

Varāhamihira's Pandiagonal Magic Square of the Order Four

TAKAO HAYASHI

Department of Mechanical Engineering, Doshisha University, Karasuma-Imadegawa, Kyoto 602, Japan

I have demonstrated in this paper that Varāhamihira (ca. 550), an Indian authority of astronomy, astrology, and divination, utilized a pandiagonal magic square of sixteen cells in prescribing how to prepare perfumes from sixteen original substances. I have also tried to reconstruct the original magic square underlying Varāhamihira's square, which consists of two sets of the series [1, 2, 3, 4, 5, 6, 7, 8], and have suggested a probable historical link between Indian and Islamic magic squares. © 1987 Academic Press, Inc.

सार्वकर्षिकं षोडशकोष्ठकं भद्रमेकं ज्योतिःशास्त्राधिकृताचार्यनराह-
मिहिरेण प्रयुक्तं सुगन्धप्रसाधनविधाने स्वविरचितायां शकुन-
परीक्षणविषयायां बृहत्संहितायां (आसन्नशतैस्तब्दाः ५५०) इति
स्फुटं निदिष्टमत्र मया । नराहमिहिरस्य यद्भद्रमेकाद्यष्टान्त-
श्रेढीद्वयमयं भवति तस्य मूलभद्रमेकादिषोडशान्तश्रेढीमयं
भवेदिति तदवस्थापि निरूपिता । तथा च भारतीयभद्र-
ताजिकीयभद्रमध्येतिहासिकसंधिरपि सूचिता ॥ © 1987 Academic
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この論文は、天竺・占星・吉凶占いの分野に於けるインド
の権威の一人、ワラハミヒラ（550年頃）が、16種の原
料から種々の香料と作り出したと記述するところ、4次元の対角
線魔方陣と利用したことを明らかにする。また彼の魔方陣には
数列 [1, 2, 3, 4, 5, 6, 7, 8] の数が二度ずつ用いられて
いるが、その元になったと思われる原魔方陣の復元を試み、
そのインドの魔方陣とイスラムの魔方陣を歴史的に結びつ
ける可能性を示唆する。 © 1987 Academic Press, Inc.

AMS 1980 subject classifications: 01A32, 10-03.
KEY WORDS: Indian mathematics.

The main purpose of this paper is to prove that a pandiagonal magic square of the order four [1] was known to Varāhamihira, an Indian authority of astronomy, astrology, and divination, who wrote his voluminous *Bṛhatsaṃhitā* consisting of 106 chapters in about A.D. 550 [2].

I. VARĀHAMIHIRA'S MAGIC SQUARE

Although the main theme of the *Bṛhatsaṃhitā* is divination, not all chapters of the work are devoted to that purpose. Thus Chapter 76 named *gandhayukti* (combinations of perfumes), which treats the composition (the word *yukti* has this meaning also) of various kinds of perfumes, contains no elements of divination. In the following passage consisting of four stanzas, Varāhamihira prescribes how to prepare the perfumes called *sarvatobhadra* (Good for All [Purposes]).

Text

- Stanza 23. dvitrindriyāṣṭabhāgair aguruḥ patraṃ turuṣkaśaileyau,
viṣayāṣṭapakṣadahanāḥ priyaṅgumustārasāḥ keśaḥ.
Stanza 24. sprṅkāvaktagarāṇāṃ māṃsyāś ca kṛtaikasaptaṣaḍbhāgāḥ,
saptartuvedacandrair malayanakhaśrikakundrukāḥ.
Stanza 25. ṣoḍaśake kacchapute yathā tathā miśrite caturdravye,
ye 'trāṣṭādaśabhāgās te 'smin gandhādayo yogāḥ.
Stanza 26. nakhatagaraturuṣkayutā jātikarpūramṛgakṛtodbodhāḥ,
guḍanakhadhūpyā gandhāḥ kartavyāḥ sarvatobhadrah.

Translation

Stanza 23. Two, three, five, and eight parts [respectively of] [3] Aguru, Patra, Turuṣka, and Śaileya [are taken]; five, eight, two, and three [parts respectively of] Priyaṅgu, Mustā, Rasa, and Keśa;

Stanza 24. four, one, seven, and six parts [respectively] of Sprṅkā, Tvac, Tagara, and Māṃsī; and [finally] seven, six, four, and one [parts respectively of] Malaya, Nakha, Śrika, and Kundruka.

Stanza 25. When four [of those sixteen] substances are mixed together as if [they were] in a *kacchapuṭa* of sixteen [cells], the combinations, [each of] which consists of eighteen parts in this case, are the first [compounds] for the perfumes [to be obtained] here.

Stanza 26. [Those compounds] are blended with Nakha, Tagara, and Turuṣka; are excited [4] with Jāti, Karpūra, and Mrga; and should be fumigated with Guḍa and Nakha. The perfumes [called] *sarvatobhadra* (Good for All [Purposes]) should be made [in this manner].

In stanzas 23 and 24, Varāhamihira gives the names of sixteen original substances together with their proportions. But he reveals in the second quarter of stanza 25 that only four of them are required at a time. How, then, should we choose four substances? The choice is certainly not free [5], as Varāhamihira specifies the combinations in the first and the third quarters of stanza 25:

as if [they were] in a *kacchapuṭa* of sixteen [cells];

and

[each of] which consists of eighteen parts in this case.

What is the *kacchapuṭa*? Sanskrit dictionaries give the meaning “a box with compartments” to this word used by Varāhamihira. This is not wrong, but it is not precise either. Following the commentator Bhaṭṭotpala (A.D. 967) [6], I take it to mean a square figure having the same number (four in the present case) of cells in each row and in each column. The same meaning fits well in stanza 29, where the phrase “navakoṣṭhāt kacchapuṭād” (from a *kacchapuṭa* having nine cells) occurs; there also are a square number of cells. The word might have also meant such a square figure with numerals in its cells, i.e., a magic square, since the concept of the magic square is indispensable for understanding stanza 25 under consideration, although it does not play any part in stanza 29 [7].

The word *kacchapuṭa* seems to have originally meant “the carapace of a turtle.” The word *kaccha*, which usually means “bank” or “margin” or “skirt” or “bordering region,” perhaps means in this compound the “limbs” of a turtle, as in the case of the compound *kacchapa* or “one which swallows its limbs,” i.e., “a turtle” [8]. The word *puṭa*, on the other hand, means “a hollow space” or “a covering.” Thus *kacchapuṭa* could have meant “the covering of the limbs,” i.e., “the carapace,” of a turtle.

The association of a turtle with magic squares immediately reminds us of the old Chinese legend that a miraculous turtle, on the back of which a diagram called the Lo Shu (洛書 Lo River Writing) was written, came out of the river Lo to help the Emperor Yü in governing the empire; the diagram, the original form of which can no longer be restored on a well-documented basis, was interpreted later, from the 10th century on, by Chinese writers as a magic square of order three with the numbers expressed with dots [Needham 1959, 56–57; Cammann 1961, 37–80].

Now, when we fill the cells of the *kacchapuṭa* with the given proportions in the given order, we obtain the diagram shown in Fig. 1. This diagram is itself a pandiagonal magic square with its constant sum 18, the number that Varāhamihira himself explicitly states in stanza 25. That is, the four numbers (1) in each row, (2) in each column, (3) in each of the two main diagonals, and (4) in each of the six broken diagonals, make the constant sum 18. Being pandiagonal, this magic square has many other tetrads whose sums are 18. Some of them have been noted by the commentator Bhaṭṭotpala [9]:

asmin kacchapuṭe gandhādaya (1) ūrdhvādahaḥkramena (2) tiryag vā (3) caturṣu koṇeṣu vā (4) madhyamacatuṣkoṇe vā (5) koṇakoṣṭhacatuṣṭaye vā (6) prākpaṅktau vā madhyamakōṣṭhadvaye vāntyapaṅktau¹ madhyamakōṣṭhadvaye vā (7) dvitīyatritīyapaṅktau vādyantakoṣṭhake vā yena tena prakāreṇa caturṣu miśriteṣu.

Thus, the tetrads pointed out by Bhaṭṭotpala are those in:

- (1) each column [(2, 5, 4, 7), etc.]
- (2) each row [(2, 3, 5, 8), etc.]
- (3) each of the four corners [(2, 3, 5, 8), etc.]

2	3	5	8
5	8	2	3
4	1	7	6
7	6	4	1

FIG. 1. Kacchapuṭa of 16 cells.

(4) the central small square [(8, 2, 1, 7)]

(5) the four corner cells [(2, 8, 7, 1)]

(6) the two central cells (3, 5) of the first row and those (6, 4) of the last row [(3, 5, 6, 4)]

(7) the two central cells (5, 4) of the first column and those (3, 6) of the last column [(5, 3, 4, 6)].

Curiously enough, Bhaṭṭotpala refers to none of the two main diagonals and the six broken diagonals. It is possible that Bhaṭṭotpala either did not know the concept of the pandiagonal magic square or gave it another equivalent definition, but this does not explain why he excluded the diagonals from his long list of tetrads.

In any case, it is now clear enough that Varāhamihira wanted the reader of his book to arrange the sixteen original substances in a square figure having sixteen cells, and to take and mix up four out of them in exactly the same manner as we take and sum up four numbers in a magic square of sixteen cells.

The use of the magic square must have affected his procedure for making the perfumes. This is confirmed by the fact that he prescribes, after the mixture of the four substances, addition of Nakha, Tagara, and Turuṣka, all of which are already included in the sixteen original substances. On the other hand, Varāhamihira's square seems to have been a result of adapting an original magic square consisting of the natural series from 1 to 16, to the problem under consideration, i.e., the problem of combining four substances in the given proportions; for the ratio, 16:1, of the largest to the smallest of the numbers used in that square would have been too large for his purpose.

2. RECONSTRUCTION OF THE ORIGINAL MAGIC SQUARE

We have in Varāhamihira's square (Fig. 1) two sets of the series [1, 2, 3, 4, 5, 6, 7, 8]. The most natural way of restoring the natural series, 1 to 16, used in the underlying magic square is to add 8 to each term of one of the two sets, with the other set kept unchanged. Which terms, then, in Varāhamihira's square should be augmented? It will be realized that, in order to keep the square magic, we have to add the same amount (8) to exactly two terms of each row, each column, and each

¹ Read cāntya-.

(a)	(b)
$\begin{array}{cccc} \underline{2} & 3 & \underline{5} & 8 \\ 5 & \underline{8} & 2 & \underline{3} \\ 4 & \underline{1} & 7 & \underline{6} \\ \underline{7} & 6 & \underline{4} & 1 \end{array}$	$\begin{array}{cccc} 2 & \underline{3} & 5 & \underline{8} \\ \underline{5} & 8 & \underline{2} & 3 \\ \underline{4} & 1 & \underline{7} & 6 \\ 7 & \underline{6} & 4 & \underline{1} \end{array}$
(c)	(d)
$\begin{array}{cccc} \underline{2} & 3 & 5 & \underline{8} \\ \underline{5} & 8 & 2 & \underline{3} \\ 4 & \underline{1} & \underline{7} & 6 \\ 7 & \underline{6} & \underline{4} & 1 \end{array}$	$\begin{array}{cccc} 2 & \underline{3} & \underline{5} & 8 \\ 5 & \underline{8} & \underline{2} & 3 \\ \underline{4} & 1 & 7 & \underline{6} \\ \underline{7} & 6 & 4 & \underline{1} \end{array}$

FIG. 2. The numbers to be increased.

of the two main diagonals. This restriction leads us to only four possible cases. See Fig. 2, where the underlined numbers are to be increased severally by 8.

The results of the augmentation are shown in Fig. 3, where squares (a) and (b) are pandiagonal, but (c) and (d) are not. It is most probable that Varāhamihira knew one of these squares. All that he had to do in order to obtain his square for the combinations of perfumes was to subtract 8 from each of the eight higher numbers, 9 to 16, in the original square.

Remarkable is the fact that the magic square shown in Fig. 3a, with a rotation of 90°, reappears in the 13th-century Islamic world as one of the most popular magic squares (see Fig. 6 below).

(a)	(b)
$\begin{array}{cccc} 10 & 3 & 13 & 8 \\ 5 & 16 & 2 & 11 \\ 4 & 9 & 7 & 14 \\ 15 & 6 & 12 & 1 \end{array}$	$\begin{array}{cccc} 2 & 11 & 5 & 16 \\ 13 & 8 & 10 & 3 \\ 12 & 1 & 15 & 6 \\ 7 & 14 & 4 & 9 \end{array}$
(c)	(d)
$\begin{array}{cccc} 10 & 3 & 5 & 16 \\ 13 & 8 & 2 & 11 \\ 4 & 9 & 15 & 6 \\ 7 & 14 & 12 & 1 \end{array}$	$\begin{array}{cccc} 2 & 11 & 13 & 8 \\ 5 & 16 & 10 & 3 \\ 12 & 1 & 7 & 14 \\ 15 & 6 & 4 & 9 \end{array}$

FIG. 3. The magic squares reconstructed.

6	7	2
1	5	9
8	3	4

FIG. 4. The magic square referred to in the *Ta Tai Li Chi*.

3. A HISTORICAL COMMENT

It is generally accepted that the idea of magic squares was born in ancient China and spread over the world, although we cannot determine the date of the birth [Needham 1959, 55–56; Cammann 1961; Cammann 1968/1969, 186 ff.]. The oldest of all the known documents that refer to magic squares is the *Ta Tai Li Chi* (『大戴禮記』 Record of rites) compiled by Tai the Elder (大戴) in about A.D. 80. This book gives the numbers of the simplest magic square, i.e., that of order three, in the order “2, 9, 4, 7, 5, 3, 6, 1, 8” [Needham 1959, 58]. When we arrange these numbers in a square having three rows of three cells each, in the order of the ordinary Chinese writing, i.e., from the top to bottom and then from the right to left, we obtain the magic square shown in Fig. 4.

After this, from time to time we come across references to the same magic square or its variations in Chinese literature [Needham 1959, 57–59; Cammann 1961] [10], but the first instance hitherto known to us of a magic square of order four occurs only in the 10th century in the *Rasā'il* of the Ikhwān al-Ṣafā' (Brethren of Purity, ca. 983), an encyclopaedic work in Arabic [Hermelink 1958, 207]. This magic square (Fig. 5), however, is not pandiagonal.

Pandiagonal magic squares of the order four, which became well known and very popular in Islam and in India after the 13th century, have long been believed to have been discovered for the first time in the Islamic world in the 12th or 13th century [Cammann 1968/1969, 202, 273–274] [11]. The most famous Islamic square of the order four is cited in Fig. 6 [Ahrens 1922, Figs. 6, 7] [12]. It is remarkable indeed that this square can be obtained by rotating by 90° one of the four magic squares (Fig. 3a) reconstructed from Varāhamihira's square. I think this cannot be a mere coincidence, since as many as 880 magic squares of the order four are known to exist [Lehmer 1933a, b]. This fact will oblige us to reconsider the history of magic squares and, in particular, the problem of their transmissions.

4	14	15	1
9	7	6	12
5	11	10	8
16	2	3	13

FIG. 5. The magic square of order four by the Ikhwān al-Ṣafā'.

8	11	14	1
13	2	7	12
3	16	9	6
10	5	4	15

FIG. 6. The famous pandiagonal magic square of the order four in Islam.

NOTES

1. A magic square of the order n is defined as a set of n^2 numbers so arranged in a square, that the sum of each row, each column, and each of the two main diagonals shall be the same amount, which we call the constant sum. If, in a magic square, the sum of each broken diagonal is equal to the constant sum, then the magic square is called pandiagonal. See, for example, Fig. 6, where the sum of 8, 12, 9, and 5 is equal to the constant sum 34. The natural numbers from 1 to n^2 are usually, but not always, employed in a magic square of the order n ; in that case, the total sum of the numbers is $n^2(n^2 + 1)/2$, and hence the constant sum $n(n^2 + 1)/2$. For magic squares in general and mathematical analysis of pandiagonal magic squares, see [Andrews 1917].

2. For an English translation, see [Bhat 1981/1982]. For its contents and historical position, see [Pingree 1981, 71–75].

3. [A] in my translation indicates that A has been supplied by me.

4. The original word is *ud-budh*, the exact value of which in this context is not clear to me. The meaning “fumigation” given in Sanskrit dictionaries to this word used by Varāhamihira is certainly not appropriate, since fumigation is separately indicated by the word *dhūp* in stanza 26 itself. It is noteworthy that the words *budh/ud-budh/pra-budh* are always associated with the substances called Jāti, Karpūra, and Mṛga. See *Brhatsamhitā*, Chap. 76, stanzas 11, 12, 16, 26, and 27.

5. The choice is free when the perfumes called Pārijāta are made, in stanza 27, from four taken from the same sixteen original substances: *bahavo'tra pārijātās caturbhir icchāparigṛhītaiḥ*. In that case, there are 1820 (= ${}_{16}C_4$) possible combinations. Incidentally, Varāhamihira gives the value of ${}_{16}C_4$ correctly (in stanza 20) in a process of computing the whole number of the perfumes that are made from four substances taken, at the ratio of 1 : 2 : 3 : 4, from sixteen original substances, although his final result, 174,720 (in stanza 17), is wrong. For this last point, see [Hayashi 1979, 163].

6. Bhaṭṭotpala, in his comment on stanzas 23 and 24, gives a square figure having four rows of four cells each, and puts into each cell the first syllables of the names of the given substances, followed by the Indian numerals indicating the given proportions.

7. A magic square was later called *jamta* (= Skt. *yantra*, lit. “a device” or “a diagram”) by Thakkura Pherū (fl. 1315), and *bhadra* (lit. “good fortune”) by Nārāyaṇa (1356).

8. Durga, cited in Apte's dictionary (s.v. *kaccha*), equates *kaccha* in *kacchapa* to *mukhasamputa*; the latter word probably means the “box-head” of a turtle.

9. Bhaṭṭotpala's *vivṛti* on *Brhatsamhitā* 76, 25–26.

10. For Chinese magic squares in general, see [Cammann 1962].

11. For Islamic magic squares in general, see [Ahrens 1917, 1922; Bergsträsser 1923; Schuster 1972]. New materials for the history of Islamic magic squares have appeared in [Sesiano 1980, 1981]. For an account of Indian magic squares, see [Cammann 1968/1969, 271–290].

12. Cammann [1968/1969, 202, 273–274] seems to hold that Indian magic squares of the order four were derived from the Islamic one shown in Fig. 6.

REFERENCES

Sanskrit Texts

- Ṭhakkura Pherū. ca. 1315. *Gaṇitasāra*, Agaracanda and B. Nāhaṭā, Eds. In *Ratnaparikṣādīsaptagrānthisamgraha*, Rājasthāna Purātana Granthamālā 44, Jodhpur 1961.
- Nārāyaṇa. 1356. *Gaṇitakaumudī*, P. Dvivedin, Ed. Princess of Wales Saraswati Bhavana Texts 57, 2 vols. Benares 1936/1942.
- Varāhamihira. ca. 550. *Brhatsamhitā*, with Bhaṭṭotpala's commentary (*vivṛti*), S. Dvivedin, Ed. Vizianagaram Sanskrit Series 10. 2 vols. Benares 1895/1897. Reprinted by A. Tripāthī, Sarasvatī Bhavana Granthamālā 97, Vārāṇasī 1968.

Other Works

- Ahrens, W. 1917. Studien über die "magischen Quadrate" der Araber. *Der Islam* 11, 186–250.
- 1922. Die "magischen Quadrate" al-Būnī's. *Der Islam* 12, 157–177.
- Andrews, W. S. 1917. *Magic squares and cubes*, 2nd ed. Chicago, 1917. Reprinted, New York: Dover, 1960.
- Apte, V. S. 1957/1959. *The practical Sanskrit–English dictionary*. Revised and enlarged edition, Poona.
- Bergsträsser, G. 1923. Zu den magischen Quadraten. *Der Islam* 13, 227–235.
- Bhat, A. K. 1981/1982. *Varāhamihira's Brhatsamhitā with an English translation, notes, and comments*. 2 vols. Delhi: Motilal Banarsidass.
- Cammann, S. 1961. The magic square of three in old Chinese philosophy and religion. *History of Religions* 1, 37–80.
- 1962. Old Chinese magic squares. *Sinologica* 7, 14–53.
- 1968/1969. Islamic and Indian magic squares. *History of Religions* 8, 181–209, 271–299.
- Hayashi, T. 1979. Indo ni okeru junretsu, kumiawase, rekkyo (Permutations, combinations, and enumerations in ancient India). *Kagakushi Kenkyu* 131, 158–171.
- Hermelink, H. 1958. Die ältesten magischen Quadrate höherer Ordnung und ihre Bildungsweise. *Sudhoffs Archiv* 42, 199–217.
- Lehmer, D. N. 1933a. A complete census of 4×4 magic squares. *Bulletin of the American Mathematical Society* 39, 764–767.
- 1933b. A census of squares of order 4, magic in rows, columns, and diagonals. *Bulletin of the American Mathematical Society* 39, 981–982.
- Needham, J. E. 1959. *Science and civilization in China*, Vol. 3. Cambridge: Cambridge Univ. Press.
- Pingree, D. 1981. *Jyotiḥśāstra: Astral and mathematical literature*. In *A history of Indian literature*, J. Gonda, Ed., Vol. 6, fasc. 4. Wiesbaden: Harrassowitz.
- Schuster, H. S. 1972. Magische Quadrate im islamischen Bereich: Ihre Entlehnung ins Abendland im Mittelalter sowie ihre Vorstufen. *Der Islam* 49, 1–80.
- Sesiano, J. 1980. Herstellungsverfahren magischer Quadrate aus islamischer Zeit (I). *Sudhoffs Archiv* 64, 187–196.
- 1981. Herstellungsverfahren . . . (II). *Sudhoffs Archiv* 65, 251–265.