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Note

Counting humps in Motzkin paths

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ABSTRACT

In this paper we study the number of humps (peaks) in Dyck, Motzkin and Schröder paths. Recently A. Regev noticed that the number of peaks in all Dyck paths of order n is one half of the number of super-Dyck paths of order n. He also computed the number of humps in Motzkin paths and found a similar relation, and asked for bijective proofs. We give a bijection and prove these results. Using this bijection we also give a new proof that the number of Dyck paths of order n with k peaks is the Narayana number. By double counting super-Schröder paths, we also get an identity involving products of binomial coefficients.

1. Introduction

A *Dyck path* of order (semilength) n is a lattice path in $\mathbb{Z} \times \mathbb{Z}$, from $(0, 0)$ to $(2n, 0)$, using up-steps $(1, 1)$ (denoted by U) and down-steps $(1, -1)$ (denoted by D) and never going below the x-axis. We use $\mathcal{D}_n$ to denote the set of Dyck paths of order n. It is well known that $\mathcal{D}_n$ is counted by the nth *Catalan number* (A000108 in [8])

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

A *peak* in a Dyck path is two consecutive steps UD. It is also well known (see, for example, [1,4,10]) that the number of Dyck paths of order n with k peaks is the *Narayana number* (A001263):

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Counting Dyck paths with restriction on the peaks has been studied by many authors; see for example [2,3,5]. Here we are interested in counting peaks in all Dyck paths of order n. By summing over the above formula over k we immediately get the following result: the total number of peaks in all Dyck paths of order n is

$$pd_n = \sum_{k=1}^{n} kN(n,k) = \binom{2n-1}{n}.$$

If we allow a Dyck path to go below the x-axis, we get a *super-Dyck path*. Let $\mathcal{SD}_n$ denote the set of super-Dyck paths of order n. By standard arguments we have

$$sd_n = \#\mathcal{SD}_n = \binom{2n}{n} = 2\binom{2n-1}{n} = 2pd_n. \tag{1.1}$$

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That is, the number of super-Dyck paths of order n is twice the number of peaks in all Dyck paths of order n. This curious relation was first noticed by Regev [7], who also noticed that a similar relation holds for Motzkin paths, which we will explain next.

A *Motzkin path of order n* is a lattice path in $\mathbb{Z}\times\mathbb{Z}$, from $(0, 0)$ to $(n, 0)$, using up-steps $(1, 1)$, down-steps $(1, -1)$ and flat-steps $(1, 0)$ (denoted by F), that never goes below the x-axis. Let $\mathcal{M}_n$ denote all the Motzkin paths of order n. The cardinality of $\mathcal{M}_n$ is the nth *Motzkin number* m_n (A001006), which satisfies the following recurrence relation:

$$m_0 = 1, \qquad m_1 = 1, \qquad m_n = m_{n-1} + \sum_{i=2}^{n} m_{i-2}m_{n-i}, \quad \text{for } n \geq 2,$$

and has the generating function

$$\sum_{n\geq 0} m_n x^n = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}.$$

A *hump* in a Motzkin path is an up-step followed by zero or more flat-steps followed by a down-step. We use hm_n to denote the total number of humps in all Motzkin paths of order n. We can similarly define *super-Motzkin paths* to be Motzkin paths that are allowed to go below the x-axis, and use $\mathcal{SM}_n$ to denote the set of super-Motzkin paths of order n. Using a recurrence relation and the WZ method [6,11], Regev [7] proved that

$$sm_n = \#\mathcal{SM}_n = \sum_{j\geq 0} \binom{n}{j}\binom{n-j}{j} = 2hm_n + 1 \tag{1.2}$$

and asked for a bijective proof of (1.1) and (1.2). The main result of this paper is such a bijective proof.

Let $\mathcal{SM}_n^{UU}(k)$ ($\mathcal{SM}_n^{UD}(k)$) denote the set of paths in $\mathcal{SM}_n$ with k peaks where the first non-flat-step is U, and the last non-flat-step is U (D). Let $\mathcal{SM}_n^{U*}$ denote all paths in $\mathcal{SM}_n$ whose first non-flat-step is U, and define

$$\mathcal{HM}_n = \{(M, P) | M \in \mathcal{M}_n, P \text{ is a hump of } M\}.$$

The main result of this paper is the following:

Theorem 1.1. *There is a bijection $\Phi\colon \mathcal{HM}_n \to \mathcal{SM}_n^{U*}$ such that if $(M, P) \in \mathcal{HM}_n$ and $L = \Phi(M, P)$, then there are k humps in M if and only if $L \in \mathcal{SM}_n^{UU}(k-1) \cup \mathcal{SM}_n^{UD}(k)$.*

The outline of the paper is as follows. In Section 2 we define the bijection Φ and prove Theorem 1.1. In Section 3 we apply Φ to Dyck paths and give a new proof of the Narayana numbers. In Section 4 we apply Φ to Schröder paths and get an identity involving products of binomial coefficients by double counting super-Schröder paths whose F steps are m-colored.

2. The bijection $\Phi\colon \mathcal{HM}_n \leftrightarrow \mathcal{SM}_n^{U*}$

Note that a Motzkin path M of order n can also be considered as a sequence $M = M_1M_2\cdots M_n$, with $M_i \in \{U, F, D\}$, and the number of U's is not less than the number of D's in every subsequence $M_1M_2\cdots M_k$ of M. Hence a hump in M is a subsequence $P = M_iM_{i+1}\cdots M_{i+k+1}$, $k \geq 0$, such that $M_i = U$, $M_{i+1} = M_{i+2} = \cdots = M_{i+k} = F$ and $M_{i+k+1} = D$. We call the end point of step M_i a *hump point*, and it will also be denoted as P. Similarly, if there exists i such that $M_i = D$, $M_{i+1} = M_{i+2} = \cdots = M_{i+k} = F$, $k \geq 0$, $M_{i+k+1} = U$, then we call the subsequence $M_iM_{i+1}\cdots M_{i+k+1}$ a *valley* of M, and the end point of M_{i+k} is called a *valley point*. The end point $(n, 0)$ of M is also considered as a valley point.

Suppose L is a path in $\mathbb{Z}\times\mathbb{Z}$ from $O(0, 0)$ to $N(n, 0)$, and A a lattice point on M; we use x_A and y_A to denote the x-coordinate and y-coordinate of A, respectively. The sub-path of L from point A to point B is denoted by L_{AB}. We use $\bar{L}$ to denote the lattice path obtained from L by interchanging all the up-steps and down-steps in L, and keep the flat-steps unchanged.

Now we are ready to define the map Φ and prove Theorem 1.1.

Proof of Theorem 1.1. (1) The map $\Phi\colon \mathcal{HM}_n \to \mathcal{SM}_n^{U*}$.

For any $(M, P) \in \mathcal{HM}_n$, we define $L = \Phi(M, P)$ by the following rules:

- Let C be the leftmost valley point in M such that $x_C > x_P$.
- Let B be the rightmost point in M such that $x_B < x_P$, $y_B = y_C$.
- Let A be the rightmost point in M such that $y_A = 0$, $x_A \leq x_B$.
- Set $L_0 = M_{OA}$, $L_1 = M_{AB}$, $L_2 = \overline{M_{BC}}$, $L_3 = M_{CN}$ (note that L_0, L_1 and L_3 may be empty).
- Define $L = \Phi(M, P) = L_0L_2\overline{L_3}L_1$.

Now we will prove that $L \in \mathcal{SM}_n^{U*}$. According to the above definition, L_0 and L_2 are both Motzkin paths; therefore $\#U = \#D$ in L_0 and L_2. And for L_1, we have $\#U - \#D = y_B - y_A = y_B = y_C$, and for L_3, $\#U - \#D = -y_C$. Therefore the total number of U's is as high as that of D's in L. Thus L is a super-Motzkin path of order n. Moreover, the first non-flat-step in L must be in L_0 (when L_0 is not empty) or in L_2 (when L_0 is empty), and L_0, L_2 are both Motzkin paths; hence the first step leaving the x-axis must be a U. Therefore we have proved that $L = \Phi(M, P) \in \mathcal{SM}_n^{U*}$.

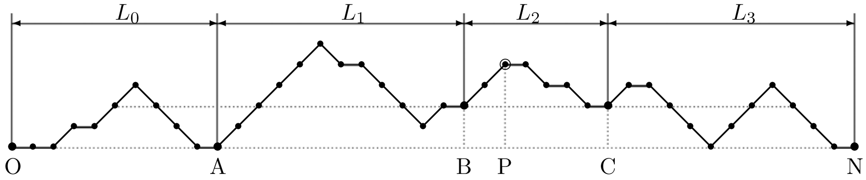

Fig. 1. A Motzkin path $M \in \mathcal{M}_{41}$ with a circled hump point P.

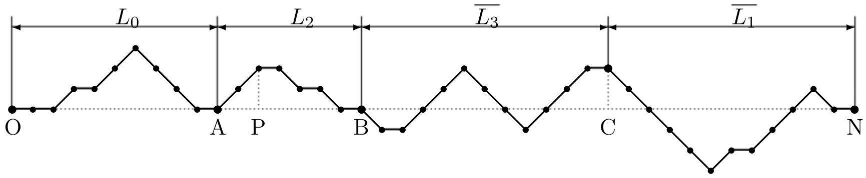

Fig. 2. A super-Motzkin path $L = \Phi(M, P)$.

(2) The inverse of Φ.

For any $L \in \mathcal{SM}_n^{U*}$, we define Ψ by the following rules:

- Let B be the leftmost point such that $y_B = 0$, and L goes below the x-axis after B. (If such a point does not exist, then set $B = N$.)
- Let A be the rightmost point in L such that $x_A < x_B$, $y_A = 0$.
- Let C be the rightmost point in L such that $x_C \geq x_B$, and $\forall G, x_G \geq x_B$ implies that $y_C \geq y_G$.
- Let P be the rightmost hump point in L such that $x_P < x_B$.
- Set $L_0 = L_{OA}, L_1 = L_{AB}, L_2 = L_{BC}, L_3 = L_{CN}$ (note that L_0, L_2 and L_3 may be empty).
- Set $M = L_0\overline{L_3}L_1\overline{L_2}$, and $\Psi(L) = (M, P)$.

Now we prove that $\Psi = \Phi^{-1}$. Since C is the highest point in L_3, and $\overline{L_3}$ and L_3 are symmetric with respect to the line $y = y_C$, C is mapped to the lowest point in $\overline{L_3}$. Moreover, L_0 and L_1 are both Motzkin paths; then $L_0\overline{L_3}L_1$ does not go below the x-axis, and the y-coordinate of the end point of $L_0\overline{L_3}L_1$ is y_C. In $\overline{L_2}$, the end point is the lowest point, and the start point of $\overline{L_2}$ is y_C higher than the end point. So $M = L_0\overline{L_3}L_1\overline{L_2}$ ends on the x-axis and never goes below it, i.e., $M \in \mathcal{M}_n$. Thus $\Psi(L) \in \mathcal{HM}_n$, and it is not hard to see that $\Psi = \Phi^{-1}$.

(3) There are k humps in M if and only if $\Phi(M, P) \in \mathcal{SM}_n^{UD}(k) \cup \mathcal{SM}_n^{UU}(k-1)$.

Since $\Phi(M) = L_0L_2\overline{L_3}\overline{L_1} = L$, the number of humps changes only in sub-paths $\overline{L_3}$ and $\overline{L_1}$ when M is converted to L. If the last step of L_1 is U, then the last step in $\overline{L_1}$ becomes D. The number of humps in L_1 is the same as the number of humps in $\overline{L_1}$, and the number of humps in $\overline{L_3}$ is 1 less than the number of humps in L_3. The last step in $\overline{L_3}$ is a U step, so concatenating $\overline{L_1}$ with $\overline{L_3}$ yields a new hump. Therefore the total number of humps in L is the same as that in M. Thus we have $\Phi(M, P) \in \mathcal{SM}_n^{UD}(k)$.

If the last step in L_1 is D, then the last step in $\overline{L_1}$ is U. The number of humps in $\overline{L_1}$ is 1 less than the number of humps in L_1, and the humps in $\overline{L_3}$ is 1 less than the number of humps in L_3. Moreover, the last step in $\overline{L_3}$ is U, so concatenating $\overline{L_1}$ with $\overline{L_3}$ yields a new hump. Therefore the total number of humps in L is 1 less than the number humps in M. Thus we have $\Phi(M, P) \in \mathcal{SM}_n^{UU}(k-1)$. □

Fig. 1 shows, as an example, a Motzkin path $M \in \mathcal{M}_{41}$ with a circled hump point P, and Fig. 2 shows a super-Motzkin path $L \in \mathcal{SM}_{41}^{U*} = \Phi(M, P)$.

From Theorem 1.1 we can easily get the following result.

Corollary 2.2. *For all $n \geq 0$, we have*

$$sm_n = 2hm_n + 1, \tag{2.1}$$

and

$$hm_n = \frac{1}{2}\left(\sum_{j\geq 0}\binom{n}{j}\binom{n-j}{j} - 1\right). \tag{2.2}$$

Proof. Eq. (2.1) follows immediately from Theorem 1.1. To prove (2.2) we count super-Motzkin paths with j U steps. We can first choose the j U steps among the total n steps, then choose j steps as D steps among the remaining $n - j$ steps. Thus we have

$$sm_n = \sum_{j\geq 0}\binom{n}{j}\binom{n-j}{j}.$$

Combining this with Eq. (2.1), we get Eq. (2.2). □

3. Counting peaks in Dyck paths and the Narayana numbers

Note that when restricted to Dyck paths, Φ is a bijection between super-Dyck paths and peaks in Dyck paths. Therefore we have the following result.

Corollary 3.3. *For all $n \geq 0$, we have*

$$sd_n = 2pd_n,$$

and

$$pd_n = \binom{2n-1}{n}.$$

Moreover, from the bijection Φ we can easily get a new proof for the Narayana numbers. To this end we need the following lemma.

Lemma 3.4. *Let $\mathcal{SD}_n^{UD}(k)$ ($\mathcal{SD}_n^{UU}(k)$) denote the set of super-Dyck paths of order n with k peaks whose first step is U and last step is D (U); then we have*

$$\#\mathcal{SD}_n^{UD}(k) = \binom{n-1}{k-1}^2, \tag{3.1}$$

$$\#\mathcal{SD}_n^{UU}(k) = \binom{n-1}{k-1}\binom{n-1}{k}, \tag{3.2}$$

and the number of super-Dyck paths with k peaks of order n is $\binom{n}{k}^2$.

Proof. Each $L \in \mathcal{SD}_n^{UD}(k)$ can be written uniquely as a word $L = U^{x_1}D^{y_1}U^{x_2}D^{y_2}\cdots U^{x_k}D^{y_k}$ such that

$$\begin{cases} x_1 + x_2 + \cdots + x_k = n, & x_1, x_2, \ldots, x_k \geq 1, \\ y_1 + y_2 + \cdots + y_k = n, & y_1, y_2, \ldots, y_k \geq 1. \end{cases}$$

The number of solutions for the x_i's and that for the y_i's are both equal to $\binom{n-k+k-1}{k-1} = \binom{n-1}{k-1}$. Hence Eq. (3.1) is proved.

Each $L' \in \mathcal{SD}_n^{UU}(k)$ can be written uniquely as a word $L' = U^{x_1}D^{y_1}U^{x_2}D^{y_2}\cdots U^{x_k}D^{y_k}U^{x_{k+1}}$ such that

$$\begin{cases} x_1 + x_2 + \cdots + x_k + x_{k+1} = n, & x_1, x_2, \ldots, x_{k+1} \geq 1 \\ y_1 + y_2 + \cdots + y_k = n, & y_1, y_2, \ldots, y_k \geq 1. \end{cases}$$

There are $\binom{n-k+k+1-1}{k} = \binom{n}{k}$ solutions for the x_i's and $\binom{n-1}{k-1}$ solutions for the y_i's. Hence Eq. (3.2) is proved.

From (3.1) and (3.2) we have that the number of super-Dyck paths with k peaks of order n is

$$\binom{n-1}{k-1}^2 + \binom{n-1}{k}^2 + 2\binom{n-1}{k-1}\binom{n-1}{k} = \binom{n}{k}^2. \qquad \square$$

Corollary 3.5. *The number of Dyck paths of order n with k peaks is*

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Proof. From Theorem 1.1 we know that each Dyck path of order n with k peaks is mapped to k super-Dyck paths, and each of the k super-Dyck paths is either in $\mathcal{SD}_n^{UU}(k-1)$ or in $\mathcal{SD}_n^{UD}(k)$. Therefore we have $kN(n,k) = \#\mathcal{SD}_n^{UU}(k-1) + \#\mathcal{SD}_n^{UD}(k)$. From Proposition 3.4 we can conclude that

$$N(n,k) = \frac{1}{k}\left(\binom{n-1}{k-1}^2 + \binom{n-1}{k-2}\binom{n-1}{k-1}\right) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}. \qquad \square$$

A bijective proof of this result can also be found in [10, Exercise 6.36(a)].

4. Humps in Schröder paths

In this section we count the number of humps in a third kind of lattice paths: Schröder paths. A *Schröder path* of order n is a lattice path in $\mathbb{Z} \times \mathbb{Z}$, from $(0,0)$ to (n,n), using up-steps $(0,1)$, down-steps $(1,0)$ and flat-steps $(1,1)$ (denoted by

U, D, F, respectively) and never going below the line $y = x$. Note that Schröder paths are different from what is obtained by rotating Motzkin paths 45° counterclockwise, since the F steps in these two kinds of paths are different. However, the bijection Φ still works when counting humps in Schröder paths. Let ss_n denote the number of super-Schröder paths of order n, and hs_n denote the number of humps in all Schröder paths of order n. We have the following result.

Corollary 4.6. *For all $n \geq 0$, we have*

$$ss_n = 2hs_n + 1, \tag{4.1}$$

and

$$hs_n = \frac{1}{2}\left(\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k} - 1\right). \tag{4.2}$$

Proof. Applying the bijection Φ to Schröder paths, we immediately get (4.1). Next we will count ss_n. Let L be a super-Schröder path of order n with k humps; then there are k U steps, k D steps, and $n-k$ F steps in L. We can first choose a super-Dyck path of order k and then "insert" $n-k$ F steps to get L. There are $\binom{2k}{k}$ ways to choose a super-Dyck paths, and $\binom{n-k+2k+1-1}{2k} = \binom{n+k}{2k}$ ways to carry out the insertion. Therefore we have

$$ss_n = \sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}.$$

From the above formula and (4.1) we get (4.2). □

The above proof inspired us to obtain the following identity, which is listed as an exercise in [9, Exercise 3(g) of Chapter 1].

Corollary 4.7. *For all $n \geq 0$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}^2(m+1)^k = \sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}m^{n-k}. \tag{4.3}$$

Proof. We will first prove (4.3) $m = 1$. From the proof of Corollary 4.6 we know that the right hand side of (4.3) is the number of super-Schröder paths of order n when $m = 1$. Now we count ss_n with a different method, to obtain the left hand side. Let L be a super-Dyck path of order n with k peaks; for each peak of L, we can either keep it invariant or change it into an F step to we get two super-Schröder paths. Hence each L is mapped to 2^k super-Schröder paths; thus the left hand side of (4.3) when $m = 1$ also equals ss_n. Therefore we have proved (4.3) for $m = 1$.

For general m we count the number of super-Schröder paths in which the F steps are m-colored. Now every super-Dyck path with k peaks is mapped to $(m+1)^k$-colored super-Schröder paths. So the total number of such paths is $\sum_{k=0}^{n}\binom{n}{k}^2(m+1)^k$. On the other hand, from the proof of Corollary 4.6 we know that the right hand side of (4.3) also counts the number of such paths, and hence we have proved (4.3). □

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