_1}g_2^{l_2}\ldots g_m^{l_m} \) and that for each \( k \in K \) there is a \((w, \lambda)\)-frame of type \( h^k \). Then there is a \((w, \lambda)\)-frame of type \((hg_1)^{l_1}(hg_2)^{l_2}\ldots(hg_m)^{l_m}\).
Lemma 3.3. For each \( v, v \equiv 0 \pmod{3} \), \( v \geq 144 \), there is a \((3, 2)\)-frame of type \( 12^a 15^b \), where \( v = 12a + 15b \), \( b \geq 4, a \geq 0 \).

Proof. Let \((n_0, n_1, \ldots)\) be the infinite sequence of integers defined as follows: \( n_0 = 11, n_j = 11 + j \) for \( j \geq 1 \). From [3, p.126], an idempotent TD \((5, 1, n_i)\) exists for each \( n_i, i > 0 \). Let \( t = v/3 \). Since \( v \geq 144 \), we have \( t \geq 48 \). There exists an integer \( n_i \) from the sequence so that \( 4n_i + 4 \leq t \leq 5n_i \). This can always be done because \( 5n_i \geq 4n_{i+1} + 4 \) for all \( i > 0 \). Let \( t = 5a + 4b \), where \( a = n_i - b \), \( 0 \leq b \leq n_i - 4 \). Form an idempotent TD \((5, 1, n_i)\) with groups \( G_1, \ldots, G_5 \) and blocks \( B_1, \ldots, B_{n_i} \) in one parallel class. Delete \( n_i - a \) points in \( G_5 \) that lie in \( B_{a+1}, \ldots, B_{n_i} \). Taking the truncated blocks \( B_1, \ldots, B_{n_i} \) as groups, we have formed a GDD of type \( 5^a 4^b \), having all blocks of size at least four. We may apply Lemma 3.2 with weight \( h = 3 \) to obtain a \((3, 2)\)-frame of type \( 12^a 15^b \), where the input \((3, 2)\)-frames of type \( 3^4 \) and \( 3^5 \) come from Theorem 3.1. \( \square \)

Lemma 3.4. Suppose that there exist a \((3, 2)\)-frame of type \( h_1^1 h_2^2 \ldots h_m^m \) and an *DF*(\( h_i, K, 2 \)) with type \( 2^4 3^u 4^l \) for each \( 1 \leq i \leq m \). Then there exist an \( \tilde{RMP}(3, 2; n, n - 3) \) and an \( \tilde{RMC}(3, 2; n, n - 2) \), where \( n = 4 + \sum_{i=1}^{m} h_i \).

Proof. For each \( 1 \leq i \leq m \), there are \( h_i \) frame parallel classes missing the group of size \( g_i \). Denote these parallel classes by \( Q_i, 1, \ldots, Q_i, h_i \). Similarly to the proof of Lemma 2.1, we can add four points \( \infty_1, \ldots, \infty_4 \) to the group and construct an \( \tilde{RMP}(3, 2; h_i + 4, h_i + 1) \) whose parallel classes are \( P_i, 1, \ldots, P_i, h_i, P_i, h_i + 1 \), each of which contains one block of size four. Let \( F_i = P_i, j \cup Q_i, j, 1 \leq i \leq m, 1 \leq j \leq h_i \). Let \( \tilde{F}' = \bigcup_{i=1}^{m} P_i, h_i + 1 \). Thus we have obtained \( 1 + \sum_{i=1}^{m} h_i = n - 3 \) parallel classes on the whole point set. It is easy to check that they form an \( \tilde{RMP}(3, 2; n, n - 3) \).

Next by Lemma 2.1, we can construct a parallel class \( P_i, h_i + 2 \) for each group of size \( h_i \). Let \( \tilde{F}'' = \bigcup_{i=1}^{m} P_i, h_i + 2 \). Add this new parallel class to the \( n - 3 \) parallel classes of the above \( \tilde{RMP}(3, 2; n, n - 3) \), we get an \( \tilde{RMC}(3, 2; n, n - 2) \). \( \square \)

Combine Lemmas 3.3, 3.4, 2.2 and 2.3, we have the following lemma.

Lemma 3.5. There exist an \( \tilde{RMP}(3, 2; n, n - 3) \) and an \( \tilde{RMC}(3, 2; n, n - 2) \) for each \( n \equiv 1 \pmod{3}, n \geq 148 \).

By Lemma 3.5, it remains to consider the orders \( 16 \leq n \leq 145 \).

Lemma 3.6. There exist an \( \tilde{RMP}(3, 2; n, n - 3) \) and an \( \tilde{RMC}(3, 2; n, n - 2) \) for each \( n \equiv 1 \pmod{3}, 88 \leq n \leq 145 \) and \( n \in \{67, 79\} \).

Proof. By Lemmas 3.2 and 3.4, we only need to construct some GDDs on \((n - 4)/3\) points having all blocks and groups of size at least four, where the required input designs *DF*(\( m, K, 2 \)) come from Lemmas 2.2 and 2.3. These GDDs can be obtained from an idempotent TD\((7, 1, 7)\) or an idempotent TD\((5, 1, 5)\). We show them in the following table.

<table>
<thead>
<tr>
<th>( n )</th>
<th>((n - 4)/3)</th>
<th>((K, 1))-GDD of type</th>
<th>(0 \leq a, b, c, d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>[124, 151]</td>
<td>[40, 49]</td>
<td>(7^a b^1 c^1)</td>
<td>(4 \leq a, b, c \leq 7)</td>
</tr>
<tr>
<td>[112, 121]</td>
<td>[36, 39]</td>
<td>(6^4 5^a 4^b)</td>
<td>(a + b = 3)</td>
</tr>
<tr>
<td>[100, 119]</td>
<td>[32, 35]</td>
<td>(7^a c^1)</td>
<td>(4 \leq a \leq 7)</td>
</tr>
<tr>
<td>[88, 97]</td>
<td>[28, 31]</td>
<td>(4^a b^1)</td>
<td>(4 \leq a \leq 7)</td>
</tr>
<tr>
<td>79</td>
<td>25</td>
<td>(5^5)</td>
<td></td>
</tr>
<tr>
<td>67</td>
<td>21</td>
<td>(4^4 5^1)</td>
<td></td>
</tr>
</tbody>
</table>

Combining Theorem 1.2(8), Lemmas 2.4, 2.5 and Lemmas 3.5, 3.6, we obtain our main results.

Theorem 3.7. There exist an \( \tilde{RMP}(3, 2; n, n - 3) \) and an \( \tilde{RMC}(3, 2; n, n - 2) \) for every \( n \equiv 1 \pmod{3} \) and \( n \geq 16 \).
4. Concluding remarks

It is not difficult to see that there does not exist an $\tilde{RMP}(3, 2; n, n - 3)$ and an $\tilde{RMC}(3, 2; n, n - 2)$ for $n \in \{1, 4, 7\}$.

Lemma 4.1. There exists an $\tilde{RMP}(3, 2; 10, 7)$.

Proof. Take the point set $V = \{1, 2, \ldots, 10\}$. We list the required seven parallel classes as follows:

<table>
<thead>
<tr>
<th>1 2 3 4</th>
<th>1 2 5 8</th>
<th>1 3 6 9</th>
<th>1 4 7 10</th>
<th>2 3 7 10</th>
<th>2 4 6 9</th>
<th>3 4 5 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 6 7</td>
<td>3 6 7</td>
<td>2 5 10</td>
<td>2 6 8</td>
<td>1 8 9</td>
<td>1 5 7</td>
<td>1 6 10</td>
</tr>
<tr>
<td>8 9 10</td>
<td>4 9 10</td>
<td>4 7 8</td>
<td>3 5 9</td>
<td>4 5 6</td>
<td>3 8 10</td>
<td>2 7 9</td>
</tr>
</tbody>
</table>

Here the leave is a 6-cycle $(5, 9, 7, 8, 6, 10)$. □

Remark. Note that with this $\tilde{RMP}(3, 2; 10, 7)$, the proof of the existence of an $\tilde{RMP}(3, 2; n, n - 3)$ can be simplified a lot, and the resulted $\tilde{RMP}(3, 2; n, n - 3)$ have a 6-cycle as its leave. But in order to deal with the two designs $\tilde{RMP}$ and $\tilde{RMC}$ together, we do not use this small design in the recursive constructions.

Lemma 4.2. There exists an $\tilde{RMC}(3, 1; 67, 33)$.

Proof. Take the point set $V = (Z_{33} \times \{1, 2\}) \cup \{\infty\}$. Instead of listing all the blocks and the parallel classes of the desired designs, we only list the blocks of the initial parallel class. The 33 parallel classes will be generated mod 33 from it. The blocks of the initial parallel class are listed below. We write $(a, i)$ as $ai$ for brevity.

| 11211232 | 15216212 | 24132142 | 91221311 | 62102312 | 52242011 |
| 6121232  | 41201301 | 2126192 | 231251182 | 101141281 | 17262191 |
| 22132271 | 31151122 | 4219202 | 5181112 | 7111292 | 7202131 |
| 82282161 | 22232218 | 272302171 | 291252\infty |

Here the excess is $\{i_1, (i + 4)_1\} \cup \{i_2, (i + 1)_2\}, i \in Z_{33}$. □

Therefore, there are five small designs, the existence of which remain undecided for $\tilde{RMP}(3, \lambda; n, m)$ and $\tilde{RMC}(3, \lambda; n, m)$, where $\lambda \in \{1, 2\}$.

<table>
<thead>
<tr>
<th></th>
<th>(3, 1)</th>
<th>(3, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{RMP}$</td>
<td>19</td>
<td>13</td>
</tr>
<tr>
<td>$\tilde{RMC}$</td>
<td>16</td>
<td>10,13</td>
</tr>
</tbody>
</table>

Acknowledgments

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References