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On shrinking and boundedly complete Schauder frames of Banach spaces ☆

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ABSTRACT

This paper studies Schauder frames in Banach spaces, a concept which is a natural generalization of frames in Hilbert spaces and Schauder bases in Banach spaces. The associated minimal and maximal spaces are introduced, as are shrinking and boundedly complete Schauder frames. Our main results extend the classical duality theorems on bases to the situation of Schauder frames. In particular, we will generalize James' results on shrinking and boundedly complete bases to frames. Secondly we will extend his characterization of the reflexivity of spaces with unconditional bases to spaces with unconditional frames.

1. Introduction

The theory of frames in Hilbert spaces presents a central tool in many areas and has developed rather rapidly in the past decade. The motivation has come from applications to engineering, i.e. signal analysis, as well as from applications to different areas of Mathematics, such as, sampling theory [1], operator theory [11], harmonic analysis [9], nonlinear sparse approximation [7], pseudo-differential operators [10], and quantum computing [8]. Recently, the theory of frames also showed connections to theoretical problems such as the Kadison–Singer problem [4].

A standard frame for a Hilbert space H is a family of vectors $x_i \in H$, $i \in \mathbb{N}$, such that there are constants $A, B > 0$ for which

$$A\|x\|^2 \leqslant \sum \left|\langle x, x_i\rangle\right|^2 \leqslant B\|x\|^2, \quad \text{whenever } x \in H.$$

In this paper we consider Schauder frames in Banach spaces, which, on the one hand, generalize Hilbert frames, and extend the notion of Schauder basis, on the other.

In [2], D. Carando and S. Lassalle consider the duality theory for atomic decompositions. In our independent work, we will mostly concentrate on properties of Schauder frames, which do not depend on the choice of associated spaces, define the concepts of minimal and maximal (associated) spaces and the corresponding minimal and maximal (associated) bases with respect to Schauder frames, and closely connect them to the duality theory. Moreover, we extend James' well-known results on characterizing the reflexivity of spaces with an unconditional bases, to spaces with unconditional frames.

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In Section 2 we recall the basic definitions and properties of Schauder frames. Then we introduce the concept of shrinking and boundedly complete frames and prove some elementary facts.

Section 3 deals with the concept of associated spaces, and introduces the definitions of minimal and maximal (associated) spaces and the corresponding minimal and maximal (associated) bases with respect to Schauder frames.

In Section 4 we extend James' results on shrinking and boundedly bases to frames [12] and prove the following theorems. All necessary definitions can be found in the following Sections 2 and 3.

Theorem A. *Let $(x_i, f_i) \subset X \times X^*$ be a Schauder frame of a Banach space X and assume that for all $m \in \mathbb{N}$*

$$\lim_{n\to\infty} \| f_m|_{\operatorname{span}(x_i:\ i\geqslant n)} \| = 0.$$

Then the following are equivalent:

(1) *(x_i, f_i) is shrinking.*
(2) *Every normalized block of (x_i) is weakly null.*
(3) *$X^* = \overline{\operatorname{span}(f_i:\ i \in \mathbb{N})}$.*
(4) *The minimal associated basis is shrinking.*

Theorem B. *Let $(x_i, f_i) \subset X \times X^*$ be a Schauder frame of a Banach space X and assume that for all $m \in \mathbb{N}$*

$$\lim_{n\to\infty} \| f_m|_{\operatorname{span}(x_i:\ i\geqslant n)} \| = 0 \quad \text{and} \quad \lim_{n\to\infty} \| x_m|_{\operatorname{span}(f_i:\ i\geqslant n)} \| = 0.$$

Then the following are equivalent:

(1) *(x_i, f_i) is boundedly complete.*
(2) *X is isomorphic to $\overline{\operatorname{span}(f_i:\ i \in \mathbb{N})}^*$ under the natural canonical map.*
(3) *The maximal associated basis is boundedly complete.*

In Section 5, we discuss unconditional Schauder frames. We obtain a generalization of James's theorem and prove that a Banach space with a locally shrinking and unconditional Schauder frame is either reflexive or contains isomorphic copies of ℓ_1 or c_0.

Theorem C. *Let $(x_i, f_i) \subset X \times X^*$ be an unconditional Schauder frame of a Banach space X and assume that for all $m \in \mathbb{N}$*

$$\lim_{n\to\infty} \| f_m|_{\operatorname{span}(x_i:\ i\geqslant n)} \| = 0.$$

Then X is reflexive if an only if X does not contain isomorphic copies of c_0 and ℓ_1.

All Banach spaces in this paper are considered to be spaces over the real number field $\mathbb{R}$. The unit sphere and the unit ball of a Banach space X are denoted by S_X and B_X, respectively. The vector space of scalar sequences (a_i), which vanish eventually, is denoted by c_{00}. The usual unit vector basis of c_{00}, as well as the unit vector basis of c_0 and ℓ_p $(1 \leqslant p < \infty)$ and the corresponding coordinate functionals will be denoted by (e_i) and (e_i^*), respectively.

Given two sequences (x_i) and (y_i) in some Banach space, and given a constant $C > 0$, we say that (y_i) *C-dominates* (x_i), or that (x_i) *is C-dominated by* (y_i), if

$$\Big\| \sum a_i x_i \Big\| \leqslant C \Big\| \sum a_i y_i \Big\| \quad \text{for all } (a_i) \in c_{00}.$$

We say that (y_i) *dominates* (x_i), or that (x_i) *is dominated by* (y_i), (y_i) C-dominates (x_i) for some constant $C > 0$.

2. Frames in Banach spaces

In this section, we give a short review of the concept of frames in Banach spaces, and make some preparatory observations.

Definition 2.1. Let X be a (finite or infinite dimensional) separable Banach space. A sequence $(x_i, f_i)_{i\in\mathbb{I}}$, with $(x_i)_{i\in\mathbb{I}} \subset X$ and $(x_i)_{i\in\mathbb{I}} \subset X^*$ with $\mathbb{I} = \mathbb{N}$ or $\mathbb{I} = \{1, 2, \ldots, N\}$ for some $N \in \mathbb{N}$, is called a (*Schauder*) *frame of* X if for every $x \in X$,

$$x = \sum_{i\in\mathbb{I}} f_i(x) x_i. \tag{1}$$

In case that $\mathbb{I} = \mathbb{N}$, we mean that the series in (1) converges in norm, that is,

$$x = \|\cdot\| - \lim_{n\to\infty} \sum_{i=1}^{n} f_i(x) x_i. \tag{2}$$

An *unconditional frame of* X is a frame $(x_i, f_i)_{i\in\mathbb{N}}$ for X for which the convergence in (1) is unconditional.
We call a frame (x_i, f_i) *bounded* if

$$\sup_i \|x_i\| < \infty \quad \text{and} \quad \sup_i \|f_i\| < \infty,$$

and *semi-normalized* if (x_i) and (f_i) both are semi-normalized, that is, if

$$0 < \inf_i \|x_i\| \leqslant \sup_i \|x_i\| < \infty \quad \text{and} \quad 0 < \inf_i \|f_i\| \leqslant \sup_i \|f_i\| < \infty.$$

Remark 2.2. Throughout this paper, it will be our convention that we only consider non-zero frames (x_i, f_i) indexed by $\mathbb{N}$, that is, the index set $\mathbb{I}$ will always be $\mathbb{N}$ and we assume that $x_i \neq 0$ and $f_i \neq 0$ for all $i \in \mathbb{N}$.

In the following proposition we recall some easy observations from [5,3].

Proposition 2.3. *(See [5,3].) Let (x_i, f_i) be a frame of X.*

(a) (i) *Using the Uniform Boundedness Principle we deduce that*

$$K = \sup_{x\in B_X} \sup_{m\leqslant n} \left\| \sum_{i=m}^{n} f_i(x) x_i \right\| < \infty.$$

We call K the projection constant of (x_i, f_i).

(ii) *If (x_i, f_i) is an unconditional frame, then it also follows from the Uniform Boundedness Principle that*

$$K_u = \sup_{x\in B_X} \sup_{\sigma_i\in\{\pm 1\}} \left\| \sum \sigma_i f_i(x) x_i \right\| < \infty.$$

We call K_u the unconditional constant of (x_i, f_i).

(b) *The sequence (f_i, x_i) is a w^*-Schauder frame of X^*, that is to say, for every $f \in X^*$,*

$$f = w^* - \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) f_i.$$

(c) *For any $f \in X^*$ and $m \leqslant n$ in $\mathbb{N}$, we have*

$$\left\| \sum_{i=m}^{n} f(x_i) f_i \right\| = \sup_{x\in B_X} \left| \sum_{i=m}^{n} f(x_i) f_i(x) \right| \leqslant \|f\| \sup_{x\in B_X} \left\| \sum_{i=m}^{n} f_i(x) x_i \right\| \leqslant K\|f\|, \tag{3}$$

and

$$\begin{aligned} \left\| \sum_{i=m}^{n} f(x_i) f_i \right\| &= \sup_{x\in B_X} \left| \sum_{i=m}^{n} f(x_i) f_i(x) \right| = \sup_{x\in B_X} \left| f\left(\sum_{i=m}^{n} f_i(x) x_i \right) \right| \\ &\leqslant \sup_{z\in \operatorname{span}(x_i:\, i\geqslant m),\, \|z\|\leqslant K} \left| f(z) \right| = K \| f|_{\operatorname{span}(x_i:\, i\geqslant m)} \|, \end{aligned} \tag{4}$$

where K is the projection constant of (x_i, f_i).

Next, we present some basic properties of frames in Banach spaces.

Proposition 2.4. *Let (x_i, f_i) be a frame of a Banach space X. Then $\overline{\operatorname{span}}(f_i:\, i \in \mathbb{N})$ is a norming subspace of X^*.*

Proof. By Proposition 2.3(b) and (c) (3), for all $f \in B_{X^*}$ and $n \in \mathbb{N}$ we have

$$f = w^* - \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) f_i, \qquad \left\| \sum_{i=1}^{n} f(x_i) f_i \right\| \leqslant K,$$

where K is the projection constant of (x_i, f_i). Thus, we obtain that

$$B_{X^*} \subset \overline{K \cdot B_{X^*} \bigcap \operatorname{span}(f_i\colon i \in \mathbb{N})}^{w^*} \subset K \cdot B_{X^*}.$$

Then it is easy to deduce that $\overline{\operatorname{span}}(f_i\colon i \in \mathbb{N})$ is norming for X. □

Definition 2.5. Let (x_i, f_i) be a frame of a Banach space X.

(x_i, f_i) is called *locally shrinking* if for all $m \in \mathbb{N}$ $\|f_m|_{\operatorname{span}(x_i\colon i \geqslant n)}\| \to 0$ as $n \to \infty$. (x_i, f_i) is called *locally boundedly complete* if for all $m \in \mathbb{N}$ $\|x_m|_{\operatorname{span}(f_i\colon i \geqslant n)}\| \to 0$ as $n \to \infty$. (x_i, f_i) is called *weakly localized* if it is locally shrinking and locally boundedly complete.

The frame (x_i, f_i) is called *pre-shrinking* if (f_i, x_i) is a frame of X^*. It is called *pre-boundedly complete* if for all $x^{**} \in X^{**}$, $\sum_{i=1}^{\infty} x^{**}(f_i)x_i$ converges.

We call (x_i, f_i) *shrinking* if it is locally shrinking and pre-shrinking, and we call (x_i, f_i) *boundedly complete* if it weakly localized and *pre-boundedly complete*.

It is clear that every basis for a Banach space is weakly localized. However, it is false for frames. The following example is an unconditional and semi-normalized frame for ℓ_1 which is not locally shrinking or locally boundedly complete. We leave the proof to the reader.

Example 2.6. Let (e_i) denote the usual unit vector basis of ℓ_1 and let (e_i^*) be the corresponding coordinate functionals, and set $\mathbf{1} = (1, 1, 1, \ldots) \in \ell_\infty$. Then define a sequence $(x_i, f_i) \subset \ell_1 \times \ell_\infty$ by putting $x_{2i-1} = x_{2i} = e_i$ for all $i \in \mathbb{N}$ and

$$f_i = \begin{cases} \mathbf{1}, & \text{if } i = 1; \\ e_1^* - \mathbf{1}, & \text{if } i = 2; \\ e_k^* - e_1^*/2^k, & \text{if } i = 2k-1 \text{ for } k \in \mathbb{N} \setminus \{1\}; \\ e_1^*/2^k, & \text{if } i = 2k \text{ for } k \in \mathbb{N} \setminus \{1\}. \end{cases}$$

Proposition 2.7. *Let (x_i, f_i) be a frame of a Banach space X. Then the space*

$$X_0 = \left\{x \in X\colon \|x|_{\operatorname{span}(f_i\colon i \geqslant n)}\| \to 0 \text{ as } n \to \infty\right\}$$

is a norm closed subspace of X. Moreover, if (x_i, f_i) is locally boundedly complete, then $X_0 = X$.

Proof. If $(x_k) \subset X_0$ with $x_k \to x$ in X, then given any $\varepsilon > 0$, there are k_0 with $\|x - x_{k_0}\| \leqslant \varepsilon$, and $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$,

$$\|x|_{\operatorname{span}(f_i\colon i \geqslant n)}\| \leqslant \|x - x_{k_0}\| + \|x_{k_0}|_{\operatorname{span}(f_i\colon i \geqslant n)}\| \leqslant 2\varepsilon,$$

which implies that $x \in X_0$.

If (x_i, f_i) is locally boundedly complete, then $x_i \in X_0$ for all $i \in \mathbb{N}$. It follows that $X = \overline{\operatorname{span}}(x_i\colon i \in \mathbb{N}) \subset X_0$. Thus, we complete the proof. □

Proposition 2.8. *Let (x_i, f_i) be a frame of a Banach space X. Then the space*

$$Y = \left\{ f \in X^*\colon f = \|\cdot\| - \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) f_i \right\},$$

is a norm closed subspace of X^*. Moreover, if (x_i, f_i) is locally shrinking, then

$$Y = \overline{\operatorname{span}}(f_i\colon i \in \mathbb{N}),$$

and, thus, (f_i, x_i) is a frame for Y.

Proof. First, define a new norm $|||\cdot|||$ on X^* as follows

$$|||f||| = \sup_{m \leqslant n} \left\| \sum_{i=m}^{n} f(x_i) f_i \right\| \quad \text{for all } f \in X^*.$$

By Proposition 2.3(c) this is an equivalent norm on $(X^*, \|\cdot\|)$. Thus, if $(g_k) \subset Y$ with $g_k \to g$ in X^*, it follows that

$$\lim_{k\to\infty} |||g - g_k||| = \lim_{k\to\infty} \sup_{m \leqslant n} \left\| \sum_{i=m}^{n} g(x_i) f_i - \sum_{i=m}^{n} g_k(x_i) f_i \right\| = 0.$$

Thus, given any $\varepsilon > 0$, there are k_0 with $|||g - g_{k_0}||| \leqslant \varepsilon$, and $m_0 \in \mathbb{N}$ such that for all $n \geqslant m \geqslant m_0$, $\| \sum_{i=m}^{n} g_{k_0}(x_i) f_i \| \leqslant \varepsilon$, and thus,

$$\left\| \sum_{i=m}^{n} g(x_i) f_i \right\| \leqslant |||g - g_{k_0}||| + \left\| \sum_{i=m}^{n} g_{k_0}(x_i) f_i \right\| \leqslant 2\varepsilon,$$

which implies that $\sum_{i=1}^{\infty} g(x_i) f_i$ converges. By Proposition 2.3(b), we get $g = \sum_{i=1}^{\infty} g(x_i) f_i \in Y$.

If (x_i, f_i) is locally shrinking, it follows from Proposition 2.3(c) that for all $i \in \mathbb{N}$, $f_i \in Y$. Hence $\overline{\text{span}}(f_i\colon i \in \mathbb{N}) \subset Y$. On the other hand, it is clear from the definition of Y that $Y \subset \overline{\text{span}}(f_i\colon i \in \mathbb{N})$. Therefore, $Y = \overline{\text{span}}(f_i\colon i \in \mathbb{N})$. □

3. Associated spaces

Definition 3.1. Let (x_i, f_i) be a frame of a Banach space X and let Z be a Banach space with a basis (z_i). We call Z an *associated space to* (x_i, f_i) and (z_i) an *associated basis*, if

$$S : Z \to X, \quad \sum a_i z_i \mapsto \sum a_i x_i \quad \text{and} \quad T : X \to Z, \quad x = \sum f_i(x) x_i \mapsto \sum f_i(x) z_i,$$

are bounded operators. We call S the *associated reconstruction operator* and T the *associated decomposition operator* or *analysis operator*.

Remark 3.2. If (x_i, f_i) is a frame of a Banach space X and Z a corresponding associated space with an associated basis (z_i), then (see [5, Definition 2.1] or [6]) (x_i, f_i) is an *atomic decomposition* of X with respect to Z. In our paper, we will mostly concentrate on frames and properties which are independent of the associated spaces.

Proposition 3.3. *Let (x_i, f_i) be a frame of a Banach space X and let Z be an associated space with an associated basis (z_i). Let S and T be the associated reconstruction operator and the associated decomposition operator, respectively.*

Then S is a surjection onto $T(X)$, and T is an isomorphic embedding from X into Z. Moreover, for all $i \in \mathbb{N}$, $S(z_i) = x_i$ and $T^*(z_i^*) = f_i$.

Proof. Note that for any $x \in X$, it follows that

$$S \circ T(x) = S \circ T\Big(\sum f_i(x) x_i \Big) = S\Big(\sum f_i(x) z_i \Big) = \sum f_i(x) x_i = x.$$

Therefore, T must be an isomorphic embedding and S a surjection onto the space $T(X) = \{\sum f_i(x) z_i\colon x \in X\}$. And the map $P : Z \to Z$, $z \mapsto T \circ S(z)$ is a projection onto $T(X)$. By Definition 3.1, it is clear that $S(z_i) = x_i$ for all $i \in \mathbb{N}$. Secondly, it follows that for any $x \in X$ and $i \in \mathbb{N}$,

$$T^*\big(z_i^*\big)(x) = z_i^* \circ T\Big(\sum f_j(x) x_j \Big) = z_i^*\Big(\sum f_j(x) z_j \Big) = f_i(x),$$

and thus, $T^*(z_i^*) = f_i$, which completes our claim. □

We now introduce the notion of minimal bases.

Definition 3.4. Let (x_i) be a non-zero sequence in a Banach space X.

Define a norm on c_{00} as follows

$$\Big\| \sum a_i e_i \Big\|_{Min} = \max_{m \leqslant n} \left\| \sum_{i=m}^{n} a_i x_i \right\|_X \quad \text{for all } \sum a_i e_i \in c_{00}. \tag{5}$$

Denote by Z_{Min} the completion of c_{00} endowed with the norm $\| \cdot \|_{Min}$. It is easy to prove that (e_i), denoted by (e_i^{Min}), is a bi-monotone basis of Z_{Min}. By the following Theorem 3.5(b), we call Z_{Min} and (e_i^{Min}) the *minimal space* and the *minimal basis with respect to* (x_i), respectively.

Note that the operator:

$$S_{Min} : Z_{Min} \to X, \quad \sum a_i e_i^{Min} \mapsto \sum a_i x_i,$$

is linear and bounded with $\|S_{Min}\| = 1$.

If (x_i, f_i) is a frame the *minimal space* (or the *minimal basis*) *with respect to* (x_i, f_i) is the minimal space (or the minimal basis) with respect to (x_i).

As the following result from [5, Theorem 2.6] shows, associated spaces always exist.

Theorem 3.5. *(See [5, Theorem 2.6].) Let (x_i, f_i) be a frame of a Banach space X and let Z_{Min} be the minimal space with the minimal basis (e_i^{Min}).*

(a) *Z_{Min} is an associated space to (x_i, f_i) with the associated basis (e_i^{Min}).*
(b) *For any associated space Z with an associated basis (z_i), (e_i^{Min}) is dominated by (z_i).*

Thus, we will call Z_{Min} and (e_i^{Min}) the minimal associated space and the minimal associated basis to (x_i, f_i), respectively.

We give a sketch of the proof.

Proof. (a) Let K be the projection constant of (x_i, f_i). It follows that the map $T_{Min} : X \to Z_{Min}$ defined by

$$T_{Min} : X \to Z_{Min}, \quad x = \sum f_i(x)x_i \mapsto \sum f_i(x)e_i^{Min},$$

is well defined, linear and bounded and $\|T\| \leqslant K$. As already noted in Definition 3.4, the operator $S_{Min} : Z \to X$ is linear and bounded.

(b) If Z is an associated space with an associated basis (z_i) and $S : Z \to X$ is the corresponding associated reconstruction operator, then it follows that for any $(a_i) \in c_{00}$,

$$\begin{aligned}\Big\|\sum a_i e_i^{Min}\Big\| &= \max_{m\leqslant n}\Big\|\sum_{i=m}^{n} a_i x_i\Big\| = \max_{m\leqslant n}\Big\|\sum_{i=m}^{n} a_i S(z_i)\Big\| \\ &\leqslant \|S\| \max_{m\leqslant n}\Big\|\sum_{i=m}^{n} a_i z_i\Big\| \leqslant K_Z\|S\|\Big\|\sum a_i z_i\Big\|, \end{aligned} \tag{6}$$

where K_Z is the projection constant of (z_i). □

Next we introduce the notion of the maximal space and the maximal basis.

Definition 3.6. Let (x_i, f_i) be a frame of a Banach space X.

Define a norm on c_{00} as follows

$$\Big\|\sum a_i e_i\Big\|_{Max} = \sup_{\substack{(b_i)\in c_{00} \\ \max_{m\leqslant n}\|\sum_{i=m}^{n} b_i f_i\|\leqslant 1}} \Big|\sum a_i b_i\Big| \quad \text{for all } \sum a_i e_i \in c_{00}. \tag{7}$$

Denote by Z_{Max} the completion of c_{00} under $\|\cdot\|_{Max}$. Clearly, (e_i) is a bi-monotone basis of Z_{Max}, which will be denoted by (e_i^{Max}). We call Z_{Max} and (e_i^{Max}) the *maximal space* and the *maximal basis with respect to* (x_i, f_i), respectively.

Theorem 3.7. *Let (x_i, f_i) be a frame of a Banach space X and let Z_{Max} be the maximal space with the maximal basis (e_i^{Max}).*

(a) *If Z is an associated space with an associated basis (z_i), then (e_i^{Max}) dominates (z_i).*
(b) *The mapping*

$$S_{Max} : Z_{Max} \to X, \quad z = \sum a_i e_i^{Max} \mapsto \sum a_i x_i, \tag{8}$$

is well defined, linear and bounded.
(c) *If (x_i, f_i) is locally boundedly complete, then Z_{Max} is an associated space to (x_i, f_i) with the associated basis (e_i^{Max}). In this case, we call Z_{Max} and (e_i^{Max}) the maximal associated space and the maximal associated basis to (x_i, f_i).*

Proof. (a) Let Z be an associated space with an associated basis (z_i), (z_i^*) is the corresponding coordinate functionals, and let $T : X \to Z$ be the associated decomposition operator. By Proposition 3.3 $T^*(z_i^*) = f_i$, for all $i \in \mathbb{N}$. Thus, for any $(a_i) \in c_{00}$, we have

$$\begin{aligned}\Big\|\sum a_i z_i\Big\| &\leqslant K_Z \sup_{\substack{(b_i)\in c_{00} \\ \|\sum b_i z_i^*\|\leqslant 1}} \Big|\Big\langle\sum a_i z_i, \sum b_i z_i^*\Big\rangle\Big| \\ &\leqslant K_Z^2 \sup_{\substack{(b_i)\in c_{00} \\ \max_{m\leqslant n}\|\sum_{i=m}^{n} b_i z_i^*\|\leqslant 1}} \Big|\sum a_i b_i\Big| \end{aligned}$$

$$\leqslant K_Z^2 \sup_{\substack{(b_i)\in c_{00} \\ \max_{m\leqslant n}\|T^*(\sum_{i=m}^n b_i z_i^*)\|\leqslant \|T^*\|}} \left|\sum a_i b_i\right|$$

$$\leqslant K_Z^2 \|T^*\| \sup_{\substack{(b_i)\in c_{00} \\ \max_{m\leqslant n}\|\sum_{i=m}^n b_i f_i\|\leqslant 1}} \left|\sum a_i b_i\right| \leqslant K_Z^2 \|T^*\| \left\|\sum a_i e_i^{Max}\right\|, \tag{9}$$

where K_Z is the projection constant of (z_i, z_i^*).

(b) Let $(Z_{Min}, (e_i^{Min}))$ be the minimal space to (x_i, f_i) and by Theorem 3.5(a) let $T_{Min} : X \to Z_{Min}$ be the corresponding associated decomposition operator. Then by (9), for any $(a_i) \in c_{00}$, we have

$$\max_{m\leqslant n}\left\|\sum_{i=m}^n a_i x_i\right\| = \left\|\sum a_i e_i^{Min}\right\| \leqslant C\left\|\sum a_i e_i^{Max}\right\|, \tag{10}$$

where $C = K_{Min}^2\|T_{Min}^*\|$ and K_{Min} is the projection constant of (e_i^{Min}). Thus, the map $S_{Max} : Z_{Max} \to X$ with $S_{Max}(e_i^{Max}) = x_i$, for $i \in \mathbb{N}$, is well defined, linear and bounded with $\|S_{Max}\| \leqslant K_{Min}^2\|T_{Min}^*\|$.

(c) If (x_i, f_i) is locally boundedly complete, then for any $x \in X$ and $l \leqslant r$, we have

$$\left\|\sum_{i=l}^r f_i(x) e_i^{Max}\right\| = \sup_{\substack{(b_i)\in c_{00} \\ \max_{m\leqslant n}\|\sum_{i=m}^n b_i f_i\|\leqslant 1}} \left|\sum_{i=l}^r b_i f_i(x)\right| \leqslant \|x|_{\mathrm{span}(f_i:\ i\geqslant l)}\|,$$

which by Proposition 2.7, tends to zero as $l \to \infty$. Thus, the map

$$T_{Max} : X \to Z_{Max}, \quad x = \sum f_i(x) x_i \mapsto \sum f_i(x) e_i^{Max}, \tag{11}$$

is well defined, linear and bounded with $\|T_{Max}\| \leqslant 1$, which completes our proof. □

The following result emphases that for every frame, that associated bases dominate (e_i^{Min}) and are dominated by (e_i^{Max}).

Corollary 3.8. *Let (x_i, f_i) be a frame of a Banach space X. Assume that (e_i^{Min}) and (e_i^{Max}) are the minimal basis and the maximal basis with respect to (x_i, f_i), respectively. Then for any associated space Z with an associated basis (z_i), there are $C_1, C_2 > 0$ such that*

$$C_1\left\|\sum a_i e_i^{Min}\right\| \leqslant \left\|\sum a_i z_i\right\| \leqslant C_2\left\|\sum a_i e_i^{Max}\right\| \quad \text{for all } (a_i) \in c_{00}. \tag{12}$$

4. Applications of frames to duality theory

The following results extend James' work on shrinking and boundedly complete bases [12] to frames. Theorem 4.1 obviously yields Theorem A and Theorem 4.2 implies Theorem B.

Theorem 4.1. *Let (x_i, f_i) be a Schauder frame of a Banach space X. Assume that Z_{Min} and (e_i^{Min}) are the minimal space and minimal basis with respect to (x_i, f_i), respectively.*

Then the following conditions are equivalent:

(a) *Every normalized block sequence of (x_i) is weakly null.*
(b) (i) *(x_i, f_i) is locally shrinking.*
  (ii) *If $(u_n) \subset B_X$ with $\lim_{n\to\infty} f_m(u_n) = 0$ for all $m \in \mathbb{N}$, then (u_n) is weakly null.*
(c) *(x_i, f_i) is locally shrinking and pre-shrinking.*
(d) (i) *(x_i, f_i) is locally shrinking.*
  (ii) *$X^* = \overline{\mathrm{span}}(f_i:\ i \in \mathbb{N})$.*
(e) (i) *(x_i, f_i) is locally shrinking.*
  (ii) *(e_i^{Min}) is a shrinking basis of Z_{Min}.*

Theorem 4.2. *Let (x_i, f_i) be a frame of a Banach space X. Then the following conditions are equivalent:*

(a) *(x_i, f_i) locally shrinking and for all $x^{**} \in X^{**}$, $\|x^{**}|_{\mathrm{span}(f_i:\ i\geqslant n)}\| \to 0$, if $n \to \infty$.*
(b) *(x_i, f_i) is locally shrinking, locally boundedly complete and pre-boundedly complete.*
(c) (i) *(x_i, f_i) is locally shrinking and locally boundedly complete.*
  (ii) *For every $x^{**} \in X^{**}$, $\sum x^{**}(f_i) x_i$ converges under the topology $\sigma(X, \overline{\mathrm{span}(f_i:\ i \in \mathbb{N})})$.*
(d) (i) *(x_i, f_i) is locally shrinking and locally boundedly complete.*

(ii) *X is isomorphic to $\overline{\operatorname{span}(f_i\colon i \in \mathbb{N})}^*$ under the natural canonical map.*

(e) (i) *(x_i, f_i) is locally shrinking and locally boundedly complete.*

(ii) *(e_i^{Max}) is a boundedly complete basis of Z_{Max}.*

For the above main theorems, we need the following results.

Proposition 4.3.

(a) *Every frame satisfying* (a) *of Theorem* 4.1 *is pre-shrinking.*
(b) *Every frame satisfying* (a) *of Theorem* 4.2 *is pre-boundedly complete.*

Proof. Assume that (x_i, f_i) is a frame of a Banach space X.

(a) Notice that every normalized block sequence of (x_i) is weakly null if and only if for all $f \in X^*$, $\|f|_{\operatorname{span}(x_i\colon i \geqslant n)}\| \to 0$, as $n \to \infty$. This easily implies our claim by Proposition 2.3(b) and (c).

(b) For $m \leqslant n$ in $\mathbb{N}$ we have

$$\begin{aligned}\left\|\sum_{i=m}^{n} x^{**}(f_i)x_i\right\| &= \sup_{f\in B_{X^*}}\left|\sum_{i=m}^{n} x^{**}(f_i)f(x_i)\right| \\ &= \sup_{f\in B_{X^*}} x^{**}\left(\sum_{i=m}^{n} f(x_i)f_i\right) \\ &\leqslant \sup_{g\in\operatorname{span}(f_i\colon i\geqslant m),\|g\|\leqslant K} x^{**}(g) = K\left\|x^{**}|_{\operatorname{span}(f_i\colon i\geqslant m)}\right\|,\end{aligned} \tag{13}$$

where K is the projection constant of (x_i, f_i). $\square$

Proposition 4.4. *Let (x_i, f_i) is a Schauder frame of a Banach space X. Assume that Z is an associated space with an associated basis (z_i) to (x_i, f_i).*

(a) *If (z_i) is shrinking, then (x_i, f_i) is pre-shrinking.*
(b) *If (z_i) is boundedly complete, then (x_i, f_i) is pre-boundedly complete.*

Proof. Assume that S and T are the corresponding associated reconstruction and decomposition operators, respectively. By Proposition 3.3, $S(z_i) = x_i$ and $T^*(z_i^*) = f_i$ for all $i \in \mathbb{N}$.

(a) If (z_i) is shrinking, we have

$$f = T^*S^*(f) = T^*\Big(\sum\langle S^*(f), z_i\rangle z_i^*\Big) = \sum\langle f, S(z_i)\rangle T^*\big(z_i^*\big) = \sum f(x_i)f_i, \tag{14}$$

which proves our claim.

(b) For any $x^{**} \in X^{**}$ and $m, n \in \mathbb{N}$ with $m \leqslant n$,

$$\begin{aligned}\left\|\sum_{i=m}^{n} x^{**}(f_i)x_i\right\| &= \left\|\sum_{i=m}^{n} x^{**}\big(T^*\big(z_i^*\big)\big)S(z_i)\right\| = \left\|S\left(\sum_{i=m}^{n} T^{**}\big(x^{**}\big)\big(z_i^*\big)z_i\right)\right\| \\ &\leqslant \|S\|\cdot\left\|\sum_{i=m}^{n} T^{**}\big(x^{**}\big)\big(z_i^*\big)z_i\right\|.\end{aligned} \tag{15}$$

Since (z_i) is boundedly complete, $\sum_{i=1}^{\infty} T^{**}(x^{**})(z_i^*)z_i$ converges, by (15), so does $\sum_{i=1}^{\infty} x^{**}(f_i)x_i$, which completes the proof. $\square$

Proposition 4.5. *Let (x_i, f_i) be a Schauder frame of a Banach space X.*

(a) *Assume that Z_{Min} and (e_i^{Min}) are the minimal space and minimal basis with respect to (x_i, f_i), respectively. If (x_i, f_i) satisfies* (a) *of Theorem* 4.1, *then (e_i^{Min}) is shrinking.*
(b) *Assume that Z_{Max} are the maximal space with the maximal basis (e_i^{Max}) with respect to (x_i, f_i). If (x_i, f_i) satisfies* (a) *of Theorem* 4.2, *then (e_i^{Max}) is boundedly complete.*

For the proof of Proposition 4.5, we will need the following result, which is a slight variation of Lemma 2.10 of [13].

Lemma 4.6. *Let X be a Banach space and a sequence $(x_i) \subset X \setminus \{0\}$, and let Z_{Min} and (e_i^{Min}) be the associated minimal space and basis, respectively.*

(a) *Let $(y_i) \subset B_{Z_{Min}}$ be a block basis of (e_i^{Min}) on Z_{Min}. Assume that the sequence $(w_i) = (S_{Min}(y_i))$ is a semi-normalized basic sequence in X. Then for $(a_i) \in c_{00}$,*

$$\left\| \sum a_i w_i \right\| \leqslant \left\| \sum a_i y_i \right\| \leqslant \left(\frac{2K}{a} + K \right) \left\| \sum a_i w_i \right\|,$$

where K is the projection constant of (w_i) and $a := \inf_{i \in \mathbb{N}} \|w_i\|$.

(b) *If every normalized block sequence of (x_i) is weakly null, then (e_i^{Min}) is shrinking.*

Proof. Let $S_{Min} : Z_{Min} \to X$ be defined as in Definition 3.4.

(a) For $i \in \mathbb{N}$, write

$$y_i = \sum_{j=k_{i-1}+1}^{k_i} \beta_j^{(i)} e_j^{Min}, \quad \text{with } 0 = k_0 < k_1 < k_2 < \cdots \text{ and } \beta_j^{(i)} \in \mathbb{R}, \text{ for } i, j \in \mathbb{N},$$

and set

$$w_i = S_{Min}(y_i) = \sum_{j=k_{i-1}+1}^{k_i} \beta_j^{(i)} x_j.$$

Let $(a_i) \in c_{00}$. We use the definition of Z_{Min} to find $1 \leqslant i_1 \leqslant i_2 + 1$ and $\ell_1 \in [k_{i_1-1}+1, k_{i_1}]$ and $\ell_2 \in [k_{i_2}+1, k_{i_2+1}]$ in $\mathbb{N}$ so that, when $i_1 \leqslant i_2 - 1$,

$$\begin{aligned}
\left\| \sum a_i w_i \right\| &\leqslant \left\| \sum a_i y_i \right\| \quad \big(\text{since } \|S_{Min}\| \leqslant 1\big) \\
&= \left\| a_{i_1} \sum_{j=\ell_1}^{k_{i_1}} \beta_j^{(i_1)} x_j + \sum_{s=i_1+1}^{i_2} a_s w_s + a_{i_2+1} \sum_{j=k_{i_2}+1}^{\ell_2} \beta_j^{(i_2)} x_j \right\| \\
&\leqslant \left\| a_{i_1} \sum_{j=\ell_1}^{k_{i_1}} \beta_j^{(i_1)} x_j \right\| + \left\| \sum_{s=i_1+1}^{i_2} a_s w_s \right\| + \left\| a_{i_2+1} \sum_{j=k_{i_2}+1}^{\ell_2} \beta_j^{(i_2)} x_j \right\| \\
&\leqslant |a_{i_1}| \|y_{i_1}\| + |a_{i_2+1}| \|y_{i_2+1}\| + K \left\| \sum a_i w_i \right\| \\
&\leqslant |a_{i_1}| + |a_{i_2+1}| + K \left\| \sum a_i w_i \right\| \leqslant \left(\frac{2K}{a} + K \right) \left\| \sum a_i w_i \right\|.
\end{aligned}$$

The other two cases $i_1 = i_2$ and $i_1 = i_2 + 1$ can be obtained in similar way.

(b) Assume that (y_i) is a normalized block sequence of (e_i^{Min}). For $i \in \mathbb{N}$, we write

$$y_i = \sum_{j=k_{i-1}+1}^{k_i} a_j e_j^{Min}, \quad \text{with } 0 = k_0 < k_1 < k_2 < \cdots \text{ and } a_j \in \mathbb{R}.$$

Then, by definition of the space S_{Min}, $(S_{Min}(y_i))$ is a bounded block sequence of (x_i). It is enough to show that (y_i) has a weakly null subsequence.

If $\liminf_{i \to \infty} \|S_{Min}(y_i)\| > 0$, then our claim follows from (a). In the case that $\lim_{i \to \infty} \|S_{Min}(y_i)\| = 0$, we use the definition of Z_{Min} to find $k_0 < m_1 \leqslant n_1 \leqslant k_1 < m_2 \leqslant n_2 < \cdots$ so that for all $i \in \mathbb{N}$, $1 = \|y_i\| = \| \sum_{j=m_i}^{n_i} a_i x_i \|$. Thus, by (a), the sequences $(w_i^{(1)})$ and $(w_i^{(2)})$ with

$$w_i^{(1)} = \sum_{j=m_i}^{n_i} a_j x_j \quad \text{and} \quad w_i^{(2)} = S_{Min}(y_i) - \sum_{j=m_i}^{n_i} a_j x_j = \sum_{j=k_{i-1}+1}^{k_i} a_j x_j - \sum_{j=m_i}^{n_i} a_j x_j \quad \text{for } i \in \mathbb{N},$$

both can, after passing to a further subsequence, be assumed to be semi-normalized and, by hypothesis, are weakly null, which implies that we can, after passing to a subsequence again, also assume that they are basic. Claim (a) implies that the sequences $(y_i^{(1)})$ and $(y_i^{(2)})$ with

$$y_i^{(1)} = \sum_{j=m_i}^{n_i} a_j e_j^{Min} \quad \text{and} \quad y_i^{(2)} = \sum_{j=k_{i-1}}^{k_i} a_j e_j^{Min} - \sum_{j=m_i}^{n_i} a_j e_j^{Min} \quad \text{for } i \in \mathbb{N},$$

are weakly null in Z_{Min}, which implies that (y_i) is weakly null. □

Proof of Proposition 4.5. (a) It can be directly obtained by Lemma 4.6(b).

(b) Denote by (e_i^*) the coordinate functionals of (e_i^{Max}). Since (x_i, f_i) is boundedly complete Proposition 3.7(c) yields that Z_{Max} is an associated space. Let $T_{Max} : X \to Z_{Max}$ be the associated decomposition operator, and recall that by Proposition 3.3, $T_{Max}^*(e_i^*) = f_i$, for $i \in \mathbb{N}$. Then for any $(a_i) \in c_{00}$,

$$\max_{m \leqslant n} \left\| \sum_{i=m}^{n} a_i f_i \right\| = \max_{m \leqslant n} \left\| \sum_{i=m}^{n} a_i T_{Max}^*\big(e_i^*\big) \right\| \leqslant \left\| T_{Max}^* \right\| \max_{m \leqslant n} \left\| \sum_{i=m}^{n} a_i e_i^* \right\| \leqslant K \left\| T_{Max}^* \right\| \left\| \sum a_i e_i^* \right\|, \tag{16}$$

where K is the projection constant of (e_i^*). Moreover,

$$\begin{aligned}
\left\| \sum a_i e_i^* \right\| &= \sup_{\substack{(b_i) \in c_{00} \\ \| \sum b_i e_i^{Max} \| \leqslant 1}} \left| \sum a_i b_i \right| \\
&\leqslant \sup_{\substack{(b_i) \in c_{00} \\ \| \sum b_i e_i^{Max} \| \leqslant 1}} \; \sup_{\substack{(c_i) \in c_{00} \\ \max_{m \leqslant n} \| \sum_{i=m}^n c_i f_i \| \leqslant \max_{m \leqslant n} \| \sum_{i=m}^n a_i f_i \|}} \left| \sum c_i b_i \right| \\
&= \max_{m \leqslant n} \left\| \sum_{i=m}^{n} a_i f_i \right\| \sup_{\substack{(b_i) \in c_{00} \\ \| \sum b_i e_i^{Max} \| \leqslant 1}} \; \sup_{\substack{(c_i) \in c_{00} \\ \max_{m \leqslant n} \| \sum_{i=m}^n c_i f_i \| \leqslant 1}} \left| \sum c_i b_i \right| \\
&= \max_{m \leqslant n} \left\| \sum_{i=m}^{n} a_i f_i \right\| \sup_{\substack{(b_i) \in c_{00} \\ \| \sum b_i e_i^{Max} \| \leqslant 1}} \left\| \sum b_i e_i^{Max} \right\| \leqslant \max_{m \leqslant n} \left\| \sum_{i=m}^{n} a_i f_i \right\|.
\end{aligned}$$

Thus, (e_i^*) is equivalent to the minimal basis with respect to $(f_i) \subset X^*$. By Proposition 2.8 (f_i, x_i) is a frame for $\overline{\operatorname{span}(f_i\colon i \in \mathbb{N})}$. (e_i^{Min}) with respect to (f_i) in X^* constructed in Lemma 4.6. Since by assumption $\|x^{**}|_{\operatorname{span}(f_i\colon i \geqslant n)}\| \to 0$, if $n \to \infty$, every normalized block sequence of (f_i) is weakly null. Therefore Lemma 4.6(b) yields that (e_i^*) is shrinking. Thus, (e_i^{Max}) is boundedly complete, which proves our claim. □

We are now ready to present a proof of our main theorems:

Proof of Theorem 4.1. (a) ⇒ (b) It is clear that (a) implies (b)(i), while (b)(ii) follows from (a) and the fact that the frame representation (1) implies that every sequence $(u_n) \subset B_X$ for which $\lim_{n\to\infty} f_i(u_n) = 0$, whenever $i \in \mathbb{N}$, has a subsequence which is an arbitrary small perturbation of a block sequence of (x_i) in B_X.

(b) ⇒ (c) By Proposition 2.3(c), every $f \in X^*$ can be written as

$$f = w^* - \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) f_i.$$

If for some f, this sum did not converge in norm, we could find a sequence $(u_k) \subset B_X$ and $m_1 \leqslant n_1 < m_2 \leqslant n_2 < \cdots$ in $\mathbb{N}$ and $\varepsilon > 0$ so that for all $k \in \mathbb{N}$,

$$f\left(\sum_{i=m_k}^{n_k} f_i(u_k) x_i \right) = \sum_{i=m_k}^{n_k} f(x_i) f_i(u_k) \geqslant \frac{1}{2} \left\| \sum_{i=m_k}^{n_k} f(x_i) f_i \right\| \geqslant \frac{\varepsilon}{2}. \tag{17}$$

By Proposition 2.3(b), $(\tilde{u}_k) \subset K \cdot B_X$, where $\tilde{u}_k = \sum_{i=m_k}^{n_k} x_i f_i(u_k)$, for $k \in \mathbb{N}$. Thus, $\tilde{u}_k$ is a bounded block sequence of (x_i), which contradicts (b)(ii).

(c) ⇒ (d) trivial.

(d) ⇒ (a) by Proposition 2.7. Thus we verified (a) ⇔ (b) ⇔ (c) ⇔ (d).

(a) ⇔ (e) by Proposition 4.5(a).

(e) ⇔ (c) by Proposition 4.4(a). □

Proof. Proof of Theorem 4.2 (a) ⇒ (b) by Proposition 4.3.

(b) ⇒ (c) trivial.

(c) ⇒ (d) Let $Y = \overline{\operatorname{span}(f_i\colon i \in \mathbb{N})}$. Define $J : X \to Y^*$ by $J(x) : f \mapsto f(x)$, which is the natural canonical map. Then we have

$$\|J(x)\| = \sup_{f\in B_Y} |J(x)f| = \sup_{f\in B_Y} |f(x)| \leqslant \|x\|,$$

which implies that J is a bounded linear operator. Next we will show that J is bijective. Since, by Proposition 2.4, Y is a norming set of X, J is injective. On the other hand, any $y^* \in Y^*$, can, by the Hahn–Banach theorem, be extended it to an element $x^{**} \in X^{**}$. Then by hypothesis, there is an $x \in X$ such that $x = \lim_{n\to\infty} \sum_{i=1}^n x^{**}(f_i)x_i$ under the topology $\sigma(X, Y)$. Thus, for any $f \in Y$,

$$J(x)(f) = f(x) = \lim_{n\to\infty} f\left(\sum_{i=1}^n x^{**}(f_i)x_i\right) = \lim_{n\to\infty} x^{**}\left(\sum_{i=1}^n f(x_i)f_i\right) = x^{**}(f), \tag{18}$$

which implies that J is surjective. Then by the Banach Open Mapping Principle, J is an isomorphism from X onto Y^*.

(d) ⇒ (a) Let $x^{**} \in X^{**}$ and put $f^* = x^{**}|_Y \in Y^*$ (i.e. $f^*(f) = x^{**}(f)$ for $f \in Y$). By assumption (d) there is an $x \in X$ so that $f(x) = f^*(f) = x^{**}(f)$ for all $f \in Y$. Thus (a) follows from Proposition 2.7.

Note we have now verified the equivalences (a) ⇔ (b) ⇔ (c) ⇔ (d).

(a) ⇒ (e) by Proposition 4.5.

(e) ⇒ (b) by Proposition 4.4(b) and Theorem 3.7(c). □

Example 4.7. The following example shows that there is a semi-normalized tight Hilbert frame for ℓ_2 satisfying (b)(ii) and (d)(ii) in Proposition 4.1 but not condition (b)(i).

Choose $c > 0$ and $(c_i) \subset (0, 1)$ so that

$$c^2 + \sum c_i^2 = 1 \quad \text{and} \quad \sum c_i = \infty. \tag{19}$$

In ℓ_2 put $x_1 = ce_1$ and for $i \in \mathbb{N}$

$$x_{2i} = \frac{1}{\sqrt{2}}e_{i+1} + \frac{c_i}{\sqrt{2}}e_1 \quad \text{and} \quad x_{2i+1} = \frac{1}{\sqrt{2}}e_{i+1} - \frac{c_i}{\sqrt{2}}e_1.$$

It follows for any $x = \sum a_i e_i \in \ell_2$ that

$$\begin{aligned}\sum_{i=1}^\infty \langle x_i, x\rangle^2 &= c^2a_1^2 + \frac{1}{2}\sum_{j=2}^\infty (a_j + c_{j-1}a_1)^2 + (a_j - c_{j-1}a_1)^2 \\ &= c^2a_1^2 + \sum_{j=2}^\infty a_j^2 + a_1^2\sum_{j=1}^\infty c_j^2 = \|x\|^2.\end{aligned}$$

Thus, (x_i) is a tight frame, which implies (b)(ii), (c)(ii) and (d)(ii).

Using the second part of (19) we can choose $0 = n_0 < n_1 < n_2 < \cdots$ so that

$$\lim_{i\to\infty} y_i = e_1, \quad \text{where } y_i = \sum_{j=n_{i-1}+1}^{n_i} (x_{2j} - x_{2j+1}) \text{ for } i \in \mathbb{N},$$

which implies that (b)(i) is not satisfied.

Proposition 4.8. *Let (x_i, f_i) be a Schauder frame of a Banach space X. Then the following conditions are equivalent*:

(a) *(x_i, f_i) is a pre-shrinking Schauder frame of X.*
(b) *(f_i, x_i) is a pre-boundedly complete Schauder frame of X^*.*
(c) *(f_i, x_i) is a pre-boundedly complete Schauder frame of $\overline{\operatorname{span}(f_i\colon i \in \mathbb{N})}$.*

Proof. (a) ⇒ (b) Assume that (f_i, x_i) is a Schauder frame of X^*. For any $x^{***} \in X^{***}$, $x^{***}|_X$ is a continuous linear functional on X. Then $\sum_{i=1}^\infty x^{***}(x_i)f_i = \sum_{i=1}^\infty x^{***}|_X(x_i)f_i$ converges in X^*, which completes the claim.

(b) ⇒ (c) is trivial.

(c) ⇒ (b) Let $Y = \overline{\operatorname{span}(f_i\colon i \in \mathbb{N})}$ and let $f \in X^*$. By Proposition 2.4 X can be isomorphically embedded into Y^* under the natural canonical map. By the Hahn–Banach theorem, extend f to an element in Y^{**} and, thus, assumption (c) yields that $\sum_{i=1}^\infty f(x_i)f_i$ converges in Y. Since this series converges in w^* to f by Proposition 2.3 this completes the proof. □

Proposition 4.9. *Let (x_i, f_i) be a Schauder frame of a Banach space X.*
If (x_i, f_i) is pre-shrinking and pre-boundedly complete, then X is reflexive.

Proof. Since (x_i, f_i) is pre-shrinking we can write every $f \in X^*$ as $f = \sum f(x_i) f_i$. Since (x_i, f_i) is pre-boundedly complete we can choose for each $x^{**} \in X^{**}$ an $x \in X$ so that $x = \sum x^{**}(f_i) x_i$. Thus for any $f \in X^*$

$$x^{**}(f) = \sum f(x_i) x^{**}(f_i) = f(x),$$

which proves our claim. □

5. Unconditional Schauder frames

The following result extends James' [12] well-known result on unconditional bases to unconditional frames.

Theorem 5.1. *Let (x_i, f_i) be an unconditional and locally shrinking Schauder frame of a Banach space X.*

(a) *If (x_i, f_i) is not pre-boundedly complete, then X contains an isomorphic copy of c_0.*
(b) *If (x_i, f_i) is not shrinking, then X contains an isomorphic copy of ℓ_1.*

Then by Proposition 4.9 and Theorem 5.1, we obtain Theorem C.
For the proof, we need the following lemma.

Lemma 5.2. *Let X be a separable Banach space and $(x_i, f_i) \subset X \times X^*$ be a locally shrinking Schauder frame of X with the projection operator K. Let Y be a finite-dimensional subspace of X. Then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|y\| \leqslant (K + \varepsilon)\|y + x\|$ whenever $x \in \operatorname{span}(x_i\colon i \geqslant N)$ and $y \in Y$.*

Proof. W.l.o.g. $\varepsilon < 1/2$. Let $(y_i)_{i=1}^n$ be an $\frac{\varepsilon}{8K^2}$-net of S_Y, and $(x_i^*)_{i=1}^n \subset S_{X^*}$ with $x_i^*(y_i) = 1$ for $1 \leqslant i \leqslant n$. For large enough k it follows that $(\tilde{x}_i^*)_{i=1}^n$, with $\tilde{x}_i^* = \sum_{j=1}^k x_i^*(x_j) f_j$, $i = 1, 2, \ldots, n$, satisfies that

$$\left\|\tilde{x}_i^*\right\| \leqslant K \quad (1 \leqslant i \leqslant n) \quad \text{and} \quad \max_{1 \leqslant i \leqslant n} \left|\tilde{x}_i^*(y)\right| \geqslant 1 - \frac{\varepsilon}{4K} \quad \text{for all } y \in S_Y.$$

It follows that $\|\tilde{x}_j\| \leqslant K$, for $j = 1, 2, \ldots, n$. Using our assumption that (x_i, f_i) is locally shrinking we can choose $N \in \mathbb{N}$, so that $\|\tilde{x}_i^*|_{\operatorname{span}(x_j\colon j \geqslant N)}\| \leqslant \frac{\varepsilon}{8K}$.

If $y \in Y$ and $x \in \operatorname{span}(x_i\colon i \geqslant N)$, then either $\|x\| \geqslant 2\|y\|$, in which case $\|y + x\| \geqslant \|x\| - \|y\| \geqslant \|y\|$. Or $\|x\| \leqslant 2\|y\|$, and then

$$K\|y + x\| \geqslant \max_{i \leqslant n} \left|\tilde{x}_i^*(y + x)\right| \geqslant \left(1 - \frac{\varepsilon}{4K}\right)\|y\| - \frac{\varepsilon}{8K}\|x\| \geqslant \left(1 - \frac{\varepsilon}{2K}\right)\|y\| \geqslant \frac{\|y\|}{1 + \varepsilon/K}. \qquad \square$$

Corollary 5.3. *Let X be a separable Banach space and $(x_i, f_i) \subset X \times X^*$ be a locally shrinking Schauder frame of X with the projection operator K. Then for every normalized block sequence (u_i) of (x_i) and every $\epsilon > 0$, there is a basic subsequence of (u_i) whose basis constant K_b is not larger than $K + \epsilon$.*

Proof. Using at each step Lemma 5.2 we can choose a subsequence basis (v_i) of (u_i), so that for all $N \in \mathbb{N}$

$$\|y + x\| \geqslant \frac{\|y\|}{K + \varepsilon} \quad \text{for all } y \in \operatorname{span}(v_i\colon i \leqslant N) \text{ and } x \in \operatorname{span}(v_i\colon i \geqslant N + 1).$$

It follows then, that (v_i) is basic and its basis constant does not exceed $K + \varepsilon$. □

Lemma 5.4. *Assume that (x_i, f_i) is an unconditional and locally shrinking frame for a Banach space X. Let K_u be the constant of unconditionality of (x_i, f_i) and let (u_i) be a block basis of (x_i). For any $\varepsilon > 0$ there is a subsequence (v_i) of (u_i) which is $K_u + \varepsilon$ unconditional.*

Proof. Without loss of generality, we assume that $\|x_n\| = 1$, for $n \in \mathbb{N}$, otherwise replace x_n by $x_n/\|x_n\|$ and f_n by $f_n\|x_n\|$. By Corollary 5.3 we can assume that (u_i) is $2K_u$-basic (note that the projection constant of (x_i, f_i) is at most K_u).

Let $(\delta_i) \subset (0, 1)$ with $\sum_{j>i} \delta_j < \delta_i$, $i \in \mathbb{N}$, and $\sum \delta_i < \varepsilon/8K_u^2$. Then we choose recursively increasing sequences (n_i) and (k_i) in $\mathbb{N}$ so that

$$\left|f_s(u_{n_i})\right| < \frac{\delta_i}{k_{i-1}} \quad \text{whenever } s \leqslant k_{i-1}, \tag{20}$$

and

$$\left\| \sum_{s=k_i}^{N} f_s\left(\sum_{j=1}^{i} \lambda_j u_{n_j} \right) x_s \right\| < \delta_{i+1} \quad \text{whenever } N \geqslant k_i \text{ and } (\lambda_j)_{j=1}^{i} \subset [-1,1]. \tag{21}$$

Indeed, assume k_{i-1} was chosen ($k_0 = 1$). Since (x_i, f_i) is locally shrinking, we can choose n_i so that (20) is satisfied. Secondly, using the compactness of the set $\{\sum_{j=1}^{i} \lambda_j u_{n_j}\colon (\lambda_j)_{j=1}^{i} \subset [-1,1]\}$, we can choose k_i so that (21) is satisfied.

We are given now $(\lambda_i) \subset c_{00}$ with $\max|\lambda_i| = 1$ and $(\varepsilon_i) \subset \{-1,1\}$. For $u = \sum \lambda_i u_{n_i}$ and $\bar{u} = \sum \varepsilon_i \lambda_i u_{n_i}$ we compute:

$$\begin{aligned}
\|\bar{u}\| &= \left\| \sum_{s=1}^{\infty} f_s(\bar{u}) x_s \right\| \\
&= \left\| \sum_{i=1}^{\infty} \sum_{s=k_{i-1}}^{k_i-1} f_s(\bar{u}) x_s \right\| \\
&\leqslant K_u \left\| \sum_{i=1}^{\infty} \varepsilon_i \sum_{s=k_{i-1}}^{k_i-1} f_s(\bar{u}) x_s \right\| \\
&\leqslant K_u \left\| \sum_{i=1}^{\infty} \sum_{s=k_{i-1}}^{k_i-1} \lambda_i f_s(u_{n_i}) x_s \right\| + K_u \sum_{i=1}^{\infty} \left\| \sum_{s=k_{i-1}}^{k_i-1} f_s\left(\sum_{j=1}^{i-1} \varepsilon_j \lambda_j u_{n_j} \right) x_s \right\| + K_u \sum_{i=1}^{\infty} \sum_{s=k_{i-1}}^{k_i-1} \sum_{j=i+1}^{\infty} \left| f_s(u_{n_j}) \right| \\
&\leqslant K_u \left\| \sum_{i=1}^{\infty} \sum_{s=k_{i-1}}^{k_i-1} \lambda_i f_s(u_{n_i}) x_s \right\| + \frac{\varepsilon}{8K_u} + K_u \sum_{i=1}^{\infty} k_i \sum_{j=i+1}^{\infty} \frac{\delta_j}{k_i} \\
&\leqslant K_u \left\| \sum_{i=1}^{\infty} \sum_{s=k_{i-1}}^{k_i-1} \lambda_i f_s(u_{n_i}) x_s \right\| + \frac{\varepsilon}{4K_u}.
\end{aligned}$$

By switching the role of u and $\bar{u}$, we compute also

$$\left\| \sum_{i=1}^{\infty} \sum_{s=k_{i-1}}^{k_i-1} \lambda_i f_s(u_{n_i}) x_s \right\| \leqslant \left\| \sum_{i=1}^{\infty} \sum_{s=k_{i-1}}^{k_i-1} f_s(u) x_s \right\| + \frac{\varepsilon}{4K_u} = \|u\| + \frac{\varepsilon}{4K_u}.$$

Since the basis constant of (u_i) does not exceed $2K_u$ it follows that $\|\bar{u}\|, \|u\| \geqslant \frac{1}{2K_u}$ and thus

$$\|\bar{u}\| \leqslant \|u\| + \frac{\varepsilon}{2K_u} \leqslant (K_u + \varepsilon)\|u\|,$$

which proves our claim. □

Proof of Theorem 5.1. (a) By assumption, there is some $x_0^{**} \in S_{X^{**}}$ such that $\sum_{i=1}^{n} x_0^{**}(f_i) x_i$ does not converge. By the Cauchy criterion, there are $\delta > 0$ and natural numbers $p_1 < q_1 < p_2 < q_2 < \cdots$ such that for $u_j = \sum_{i=p_j}^{q_j} x_0^{**}(f_i) x_i$ we have $\|u_j\| \geqslant \delta$ for every j. By Corollary 5.3, we can find a basic subsequence (u_{n_j}) of (u_j) with the basis constant $C > 1$. Then for every sequence $(\lambda_j)_{j=1}^{m}$ of scalars and every $i \in \{1, \ldots, m\}$, we have $\|\sum_{j=1}^{m} \lambda_j u_{n_j}\| \geqslant \frac{1}{2C} \|\lambda_i u_{n_i}\| \geqslant \frac{\delta}{2C} |\lambda_i|$. That is, $\|\sum_{j=1}^{m} \lambda_j u_{n_j}\| \geqslant \frac{\delta}{2C} \|(\lambda_j)\|_\infty$.

Recall that the unconditional constant of (x_i, f_i) is defined by

$$K_u = \sup_{x \in B_X} \sup_{(\varepsilon_i) \subset \{\pm 1\}} \left\| \sum_{i=1}^{\infty} \varepsilon_i f_i(x) x_i \right\| = \sup_{x \in B_X} \sup_{(\lambda_i) \subset [-1,1]} \left\| \sum_{i=1}^{\infty} \lambda_i f_i(x) x_i \right\| < \infty$$

(the second "=" follows from a simple convexity argument). Secondly we compute

$$\begin{aligned}
\sup_{(\lambda_i) \in c_{00} \cap [-1,1]^{\mathbb{N}}} \left\| \sum \lambda_i u_i \right\| &= \sup_{(\lambda_i) \in c_{00} \cap [-1,1]^{\mathbb{N}}} \left\| \sum_i \sum_{s=p_i}^{q_i} \lambda_i x_0^{**}(f_i) \right\| \\
&\leqslant \sup_{x^{**} \in B_{X^{**}}} \sup_{(\lambda_s) \in c_{00} \cap [-1,1]^{\mathbb{N}}} \left\| \sum_s \lambda_s x^{**}(f_i) x_i \right\| \\
&= \sup_{x \in B_X} \sup_{(\lambda_s) \in c_{00} \cap [-1,1]^{\mathbb{N}}} \left\| \sum \lambda_s f_s(x) x_i \right\| = K_u.
\end{aligned}$$

(b) Since (x_i, f_i) is not shrinking, there exists $f \in S_{X^*}$ and a normalized block basis (u_n) of (x_n) and a $\delta > 0$, so that $f(u_n) \geqslant \delta$, for $n \in \mathbb{N}$. Since by Lemma 5.4 we can assume that (u_n) is $2K_u$-unconditional, it follows $(\lambda_i) \in c_{00}$ that

$$\Big\| \sum \lambda_i u_i \Big\| \geqslant \frac{1}{2K_u} \Big\| \sum |\lambda_i| u_i \Big\| \geqslant f\Big(\sum |\lambda_i| u_i \Big) \geqslant \frac{\delta}{K_u} \sum |\lambda_i|. \qquad \square$$

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