Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 326 (2007) 192–208
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Absolutely continuous spectrum of Schrödinger operators with potentials slowly decaying inside a cone ☆

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Received 14 September 2005
Available online 3 April 2006
Submitted by F. Gesztesy

Abstract

For a large class of multi-dimensional Schrödinger operators it is shown that the absolutely continuous spectrum is essentially supported by $[0, \infty)$. We require slow decay and mildly oscillatory behavior of the potential in a cone and can allow for arbitrary non-negative bounded potential outside the cone. In particular, we do not require the existence of wave operators. The result and method of proof extends previous work by Laptev, Naboko and Safronov.

Keywords: Schrödinger operators; Absolutely continuous spectrum

1. Introduction

In this paper we consider Schrödinger operators

$$-\Delta + V(x), \quad V \in L^\infty(\mathbb{R}^d),$$

☆ This work was partially supported through NSF-grant DMS-0245210.
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.01.093

acting in the space $L^2(\mathbb{R}^d)$. If $V = 0$ then the operator has purely absolutely continuous spectrum on $[0, +\infty)$. We find conditions on V which guarantee that the absolutely continuous spectrum of the operator $-\Delta + V$ is essentially supported by $[0, \infty)$. This means that the spectral projection associated to any subset of positive Lebesgue measure is not zero.

Traditionally, investigations of the positive spectrum of Schrödinger operators $-\Delta + V(x)$ in $L^2(\mathbb{R}^d)$ with decaying potential V have mostly been done through methods of scattering theory, i.e., by studying existence and completeness properties of wave operators. In particular, as long as no additional smoothness assumptions on V are made, proving existence of the wave operators $\mathfrak{W}_\pm = \text{s-lim}_{t\to\mp\infty} e^{it(-\Delta+V)}e^{-it(-\Delta)}$ by some version of Cook's method is the standard way of showing that the spectrum of $-\Delta + V$ has an absolutely continuous component covering $[0, \infty)$. In fact, existence of the wave operators yields the stronger result that the absolutely continuous part $E^{\mathrm{ac}}_{-\Delta+V}$ of the spectral resolution for $-\Delta + V$ is "equivalent" to Lebesgue measure on $[0, \infty)$ in the sense that for Borel sets $\omega \subset [0, \infty)$,

$$E^{(\mathrm{ac})}_{-\Delta+V}(\omega) > 0 \quad \text{if and only if} \quad |\omega| > 0. \tag{1.1}$$

We will refer to this also by saying that the absolutely continuous spectrum of $-\Delta + V$ is essentially supported by $[0, \infty)$. A natural question to ask is whether the latter spectral theoretic fact can be proven without requiring the seemingly stronger scattering theoretic input. In other words, one would like to have methods which allow to show the existence of absolutely continuous spectrum in situations where wave operators are not known or not even likely to exist. A breakthrough in this direction was achieved by P. Deift and R. Killip in [3]. They showed that (1.1) holds for one-dimensional Schrödinger operators $-d^2/dx^2 + V$ with $V \in L^2(\mathbb{R})$. They did not use results on wave operators nor—at least in $d = 1$ related—results on asymptotics of solutions of the time-independent Schrödinger equation (which are still unknown for this class). Instead, their approach is based on a bound for the entropy of the spectral measure in terms of the L^2-norm of the potential, which was known from inverse spectral theory. Subsequently, the idea of using entropy bounds for the spectral measure to study the a.c. spectrum has been used in a number of other works on one-dimensional Schrödinger operators and Jacobi matrices, for example [6,7,9]. The extension of these ideas to $d \geqslant 2$ was not straightforward, but recently A. Laptev, S. Naboko and O. Safronov [8,11] have found a method which allows to estimate suitable spectral measures of $-\Delta + V$ in terms of (negative) eigenvalue sums of $-\Delta + V$ and $-\Delta - V$. "Suitable" means to work with spectral measures $d(E_{-\Delta+V}(\lambda)f, f)$ for radially symmetric functions f.

One writes the Schrödinger operator as a 2×2-block-matrix operator with respect to the radially symmetric functions in $L^2(\mathbb{R}^d)$ and their complement and uses a Feshbach-type argument to essentially reduce the problem to studying a Schrödinger operator on a half line. The main technical difficulties arise from the fact that the "potential" of the 1-d operator is operator-valued, non-selfadjoint, as well as energy-dependent, requiring the extension of various 1-d concepts (e.g. Jost functions, m-functions, scattering coefficients) to this new setting. Based on the method from [8] it was proven in [11] that (1.1) holds in $d \geqslant 3$ if $V \in L^{d+1}(\mathbb{R}^d)$ and its Fourier transform $\hat{V}$ (defined as a distribution) is locally square-integrable near the origin, i.e., $\int_{|\xi|<\delta} |\hat{V}(\xi)|^2\, d\xi < \infty$ for some $\delta > 0$. We refer to [11] for results in $d = 2$ and also for a discussion of the meaning of the condition on $\hat{V}$ (requiring oscillatory behavior of V). Our main goal here is to extend the results of [8,11] by observing that the methods used only require decay of the potential in a cone $\Omega \subset \mathbb{R}^d$. More specifically, we consider a Schrödinger operator $-\Delta + V$ in $L^2(\mathbb{R}^d)$, $d \geqslant 3$, where

$$V \in L^\infty(\mathbb{R}^d), \quad V \text{ real valued.} \tag{1.2}$$

For any measurable set Ω let χ_Ω be the characteristic function of this set. Suppose now, that Ω is a cone in $\mathbb{R}^d$ and

$$V = V_1 + V_2, \quad \chi_\Omega V_2 = 0. \tag{1.3}$$

Finally, let $\hat{V}_1$ be the Fourier transform of V_1,

$$\hat{V}_1(\xi) = \int_{\mathbb{R}^d} e^{-i\xi x} V_1(x)\, dx, \tag{1.4}$$

which can be understood in the sense of distributions.

Theorem 1.1. *Let $d \geqslant 3$ and let V be a real valued function on $\mathbb{R}^d$ obeying* (1.2), (1.3) *for some cone Ω in $\mathbb{R}^d$ with $V_2 \geqslant 0$. Suppose that for some $\delta > 0$,*

$$\int_{|\xi|<\delta} \left|\hat{V}_1(\xi)\right|^2 d\xi < \infty \quad \text{and} \quad \int_{\mathbb{R}^d} \left|V_1(x)\right|^{d+1} dx < \infty.$$

Then the absolutely continuous spectrum of the operator $-\Delta + V$ is essentially supported by $[0, \infty)$.

Remark. Since (1.2) and $V_2 \geqslant 0$ are the only condition on V_2, the potential V in Theorem 1.1 does not have to decay outside of the cone Ω.

We prove this with a modification of the method of Laptev–Naboko–Safronov. One introduces an additional conical cut-off in the radially symmetric test function f, whose spectral measure is considered. In doing so we encounter a number of new technical difficulties and therefore provide full details of the proof. In both methods, the ones of Deift–Killip and of Laptev–Naboko–Safronov, bounds on spectral measures are first proven for compactly supported approximations of the potentials. As these bounds hold uniformly for all approximations, they carry over to the limit. Then we prove and use an approximation result where convergence is only required in L^2_{loc}-sense (as this implies strong resolvent convergence). This allows for applications like ours, i.e., to consider V which do not decay in all directions.

We would like to mention that further applications of the techniques employed in [8] and here are given in the recent paper [12]. A similar technique can also be applied to study the Dirac operator. Here one can prove a certain trace formula which implies the existence of the a.c. spectrum on the whole real line (see [4]). Additional results for Schrödinger operators with slowly decaying potentials can be found in [5], where also applications to random Schrödinger operators are given.

This article is a continuation and extension of the work presented in [8] and uses many of the same techniques. For the sake of completeness we will recall the corresponding arguments in the proofs.

In Section 2 below we will reduce Theorem 1.1 to our main technical result, Theorem 2.1. This result states a quantitative property of suitable spectral measures of an operator closely related to $-\Delta + V$ (via trace class perturbation theory). This property provides an analog of the Szegő condition which has been studied for spectral measures associated with Jacobi matrices. The remaining Sections 3 to 7 are devoted to the proof of Theorem 2.1. The Feshbach-type argument to reduce the problem to studying a generalized half line Schrödinger operator is provided in Section 3, where we also introduce a number of regularizations of the underlying operator, which are used in Sections 4–6. In particular, we work with smooth compactly supported potentials and also assume that the potential has only finitely many spherical modes. In Sections 4 and 5 we introduce Jost solutions, Green functions, Wronskians and M-functions for the regularized operator. Section 6 provides bounds of spectral measures in terms of traces of the negative eigenvalues of operators related to $-\Delta + V$. At the end of Section 6 and throughout Section 7 we carry out the necessary limiting arguments to remove the regularizations and complete the proof of Theorem 2.1.

2. Reduction of Theorem 1.1 to the main technical Theorem 2.1

We will not directly study the operator $-\Delta + V$ from Theorem 1.1, but instead a related operator H which arises from $-\Delta + V$ through trace class perturbations and a change to spherical coordinates. Thus $-\Delta + V$ and H will have unitarily equivalent absolutely continuous parts and Theorem 1.1 will follow from our main technical Theorem 2.1, which gives certain quantitative characteristics of the spectral measures of H and, in particular, the required absolute continuity properties.

First, we observe that we may work with a Dirichlet operator on the exterior of the unit ball B_1 in $\mathbb{R}^d$. More precisely, let

$$H_1 = -\Delta + V \quad \text{in } L^2\big(\mathbb{R}^d \setminus B_1\big) \tag{2.1}$$

with Dirichlet boundary conditions on $\mathbb{S}^{d-1} = \partial B_1$. Here the potential V is given as in Theorem 1.1, but as will be convenient for our later switch to spherical coordinates, we also assume that for some $c_1 > 1$ it holds that

$$V(x) = -\frac{\alpha_d}{|x|^2} \quad \text{for } 1 < |x| < c_1, \tag{2.2}$$

where $\alpha_d = \frac{(d-1)^2}{4} - \frac{d-1}{2}$. By general facts from trace class perturbation theory, e.g. [1,2], the absolutely continuous parts of the original operator $-\Delta + V$ and H_1 are unitarily equivalent.

We now switch to spherical coordinates $x = r\theta$, $r \in (1, \infty)$, $\theta \in \mathbb{S}^{d-1}$, i.e., consider the unitarily equivalent operator to H_1 given by

$$H_2 = -\frac{d^2}{dr^2} - \frac{d-1}{r}\frac{d}{dr} - \frac{1}{r^2}\Delta_\theta + V(r,\theta) \tag{2.3}$$

with Dirichlet boundary condition at $r = 1$ in $L^2((1,\infty) \times \mathbb{S}^{d-1}, r^{d-1}\, dr\, d\theta)$. Here Δ_θ is the Laplace–Beltrami operator on $L^2(\mathbb{S}^{d-1})$ and, slightly abusing notation, $V(r,\theta) = V(r\theta)$.

One further unitary transformation $H_3 = U^* H_2 U$, where U from $L^2((1,\infty), dr)$ to $L^2((1,\infty), r^{d-1}\, dr)$ is given by $(Uf)(r) = r^{-(d-1)/2} f(r)$, leads to

$$H_3 = -\frac{d^2}{dr^2} + \frac{1}{r^2}(-\Delta + \alpha_d) + V(r,\theta), \tag{2.4}$$

again with Dirichlet boundary condition at $r = 1$, in $L^2((1,\infty) \times \mathbb{S}^{d-1}, dr\, d\theta) = L^2((1,\infty), L^2(\mathbb{S}^{d-1}))$.

In order to be able to exploit the decay of V inside the cone Ω, we introduce a fixed real-valued smooth function ϕ on the sphere such that

$$\phi\big(x/|x|\big) = 0 \quad \text{for all } x \notin \Omega, \tag{2.5}$$

supposing that $\int_{\mathbb{S}^{d-1}} |\phi|^2\,d\theta = 1$. Also, let P_0 denote the orthogonal projection onto ϕ in $L^2(\mathbb{S}^{d-1})$, i.e.,

$$P_0\psi = \phi \cdot \int\limits_{\mathbb{S}^{d-1}} \phi\psi\,d\theta.$$

The operator $K_\theta = 2\,\mathrm{Re}(I-P_0)\Delta_\theta P_0 + P_0\Delta_\theta P_0 = [(I-P_0)\Delta_\theta P_0]^* + \Delta_\theta P_0$ is self-adjoint and finite rank in $L^2(\mathbb{S}^{d-1})$. Consider the bounded self-adjoint operator

$$K = \frac{\xi(r)}{r^2} \otimes K_\theta$$

in $L^2((1,\infty)\times\mathbb{S}^{d-1})$, where $\xi(r)$ is a smooth compactly supported function which is equal to 1 on $(1,c_1)$. For later use, note that there is a constant $C_1>0$ such that, in form sense,

$$-C_1\frac{\xi(r)}{r^2} \leqslant K \leqslant C_1\frac{\xi(r)}{r^2}. \tag{2.6}$$

For technical reasons, which have the same origin as (2.2) and will become clear at the end of the next section, we will not consider H_3 but instead

$$H = H_3 + K \tag{2.7}$$

in our proof of absolute continuity. They have unitarily equivalent absolutely continuous parts, which is seen as follows:

Denote by I_θ the identity operator in $L^2(\mathbb{S}^{d-1})$ and observe that

$$\left(-\frac{d^2}{dr^2}\otimes I_\theta - z\right)^{-1} K \left(-\frac{d^2}{dr^2}\otimes I_\theta - z\right)^{-1}$$

is a trace class, being the tensor product of the trace class operator $(-d^2/dr^2 - z)^{-1}\frac{\xi(r)}{r^2}(-d^2/dr^2 - z)^{-1}$ in $L^2(1,\infty)$ and the finite rank operator K_θ in $L^2(\mathbb{S}^{d-1})$.

Furthermore,

$$\left(-\frac{d^2}{dr^2}\otimes I_\theta - z\right)\left(-\frac{d^2}{dr^2} - \frac{1}{r^2}\Delta_\theta - z\right)^{-1}$$

is bounded, implying that $(-d^2/dr^2\otimes I_\theta - z)(H-z)^{-1}$ and $(-d^2/dr^2\otimes I_\theta - z)(H_3-z)^{-1}$ are bounded.

This combines to yield that $(H-z)^{-1} - (H_3-z)^{-1}$ is a trace class, giving the equivalence of a.c. parts.

We are now ready to state our main technical result for the operator H. It provides quantitative characteristics of spectral measures of H, which are a multi-dimensional continuous analog of the well-known Szegő condition for orthogonal polynomials and Jacobi matrices (compare with [8]).

Theorem 2.1. *Let the operator H in $L^2((1,\infty)\times\mathbb{S}^{d-1})$ be defined as above, $E_H(\cdot)$ its spectral resolution and let f_0 be any bounded non-zero function with* supp $f_0 \subset \{r\colon 1<r<c_1\}$. *Then*

$$\int\limits_0^\infty \frac{\log[\frac{d}{d\lambda}(E_H(\lambda)f,f)]\,d\lambda}{(1+\lambda^{3/2})\sqrt{\lambda}} > -\infty, \tag{2.8}$$

where $f(r,\theta) = f_0(r)\phi(\theta)$.

The inequality (2.8) guarantees that the a.c. spectrum of H is essentially supported by $[0,\infty)$, since $d(E_H(\lambda)f,f)/d\lambda > 0$ almost everywhere and gives quantitative information about the measure $(E_H(\lambda)f,f)$. Through the series of unitary equivalences and trace class perturbations discussed above this immediately yields Theorem 1.1. Thus it remains to prove Theorem 2.1, which is the content of the rest of this paper.

3. Reduction to a one-dimensional problem with operator-valued potential

The main technical tool in the approach of [8] is to use a Feshbach-type reduction of the multi-dimensional Schrödinger-type operator H to the subspace generated by the radially symmetric functions and to find bounds for suitable spectral measures of H by studying the reduced, essentially one-dimensional, problem. In this section we will adapt this method to our situation.

At this point we will take advantage of being able to use a strongly regularized version of H, to be described below. The entropy bounds of spectral measures for the regularized operator, found in Section 6, are sufficiently uniform to allow for the necessary limiting arguments to eventually prove Theorem 2.1 for the general case (Section 7).

Let $\{Y_j\}_{j=0}^{\infty}$ be the orthonormal basis in $L^2(\mathbb{S}^{d-1})$ of (real) spherical functions, i.e., eigenfunctions of the Laplace–Beltrami operator $-\Delta_\theta$. Let S_j be the orthogonal projection onto Y_j in $L^2(\mathbb{S}^{d-1})$ given by

$$S_j\psi = Y_j \int_{\mathbb{S}^{d-1}} Y_j \psi \, d\theta.$$

We will also frequently use S_j to denote the corresponding projection $I \otimes S_j$ in $L^2((1,\infty)\times \mathbb{S}^{d-1})$, and do the same with P_0 introduced in Section 2.

The regularized operator for which we will do the Feshbach reduction is given by

$$H_\varepsilon = -\frac{d^2}{dr^2} + \frac{\zeta_\varepsilon(r)}{r^2}(-\Delta_\theta + \alpha_d) + \tilde{K} + V_*. \tag{3.1}$$

Here we assume

(i) $V \in C_0^\infty(\mathbb{R}^d)$, satisfying (2.2).
(ii) $V_* = SVS$, where for some fixed $n \in \mathbb{N}$, $S = \sum_{j=0}^n S_j$.
(iii) For fixed $i \in \mathbb{N}$, $i \leqslant n$, let $\tilde{\phi}$ be a real linear combination of the first $i+1$ spherical harmonics, i.e., $\tilde{\phi} = \sum_{j=0}^i S_j\tilde{\phi}$. Suppose that $\int_{\mathbb{S}^{d-1}} |\tilde{\phi}|^2\, d\theta = 1$. Let $\tilde{P}_0$ and $\tilde{K}$ be defined as P and K in Section 2, but with ϕ replaced by $\tilde{\phi}$.
(iv) $\zeta_\varepsilon(r) = \zeta(r-\varepsilon^{-1})$, where ζ is a smooth function in $\mathbb{R}$ such that

$$\zeta(r) = \begin{cases} 1, & \text{if } r<0; \\ 0, & \text{if } r>1; \\ \in [0,1] & \text{otherwise.} \end{cases}$$

Thus $\zeta_\varepsilon \to 1$ as $\varepsilon \to 0$.

We introduce the following operators:

$$V^{(\ell)} = \tilde{P}_0\left(V_* + \tilde{K} + \frac{\zeta_\varepsilon(r)}{r^2}(-\Delta_\theta + \alpha_d)\right)\tilde{P}_0, \qquad H^{(0,\ell)} = -\frac{d^2}{dr^2}\tilde{P}_0,$$

$$V^{(\ell,h)} = \tilde{P}_0\left(V_* + \tilde{K} + \frac{\zeta_\varepsilon(r)}{r^2}(-\Delta_\theta)\right)(I - \tilde{P}_0), \qquad V^{(h,\ell)} = \left(V^{(\ell,h)}\right)^*,$$
$$V^{(h)} = (I - \tilde{P}_0)\left(V_* + \frac{\zeta_\varepsilon(r)}{r^2}(-\Delta_\theta + \alpha_d)\right)(I - \tilde{P}_0), \qquad H^{(0,h)} = -\frac{d^2}{dr^2}(I - \tilde{P}_0).$$

We use the superscripts ℓ and h here to distinguish between lower and higher spherical harmonics. It is important to note that $V^{(h,\ell)}$ vanishes on the orthogonal complement to the first $n+1$ spherical functions, which in turn is an invariant subspace for $V^{(h)}$. Formally, the operator $H_\varepsilon - z$ can be represented as a matrix:

$$H_\varepsilon - z = \begin{pmatrix} H^{(0,\ell)} + V^{(\ell)} - z & V^{(\ell,h)} \\ V^{(h,\ell)} & H^{(0,h)} + V^{(h)} - z \end{pmatrix},$$

and the equation

$$(H_\varepsilon - z)u = \tilde{P}_0 f, \quad \operatorname{Im} z \neq 0, \tag{3.2}$$

is equivalent to

$$\left(H^{(0,\ell)} + Q_z - z\right)\tilde{P}_0 u = \tilde{P}_0 f, \qquad \left(H^{(0,h)} + V^{(h)} - z\right)^{-1} V^{(h,\ell)} u = (\tilde{P}_0 - I)u, \tag{3.3}$$

where the operator Q_z in the space $L^2((1,\infty))$ is given by

$$Q_z = V^{(\ell)} - V^{(\ell,h)}\left(H^{(0,h)} + V^{(h)} - z\right)^{-1} V^{(h,\ell)}. \tag{3.4}$$

Therefore the problem is reduced to the study of the properties of the one-dimensional operator

$$L_z u(r) = -\frac{d^2 u}{dr^2} + Q_z u, \quad u \in L^2(1,\infty),\ u(1) = 0. \tag{3.5}$$

In (3.4) and (3.5) we identify Q_z acting on multiples of $\tilde{\phi}$ with the unitary equivalent operator $\tilde{\phi}^{-1} Q_z \tilde{\phi}$ in $L^2(1,\infty)$.

From (3.2), (3.3) we obtain

$$\tilde{P}_0 (H_\varepsilon - z)^{-1} \tilde{P}_0 = (L_z - z)^{-1}. \tag{3.6}$$

Let $\operatorname{supp} V \cup \operatorname{supp} \zeta_\varepsilon(|\cdot|) \subset \{x \in \mathbb{R}^d\colon\ |x| < c_2\}$, $c_2 > c_1$ and let χ be the operator of multiplication by the characteristic function of the interval (c_1, c_2). Then for the operator (3.4) we have

$$Q_z = Q_z\chi = \chi Q_z.$$

Note that one has to include (2.2) and (2.7) to have this and other convenient properties of the considered operators. In particular, we will have a simple representation for the spectral measure of H.

We observe that the imaginary part of the analytic operator valued function Q_z (as a function of z) is negative in the upper half plane and positive in the lower half plane.

4. Jost solutions and the Green function

Note that this section is rather similar to the corresponding section in [8]. We repeat it mostly in order to introduce the reader to the main ideas.

In Sections 4–6 instead of the potential V we deal with $V_* = SVS$, $S = \sum_{j=0}^{n} S_j$, which approximates V for large n. It can be interpreted as an operator of multiplication by a matrix valued function of r. Since S_j are projections on real spherical functions, this matrix is real. Recall also that the factor $1/r^2$ in front of $-\Delta_\theta$ and α_d is substituted by a smooth compactly supported function $\zeta_\varepsilon(r)/r^2$.

Recall that our problem has been reduced to the study of the equation

$$-\frac{d^2}{dr^2}\psi(r) + (Q_z\psi)(r) = z\psi(r), \quad r \geqslant 1,\ z \in \mathbb{C}, \tag{4.1}$$

with the operator Q_z given by (3.4). Let $\psi_k(r)$ be the solution of Eq. (4.1) satisfying

$$\psi_k(r) = \exp(ikr), \quad k^2 = z,\ \operatorname{Im} k > 0,\ \forall r > c_2.$$

We will shortly discuss the the question of the existence and uniqueness of this solution, but first note that this solution also satisfies the "integral" equation

$$\psi_k(r) = e^{ikr} - k^{-1}\int_r^\infty \sin k(r-s)(Q_z\psi_k)(s)\,ds. \tag{4.2}$$

In order to establish existence of $\psi_k(r)$ and study its properties we use the following analytic Fredholm theorem (see, for example, [10, Theorem VI.14] or [13, Chapter I, Section 8]):

Theorem 4.1. *Let $D \subset \mathbb{C}$ be an open connected set and let $\mathfrak{T}(k)$ be an analytic operator valued function on D such that $\mathfrak{T}(k)$ is a compact operator in a Hilbert space for each $k \in D$. Then*

(1) *either $(I - \mathfrak{T}(k))^{-1}$ exists for no $k \in D$,*
(2) *or $(I - \mathfrak{T}(k))^{-1}$ exists for all $k \in D \setminus D_0$, where D_0 is a discrete subset of D. In this case $(I - \mathfrak{T}(k))^{-1}$ is meromorphic in D with possible poles belonging to D_0.*

The first application of this theorem is in the following statement about Q_z. Note that the proof given here is a little bit longer compared to the proof in [8], since we establish invertibility of a certain operator for large real k.

Lemma 4.1. *The operator Q_z has a meromorphic continuation into the second sheet of the complex plane. Moreover, $Q_z \to V^{(\ell)}$ in the operator norm as $z = \lambda + i0$, $\lambda \to +\infty$.*

Proof. Consider an operator $-d^2/dr^2 + q(r)$ in $L^2((1,\infty), \mathbb{C}^n)$ with Dirichlet boundary condition at 1. Here q is a matrix valued function, compactly supported on some $[c_1, c_2] \subset (0, +\infty)$. Let $f \in C_0^\infty(\mathbb{R}_+)$ be a function which is identically equal to one on the support of the matrix-function q and let F be the multiplication by f. Then

$$\begin{aligned} &F\big(-d^2/dr^2 + q(r) - z\big)^{-1}F \\ &\quad = \big(I + F\big(-d^2/dr^2 - z\big)^{-1}q\big)^{-1}F\big(-d^2/dr^2 - z\big)^{-1}F. \end{aligned}$$

Clearly both operators $F(-d^2/dr^2 - z)^{-1}q$ and $F(-d^2/dr^2 - z)^{-1}F$ have analytic continuations into the second sheet of the complex plane through the positive semi-axis and they tend to zero when $z = \lambda + i0$, $\lambda \to +\infty$. By using Theorem 4.1 we obtain that the operator

$$\big(I + F\big(-d^2/dr^2 - z\big)^{-1}q\big)^{-1}$$

tends to I when $z = \lambda + i0$, $\lambda \to +\infty$. Thus the operator Q_z defined in (3.4) has a meromorphic continuation into the second sheet of the complex plane and tends to $V^{(\ell)}$. $\square$

Our second application of Theorem 4.1 is devoted to the study of the operator

$$\mathfrak{T}(k)\psi(r) = -k^{-1}\int_r^\infty \sin k(r-s)(Q_z\psi)(s)\,ds$$

in $L^2(1,c_2)$. Observe that $\mathfrak{T}(k)$ is small in norm and therefore $I-\mathfrak{T}(k)$ is invertible for large real k. We conclude that Eq. (4.2) is uniquely solvable for all k except perhaps a discrete sequence of points and that its solution ψ_k is a meromorphic function with respect to k, with values in $L^2(1,c_2)$, in a neighborhood of every $\operatorname{Im} k \geqslant 0$, $k\neq 0$. For $1<x<c_1$ the solution is a linear combination of two exponential functions

$$\psi_k(x) = a(k)e^{ikx} + b(k)e^{-ikx}, \quad 1<x<c_1, \tag{4.3}$$

where both $a(k)$ and $b(k)$ are meromorphic functions (even in some neighborhoods of points $k\neq 0$ of the real axis).

The operator $R(z)=(L_z-z)^{-1}$, where L_z is defined in (3.5), plays a very important role in our arguments. Let χ_{c_1} be the operator of multiplication by the characteristic function of $(1,c_1)$. Then $R(z)\chi_{c_1}$ is an integral operator whose kernel is given by

$$G_z(r,s) = \begin{cases} \frac{\psi_k(s)}{\psi_k(1)}\frac{\sin(k(r-1))}{k}, & \text{for } r<s<c_1, \\ \frac{\psi_k(r)}{\psi_k(1)}\frac{\sin(k(s-1))}{k}, & \text{for } s<\min\{c_1,r\}. \end{cases} \tag{4.4}$$

The proof of this assertion can be found in [8].

We should also mention that since $\psi_k(1)$ is meromorphic in k in a neighborhood of any $k\neq 0$, we conclude that $\psi_k(1)=0$ only on a discrete subset of the closed upper half plane, having no accumulation points except perhaps zero.

5. Wronskian and properties of the M-function

The material of this section is of common knowledge in the theory of one-dimensional operators. For an extension to a situation like the one considered here, see [8].

Let Q_z be defined as in (3.4). For all k, except perhaps a discrete sequence, the function

$$M(k) = \frac{\psi_k'(1)}{\psi_k(1)}, \quad \operatorname{Im} k \geqslant 0, \tag{5.1}$$

is well defined and called the Weyl M-function of the operator (4.1). It is proven in [8] that for all real k one has the following inequality:

$$\frac{k}{\operatorname{Im} M(k)} \leqslant \left|\psi_k(1)\right|^2.$$

Moreover, if we represent the solution ψ_k for real k in the form

$$\psi_k(x) = a(k)e^{ikx} + b(k)e^{-ikx}, \quad x<c_1,$$

then (see [8])

$$|a|^2 - |b|^2 \geqslant 1, \qquad k=\bar{k}.$$

On the other hand, it is obvious that

$$\frac{k}{\operatorname{Im} M(k)} \leqslant 2|a|^2 + 2|b|^2$$

and therefore

$$\left|a(k)\right|^{-2} \leqslant 4k^{-1}(\operatorname{Im} M), \quad k > 0. \tag{5.2}$$

One can also show that

$$\operatorname{Im} M(k) > 0 \quad \text{if } \operatorname{Im} k^2 > 0. \tag{5.3}$$

However, it is not clear why M is an analytic function of $z = k^2$ in the upper half plane. The reason for this property is that due to (4.4) and (5.1) the function M can also be defined as $M(k) = \frac{\partial^2}{\partial r \partial s} G_z(r,s)|_{(1,1)}$, where G_z is the integral kernel of the operator $\tilde{P}_0(H_\varepsilon - z)^{-1}\tilde{P}_0$ (recall that H_ε is a one dimensional Schrödinger operator with a smooth compactly supported matrix potential).

Thus, there are constants $C_0 \in \mathbb{R}$ and $C_1 \geqslant 0$ and a positive measure μ, such that

$$\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2} < \infty,$$

where

$$M(k) = C_0 + C_1 z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu(t), \quad k^2 = z. \tag{5.4}$$

6. Trace inequalities

In this section we continue to modify the method of [8]. The coefficient $a(k)$ introduced in (4.3) depends on ε and here we shall write $a_\varepsilon(k)$ instead of $a(k)$. In the same way as in [8] one can obtain from (4.2) and (3.4),

$$a_\varepsilon(k) = \lim_{r \to -\infty} \exp(-ikr)\psi_k(r) = 1 - \frac{1}{2ik} \int V^{(\ell)}\, dr + o(1/k), \tag{6.1}$$

as $k \to \infty$. Now let $i\beta_m$ and γ_j be zeros and poles of $a_\varepsilon(k)$ in the open upper half plane. Note that $-\overline{\gamma_j}$ are also poles of $a_\varepsilon(k)$ (this will follow from (6.7)). We shall show in Proposition 6.1 that $\{-\beta_m^2\}$ are the eigenvalues of a certain self-adjoint operator of Schrödinger type. Therefore we choose $\beta_m > 0$. Thus, for the corresponding Blaschke product

$$\mathfrak{B}(k) = \prod_m \frac{(k - i\beta_m)}{(k + i\beta_m)} \prod_j \frac{(k - \overline{\gamma_j})}{(k - \gamma_j)}$$

we have $|\mathfrak{B}(k)| = 1$, $\overline{\mathfrak{B}(k)} = \mathfrak{B}(-k)$, $k \in \mathbb{R}$. Hence, we obtain

$$\int_{-\infty}^{+\infty} \log\bigl(a_\varepsilon(k)/\mathfrak{B}(k)\bigr)\, dk = \frac{\pi}{2} \int V^{(\ell)}(r)\, dr + 2\pi\left(\sum \beta_m - \sum \operatorname{Im} \gamma_j\right), \tag{6.2}$$

provided that for some integer $l \geqslant 0$ the coefficient $a_\varepsilon(k)$ has an expansion $a_\varepsilon(k) = \sum_{j \geqslant -l} c_j k^j$ at zero. The existence of such an expansion as well as the condition $|a_\varepsilon(k)| - 1 = O(1/|k|^2)$ as $k \to \pm\infty$ is proven as in the Appendix of [8]. For this, let us only note that the matrices $A(k)$ and $B(k)$ in $SL^2(\mathbb{S}^{d-1})$ now take the role of scattering coefficients for the equation

$$-\frac{d^2\Phi}{dr^2} + \tilde{K}\Phi + \frac{\zeta_\varepsilon}{r^2}(-\Delta_\theta \Phi + \alpha_d \Phi) + V_* \Phi = k^2 \Phi, \tag{6.3}$$

i.e., are defined through the requirement that a (matrix-valued) solution Φ satisfies

$$\Phi = \begin{cases} \exp(ikr)S & \text{for } r > c_2, \\ \exp(ikr)A(k) + \exp(-ikr)B(k) & \text{for } r < c_1. \end{cases}$$

With these $A(k)$, $B(k)$ and $\tilde{P}_0$ the projection onto a finite linear combination $\tilde{\phi}$ of spherical harmonics as above (rather than the projection onto the first, constant, spherical harmonic as in [8]) we get the identity (10.2) from [8], i.e.,

$$\frac{1}{a_\varepsilon(k)}\tilde{P}_0 = \tilde{P}_0\big(A(k) + (I - \tilde{P}_0)e^{-2ik}B(k)\big)\tilde{P}_0$$

with virtually the same proof. The above claimed properties of $a_\varepsilon(k)$ are now proven as in the second subsection of the Appendix of [8].

Now for any pair of finite numbers $r_2 > r_1 \geqslant 0$ we have

$$\frac{1}{2}\int_{r_1}^{r_2} \log\frac{k}{4\operatorname{Im} M(k)}\,dk \leqslant \int_{-\infty}^{+\infty} \log\big|a_\varepsilon(k)\big|\,dk. \tag{6.4}$$

Hence,

$$\frac{1}{2}\int_{r_1}^{r_2} \log\frac{k}{4\operatorname{Im} M(k)}\,dk \leqslant \frac{\pi}{2}\int V^{(\ell)}(r)\,dr + 2\pi\sum \beta_m. \tag{6.5}$$

With S as before and χ_* the characteristic function of $[1,\infty)$, let $\hat{H}_\varepsilon$ be the operator in $L^2(\mathbb{R}, SL^2(\mathbb{S}^{d-1}))$ such that

$$\hat{H}_\varepsilon u = -\frac{d^2u}{dr^2} + \chi_*\left(\tilde{K}u + \zeta_\varepsilon\left[\frac{(-\Delta_\theta + \alpha_d)u}{r^2}\right]\right),$$
$$(I - \tilde{P}_0)u(1,\cdot) = 0, \quad u(r,\cdot) \in SL^2\big(\mathbb{S}^{d-1}\big),\ \forall r, \tag{6.6}$$

where ζ_ε is the same as above and the function $(I - \tilde{P}_0)u'(r,\cdot)$ is allowed to have a jump at 1.

Proposition 6.1. *Each $-\beta_m^2$ is one of the eigenvalues $-\beta_m^2(V_*)$ of the operator $\hat{H}_\varepsilon + V_*$. Multiplicities of zeros of the function $a_\varepsilon(k)$ are equal to one.*

Proof. Suppose $s < c_1 < c_2 < r$. Then the kernel of the operator $\tilde{P}_0(\hat{H}_\varepsilon + V_* - z)^{-1}\tilde{P}_0$ for such s, r equals

$$g(r,s,k) = -\frac{\exp ik(r-s)}{2ika_\varepsilon(k)}. \tag{6.7}$$

The proof of the latter relation is a counterpart of the proof of (4.4). This means that if χ_{c_1} and χ_{c_2} are the characteristic functions of $(1, c_1)$ and (c_2, ∞) then $\chi_{c_1}\tilde{P}_0(\hat{H}_\varepsilon + V_* - z)^{-1}\tilde{P}_0\chi_{c_2}$ has poles only at zeros of $a_\varepsilon(k)$ and each zero of $a_\varepsilon(k)$ is a pole of the resolvent $\chi_{c_1}\tilde{P}_0(\hat{H}_\varepsilon + V_* - z)^{-1}\tilde{P}_0\chi_{c_2}$. But the poles of the latter operator are eigenvalues of $\hat{H}_\varepsilon + V_*$. Moreover, due to the spectral theorem, we can consider the expansion of $\tilde{P}_0(\hat{H}_\varepsilon + V_* - z)^{-1}\tilde{P}_0$ near the eigenvalue $-\beta_m^2$,

$$\chi_{c_1}\tilde{P}_0(\hat{H}_\varepsilon + V_* - z)^{-1}\tilde{P}_0\chi_{c_2} = -\chi_{c_1}\frac{\tilde{P}_0 E_{\hat{H}_\varepsilon + V_*}(\{-\beta_m^2\})\tilde{P}_0}{k^2 + \beta_m^2}\chi_{c_2} + Z(k), \tag{6.8}$$

where $\|Z(k)\| = O(1)$, as $k \to i\beta_m$. Comparing (6.7) and (6.8) we find that the multiplicities of these zeros are equal to one. $\square$

Let $E_{H_\varepsilon}(\omega)$, $\omega \subset \mathbb{R}$, be the spectral projection of the operator H_ε. Then for bounded spherically symmetric functions f_0 supported inside $(1, c_1)$ and $f(x) = \tilde{\phi}(x/|x|) f_0(|x|)$ we have

$$\big(E_{H_\varepsilon}(\omega) f, f\big) = \int\limits_\omega \big|F(\lambda)\big|^2 \, d\mu(\lambda), \quad \omega \subset \mathbb{R}_+ = (0, \infty), \tag{6.9}$$

where

$$F(\lambda) = \frac{1}{k} \int\limits_0^{c_1} \sin\big(k(r-1)\big) f_0(r)\, r^{(d-1)/2} \, dr, \tag{6.10}$$

$\operatorname{supp} f_0 \subset \{x\colon\ 1 < |x| < c_1\}$ and $k^2 = \lambda > 0$. These relations can be obtained from the formula for the kernel of $\tilde{P}_0 (H_\varepsilon - z)^{-1} \tilde{P}_0$ and the fact that the spectral measure is related to boundary value of the imaginary part of $\tilde{P}_0 (H_\varepsilon - z)^{-1} \tilde{P}_0$ when z approaches the real line. If F is defined as in (6.10), then the function $\log(|F(\lambda)|)$ is in $L^1_{\mathrm{loc}}(0, +\infty)$ and therefore the integral of the function

$$\log\left[\frac{d}{d\lambda}\big(E_{H_\varepsilon}(\lambda) f, f\big)\right] = 2 \log\big(|F(\lambda)|\big) + \log\big(\mu'(\lambda)\big)$$

can be estimated as follows:

$$\int\limits_{r_1}^{r_2} \log\left[\frac{d}{d\lambda}\big(E_{H_\varepsilon}(\lambda) f, f\big)\right] d\lambda \geqslant \int\limits_{r_1}^{r_2} \log\big[\mu'(\lambda)\big] \, d\lambda - C(f_0, r_1, r_2) \tag{6.11}$$

for all $r_2 > r_1 > 0$. The integral in the left-hand side of (6.11) can diverge only to $-\infty$ due to the Jenssen inequality:

$$\int\limits_0^{r_2} \log\left[\frac{d}{d\lambda}\big(E_{H_\varepsilon}(\lambda) f, f\big)\right] d\lambda \leqslant r_2 \log\left[\int\limits_0^{r_2} r_2^{-1} \frac{d}{d\lambda}\big(E_{H_\varepsilon}(\lambda) f, f\big) \, d\lambda\right].$$

So, our problem is to estimate it from below.

The passage from the Weyl function M to the spectral measure E allows us to pass to the case when ζ_ε is substituted by 1. In this case the estimate (6.4) should be replaced by a relation which follows from [6, Corollary 5.3]

$$-\int\limits_{r_1}^{r_2} \log\big(d\big(E_{SHS}(\lambda) f, f\big)/d\lambda\big) \frac{d\lambda}{4\sqrt{\lambda}} \leqslant \limsup_{\varepsilon \to 0} \int\limits_{-\infty}^{+\infty} \log\big|a_\varepsilon(k)\big| \, dk + C(r_2, r_1, f_0). \tag{6.12}$$

Observe that when $\varepsilon \to 0$ the eigenvalues of $\hat{H}_\varepsilon + V_*$ converge to the eigenvalues of the operator $\hat{H} + V_*$, where $\hat{H}$ is the following operator in $L^2(\mathbb{R}, L^2(\mathbb{S}^{d-1}))$:

$$\hat{H} = -\frac{d^2 u}{dr^2} + \tilde{K} u + \frac{\chi_*}{r^2}(-\Delta_\theta u + \alpha_d u), \qquad (I - \tilde{P}_0) u(1, \cdot) = 0.$$

Denote the eigenvalues of $\hat{H} + V_*$ by $-(\beta_m^{(0)})^2$, where $\beta_m^{(0)} > 0$. Then (6.2) and (6.12) imply

Proposition 6.2. *For any pair of finite numbers $r_2 > r_1 \geqslant 0$,*

$$-\int_{r_1}^{r_2} \log \frac{d(E_{SHS}(\lambda)f,f)}{d\lambda} \frac{d\lambda}{2\sqrt{\lambda}}$$
$$\leqslant \pi\left(\int_1^\infty \int_{\mathbb{S}^{d-1}} V(r,\theta)\bigl|\tilde\phi(\theta)\bigr|^2\,d\theta\,dr + 2\sum \beta_m^{(0)} + \int_{\mathbb{S}^{d-1}} \bigl|\nabla_\theta \tilde\phi(\theta)\bigr|^2\,d\theta + C(d,r_1,r_2,f_0)\right). \tag{6.13}$$

In order to control the contribution of roots $\beta_m^{(0)}$ of eigenvalues into the right-hand side of (6.13) we need

Proposition 6.3. *Let C_1 be the constant from* (2.6). *Let $\Lambda_j(V)$ be negative eigenvalues of $-\Delta + V - C_1\frac{\xi(r)}{r^2}$ on the whole space. Then there is a constant $C_d > 0$ such that*

$$\sum \beta_m^{(0)} \leqslant \sum \bigl|\Lambda_j(V)\bigr|^{1/2} + \|V\|_\infty^{1/2} + C_d. \tag{6.14}$$

Proof. Recall that H_0 is unitary equivalent to the operator

$$-\frac{d^2u}{dr^2} - \frac{\Delta_\theta u}{r^2} + \frac{\alpha_d}{r^2}u, \qquad u(1,\cdot) = 0,$$

whose resolvent differs from the resolvent of $\hat H - \tilde K$ by an operator of rank 1. For any self-adjoint operator T and $t > 0$ denote $N(t,T) = \operatorname{rank} E_T(-\infty,-t)$. Then

$$\begin{aligned}\sum \beta_m^{(0)} &= \int_0^{\|\tilde K\|+\|V\|_\infty} N(t,\hat H + V_*)\frac{dt}{2\sqrt{t}}\\ &\leqslant \int_0^{\|\tilde K\|+\|V\|_\infty} \bigl(1 + N(t, H_0 + \tilde K + V)\bigr)\frac{dt}{2\sqrt{t}}\\ &\leqslant \operatorname{tr}(H_0 + \tilde K + V)_-^{1/2} + 2\|V\|_\infty^{1/2} + 2\|\tilde K\|^{1/2}.\end{aligned}$$

Now the assertion follows from the variational principle since

$$H_0 \geqslant -\Delta - C_1\frac{\xi(r)}{r^2}. \qquad \square$$

By multiplying the potential with a constant $\epsilon_0 > 1$, we can absorb the term $-C_1\xi(r)/r^2$ and arrive at a trace bound is terms of the negative eigenvalues of $-\Delta + \epsilon_0 V$:

Corollary 6.1. *Let $\epsilon_0 > 1$ and let $\lambda_j(\epsilon_0 V)$ be the negative eigenvalues of $-\Delta + \epsilon_0 V$ on the whole space. Then for any $\epsilon_0 > 1$ there is a constant $C(d,\epsilon_0) > 0$ such that*

$$\sum \beta_m^{(0)} \leqslant \sum \bigl|\lambda_j(\epsilon_0 V)\bigr|^{1/2} + \|V\|_\infty^{1/2} + C(d,\epsilon_0). \tag{6.15}$$

7. Spectral measures converge weakly

Much of the work which remains to complete the proof of Theorem 2.1 is to remove the remaining three restrictive assumptions (i)–(iii) which were made in Section 3, which will all be done through arguments involving weak convergence of spectral measures. Note that the limiting argument to remove (iv) was already incorporated in Section 6.

First we prove the following result which allows us to pass from $V_* = \sum_{j=0}^{l} S_j V \sum_{j=0}^{l} S_j$ to V.

Proposition 7.1. *Let V be a compactly supported smooth function and $f \in L^2(\mathbb{R}^d \setminus B_1)$. Then the quadratic forms of resolvents $((H_l - z)^{-1} f, f)$ corresponding to the "potential" $\sum_{j=0}^{l} S_j V \sum_{j=0}^{l} S_j$ converge uniformly in z to $((H - z)^{-1} f, f)$ when z belongs to any compact subset of the upper half plane. Therefore the sequence of measures $(E_{H_l}(\cdot) f, f)$ converges weakly to the spectral measure $(E_H(\cdot) f, f)$.*

Proof. Fix $f \in L^2$ and let $\epsilon > 0$ be given. Assume that $\mathcal{K}$ is a compact subset in the upper half plane. Then for any self-adjoint operator $T = T^*$,

$$\left\|(T - z)^{-1}\right\| \leqslant C, \quad \forall z \in \mathcal{K},$$

with a constant C independent of T and z. One can also choose a finite number of points $z_j \in \mathcal{K}$ so that any $z \in \mathcal{K}$ satisfies

$$\left\|(T - z)^{-1} - (T - z_j)^{-1}\right\| \leqslant \epsilon, \quad \forall T = T^*, \tag{7.1}$$

for some z_j. Now suppose z is near z_j and (7.1) is fulfilled. Choose $u_0 \in C_0^\infty(\mathbb{R}^d \setminus \bar{B}_1)$ so that the function $u = (H - z_j)u_0$ is a good approximation of f:

$$\|u - f\| \leqslant \epsilon.$$

Then

$$\left\|\left[(H_l - z_j)^{-1} - (H - z_j)^{-1}\right](f - u)\right\| \leqslant 2C\epsilon.$$

On the other hand,

$$\left[(H_l - z_j)^{-1} - (H - z_j)^{-1}\right]u = (H_l - z_j)^{-1}(V - V_*)u_0,$$

and

$$(V - V_*)u_0 = \left(I - \sum_{j=0}^{l} S_i\right) V u_0 + \sum_{i=0}^{l} S_i V \left(I - \sum_{m=0}^{l} S_m\right) u_0$$

converges to zero in L^2 as l tends to ∞. That proves that for each j we can find l so that

$$\left\|\left[(H_l - z_j)^{-1} - (H - z_j)^{-1}\right] f\right\| \leqslant (2C + 1)\epsilon.$$

Since we deal with a finite number of indices j the uniform convergence follows from (7.1). $\square$

Now let ϕ be an arbitrary normalized smooth function on the sphere $\mathbb{S}^{d-1}$ which satisfies (2.5). Consider the measure $(E_H(\cdot) f, f)$ for f as in Theorem 2.1. We can find a sequence ϕ_n of finite linear combinations of Y_j with norm 1 which converges to ϕ in $C^2(\mathbb{S}^{d-1})$. Note that the operator K depends continuously on ϕ and therefore we can assume that the operator H is defined with a fixed but already arbitrary ϕ. Obviously the sequence of functions $f_n(r,\theta) = \phi_n(\theta) f_0(r)$ will converge to f in L^2. Therefore the sequence $((H-z)^{-1} f_n, f_n)$ converges uniformly in z to $((H-z)^{-1} f, f)$ when z belongs to any compact subset of the upper half plane. Thus, using also Proposition 7.1 and taking limits in the estimate found from combining Proposition 6.2 and Corollary 6.1,

Proposition 7.2. *For any pair of finite numbers $r_2 > r_1 \geqslant 0$ and for any $\epsilon_0 > 1$,*

$$
\begin{aligned}
&-\int_{r_1}^{r_2} \log\left[\frac{d(E_H(\lambda) f, f)}{d\lambda}\right] \frac{d\lambda}{\sqrt{\lambda}} \\
&\quad \leqslant 2\pi\left(\int_1^\infty \int_{\mathbb{S}^{d-1}} V(r,\theta)\left|\phi(\theta)\right|^2 d\theta\, dr + 4\|V\|_\infty^{1/2} + 2\sum\left|\lambda_j(\epsilon_0 V)\right|^{1/2}\right. \\
&\qquad \left. + \int_{\mathbb{S}^{d-1}} \left|\nabla_\theta \phi(\theta)\right|^2 d\theta + C(\epsilon_0, d, f_0, r_1, r_2)\right),
\end{aligned} \tag{7.2}
$$

where $\lambda_j(\epsilon_0 V)$ are the eigenvalues of $-\Delta + \epsilon_0 V$.

Let χ_Ω be the characteristic function of the cone Ω. We decompose a compactly supported potential V into a suitable sum $V = V_1 + V_2$ as in Theorem 1.1, so that $\chi_\Omega V_2 = 0$ and $V_2 \geqslant 0$. Let us also introduce $V_\Omega = V_2 - V_1$ and denote by H_Ω the operator corresponding to the potential V_Ω, i.e., $H_\Omega = H - V + V_\Omega$.

When used for H_Ω, the trace inequality (7.2) has a term depending on V_Ω which is of opposite sign compared to the corresponding term in the trace formula for H (note that ϕ is supported in Ω and thus V_2 does not contribute to (7.2)). Therefore by adding the trace formula for V to the trace formula for V_Ω we obtain after cancellation

$$
\begin{aligned}
&-\int_{r_1}^{r_2} \log\left[\frac{d(E_H(\lambda) f, f)}{d\lambda}\right] \frac{d\lambda}{\sqrt{\lambda}} - \int_{r_1}^{r_2} \log\left[\frac{d(E_{H_\Omega}(\lambda) f, f)}{d\lambda}\right] \frac{d\lambda}{\sqrt{\lambda}} \\
&\quad \leqslant 4\pi\left(4\|V\|_\infty^{1/2} + \sum\left|\lambda_j(\epsilon_0 V)\right|^{1/2} + \sum\left|\lambda_j(\epsilon_0 V_\Omega)\right|^{1/2}\right. \\
&\qquad \left. + \int_{\mathbb{S}^{d-1}} \left|\nabla_\theta \phi(\theta)\right|^2 d\theta + C\right),
\end{aligned} \tag{7.3}
$$

where $C = C(\epsilon_0, r_1, r_2, f_0, d)$.

Note that the second term on the left-hand side of (7.3) is bounded from below due to Jenssen's inequality by a constant $C(r_1, r_2)$ depending only on r_1 and r_2. The right-hand side of (7.3) can be estimated by using the technique of [11]. We can easily show that if $V + V_\Omega$ is positive then

$$
\begin{aligned}
&\sum_j \sqrt{\left|\lambda_j(\epsilon_0 V)\right|} + \sum_j \sqrt{\left|\lambda_j(\epsilon_0 V_\Omega)\right|} \\
&\quad \leqslant C(\epsilon_0, r)\left(\left[\int_{B_r} \left|\hat{V}_1(\xi)\right|^2 d\xi\right]^{(d+1)/4} + \int_{\mathbb{R}^d} \left|V_1(x)\right|^{d+1} dx\right),
\end{aligned} \tag{7.4}
$$

for all $V \in C_0^\infty(\mathbb{R}^d)$, $d \geqslant 3$, where B_r is the ball of radius r around zero. If $V = -V_\Omega$, then this is a simple consequence of the results of [11]. For the general case it follows from the variational principle. Due to the (7.4) the integrals of the function

$$\log\left[\frac{d}{d\lambda}\big(E_H(\lambda)f, f\big)\right]$$

can be estimated as follows:

$$\begin{aligned}
&\int_{r_1}^{r_2} \log\left[\frac{d}{d\lambda}\big(E_H(\lambda)f, f\big)\right] \\
&\quad \geqslant -C(\epsilon_0, r)\left(\left[\int_{B_r} \big|\hat{V}_1(\xi)\big|^2\, d\xi\right]^{(d+1)/4} + \|V\|_\infty^{1/2} + \int_{\mathbb{R}^d} \big|V_1(x)\big|^{d+1}\, dx\right) \\
&\qquad - C(\epsilon_0, f_0, \phi, r_1, r_2)
\end{aligned} \tag{7.5}$$

for all $r > 0$ and $r_2 > r_1 > 0$. In the next proposition we finally remove the assumption of compact support and smoothness for V.

Proposition 7.3. *Let V satisfy the conditions of Theorem* 2.1. *Then there exists a sequence V_n of compactly supported smooth functions converging to V locally in L^2 with the properties $V_n = V_{1,n} + V_{2,n}$,*

$$\int |V_{1,n}|^{d+1}\, dx < C(V), \qquad \|V_n\|_\infty < C(V), \tag{7.6}$$

$$\chi_\Omega V_{2,n} = 0, \quad V_{2,n} \geqslant 0, \ \forall n, \tag{7.7}$$

and

$$\int_{|\xi| \leqslant \delta/2} \big|\hat{V}_{1,n}(\xi)\big|^2\, d\xi < C(V), \tag{7.8}$$

such that the quadratic forms of the resolvents $((H_n - z)^{-1} f, f)$ corresponding to V_n converge uniformly in z to $((H - z)^{-1} f, f)$ when z belongs to any compact subset of the upper half plane. Therefore the sequence of measures $(E_{H_n}(\cdot) f, f)$ converges weakly to the spectral measure $(E_H(\cdot) f, f)$ for a fixed function $f \in L^2(\mathbb{R}^d \setminus B_1)$.

Proof. The existence of a sequence V_n satisfying (7.6)–(7.8) and converging locally in L^2 to V follows from standard arguments. Using this, the proof of resolvent convergence is virtually identical to the proof of Proposition 7.1, in fact slightly simpler. So we omit it. □

Finally according to inequality (6.13) and Propositions 7.3 and 7.1 we observe that there exists a sequence of operators H_l and a function f specified in the conditions on Theorem 2.1 such that the measures $(E_{H_l}(\cdot) f, f)$ weakly converge to $(E_H(\cdot) f, f)$ and for any fixed $c > 0$,

$$\int_0^c \frac{\log(d(E_{H_l}(\lambda) f, f)/d\lambda)\, d\lambda}{(1 + \lambda^{3/2})\sqrt{\lambda}} > -C(V), \quad \forall l,$$

where $C(V)$ is independent of c. Therefore due to the statement on the upper semi-continuity of an entropy (see [6]) we obtain

$$\int_0^c \frac{\log(d(E_H(\lambda)f, f)/d\lambda)\,d\lambda}{(1+\lambda^{3/2})\sqrt{\lambda}} > -C(V).$$

The proof of Theorem 2.1 is complete.

Acknowledgments

O.S. is grateful to S. Denissov for sharing the results of his work [5] at an early stage. The authors thank R. Shterenberg for reading a preliminary version of this manuscript. We also express our gratitude to the referee of this paper for valuable suggestions which helped to substantially improve the presentation of our work.

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