Hardy's Inequality and Fractal Measures

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Hardy's inequality and the subsequent improvement by McGehee, Pigno, and Smith are generalized from the positive integers to sets of dimension 0, dimension 1, and in between. The asymptotic estimate obtained for the Fourier transform of fractal measures is much in the spirit of recent work by Strichartz. © 1992 Academic Press, Inc.

1. INTRODUCTION

An interesting problem in Fourier analysis is to extend the classical inequalities of the Fourier transform, or what Hardy and Littlewood refer to as the theory of Fourier constants [5], to tempered distributions that correspond to lower-dimensional sets. Particularly important theorems are the $L^1$ inequality known as Hardy's inequality with the McGehee-Pigno-Smith (henceforth M.P.S.) generalization [7], the Plancherel theorem for $L^2$, and Payley's theorem [12] with the Pitt-Stein [8] generalizations for $L^p$, $1 < p < 2$. Extensions of the Plancherel theorem for measures supported on manifolds in $\mathbb{R}^n$ have been established by Agmon and Hormander [11] and more recently by Strichartz [9] for measures on $\mathbb{R}^n$ of dimension $0 \leq \alpha \leq n$, $\alpha$ not necessarily an integer. This paper proves a generalized Hardy inequality (henceforth g.h.i.) for fractal measures on $\mathbb{R}$ of dimension $0 \leq \alpha \leq 1$. This result includes the M.P.S. version as the periodic case for $\alpha = 0$. Each of the results above for $\alpha < n$ involves a limit on the Fourier transform side and provides information in the form of an asymptotic growth estimate for the transform.

Some regularity will be required of the support of the fractal measure. Classically, Hardy's inequality and the M.P.S. version hold only for measures supported on a well-ordered set of integers, which means the transform of the measure is in $H^1$ of the unit circle, at least up to a multiplicative factor of $e^{inx}$. The well-known inequalities above, in which $p > 1$, are rearrangement-invariant, while Hardy's inequality is not. This implies
that the nature of the support of the measure when $\alpha = 0$ or $\alpha = 1$ is far more important in Hardy's inequality than in the others. Likewise, when $0 < \alpha < 1$, it is natural to expect the support of the measure to play a greater role in g.h.i. than in the other inequalities. This point may be clarified by the last result of the paper, an extension of Paley's theorem for 0-dimensional measures; this is a $p > 1$ analogue of g.h.i. in which the support is quite arbitrary.

The term fractal measure here means a measure $\nu$ supported on a set $E \subset \mathbb{R}^1$, that is $\mu_\alpha$-measurable, where $d\mu_\alpha$ is a $\alpha$-dimensional Hausdorff measure and $0 \leq \alpha \leq 1$. Certain classes of such measures will be studied, including measures supported on self-similar sets such as the Cantor set.

It is assumed throughout that $\nu$ is finite, so it is a tempered distribution with a Fourier transform locally in $L^1(\mathbb{R})$. It is also assumed that $\nu$ is either positive, or is of the form $f d\mu_\alpha$. In the latter case, let $E = \{x : f(x) \neq 0\}$, which we will refer to as supp $f$. Define

$$\sigma_\alpha(E, x) = \mu_\alpha(E \cap (-\infty, x]) .$$  \hspace{1cm} (1.1)

Consider the following generalized Hardy inequality (g.h.i.) for $0 \leq \alpha \leq 1$,

$$\int_{-\infty}^{\infty} \frac{|f(x)|}{\sigma_\alpha(E, x)} d\mu_\alpha(x) \leq C \liminf_{L \to \infty} L^{1-\alpha} \int_{-L}^{L} |f d\mu_\alpha(x)| dx, \hspace{1cm} (1.2)$$

where $C$ is a constant that may depend on $E$ but not $f$. This inequality does not hold for general fractal measures. The collective statement of Theorems 1 and 3 is that g.h.i. holds whenever $E$ is $\alpha$-coherent; see Definition 1 below. Theorem 3 also holds for quasi-regular sets; see Definition 2.

Before defining coherence, certain problems with sets of measure zero must be dealt with. For $x \in \mathbb{R}$ and $\delta > 0$, let $I_\delta(x)$ be the open interval $(x - \delta, x + \delta)$ and let $I_\delta = I_\delta(0)$. Suppose that $E \subset \mathbb{R}$ is $\mu_\alpha$-measurable, with $0 < \mu_\alpha(E) < \infty$. The upper density of $E$ at $x$ is defined by

$$\bar{D}_\alpha(E, x) = \limsup_{r \to 0} \frac{\mu_\alpha(E \cap I_r(x))}{(2r)^\alpha}. \hspace{1cm} (1.3)$$

Then $\bar{D}_\alpha(E, x) = 0$ for $\mu_\alpha$-a.e. $x \notin E$. And for $\mu_\alpha$-a.e. $x \in E$, one has that $2^\alpha \leq \bar{D}_\alpha(E, x) \leq 1$ (see Falconer [3, pp. 22-25]). So, $E$ agrees $\mu_\alpha$-a.e. with its "Lebesgue set"

$$E^* = \{x \in E : 2^\alpha \leq \bar{D}_\alpha(E, x) \leq 1\}.$$

It is not really necessary that $E$ have finite measure. Given $x \in \mathbb{R}$, let $E_x = E \cap (-\infty, x]$. It will always be assumed that $\mu_\alpha(E_x) < \infty$ for some $x$, for otherwise g.h.i. is trivial. Let $s = \sup\{x : \mu_\alpha(E_x) < \infty\}$. Notice $E_s$
is $\sigma$-finite with respect to $\mu_x$, so the results above still apply; $E_x$ agrees $\mu_x$-a.e. with $(E_x)^*$. Let $E^0 = (E_x)^*$ and $E^0_x = E^0 \cap (-\infty, x]$. Given sets $A$ and $B$, let $A + B = \{a + b: a \in A, b \in B\}$.

**Definition 1.** $E \subset \mathbb{R}$ is coherent if there is a constant $C$ such that for all $x \leq s$,

$$\limsup_{s \to 0} |E^0_x + I_\delta| \delta^{2-1} \leq C \cdot \mu_x(E^0_x). \tag{1.4}$$

This definition depends on the value of $\alpha$, which will normally be understood. If there is any risk of confusion we will call the set $\alpha$-coherent. The inequality in the definition can always be reversed (if $C = 1$) by the definition of Hausdorff measure. The right-hand side is equal to $C \cdot \mu_x(E_x)$ and to $C \cdot \sigma_x(E, x)$. It is necessary to use $E^0_x$ rather than $E_x$ because sets of measure zero could greatly affect the left-hand side.

Theorem 4 shows that the Cantor set and its variations for $0 < \alpha < 1$ are coherent if the construction is not too wild, that is, if its dissection numbers are bounded. Theorem 5 constructs a Cantor-like set for which g.h.i. fails. If $\alpha = 0$, any well-ordered set is coherent, with $C = 1$. If $\alpha = 1$, any compact set is coherent, also with $C = 1$. However, in this case coherence is not necessary; see Theorem 2.

The results in this paper appear with the Fourier transform on the right-hand side, though it is more usual to have it on the left. It makes little difference when $\alpha = 0$, at least in the periodic case, or when $\alpha = 1$, but for dimensions in between it matters, because Fourier inversion is not clear. Also, in the case $\alpha = 0$, it matters for almost-periodic functions. Each of these functions defines a unique Fourier series, but that series does not converge to a unique function in the $B^\alpha.a.p.$ pseudonorm [2].

The fundamental case is $\alpha = 0$. The M.P.S. result is the important subcase in which the Fourier transform of the zero-dimensional measure is periodic. The immediate corollary is a proof of the celebrated Littlewood conjecture for trigonometric polynomials. In the same way, an immediate corollary of g.h.i. is an $\alpha$-dimensional version of Littlewood's conjecture.

The right-hand side of (1.2) is a natural substitute for the $L^1$ norm of the Fourier transform of an $\alpha$-dimensional measure. It resembles terms studied in [2, 9, 10], for example. However, it is usually impossible to compute exactly, and difficult even to determine whether it is finite. For a simple application of g.h.i., let $f = \chi_E$, where $E$ is an $\alpha$-coherent or quasi-regular set; for example, a Cantor set contained in the unit interval of dimension $\alpha$. Then (1.2) shows that

$$\liminf_{L \to \infty} \int_{-L}^{L} |\chi_E d\mu_\alpha| \, dx = +\infty. \tag{1.5}$$
This result is nontrivial; the lim inf can converge to 0 if $E$ is not coherent; see (4.3). However, it may not be best-possible in the sense that a smaller exponent of $L$ on the left-hand side might produce the same result. The question of sharpness seems to be more complicated in this context than in the $L^2$ setting (see [9]). This is discussed further in Section 4.

The results of this paper are organized in the following manner. Section 2 is devoted to establishing g.h.i. for the integer dimensions $\alpha = 0$ and $\alpha = 1$, which are Theorems 1 and 2, respectively. It should be noted that lim inf can be replaced by lim in these dimensions. Section 3 discusses g.h.i. for the more difficult cases $0 < \alpha < 1$.

In Section 4, several results clarify the role of coherence. First, self-similar sets such as Cantor sets are proved to be coherent. Theorem 5 constructs a set for which g.h.i. fails. This set resembles a Cantor set, but it is neither coherent nor quasi-regular.

Finally, Theorem 6, an extension of Paley's Theorem to the class of $B^p.a.p.$ almost-periodic functions (where $1 < p \leq 2$ and $\alpha = 0$ again) shows that coherence is unnecessary when $p > 1$. The dominated side of this inequality is rearrangement-invariant; the natural order of the real numbers plays no essential role. This theorem is similar in appearance to Theorem 1 but applies, for example, to a zero-dimensional measure supported on the rationals whose coefficients are absolutely summable. The rationals make up a fairly acceptable set in most regards, but they are not well-ordered, and not coherent. So, this theorem suggests that g.h.i. is more delicate than the other inequalities that may be extended, in the sense that the support of the measure must be more restricted in g.h.i. than the others.

The extensions of Plancherel's theorem by Strichartz [9] involve smoothing out the distribution, applying the classical Plancherel theorem and approximation arguments. Approximation arguments are used in Theorems 1 and 6 of this paper to handle the 0-dimensional case. The proofs for the general cases, Theorems 2 and 3, are from the ground up in the sense that the M.P.S. machinery is modified for this setting while the M.P.S. result is not used directly.

2. INTEGRAL DIMENSIONS

The first theorem uses the class $B.a.p.$ of almost-periodic functions discussed in Besicovitch [2]. These are the almost-periodic functions $u$ for which the pseudonorm $\limsup_{L \to \infty} L^{-1} \int_{-L}^{L} |u| \, dx$ is finite. If $u$ is almost-periodic, then the limit of the right-hand side exists, so lim sup may be replaced by lim. Every trigonometric polynomial is almost-periodic and is in $B.a.p.$ The Fourier series of a $B.a.p.$ function $u$ converges to $u$ in the
pseudonorm above, but may also converge to other B.a.p. functions—the series does not determine \( u \).

**Theorem 1.** Let \( f d\mu_0 \) be a zero-dimensional measure defined by \( f(x) = \sum_{k=-\infty}^{\infty} c_k \delta(x-a_k) \) where \( a_1 < a_2 < \cdots \) and \( \delta \) is the usual Dirac measure at zero. Assume \( \overline{f d\mu_0} = \sum c_k e^{iakx} \) belongs to B.a.p. Then,

\[
\sum_{k=1}^{\infty} \frac{|c_k|}{k} \leq C \lim_{L \to \infty} L^{-1} \int_{-L}^{L} |\overline{f d\mu_0}(x)| \, dx.
\] (2.1)

**Proof.** First assume that \( \{a_k\} \) is a finite sequence with \( N \) terms so that \( \overline{f d\mu_0} \) is a polynomial. Let \( \varepsilon > 0 \). By a lemma of Dirichlet (Zygmund [12, p. 235]), there are infinitely many integers \( L_j \) with numbers \( \{a_k\} \in \mathbb{Z}/L_j \) such that \( |a_k - a'_k| < \varepsilon/L_j \) for all \( 1 \leq k \leq N \).

Let \( u_j(x) = \sum c_k e^{iakx} \). Then for \( x \in [-L_j, L_j] \),

\[
|\overline{f d\mu_0}(x) - u_j(x)| \leq C \sum_k |c_k| |a_k - a_k'| |x| \leq Ce \sum_k |c_k|.
\]

Since \( u_j \) is periodic we may apply M.P.S.

\[
\sum_{k=1}^{\infty} \frac{|c_k|}{k} \leq C L_j^{-1} \int_{-L_j}^{L_j} |u_j(x)| \, dx
\]

\[
\leq C L_j^{-1} \int_{-L_j}^{L_j} \overline{|f d\mu_0(x)|} \, dx + Ce \sum |c_k|.
\]

Taking limits as \( j \to \infty \) and then as \( \varepsilon \to 0 \) proves (2.1) in this case.

For the general case, we will approximate using Bochner–Fejer polynomials [2]. Given \( u = \overline{f d\mu_0} \in \text{B.a.p.} \), there exists a sequence of polynomials \( \{\sigma_n\} \) of the form

\[
\sigma_n(x) = \sum_{k=-N(n)}^{N(n)} c_k^{(n)} e^{iakx}
\]

(where the frequencies \( a_k \) are the same as those of \( \overline{f d\mu_0} \)) such that

\[
\|u - \sigma_n\|_{\text{B.a.p.}} = \limsup_{L \to \infty} L^{-1} \int_{-L}^{L} |u - \sigma_n| \, dx \leq 2^{-n}
\] (2.2)

and

\[
\lim_{n \to \infty} c_k^{(n)} = c_k \quad \text{for each } k.
\] (2.3)
We have proven the theorem for such polynomials. So using Fatou's lemma for sums, and the fact that limits exist for B.a.p. functions,

\[
\sum_{k=1}^{\infty} \frac{|c_k|}{k} \leq \lim_{n \to \infty} \sum_{k=1}^{N(n)} \frac{|c_k^{(n)}|}{k} \\
\leq C \lim_{n \to \infty} \lim_{L \to \infty} L^{-1} \int_{-L}^{L} (|u| + |u - \sigma_n|) \\
= C \lim_{L \to \infty} L^{-1} \int_{-L}^{L} |u| \, dx.
\]

This proves the theorem.

Given \( x \in R, 0 < a < 1 \), and a set \( E \subset R \), let

\[
\sigma_{a}(E, x) = \mu_a(E \cap (-\infty, x]),
\]

where \( \mu_a \) is Hausdorff measure. In the last theorem, the index \( k \) could be written as \( \sigma_0(\{a_k\}, x) \). The next theorem provides a one-dimensional analog. This result should be compared to those of \([6, 11]\) for \( p = 1 \).

**Theorem 2.** There is an absolute constant \( C \) such that if \( \hat{u} \in L^1(\mathbb{R}) \)

\[
\int_{\mathbb{R}} \frac{|u(x)| \, dx}{\sigma_1(E, x)} \leq C \|\hat{u}\|_{L^1},
\]

where \( E = \text{supp } u \).

**Proof.** We claim that it is enough to prove (2.4) with \( \sigma_1(E, x) \) replaced by \( \sigma_1(E, x) + 1 \). To prove this claim, assume (2.4) with \( \sigma_1(x) + 1 \) in the denominator. Given \( \hat{u} \in L^1 \) and \( \gamma > 0 \), let \( \hat{u}(x) = \gamma \hat{u}(\gamma x) \). So, \( v(x) = u(x/\gamma) \) and \( \sigma_1(\text{supp } v, \gamma x) = \gamma \sigma_1(E, x) \). By changing variables and applying (2.4) to \( \hat{u} \), we get

\[
\int_{\mathbb{R}} \frac{|u(x)| \, dx}{\gamma^{-1} + \sigma_1(E, x)} \leq \int_{\mathbb{R}} \frac{|v(x)| \, dx}{1 + \sigma_1(\text{supp } v, x)} \leq C \|\hat{v}\|_{L^1},
\]

and let \( \gamma \to \infty \) to get (2.4) for \( \hat{u} \) without the +1 in the denominator.

Now the idea of the proof is the same as in M.P.S. We will construct functions \( F_m \) on \( \mathbb{R} \) such that

1. \( \hat{F}_m \) is supported in \( (-\infty, N(m)] \) where \( N(m) \to \infty \) as \( m \to \infty \)
2. \( \|F_m\|_{\infty} \leq 1 \)
3. \( 30 \cdot \text{Re} \hat{F}_m(x) u(x) \geq |u(x)| / [1 + \sigma_1(E, x)] \) for all \( x \in E \cap (-\infty, N(m)] \).
Given such $F_m$, the theorem follows easily; if $\text{supp } u \subset (-\infty, N(m)]$ for some $m$, then

$$\int \frac{|u(x)|}{1 + \sigma_1(E, x)} \, dx \leq C \cdot \text{Re} \int u(x) \hat{F}_m(x) \, dx$$

$$= C \cdot \text{Re} \int \hat{u} \ast F_m(x) \, dx$$

$$= C \cdot \text{Re}[\hat{u} \ast F_m(0)]$$

$$\leq C \|\hat{u}\|_{L^1} \|F_m\|_{L^\infty}$$

$$\leq C \|\hat{u}\|_{L^1}.$$

An approximation argument shows that this inequality holds for all $\hat{u} \in L^1(\mathbb{R})$.

The construction of $F_m$ follows M.P.S., but here $F_m$ is a function on $\mathbb{R}$ instead of a sequence. For the sake of completeness, an outline of the construction follows.

Let $S_0 = \{x \in E: 0 \leq \sigma_1(x) \leq 1\}$. Let $S_1 = \{x \in E: 1 < \sigma_1(x) \leq 5\}$. Define $\{S_j\}$ for $j > 1$ in this manner so that $|S_j| = 4^j$ and $E = \bigcup S_j$. Let $N(m) = \sup S_m$.

Define $f_j \in L^2(\mathbb{R})$ by

$$f_j(x) = 0 \quad \text{for } x \notin S_j$$

$$|f_j(x)| = 4^{-j} \quad \text{for } x \in S_j$$

$$f_j(x) u(x) \geq 0 \quad \text{for all } x.$$

Let

$$h_j = \frac{1}{4} (I + iH) |f_j| \in L^2,$$

where $H$ is the Hilbert transform. Notice that $\text{Re } h_j = |f_j|/4$ and $\|h_j\|_2 \leq (3/8) \|f_j\|_2 = 3 \cdot 2^{-j-3}$. Also, $\text{supp } h_j \subset (-\infty, 0]$.

Let $F_{-1}(x) = 0$ and for $m \geq 0$ let

$$F_m(x) = F_{m-1}(x) \cdot \exp(-h_m(x)) + \frac{f_m(x)}{5}.$$

This is a continuous function in $L^2(\mathbb{R})$. It is supported on the union of the supports of the $f_j$ for $0 \leq j \leq m$ so condition (1) holds. Because $\exp(-x) + x/5 \leq 1$ whenever $0 \leq x \leq 1$ and since $\|f_m\|_\infty \leq \|f_m\|_1 = 1$, induction proves condition (2);

$$|F_m(x)| \leq 1 \cdot \exp(-|f_m(x)|) + \frac{|f_m(x)|}{5} \leq 1.$$
Claim. For \( j \leq m < \infty \) and for all \( x \in S_j \),

\[
|\hat{F}_m(x) - \hat{f}_j(x)/5| \leq \frac{1}{10}|\hat{f}_j(x)|.
\]

(2.5)

This inequality is proved in M.P.S. in a slightly different context. Since no real modifications are needed, we omit the proof here.

Now for \( x \in S_j \), we have \( 1 + \sigma_1(x) > 4j/3 \). So, (2.5) shows that

\[
\Re \hat{F}_m u(x) - \hat{f}_j u(x)/5 \geq \hat{f}_j u(x)/10
\]

and so

\[
\Re \hat{F}_m u(x) \geq \hat{f}_j u(x)/10 \geq \frac{|u(x)|}{30(1 + \sigma_1(x))}.
\]

This holds for all \( x \in \bigcup_{j=0}^\infty S_j \) which is \( E \cap (-\infty, N(m)] \). This proves condition (3) on \( F_m \) and completes the proof of Theorem 2.

3. Fractional Dimensions

The next theorem generalizes the previous ones to dimensions between 0 and 1. It requires that the measure is supported on a coherent set. Coherence can be replaced by quasi-regularity, but the theorem fails unless some kind of regularity is imposed; see Corollary 2 and Theorem 5. Also, the proof shows that the measure needn't be of the form \( f \, d\mu_\alpha \) if it is positive; see Corollary 1. The exponent \( \alpha - 1 \) is clearly sharp only when \( \alpha \) is an integer. This is briefly discussed in Section 4.

**Theorem 3.** Suppose \( 0 < \alpha < 1 \), \( f \in L^1(\mu_\alpha) \) is supported on \( E \), and \( E \) is \( \alpha \)-coherent. Then there is a constant \( C \) independent of \( f \) such that

\[
\int \frac{|f(x)| \, d\mu_\alpha(x)}{\sigma_\alpha(E, x)} \leq C \liminf_{L \to \infty} L^{\alpha - 1} \int_{-L}^L |f \, d\mu_\alpha| \, dx.
\]

(3.1)

**Proof of Theorem 3.** The idea of the proof is to construct an auxiliary function \( F \) as in Theorem 2. This seems impossible to do on the given fractal set \( E \). So instead, the given measure is approximated using convolution with a Schwartz function \( \phi_L \). Then a sequence \( \hat{F} \) is constructed for the new smoothed-out measure, \( \phi_L \ast f \, d\mu_\alpha \), on a dense dilation of the integers. After this modified M.P.S. construction, we take the lim inf as \( L \to \infty \). Most of the work occurs at this stage, in Lemmas 4 and 5.
HARDY'S INEQUALITY

Let \( \phi \) be an even Schwartz function such that
\[
\int \phi \, dx = 1
\]
and
\[
\text{supp } \phi \subseteq [-1, 1].
\]

We can not arrange that \( \phi \) have compact support, but the following lemma is a substitute.

**Lemma 1.** There are constants \( C \) and \( K_0 \) such that for all \( x \), and for all \( K > K_0 \),
\[
\sum_{|n + x| > K} |\phi(n + x)| < \frac{C}{K}.
\]

**Proof.** Since the sum is over a set of integers, we may assume \( 0 < x < 1 \). Also, the condition \( |n + x| > K \) can be replaced by \( n + x > K \). Since \( \phi \) is a Schwartz function, \( \phi(x) = O(x^{-2}) \). So, there is a \( C \) and \( K_0 \) such that \( \int_0^\infty |\phi(x)| \, dx < C/K \) for \( K > K_0 \). We may assume the same inequality holds for the Schwartz function \( \phi' \). Then for any \( x \in [0, 1) \),
\[
\int_1^{n+1} |\phi(t) - \phi(n + x)| \, dt \leq \sup_{n \leq t < n + 1} |\phi(t) - \phi(n + x)| \leq \int_n^{n+1} |\phi'(t)| \, dt.
\]
So, by the triangle inequality,
\[
\sum_{n + x > K} |\phi(n + x)| \leq \sum_{n > K} \int_n^{n+1} |\phi(t)| + |\phi'| \, dt \leq \frac{2C}{K}
\]
which proves the lemma.

Since \( E \) is coherent, it is bounded below. Let \( m = \inf E \). Fix \( \varepsilon > 0 \). It is easy to construct a Cantor set with \( \mu_\alpha \) measure 1. Such a set is coherent by Theorem 4. So, by dilation and translation, there is an \( \alpha \)-coherent set \( C_\varepsilon \subset [m - 2, m - 1] \) such that \( \mu_\alpha(C_\varepsilon) = \varepsilon \). Notice that the constant in (1.4) is not affected by dilation of the set. Let \( E' = E \cup C_\varepsilon \). This set is also coherent, with a (1.4) constant independent of \( \varepsilon \). It will replace \( E \) until the very last step of the proof of Theorem 3, in which \( \varepsilon \to 0 \). We will use the new notation \( E_x = E' \cap (-\infty, x] \) and \( \sigma_\alpha(x) = \mu_\alpha(E_x) \).

Suppose a real number \( M \) has been chosen such that \( \varepsilon \leq \sigma_\alpha(M) < \infty \). The next lemma provides a kind of uniformity in the limit in (1.4) that will be useful later.
Lemma 2. There is a $\delta_0 > 0$ such that for all $x \leq M$ and all $0 < \delta < \delta_0$,

$$|E_x + I_\delta| \delta^{x-1} < C_E(\sigma_a(x) + \varepsilon)$$

and

$$|E_x + I_\delta| \delta^{x-1} > (1/2)(\sigma_a(x) - \varepsilon/2),$$

where $C_E$ is the same constant as in the definition of coherent; it depends only on $E$, and $a$.

Proof: Since $\mu_a(E' \cap (-\infty, M]) < \infty$, there is a finite increasing sequence of points $\{x_i\}$ in $E_M$ such that

$$\mu_a(\{x_i, x_{i+1}\}) < \varepsilon/4$$

for all $i$, including the case $i = 0$ for which we adopt the convention $x_0 = -\infty$. We set the last term $x_N = M$.

For each $x_i$, $1 \leq i \leq N$, there is by the definition of coherent (1.4), a $\delta_i < \varepsilon$ such that

$$|E_{x_i} + I_\delta| \delta^{x_i-1} < C_E(\sigma_a(x_i) + \varepsilon/4)$$

whenever $\delta < \delta_i$. Let $Q(\delta, E_x)$ be the minimal number of intervals of length exactly $\delta$ required to cover $E_x$. Then from the definition of Hausdorff measure,

$$\mu_a(E_x) \leq \lim_{\delta \to 0} \delta^x Q(\delta, E_x) \leq 2 \lim_{\delta \to 0} \delta^{x-1} |E_x + I_\delta|.$$ 

So, $\delta_i$ can be chosen small enough that $\delta < \delta_i$ implies

$$\mu_a(E_{x_i}) \leq 2\delta^{x-1} |E_{x_i} + I_\delta| + \varepsilon/4.$$ 

Let $\delta_0 < \varepsilon$ be the smallest of the $\delta_i$. Suppose $\delta < \delta_0$ and $x \leq M$. Then for some $i \geq 0$, we have $x_i < x \leq x_{i+1}$. So,

$$|E_x + I_\delta| \delta^{x-1} \leq |E_{x_{i+1}} + I_\delta| \delta^{x_{i+1}-1} \leq C_E[\sigma_a(x_{i+1}) + \varepsilon/4]$$

$$\leq C_E[\sigma_a(x) + \varepsilon/4] + C_E \mu_a(E' \cap [x, x_{i+1}])$$

$$\leq C_E[\sigma_a(x) + \varepsilon].$$
Likewise,
\[
|E_x + I_\delta| \delta^{x-1} \geq |E_x + I_\delta| \delta^{x-1} \\
> (1/2)[\sigma_a(x_i) - \varepsilon/4] \\
\geq (1/2)[\sigma_a(x) - \varepsilon/2]
\]
which proves the lemma.

Now fix \( \delta \) such that \( 0 < \delta < \delta_0 \). Fix \( K > K_0 \) as defined by Lemma 1. We also assume \( KC_4 > 2C_5 \), where \( C_4 \) and \( C_5 \) are absolute constants that arise in Lemmas 4 and 5, respectively. Let \( L = K/\delta \) and let \( \phi_L(x) = \phi(Lx) \). Let \( S = [(E' + I_\delta) \cap \mathbb{Z}/L] \cap (-\infty, M) \).

**LEMMA 3.** There is a sequence \( \tilde{f}: \mathbb{Z}/L \to \mathbb{C} \) such that
\[
\text{Re}[\tilde{f}(n/L)\phi_L * f d\mu_a(n/L)] \\
\geq CK^{x-1} \frac{|\phi_L * f d\mu_a(n/L)|}{L^x(\varepsilon + \sigma_a(n/L))} \quad \text{for all } n/L \in S \tag{3.2}
\]
\[
|\tilde{f}(n/L)| \leq \frac{CK^{x-1}}{L^x(\sigma_a(n/L) + \varepsilon)} \quad \text{for } n/L > m - 1/2 \tag{3.3}
\]
\[
|\tilde{f}(n/L)| \leq Ce \quad \text{for all } n \tag{3.4}
\]
\[
\|F\|_\infty \leq 1. \tag{3.5}
\]

**Proof.** Let \( S = \{n_1/L, n_2/L, \ldots, n_m/L\} \). Choose \( j_0 \) such that
\[
4^{-j_0} < \varepsilon \leq 4^{-j_0 + 1}. \tag{3.6}
\]
We can ensure that \( S \) has at least \( 4^{j_0} \) terms by choosing \( \delta_0 \) small enough (to see this, consider Lemma 2 and inequality (3.10) below). In fact, we can assume that the first \( 4^{j_0} \) terms come from \( C_\varepsilon \), and are all less than \( m \). Let \( S_0 \) be the set of the first \( 4^{j_0} \) terms of \( S \). Let \( S_1 \) be the set of the next \( 4^{j_0+1} \) terms, etc., until \( S \) is exhausted. If there are terms left over when this construction stops, they are included in the last set, \( S_r \). So, \( S = \bigcup_{j=0}^{r} S_j \), where each \( S_j, j < r \), has \( 4^{j_0+j} \) elements. Then, construct functions \( \tilde{f}_j \) and \( F \) as in M.P.S. (using the function \( \phi_L * f d\mu_a \) instead of the function referred to there as \( \tilde{\mu} \)), so that the following inequality holds
\[
\left| \tilde{f}(n/L) - \tilde{f}_j(n/L) \right| \leq (1/10)4^{-(j_0+j)} \quad \text{for } n/L \in S_j. \tag{3.7}
\]

The calculations in M.P.S. actually prove something a little more general. If \( n < n_0 \) and \( n/L \notin S \), define \( i = i(n) \) by
\[
n_i/L = \min\{n_k/L \in S : n_k > n\}.
\]
Define \( j = j(n) \) by the condition \( n/L \in S_j \). Then inequality (3.7) also holds for this \( n \) and \( j \). However, in this case \( \int_j(n/L) = 0 \). So,

\[
|\hat{F}(n/L)| \leq (1/10)4^{-(j_0 + j)} \quad \text{for} \quad n/L \notin S, \ n < n_q, \ j = j(n). \tag{3.8}
\]

From the construction of \( S \) above, if \( n_k/L \in S_j \) with \( j > 0 \), then

\[
c_14^{-(j_0 + j)} \leq 1/k \leq c_24^{-(j_0 + j)}
\]

and since \( E_{n_k/L} + I_\delta \) is made up of intervals of length at least \( \delta > 1/L \),

\[
c_3k/L \leq |E_{n_k/L} + I_\delta| \leq c_4k/L,
\]

where the \( c_i \) in (3.9) and (3.10) are absolute constants.

From M.P.S.,

\[
|\hat{F}(n/L)| = 4^{-(j_0 + j)} \quad \text{for} \quad n/L \in S_j. \tag{3.11}
\]

For \( n/L > m - 1/2 \), \( \sigma_z(n/L) \geq \epsilon \) so that the \(-\epsilon/2\) in Lemma 2 may be replaced by \(+\epsilon\). Also, \( j > 0 \) for these \( n \) so (3.9) applies. From these, and (3.10), we get (3.3) for \( n/L \in S \);

\[
|\hat{F}(n/L)| \leq 4^{-(j_0 + j)} \leq 1/k \leq \frac{1}{L |E_{n/L} + I_\delta|} \leq \frac{1}{K^{1-\alpha}L^2(\sigma_z(n/L) + \epsilon)},
\]

where we have omitted absolute constants.

This inequality also applies off \( S \) as follows; (3.3) is trivial for \( n > n_q \) because \( \hat{F} \) is zero there. For \( n/L \notin S \) and \( n < n_q \), the first two inequalities of (3.12) hold with \( j = j(n) \) and \( k = k(n) \). The third then holds with the subscript \( n_k/L \). This change is harmless because

\[
|E_{n_k/L} + I_\delta| - |E_{n/L} + I_\delta| \leq \delta.
\]

So the error in the denominator is at most \( L\delta = K \) which is much smaller than \( L^2\epsilon \); we can assume \( L \) is quite large through proper choice of \( \delta_0 \). So, the error is negligible and we have (3.3) for all \( n/L > m - 1/2 \).

Inequality (3.4) follows from (3.11) and (3.7) on \( S \), and from (3.8) off \( S \).

Part of the M.P.S. construction is that \( \int_j(\phi_L * f \, d\mu_z) \geq 0 \). This together with (3.11), (3.7), and (3.9) imply that

\[
\text{Re}[\hat{F}(n/L)\phi_L * f \, d\mu_z(n/L)] \geq \frac{|\phi_L * f \, d\mu_z(n/L)|}{c_5k} \quad \text{for} \quad n/L \in S \tag{3.13}
\]
except that for \( n/L \in S_0 \) we must replace \( c_3k \) by \( 4^{3/2} \). With (3.10) and Lemma 2, this shows that the left side of (3.2) is at least

\[
\frac{|\phi_L \ast f \, d\mu_2(n/L)|}{L |E_{n/L} + I_3|} \geq \frac{|\phi_L \ast f \, d\mu_2(n/L)|}{K^{1-\alpha} L^\alpha (\sigma_\omega(n/L) + \varepsilon)}
\]

for \( n/L \in S - S_{\hat{c}_3} \). For \( n/L \in S_{\hat{c}_3} \), use (3.6) instead. In this case we need the inequality \( \varepsilon > 1/(K^{1-\alpha} L^\alpha \varepsilon) \), which holds for large enough \( L \). This proves (3.2) for all \( n/L \in S \).

Inequality (3.5) is part of the M.P.S. construction. This proves Lemma 3.

\( \hat{F} \) must be slightly modified off \( S \) before proceeding with the proof of the theorem. For \( n/L \in S \), define \( G(n/L) = \hat{F}(n/L) \). For \( n/L > M \), let \( G(n/L) = 0 \). For \( n/L \leq M \) with \( n/L \notin S \), let

\[
G(n/L) = \phi(n/L).
\]

Let

\[
A(L) = \sum_{n \in \mathbb{Z}} G(n/L) \phi_L \ast f \, d\mu_2(n/L)
\]

\[
B(L) = \sum_{n \in \mathbb{Z}} (\hat{F} - G)(n/L) \phi_L \ast f \, d\mu_2(n/L).
\]

Since \( G \) may be viewed as a substitute for \( \hat{F} \), the term \( B(L) \) may be viewed as an error term. \( L^\alpha |A| \) is supposed to approximate the left-hand side of (3.1) for large enough \( L \). This will be the content of Lemma 4. The next calculation shows the relation to the right-hand side of (3.1).

\[
L^\alpha |A| - L^\alpha |B| \leq L^\alpha \left| \sum \hat{F}(n/L) \phi_L \ast f \, d\mu_2(n/L) \right|
\]

\[
= L^\alpha |F \ast (\phi_L \ast f \, d\mu_2)(0)|
\]

\[
\leq L^{\alpha - 1} \int |\hat{\phi}(x/L) \hat{f} \, d\mu_2(x)| \, dx
\]

\[
\leq L^{\alpha - 1} \|\hat{\phi}\|_\infty \int_{-L}^{L} |\hat{f} \, d\mu_2(x)| \, dx.
\]

**Lemma 4.** There is an absolute constant \( C_4 \) such that

\[
\liminf_{L \to \infty} L^\alpha |A| \geq C_4 \int_{-\infty}^{M} \frac{|f| \, d\mu_2}{\varepsilon + \sigma_\omega(x)}.
\]
LEMMA 5. There is an absolute constant \( C_5 \) such that

\[
\limsup_{L \to \infty} L^a |B| \leq \frac{C_5}{K} \int_{-\infty}^{M} \frac{|f|}{\varepsilon + \sigma_a(x)} \, d\mu_a.
\]

These will be proved below. First notice that together with the last calculation, they prove the theorem. Since \( K \) was chosen so that \( C_5/K < C_4/2 \), we can combine the lemmas to get

\[
(C_4 - C_4/2) \int_{-\infty}^{M} \frac{|f|}{\varepsilon + \sigma_a(x)} \, d\mu_a \leq \liminf_{L \to \infty} (L^a |A| - L^a |B|) 
\leq \|\phi\|_\infty \liminf_{L \to \infty} L^{a-1} \int_{-L}^{L} |f \, d\mu_a| \, dx.
\]

Notice that \( \sigma_a(x) = \sigma_a(E, x) + \varepsilon \) for all \( x \) in the support of \( f \). Then let \( \varepsilon \to 0 \) and let \( M \to \sup\{x: \sigma_a(x) < \infty\} \) to get (3.1).

Proof of Lemma 4. Let \( \varepsilon_1 \) be arbitrary \( 0 < \varepsilon_1 < \varepsilon \). Let

\[
I_j = \{x \leq M: j\varepsilon_1 \leq \sigma_a(x) < (j+1)\varepsilon_1\}
\]

for \( j = 0, 1, \ldots, J \) where \( M \in I_j \). Notice that \( \mu_a(E \cap I_j) = \varepsilon_1 \) for each \( 0 \leq j < J \). Then by (3.2) and (3.14),

\[
L^a |A| = \left| \sum_{n \in \mathbb{Z}} G(n/L) \phi_L * f \, d\mu_a(n/L) \right|
\geq C \sum_{n < LM} \frac{|\phi_L * f \, d\mu_a(n/L)|}{\varepsilon + \sigma_a(n/L)}
\geq C \sum_{j=0}^{J} \sum_{n/L \in I_j} \frac{|\phi_L * f \, d\mu_a(n/L)|}{\varepsilon + (j+1)\varepsilon_1}
= C \sum_{j=0}^{J} \frac{A_j}{\varepsilon + (j+1)\varepsilon_1}
\]

the last equation being a definition of \( A_j \). We now claim that, for all \( j \),

\[
\liminf_{L \to \infty} A_j \geq \left| \int_{I_j} f \, d\mu_a \right|.
\]
To prove (3.16) notice that for all $j$, $L$,

$$A_j \geq \left| \sum_{n/L \in I_j} \phi_L \ast f \, d\mu_a(n/L) \right|$$

$$\geq - \left| \sum_{n/L \in I_j} \phi_L \ast \chi_R \ast f \, d\mu_a(n/L) \right| + \sum_{n \in \mathbb{Z}} \phi_L \ast \chi_{I_j} \ast f \, d\mu_a(n/L)$$

$$- \left| \sum_{n/L \notin I_j} \phi_L \ast \chi_{I_j} \ast f \, d\mu_a(n/L) \right|$$

$$= - A_j^a + A_j^b - A_j^c.$$

Now since $\text{supp } \phi \subseteq [-1, 1]$, the Poisson summation formula shows that for all $x$,

$$\sum_{z} \phi(n-x) = \hat{\phi}(0) = 1.$$

So,

$$A_j^b = \left| \sum_{I_j} \int_{I_j} \phi_L(n/L - x) \, f(x) \, d\mu_a(x) \right|$$

$$= \left| \int_{I_j} f \, d\mu_a \right|.$$

So, we must show that $A_j^a$ and $A_j^c$ approach zero as $L \to \infty$. We will assume $I_j = (-\infty, 0]$, other cases being similar.

Since $\phi$ is a Schwartz function, there is a constant $R$ such that for all real $x$

$$\sum_{n \in \mathbb{Z}} |\phi(n-x)| < R.$$

Let $\epsilon_2 > 0$. Since $f \in L^1(d\mu_a)$ and $\alpha > 0$, there is a $\delta' > 0$ such that

$$\int_{I_{\delta}(0)} f \, d\mu_a < \epsilon_2/R. \quad (3.17)$$

Of course, if $I_j$ has a boundary point at some $x_0 \neq 0$, then $I_{\delta}(0)$ must be replaced by $I_{\delta}(x_0)$. Since $\phi$ is a Schwartz function, there is a constant $C$ such that

$$\phi(L(x-n/L)) \leq \frac{C}{|L(x-n/L)|^2 + 1}.$$
Since $f \in L^1(du_a)$,

$$
\sum_{n \leq 0} |\phi_L (\chi_{[0, \infty]}(n/L))f(du_a)(n/L)| \leq \sum_{n \leq 0} \frac{C}{|L(n/L - n/L)|^2 + 1}
$$

which approaches zero as $L$ approaches infinity. Also,

$$
\sum_{n \leq 0} |\phi_L (\chi_{[0, \infty]}(n/L))f(du_a)(n/L)| \leq \sum_{n \leq 0} |\phi_L (n/L - x)|f(du_a)(x)
$$

which shows that

$$
\lim sup A^\delta_j < \lim sup \sum_{n \leq 0} |\phi_L (\chi_{[0, \infty]}(n/L))f(du_a)(n/L)| \leq \lim sup \sum_{n \leq 0} |\phi_L (n/L - x)|f(du_a)(x) \leq R(\varepsilon_2/R) = \varepsilon_2
$$

as $L \to \infty$. Since $\varepsilon_2 > 0$ was arbitrary, this proves that $\lim A^\delta_j = 0$. This proof works because $n \in (-\infty, 0]$ and $x \in (0, \infty)$ range over disjoint intervals. So, it works for arbitrary intervals $I_j$ and for $A^\delta_j$ as well. The claim is proved, but it is not exactly what we need; the absolute value should be inside the integral. We now show that the error is small. Define

$$
e_j = \int_{l_j} |f| \, du_a - \int_{l_j} f \, du_a \geq 0.
$$

We will show that $\sum e_j \to 0$ as $\varepsilon_1 \to 0$. This will complete the proof of Lemma 4;

$$
\lim inf_{L \to \infty} L^2 |A| \geq C \lim inf_{L \to \infty} \sum_{j} \frac{A_j}{\varepsilon + (j + 1)\varepsilon_1}
$$

$$
\geq C \sum \lim inf_{L \to \infty} \frac{A_j}{\varepsilon + j\varepsilon_1}
$$

$$
\geq C \sum \int_{l_j} |f| \, du_a - e_j
$$

$$
\geq C \sum \int_{-\infty}^{\infty} |f| \, du_a - \sigma_\varepsilon(x) - \sum_{j} \frac{e_j}{\varepsilon}.
$$

(3.18)
Let $\mu = \chi_{E_M} \mu_x$, and for each interval $I \subset R$ define

$$\text{avg}_I(f) = \frac{1}{\mu(I)} \int_I f \, d\mu$$

and

$$Uf(x) = (\varepsilon_1)^{-1} \sup_I \int_I |f - \text{avg}_I(f)| \, d\mu$$

(3.19)

taken over all intervals $I$ containing $x$ such that $\mu(I) = \varepsilon_1$. It is easy to check that $U$ is a sublinear operator on $L^1(d\mu)$ and that for $x \in I_j$, $Uf(x) \geq \varepsilon_j/\mu(I_j)$. So,

$$\sum e_j = \sum \frac{1}{\mu(I_j)} \int_{I_j} e_j \, d\mu \leq \int_R Uf(x) \, d\mu.$$  

(3.20)

If $f$ is continuous with compact support, then the right-hand side of (3.20) goes to zero with $\varepsilon_1$. For in that case, $f$ is uniformly continuous and $|f(x) - \text{avg}_I(f)| \to 0$ uniformly in $x$ and $I$ as $|I| = \varepsilon_1 \to 0$. Therefore, $Uf \to 0$ uniformly on its support, which is bounded. Since $\mu$ is finite on any bounded set, $\int Uf \, d\mu \to 0$ in this case.

Now we show that $U$ is bounded on $L^1(d\mu)$ independent of $\varepsilon_1$. Given any $x \in E$, there is a $j = j(x)$ such that $x \in I_j$. Let $I_j^* = I_{j-1} \cup I_j \cup I_{j+1}$ (where $I_{-1}$ is the empty set). Applying the triangle inequality to (3.19), $Uf(x)$ splits naturally into two parts, each of which is at most $Vf(x) = (\varepsilon_1)^{-1} \int_{I_j^*} |f| \, d\mu$. But $Vf(x)$ is constant on each $I_j$ so,

$$\frac{1}{2} \int_R Uf(x) \, d\mu \leq \int_R Vf(x) \, d\mu$$

$$= \sum \int_{I_j} Vf(x) \, d\mu$$

$$= \sum \int_{I_j^*} |f| \, d\mu$$

$$= 3 \sum \int_{I_j} |f| \, d\mu$$

$$= 3 \int_R |f| \, d\mu$$

which shows $U$ is bounded on $L^1$. 

For any $f \in L^1(d\mu)$, there is a continuous function $g$ with compact support that is arbitrarily close to $f$ in the $L^1(d\mu)$ norm. So,

$$
\int Uf(x) \, d\mu \leq \int Ug(x) + U(f - g)(x) \, d\mu \\
\leq \int Ug(x) \, d\mu + 2 \int |(f - g)(x)| \, d\mu. \tag{3.21}
$$

The first term of the last expression goes to zero with $\epsilon_1$ because $g$ is continuous. The second term can be made arbitrarily small by proper choice of $g$. With (3.20), this completes the proof that $\sum \epsilon_j \to 0$, and also the proof of Lemma 4.

**Proof of Lemma 5.** Let $H(n/L) = (\hat{f} - G)(n/L)$, which is zero for $n/L \in S$. For $n/L > m - 1/2$, $L^\infty |H(n/L)| \leq C(\epsilon + \sigma_a(n/L))^{-1}$ by (3.3) and (3.14). Also, assuming $L$ is large enough, $|H(n/L)| \leq \epsilon$ for all $n$.

Let $\{I_j\}$ be the partition defined in Lemma 4, except that now $\epsilon_1 = \epsilon$. Similar to the proof that $A^q \to 0$, we see that for each $j$

$$
\limsup_{L \to \infty} \sum_{n/L \in I_j} |\phi_L \ast \chi_{I_j} f \, d\mu_a| = 0.
$$

Using Lemma 1,

$$
\sum_{n/L \notin I_j \in \mathcal{L}} |\phi_L \ast \chi_{I_j} f \, d\mu_a| \leq \left\| \sum_{n/L \notin S} \phi_L(n/L - x) \right\|_{L^\infty(E)} \|\chi_{I_j} f\|_{L^1(d\mu_a)} \\
\leq \frac{C}{K} \|\chi_{I_j} f\|_{L^1(d\mu_a)}
$$

because $n/L \notin S$ and $x \in E$ implies $|n/L - x| > \delta$ so that $|n - Lx| > L\delta = K$. So, for each $j$,

$$
\limsup_{L \to \infty} \sum_{n/L \notin I_j \in \mathcal{L}} |\phi_L \ast f \, d\mu_a| \leq \frac{C}{K} \int_{I_j} \left| f \right| \, d\mu_a.
$$

Notice that $H$ is zero above $M = \sup I_j$, so summing over $j$ gives

$$
\limsup_{L \to \infty} L^\ast \sum_{n/L > m - 1/2} |H(n/L)| \cdot |\phi_L \ast f \, d\mu_a| \\
\leq \frac{C}{K} \sum_{j=0}^{J} \frac{\int_{I_j} \left| f \right| \, d\mu_a}{\epsilon + \sigma_a(\inf I_j)} \\
\leq \frac{C}{K} \int_{-\infty}^{M} \frac{\left| f \right| \, d\mu_a}{\epsilon + \sigma_a(x)}
$$
because \( \varepsilon + \sigma(x) \) is roughly constant on each \( I_j \) (except when \( j = 0 \), in which case the numerator is zero).

For \( n/L \leq m - 1/2 \), we use the fact that \( |H(n/L)| \leq \varepsilon \) and that for \( x \in \text{supp} f, x - n/L > 1/2 \). So,

\[
L^x \sum_{n/L \leq m - 1/2} |H(n/L)| \cdot |\phi_L * f \mu_x|
\]

\[
\leq \varepsilon L^x \sum_{n/L \leq m - 1/2} |\phi_L| (n/L - x) \left\| f \right\|_1
\]

\[
\leq \varepsilon L^x \sum_{|n - Lx| \geq L/2} |\phi| (n - Lx) \left\| f \right\|_1
\]

which goes to zero as \( L \to \infty \), because \( \| \sum_{|n - Lx| \geq L/2} |\phi(n - Lx)| \|_{L^\infty(E)} < C/L \) by Lemma 1. This proves Lemma 5, and thus Theorem 3.

**COROLLARY 1.** The measure \( f \mu_x \) in Theorem 3 may be replaced by any finite positive nonatomic Borel measure \( \nu \) supported on \( E \).

The proof of the Corollary is the same as that of the theorem. The form \( f \mu_x \) is used only in (3.17) and to show that \( \sum e_j \to 0 \) in Lemma 4. Since \( \nu \) has no atoms it satisfies (3.17) at every point. Since \( \nu \) is positive, each \( e_j = 0 \).

**DEFINITION 2.** A set \( E \) is quasi-regular if its lower density

\[
D = \liminf_{r \to 0} (2r)^{-2} \mu_x(E \cap I_r(x))
\]

is bounded away from zero \( \mu_x \)-a.e. on \( E \).

**COROLLARY 2.** Suppose that \( E \) is quasi-regular and that \( \mu_x(E) < \infty \), but that \( E \) is not necessarily \( \alpha \)-coherent. Then Theorem 3 holds on the set \( E \).

**Proof.** Assume \( E \) is quasi-regular. Then there is a constant \( C \) and a positive measurable function \( r: E \to \mathbb{R} \) such that for \( \mu_x \)-a.e. \( x \in E \), and for all \( 0 < r < r(x) \),

\[
\mu_x(E \cap I_r(x)) > Cr^2.
\]

Let \( E^\delta = \{ x \in E : r(x) \geq \delta \} \). Let \( G^\delta = E - E^\delta \). Since \( \mu_x(E) < \infty \), \( \mu_x(G^\delta) \to 0 \) as \( \delta \to 0 \). Likewise, \( \| \chi_{G^\delta} f \mu_x \| \to 0 \) as \( \delta \to 0 \). Also, for each fixed \( \delta \), \( E^\delta \) is bounded below.
The proof of Theorem 3 requires little modification. We may assume \( \delta \) is small enough that \( \mu_\sigma(G^\delta) < \varepsilon/4 \). So, replacing \( E \) by \( E^\delta \) does not significantly affect terms such as \( \sigma_\sigma(x) + \varepsilon \) that appear in the proof. Change notation slightly and define \( E' = E^\delta \cup C_\varepsilon \). Lemma 1 and the second inequality of Lemma 2 are unaffected by this change.

Consider the first inequality of Lemma 2. \( E_x + I_\delta \) can be covered by \( J \) intervals \( \{I_j\} \) of length \( 2\delta \) in which the measure of \( E_x \) is greater than \( \delta^x \) (except perhaps for the interval containing \( x \)). We may assume all the intervals are necessary for the cover, so that no three overlap at any point. Summing over the intervals beneath a given \( x \) gives

\[
|E_x + I_\delta| \delta^{x-1} \leq 2J\delta^x
\]

\[
\leq C \sum \mu_a(E_x \cap I_j)
\]

\[
\leq C\sigma_\sigma(x)
\]

in which there is a possible error of at most \( \delta^x \) from the interval that contains \( x \). Since we may assume this is less than \( \varepsilon \), this gives the first inequality in Lemma 2.

In Lemma 3, and in the definitions of \( S, A(L), \) and \( B(L) \), continue to use \( E^\delta \) in place of \( E \). Also, replace \( f \, d\mu_\sigma \) by \( \chi_{E^\delta} f \, d\mu_\sigma \). Define

\[
D(I) = \sum_{n \in \mathbb{Z}} \hat{F}(n/L) \phi_L \ast \chi_{E^\delta} f \, d\mu_\sigma(n/L)
\]

so that \( A(L) + B(L) + D(L) = \sum \hat{F}(n/L) \phi_L \ast f \, d\mu_\sigma(n/L) \). Since Lemmas 4 and 5 still hold, the rest of the proof of Theorem 3 depends only on showing that

\[
\limsup_{L \to \infty} L^x |D(L)| = 0.
\]

The values of \( n \) such that \( n/L < m - 1/2 \) contribute little to \( D(L) \) because \( \phi \) is a Schwartz function. For \( n/L > m - 1/2 \), note that \( \sigma_\sigma > \varepsilon \) and apply (3.3),

\[
L^x |D(L)| \leq \sum |\phi_L \ast \chi_{E^\delta} f \, d\mu_\sigma(n/L)|.
\]

For large enough \( L \), this is less than \( \int_{E^\delta} |f| \, d\mu_\sigma \), which approaches zero as \( \delta \to 0 \).
4. On Coherent Sets

Coherence seems a bit stronger than quasi-regularity. However, these conditions are, strictly speaking, independent of each other. One can easily construct a quasi-regular set $E$ such that the Lebesgue set of $E$ is dense in $[0, 1]$; so $E$ is not coherent if $\alpha < 1$. Coherence does not imply quasi-regularity, because coherence is not really a local property.

**Proposition 1.** Given $0 < \alpha < 1$, there is a set $E \subset [0, 1]$ that is $\alpha$-coherent but not quasi-regular.

**Proof.** Given a positive integer $k$, construct a Cantor set, $C(2^k, 3^k)$, as follows. Remove $2^k - 1$ intervals of equal length from $[0, 1]$ leaving $2^k$ subintervals, each of length $3^{-k}$. Repeat the excision on each of the $2^k$ subintervals leaving $2^{2k}$ subintervals of length $3^{-2k}$. Repeat ad infinitum, so that after stage $l$ the set $C_l$ has $2^{kl}$ subintervals, each of length $3^{-kl}$. Let $C(2^k, 3^k) = \cap C_l$. For every $k$, this set has dimension $\alpha = \ln 2/\ln 3$.

These sets are all coherent and quasi-regular. As $k \to \infty$, the (1.4) constant $C$ increases like $2^{k(1-\alpha)}$. The lower density (see Definition 2) does not really depend on $x$ in these sets. It decreases like $(2/3)^k$.

Now, let $E_1 = C(2, 3)$. Replace $E_1 \cap [2/3, 1]$ by $C(4, 9)$ (naturally it must be translated by $2/3$ and dilated by $1/3$, so that it fits into $[2/3, 1]$). Call the new set $E_2$. Replace $E_2 \cap [26/27, 1]$ by $C(8, 27)$. Notice that $[26/27, 1]$ arises as the last subinterval of the first stage of the construction of $C(4, 9)$. Continue, replacing the last “segment” of $E_{n-1}$ by $C(2^n, 3^n)$.

Let $E$ be the limit set. It is $\alpha$-dimensional. It is not quasi-regular because the lower density approaches zero as $x$ approaches 1 from the left. To see that it is coherent, consider the worst case, $x = 1$. The constant in (1.4) can be computed as a sum over $k$ of the contribution from each $C(2^k, 3^k)$. This contribution is the measure of the dilated version times its (1.4) constant, which is at most $2^{-k}$ times $2^{k(1-\alpha)}$. Since $2^{-k\alpha}$ decreases geometrically, the series is summable, so the (1.4) constant of $E$ is finite. This proves the proposition.

We now show that Cantor sets are coherent. The type of set discussed below is a little more general than the sets constructed in [4] in the sense that the pieces need not be regularly spaced. It is a little less general in the sense that the dissection number is a constant, denoted $m$.

A measure $\mu$ is self-similar and equicontractive means that there are linear contractions $S_j(x) = \rho x + b_j$, $0 < \rho < 1$, $b_j \in \mathbb{R}$, such that

$$\mu = \sum_{j=1}^{m} a_j \mu \circ S_j^{-1}.$$ 

See Strichartz [10]. In the theorem below, $a_j = 1/m$ for all $j$. 

THEOREM 4. Suppose $E \subseteq [0, 1]$ has finite $\mu_\alpha$ measure, and that $\mu = \chi_E \, d\mu_\alpha$ is self-similar and equicontractive. Then $E$ is coherent. In particular, the usual Cantor set is coherent.

**Proof.** We will assume that $E$ has been normalized s.t. $\mu_\alpha(E) = 1$. So, $m_p^\alpha = 1$. We will prove (1.4) in the case $x = 1$; other cases are the same. Let $\delta_j = \rho^j$. We can cover $E$ with $m_j$ intervals of length $\delta_j$ (the proof is by induction, and self-similarity). So, $E \cup I_{\delta_j}$ has measure at most $2m_j \delta_j$. So,

$$|E \cup I_{\delta_j}| \delta_j^{\alpha/2} \leq 2m_j \delta_j = 2$$  \hspace{1cm} (4.1)

for each $j$. Now if $\delta > 0$ lies between, say, $\delta_{j+1}$ and $\delta_j$ we observe that $E \cup I_{\delta_{j+1}} \subseteq E \cup I_{\delta} \subseteq E \cup I_{\delta_j}$. This establishes (4.1) for all $\delta$ (but with the constant 2 replaced by a larger constant), which shows that $E$ is coherent.

Theorem 3 fails without some restriction, such as coherence, on the set $E$. Theorem 5 demonstrates this by an example. The set $E$ that arises below could be defined using the usual Cantor excision process described in [4], though we prefer to build it up from zero-dimensional sets.

**Theorem 5.** Given $0 < \alpha < 1$ and $0 < \beta < 1$, there exists an $\alpha$-dimensional set $E$ contained in $[-2, 2]$ such that

$$\int \frac{\chi_E \, d\mu_\alpha(x)}{\sigma_\alpha(E, x)} = + \infty$$

and

$$\liminf_{L \to \infty} L^{\beta - 1} \int_{-L}^{+L} |\chi_E \, d\mu_\alpha(x)| \, dx = 0.$$  \hspace{1cm} (4.2)

**Proof.** We will define a measure $\mu = \chi_E \, d\mu_\alpha$ as a weak limit of a sequence of zero-dimensional measures $\{f_j \, d\mu_0\}$. We need to define some sequences of constants first. The reader may wish to keep the Cantor 2/3’s set in mind (in which $n_j = 2$ for all $j$ and $\delta_j = 3^{-j}$). Let $n_1 = m_1 = \delta_1 = 1$. For $j > 1$ we define $n_j$ as a large odd integer that depends on previously defined constants such as $\delta_{j-1}$ and $m_{j-1}$; we will make this precise later. For $j \geq 1$ let $m_j = n_j n_{j-1} \cdots n_1$ and

$$\delta_j = m_j^{-1/\alpha}.$$  

Of course, $\delta_j \to 0$ rapidly for proper choice of $n_j$. Let $\gamma_j = \delta_j - \delta_{j+1} \approx \delta_j$. Note that $\sum_{i \geq j} \gamma_i = \delta_j$. Let $k_j(x) = 1/n_j$ if

$$|\frac{(n_j - 1)x}{\gamma_j}|.$$
is an integer less than or equal to \( (n_j - 1)/2 \). Let \( k_j(x) = 0 \) otherwise. Then \( k_j(x) \, d\mu_0 \) is a positive measure which is supported in \([ -\gamma_j/2, +\gamma_j/2] \) and has norm \( \|k_j \, d\mu_0\| = 1 \). Its Fourier transform resembles the Dirichlet kernel, but has period \( 2\pi(n_j - 1)/\gamma_j \). Now let \( f_1 = k_1 \, d\mu_0 \) and for \( j > 1 \) define a zero-dimensional measure by

\[
f_j = f_{j-1} * k_j,
\]

which has norm \( \|f_j\| = \|f_{j-1}\| \|k_j \, d\mu_0\| = 1 \). Notice \( f_j \) is supported on the sum of the supports of the \( k_j \). This set has \( m_j \) elements and is contained in \([-1/2, 1/2]\) because \( \sum \gamma_j = \delta_1 = 1 \). These measures have a subsequence that converges weakly to some measure \( \mu \) supported on a set \( E \) which we will describe below.

**Lemma 6.** There is a set \( E \subset [-1/2, 1/2] \) such that \( d\mu = \chi_E \, d\mu_x \), so \( \mu_x(E) = 1 \).

**Lemma 7.** For each \( j \geq 1 \) there is a positive measure \( \mu_j \) supported in the interval \([ -\delta_{j+1}/2, \delta_{j+1}/2] \) such that \( \|\mu_j\| = 1 \) and

\[
f_j * \mu_j = \mu
\]

so, for all \( x \in \mathbb{R} \), \( |\hat{\mu}(x)| \leq |\hat{f}_j(x)| \).

**Lemma 8.** Let \( L_j = \pi(n_j - 1)/\gamma_j \) which is half the period of \( \hat{k}_j \). Then as \( j \to \infty \),

\[
L_j^{\theta - 1} \int_{-L_j}^{L_j} |\hat{f}_j(x)| \, dx \to 0. \tag{4.4}
\]

Assume these for the moment and we will prove the theorem. Consider the integral in (4.2) only over the set of \( x \) such that

\[
\sigma_x(E, x) \approx 2^{-j}
\]

(which is nonempty for \( j > 0 \) by Lemma 6, and because \( x > 0 \)). Each of these integrals is approximately equal to 1, so (4.2) is clear.

By Lemmas 7 and 8,

\[
L_j^{\theta - 1} \int_{-L_j}^{L_j} |\hat{\mu}(x)| \, dx \leq L_j^{\theta - 1} \int_{-L_j}^{L_j} |\hat{f}_j(x)| \, dx
\]

which approaches zero as \( j \) approaches infinity. This proves (4.3), so we need only prove the lemmas.
Proof of Lemma 6. First notice that $E$ could also be constructed by the deletion method which is commonly used to define the Cantor $2/3$’s set. Let $E_j = \text{supp} f_{j-1} + [-\delta_j/2, \delta_j/2]$. Note $\text{supp} f_i \subset E_j$ for all $i$ and that $E_{j+1} \subset E_j$. Then let $E$ be the closure of $\bigcup \text{supp} f_j$, which is the same as $\bigcap E_j$.

To show that $d\mu = d\mu_\omega$ on $E$, it is enough to test both measures on an interval $S = [a, b]$ with the property that $a$ and $b$ belong to consecutive components of the complement of some $E_j$. That is, $S$ contains exactly one subinterval of some $E_j$. In this case,

$$\mu(S) = f_j(S) = f_{j-1}(0) = \frac{1}{m_{j-1}}.$$

Likewise, we can compute $\mu_\omega(S \cap E)$ using the methods of Federer [4]. As a slight oversimplification of these methods, we cover $S \cap E$ by sub-intervals of any $E_i$ with $i \geq j$. These have length $\delta_i$ and there are $n_j n_{j+1} \cdots n_{i-1} = m_{i-1}/m_{j-1}$ of them. Using these to calculate $\mu_\omega(S \cap E)$ gives

$$\delta_i^2 m_{i-1}/m_{j-1} = 1/m_{j-1} = \mu(S).$$

This proves Lemma 6.

Proof of Lemma 7. The measure $\mu$ was defined as the weak limit as $i \to \infty$ of $k_1 \ast k_2 \ast \cdots \ast k_i$. The measure $\mu_j$ is defined similarly. It is the weak limit as $i \to \infty$ of $k_{j+1} \ast k_{j+2} \ast \cdots \ast k_i$. It is supported in the sum of the supports of the $k_{j+1}$, $k_{j+2}$, $\ldots$ which has diameter $2 \sum_{i \geq j} \delta_i/2 = \delta_{j+1}$ and is centered at zero.

Proof of Lemma 8. We will first study the integral in (4.4) from 0 to $L_j/2$ and show that it can be made arbitrarily small by proper choice of $n_j$. The estimates obtained apply to any interval of similar length. The proof uses induction on $j$—we will assume the $n_k$ for $k < j$ have been chosen to make the left-hand side of (4.4) less than $100 \cdot 2^{-k}$. We also assume the same inequality holds for all intervals of length $2L_k$. Partition $[0, L_j/2]$ into intervals $\{I_i\}$ each of length $2L_{j-1}$; the last interval may lie partially outside $[0, L_j/2]$. So,

$$\int_0^{L_j/2} |\tilde{f}_j(x)| \, dx \leq \sum I_i |\tilde{f}_j(x)| \, dx = \sum I_i |\tilde{f}_{j-1}(x)| \, |\tilde{k}_j(x)| \, dx.$$

We want to dominate $\tilde{k}_j(x)$ by a small constant on each $I_i$. 
For $0 \leq x \leq L_j/2$, 

$$
\hat{k}_j(x) = \sin[(2n_j + 1) \pi x/2L_j]/n_j \sin(\pi x/2L_j) \leq 3 \min \{1, (\delta_j x)^{-1}\} = g(x).
$$

The last equation is a definition. Since $|I_i| = 2L_{j-1}$ is less than $1/\delta_j$, the largest value of $g(x)$ on each $I_i$ is less than twice its average value (or even its smallest value) on $I_i$. So, we may replace $\hat{k}_j(x)$ above by the average of $g$ over $I_i$ 

$$
\int_{I_i} |\hat{f}_{j-1}(x)| |\hat{k}_j(x)| \, dx \leq 2 \int_{I_i} |\hat{f}_{j-1}(x)| \, dx \frac{1}{|I_i|} \int_{I_i} |g(x)| \, dx.
$$

We have assumed inductively that $\int_{I} |\hat{f}_{j-1}(x)| \, dx \leq CL_j^{-\beta}$ for every $i$. So, 

$$
L_j^{\beta-1} \int_{0}^{L_j/2} |\hat{f}_j(x)| \, dx \leq CL_j^{\beta-1} L_j^{-\beta} \sum_i \int_{I_i} |g(x)| \, dx
$$

$$
\leq CL_j^{\beta-1} L_j^{-\beta} \int_{-L_j}^{L_j} |g(x)| \, dx
$$

$$
\leq CL_j^{\beta-1} L_j^{-\beta} \delta_j^{-1} [\ln(\delta_j L_j) + 1]
$$

which approaches 0 as $n_j \to \infty$ (as $L_j \to \infty$).

Since $\hat{k}_j$ has period $2L_j$ and is symmetric with respect to $x = 0$ and with respect to $x = L_j/2$, we get the same kind of estimates on the other parts of $[-L_j, L_j]$. So, we can choose $n_j$ to make $L_j^{\beta-1} \int_{-L_j}^{L_j} |\hat{f}_j(x)| \, dx$ as small as we like. In fact, $[-L_j, L_j]$ could be replaced by any interval of the same length without significantly changing the estimates above. This completes Lemma 8 and Theorem 5.

The set $E$ in Theorem 5 is neither quasi-regular nor coherent. It shows that the results in Section 3 are fairly sharp in terms of their restrictions on $E$. It is not clear whether they are sharp in terms of the exponent of $L$. For a given fractal measure $\nu$, there is a critical number $\beta_\nu$ such that 

$$
\lim inf_{L \to \infty} L_j^{\beta-1} \int_{-L_j}^{L_j} |\hat{\nu}(x)| \, dx
$$

converges to 0 when $\beta < \beta_\nu$ and diverges when $\beta > \beta_\nu$.

This number is generally quite difficult to determine. Theorem 3 shows that $\beta_\nu = \alpha$ when $E$ is coherent. It is not known whether equality, the most interesting case, can occur.
For an example, let $v = \chi_A \, d\mu$ where $A$ is the Cantor $2/3$'s set and $\alpha = \ln(2)/\ln(3) \approx 0.63$. Computer experiments suggest that $\beta_v \approx 0.4$.

Exact values are possible in the $L^2$ setting. Continuing the example above, let $\mu = v * v$, so that $\hat{\mu} = \hat{v}^2$. Then the Strichartz–Plancherel theorem [9] shows that $\beta_\mu$ is the $\alpha$ given above. However, $\mu$ is supported on a rather strange set $G$ properly contained in $\text{supp } v + \text{supp } v$. $G$ is has dimension $\alpha_2 = \ln(8/3)/\ln(3)$ (see [10]), which is bigger than $\alpha = \beta_\mu$. This seems a bit peculiar, because $\mu$ is self-similar and equicontractive. But by Theorem 3, $G$ is not coherent; by Theorem 4, $\chi_G \, d\mu_{\alpha_2}$ is not both self-similar and equicontractive. Likewise, $G$ must not be quasi-regular.

Now we consider the case $p > 1$, in which the order of the real numbers is less important and there are fewer pathologies. The following theorem generalizes a result of Paley.

**Theorem 6.** Let $c_k$ be a sequence of complex numbers and $a_k$ be a sequence of real numbers, not necessarily increasing. Let $f \, d\mu_0$ be the zero-dimensional measure

$$f(x) = \sum_{k=1}^{\infty} c_k \delta(x - a_k)$$

and let $1 < p \leq 2$. Assume that

$$u(x) = \overline{\int f \, d\mu_0(x)} \sim \sum_{k=1}^{\infty} c_k e^{iax} \in B^p. a.p.$$ Then

$$\sum_{k=1}^{\infty} \frac{|c_k|^p}{k^{2-p}} \leq \sum_{k=1}^{\infty} \frac{|c_k^*|^p}{k^{2-p}} \leq C \|u\|_{B^p. a.p.}^p,$$

where $c_k^*$ is the nonincreasing rearrangement of the sequence $|c_k|$.

As usual, $\|u\|_{B^p. a.p.} = \lim L^{-1} \int_{-L}^L |u|^p \, dx$ and the symbol $\sim$ indicates that the sum converges in $B^p. a.p$. This condition implies that $c_k \to 0$, so that the sequence has a well-defined rearrangement $c_k^*$.

This result was proved by Paley [12] in the case where the $a_k$ are integers. In that case the right-hand side is simply the $L^p([0, 2\pi])$ norm of the periodic function $\overline{\int f \, d\mu_0}$.

**Proof.** The first inequality in (4.5) is obvious. The proof of the second uses approximation as in Theorem 1. First assume that $u$ is a polynomial.
Define $L_j$, $a_k$, and $u_j$ as in Theorem 1. Then by Paley’s theorem,

$$\left( \sum \frac{|c_k|_d}{k^{2-p}} \right)^{1/p} \lesssim C \left( L_j^{-1} \int_{-L_j}^{L_j} |u_j(x)|^p \, dx \right)^{1/p} \leq C \left( L_j^{-1} \int_{-L_j}^{L_j} |u(x)|^p \, dx \right)^{1/p} + CE \sum |c_k^*|$$

and taking limits proves the result.

Let $c_n \to u \in B^p.a.p.$ be polynomials, for which the second inequality has just been proved. The method of Theorem 1 shows that it holds for $u$ as well. Only condition (2.3) on the convergence of the rearranged sequences of coefficients requires justification. Convergence in $B^p.a.p.$ implies convergence in $B.a.p.$ which implies the uniform convergence of $\lim c_k^{(n)} = c_k$ (Holder’s inequality). By Lemma 9 below, this implies the uniform convergence of $\lim (c_k^{(n)})^* = c_k^*$. This completes the argument for Theorem 6.

**Lemma 9.** Given two bounded sequences $h$ and $g$ with rearrangements $h^*$ and $g^*$, we have

$$\|h^* - g^*\|_\infty \leq \|h - g\|_\infty.$$

**Proof.** Given a positive integer $t$, we will show that $|h^*(t) - g^*(t)| \leq \|h - g\|_\infty$. We may assume $h^*(t) \geq g^*(t)$. We have that

$$h^*(t) = \sup_{s \in E} |h(s)|,$$

where the sup is taken over all sets $E$ of exactly $t$ positive integers. The sup is attained for some set which we will call $E$. Then

$$g^*(t) \geq \min_{s \in E} |g(s)| = |g(s_0)|$$

for some $s_0 \in E$. So,

$$0 \leq h^*(t) - g^*(t) \leq \min_{s \in E} |h(s)| - |g(s_0)| \leq |h(s_0)| - |g(s_0)| \leq \|h - g\|_\infty$$

and this proves Lemma 9.

**References**
