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Controllability of Second-Order Semilinear Neutral Functional Differential Systems in Banach Spaces

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Abstract—Sufficient conditions for controllability of semilinear second-order neutral functional differential systems in Banach spaces are established using the theory of strongly continuous cosine families. The results are obtained by using the Leray-Schauder alternative.

Keywords—Controllability, Semilinear neutral functional differential system, Leray-Schauder alternative.

1. INTRODUCTION

Controllability of linear and nonlinear systems represented by ordinary differential equations in finite-dimensional space has been extensively studied. Several authors have extended the concept to infinite-dimensional systems in Banach Spaces with bounded operators. Chukwu and Lenhart [1] have studied the controllability of nonlinear systems in abstract spaces. Naito [2,3] has studied the controllability for semilinear systems and nonlinear Volterra integrodifferential systems. Quinn and Carmichael [4] have shown that the controllability problem in Banach spaces can be converted into one of a fixed-point problem for a single-valued mapping. Balachandran *et al.* [5,6] established sufficient conditions for controllability of nonlinear integrodifferential systems in Banach spaces.

In many cases, it is advantageous to treat the second-order abstract differential equations directly rather than to convert them to first-order systems. For example, Fitzgibbon [7] used the second-order abstract differential equations for establishing the boundedness of solutions of the equation governing the transverse motion of an extensible beam. A useful tool for the study of abstract second-order equations is the theory of strongly continuous cosine families. We will make use of some of the basic ideas from cosine family theory [8,9]. Motivation for second-order systems can be found in [7,10]. Park *et al.* [11] and Balachandran *et al.* [12] have discussed the controllability of second-order nonlinear systems in Banach spaces with the help of the Schauder fixed-point theorem. The purpose of this paper is to study the controllability of semilinear second-order neutral functional differential systems in Banach spaces by using the Leray-Schauder alternative [13].

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2. PRELIMINARIES

We consider the semilinear second-order neutral control system

$$\frac{d}{dt}\left[x'(t) - g\left(t, x_t\right)\right] = Ax(t) + Bu(t) + f\left(t, x_t, x'(t)\right), \qquad t \in J = [0,T], \quad x_0 = \phi, \qquad x'(0) = y_0, \tag{1}$$

where the state $x(.)$ takes values in the reflexive Banach space X, $y_0 \in X$, A is the infinitesimal generator of the strongly continuous cosine family $C(t)$, $t \in R$, of bounded linear operators in X, $f : J \times C \times X \to X$ and $g : J \times C \to X$ are given functions, B is a bounded linear operator from U to X, and the control function $u(.)$ is given in $L^2(J, U)$, a Banach space of admissible control functions, with U also being a Banach space and $\phi \in C$.

Here $C = C([-r, 0], X)$ is the Banach space of all continuous functions $\phi : [-r, 0] \to X$ endowed with the sup-norm

$$\|\phi\| = \sup\{|\phi(\theta)| : -r \le \theta \le 0\}.$$

Also, for $x \in C([-r,T], X)$, we have $x_t \in C$ for $t \in J$, $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r, 0]$.

DEFINITION 1. *(See [8].) A one-parameter family $C(t)$, $t \in R$, of bounded linear operators in the Banach space X is called a strongly continuous cosine family iff*

(i) $C(s+t) + C(s-t) = 2C(s)C(t)$, *for all* $s, t \in R$;
(ii) $C(0) = I$;
(iii) $C(t)x$ *is continuous in* t *on* R *for each fixed* $x \in X$.

Define the associated sine family $S(t)$, $t \in R$, by

$$S(t)x = \int_0^t C(s)x\, ds, \qquad x \in X, \quad t \in R.$$

Assume the following conditions on A.

(H_1) A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in R$, of bounded linear operators from X into itself and the adjoint operator A^* is densely defined i.e., $\overline{D(A^*)} = X^*$ (see [14]).

The infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in R$, is the operator $A : X \to X$ defined by

$$Ax = \left.\frac{d^2}{dt^2} C(t)x\right|_{t=0}, \qquad x \in D(A),$$

where $D(A) = \{x \in X : C(t)x$ is twice continuously differentiable in $t\}$. Define $E = \{x \in X$: $C(t)x$ is once continuously differentiable in $t\}$.

To establish our main theorem, we need the following lemmas.

LEMMA 1. *(See [8].) Let (H_1) hold. Then*

(i) *there exist constants $M \ge 1$ and $\omega \ge 0$ such that*

$$|C(t)| \le Ne^{\omega|t|} \text{ and } |S(t) - S(t^*)| \le N\left|\int_t^{t^*} e^{\omega|s|}\, ds\right|, \qquad \text{for } t,\ t^* \in R;$$

(ii) $S(t)X \subset E$ *and* $S(t)E \subset D(A)$, *for* $t \in R$;
(iii) $\frac{d}{dt}C(t)x = AS(t)x$, *for* $x \in E$ *and* $t \in R$;
(iv) $\frac{d^2}{dt^2}\, C(t)x = AC(t)x$, *for* $x \in D(A)$ *and* $t \in R$.

LEMMA 2. *(See [8].) Let (H_1) hold, let $v : R \to X$ such that v is continuously differentiable, and let $q(t) = \int_0^t S(t-s)v(s)\,ds$. Then*

$$q \text{ is twice continuously differentiable and for } t \in R,\ q(t) \in D(A),$$
$$q'(t) = \int_0^t C(t-s)v(s)\,ds \qquad \text{and} \qquad q''(t) = Aq(t) + v(t).$$

LEMMA 3. LERAY-SCHAUDER ALTERNATIVE. *(See [13].) Let S be a convex subset of a normed linear space Y and assume $0 \in S$. Let $F : S \to S$ be a completely continuous operator, and let*

$$\xi(F) = \{x \in S : x = \lambda F x \text{ for some } 0 < \lambda < 1\}.$$

Then either $\xi(F)$ is unbounded or F has a fixed point.

We make the following assumptions.

(H_2) $C(t)$, $t > 0$ is compact.
(H_3) $Bu(t)$ is continuous in t.
(H_4) The linear operator $W : L^2(J,U) \to X$ defined by

$$Wu = \int_0^T S(T-s)Bu(s)\,ds$$

induces a bounded invertible operator $\tilde{W} : L^2(J,U)/\ker W \to X$. (See Remark for construction of $\tilde{W}^{-1}$.)
(H_5) $g : J \times C \to X$ is completely continuous and for any bounded set K in $C([-r,T],X)$, the set $\{t \to g(t,x_t) : x \in K\}$ is equicontinuous in $C([0,T]),X)$.
(H_6) There exists constants c_1 and c_2 such that

$$|g(t,\phi)| \le c_1(\|\phi\|) + c_2, \qquad t \in J, \quad \phi \in C.$$

(1) (H_7)" The function $f(t,.,.)$: $C \to X$ is continuous for each $t \in J$.
(H_8) The function $f(.,x,y) : J \to X$ is strongly measurable for each $x \in C$ and $y \in X$.
(H_9) For every positive constant k, there exists $\alpha_k \in L^1(J)$ such that

$$\sup_{\|x\|,|y| \le k} |f(t,x,y)| \le \alpha_k(t), \qquad \text{for a.a. } t \in J.$$

Then the integral equation formulation of (1) can be written as (see [15,16])

$$\begin{aligned} x(t) &= \phi(t), \qquad -r \le 0 \le T, \\ x(t) &= C(t)\phi(0) + S(t)\left[y_0 - g(0,\phi)\right] + \int_0^t C(t-s)g(s,x_s)\,ds \\ &\quad + \int_0^t S(t-s)Bu(s)\,ds + \int_0^t S(t-s)f\left(s,x_s,x'(s)\right)\,ds, \qquad t \in J. \end{aligned} \tag{2}$$

DEFINITION 2. *System (1) is said to be controllable on J if for every $\phi \in C$ with $\phi(0) \in D(A)$, $y_0 \in E$, and $x_1 \in X$, there exists a control $u \in L^2(J,U)$ such that the solution $x(.)$ of (1) satisfies $x(T) = x_1$.*

3. MAIN RESULT

THEOREM. *Suppose (H_1)–(H_9) hold and there exists a continuous function $p : J \to [0, \infty)$ such that*

$$|f(t, x, y)| \leq m(t)\Omega(\|x\| + |y|), \qquad t \in J, \quad x \in C, \quad \text{and} \quad y \in X,$$

where $\Omega : (0, \infty) \to (0, \infty)$ is a continuous nondecreasing function and

$$\int_0^T m(s)\, ds < \int_c^\infty \frac{ds}{s + \Omega(s)},$$

where

$$m(t) = \max\{c_1 [Mc_1 + M + M^*], M(c_1 T + T + 1)p(t)\},$$
$$M = \sup\{|C(t)| : t \in J\}, \qquad M^* = \sup\{|AS(t)| : t \in J\},$$

$$c = (M + M^*)\|\phi\| + (1 + T)M\{|y_0| + c_1\|\phi\| + c_2\} + (M + M^*) c_2 T + c_1\|\phi\| + c_2 + (T^2 + T) MN,$$

$$N = |B| \left|\tilde{W}^{-1}\right| \left[|x_1| + M\|\phi\| + MT\{|y_0| + c_1\|\phi\| + 2c_2\} + Mc_1 \int_0^T \|x_\tau\|\, d\tau + MT \int_0^T p(s)\Omega(\|x_s\| + |x'(s)|)\, ds\right].$$

Then system (1) is controllable on J.

PROOF. Consider the space $Z = C([-r, T], X) \cap C^1(J, X)$ with norm

$$\|x\|^* = \max\{\|x\|_r, \|x\|_0\},$$

where

$$\|x\|_r = \sup\{|x(t)| : -r \leq t \leq T\}, \qquad \|x\|_0 = \sup\{|x'(t)| : 0 \leq t \leq T\}.$$

Using (H_4), for an arbitrary function $x(.)$, we define the control

$$u(t) = \tilde{W}^{-1}\left[x_1 - C(T)\phi(0) - S(T)[y_0 - g(0, \phi)] - \int_0^T C(T - s)g(s, x_s)\, ds - \int_0^T S(T - s)f(s, x_s, x'(s))\, ds\right](t).$$

Using this control, we will show that the operator $F : Z \to Z$ defined by

$$(Fx)(t) = \phi(t), \qquad -r \leq t \leq 0,$$
$$(Fx)(t) = C(t)\phi(0) + S(t)[y_0 - g(0, \phi)] + \int_0^t C(t - s)g(s, x_s)\, ds + \int_0^t S(t - s)B\tilde{W}^{-1}\left[x_1 - C(T)\phi(0) - S(T)[y_0 - g(0, \phi)] - \int_0^T C(T - \tau)g(\tau, x_\tau)\, d\tau - \int_0^T S(T - \tau)f(\tau, x_\tau, x'(\tau))\, d\tau\right](s)\, ds + \int_0^t S(t - s)f(s, x_s, x'(s))\, ds, \qquad t \in J,$$

has a fixed point. This fixed point is then a solution of equation (2).

Clearly, $(Fx)(T) = x_1$, which means that the control u steers the system from the initial state x_0 to x_1 in time T, provided we obtain a fixed point of the nonlinear operator F.

In order to study the controllability problem for system (1), we apply Lemma 3 to the following system:

$$\frac{d}{dt}\left[x'(t) - \lambda g\left(t, x_t\right)\right] = \lambda Ax(t) + \lambda Bu(t) + \lambda f\left(t, x_t, x'(t)\right), \qquad t \in J, \quad \lambda \in (0,1). \tag{3}$$

Let x be a mild solution of system (3). From

$$\begin{aligned} x(t) &= \lambda C(t)\phi(0) + \lambda S(t)\left[y_0 - g(0,\phi)\right] + \lambda \int_0^t C(t-s)g(s,x_s)\,ds \\ &\quad + \lambda \int_0^t S(t-s)B\tilde{W}^{-1}\Bigg[x_1 - C(T)\phi(0) - S(T)\left[y_0 - g(0,\phi)\right] \\ &\quad - \int_0^T C(T-\tau)g(\tau,x_\tau)\,d\tau - \int_0^T S(T-\tau)f\left(\tau, x_\tau, x'(\tau)\right)\,d\tau\Bigg](s)\,ds \\ &\quad + \lambda \int_0^t S(t-s)f\left(s, x_s, x'(s)\right)\,ds, \qquad t \in J, \end{aligned}$$

we have

$$\begin{aligned} |x(t)| \le M\|\phi\| + MT\{|y_0| + c_1\|\phi\| + 2c_2\} + Mc_1 \int_0^t \|x_s\|\,ds + MT^2N \\ + MT\int_0^t p(s)\Omega\left(\|x_s\| + |x'(s)|\right)\,ds, \qquad t \in J. \end{aligned}$$

Consider the function μ defined by

$$\mu(t) = \sup\{|x(s)| : -r \le s \le t\}, \qquad t \in J.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |x(t^*)|$. If $t^* \in [0,t]$, by the previous inequality, we have

$$\begin{aligned} \mu(t) \le M\|\phi\| + MT\left\{|y_0| + c_1\|\phi\| + 2c_2\right\} + Mc_1 \int_0^t \mu(s)\,ds + MT^2N \\ + MT\int_0^t p(s)\Omega(\mu(s) + |x'(s)|)\,ds, \qquad t \in J, \end{aligned}$$

If $t^* \in [-r,0]$, then $\mu(t) = \|\phi\|$ and the previous inequality holds since $M \ge 1$.

Denoting by $v(t)$, the right-hand side of the above inequality, we have

$$\mu(t) \le v(t), \quad t \in J, \qquad v(0) = M\|\phi\| + MT\left\{|y_0| + c_1\|\phi\| + 2c_2\right\} + MT^2N,$$

and

$$\begin{aligned} v'(t) &= Mc_1\mu(t) + MTp(t)\Omega\left(\mu(t) + |x'(t)|\right), \qquad t \in J, \\ &\le Mc_1 v(t) + MTp(t)\Omega\left(v(t) + |x'(t)|\right), \qquad t \in J. \end{aligned}$$

By

$$\begin{aligned} x'(t) &= \lambda AS(t)\phi(0) + \lambda C(t)\left[y_0 - g(0,\phi)\right] + \lambda g\left(t, x_t\right) + \lambda \int_0^t AS(t-s)g\left(s, x_s\right)\,ds \\ &\quad + \lambda \int_0^t C(t-s)B\tilde{W}^{-1}\Bigg[x_1 - C(T)\phi(0) - S(T)\left[y_0 - g(0,\phi)\right] \\ &\quad - \int_0^T C(T-\tau)g(\tau,x_\tau)\,d\tau - \int_0^T S(T-\tau)f\left(\tau, x_\tau, x'(\tau)\right)\,d\tau\Bigg](s)\,ds \\ &\quad + \lambda \int_0^t C(t-s)f\left(s, x_s, x'(s)\right)\,ds, \qquad t \in J, \end{aligned}$$

we obtain

$$|x'(t)| \leq M^*\|\phi\| + M\{|y_0| + c_1\|\phi\| + c_2\} + c_1\|x_t\| + c_2 + M^*\left\{c_2T + c_1\int_0^t \|x_s\|\,ds\right\} + MTN + M\int_0^t p(s)\Omega(\|x_s\| + |x'(s)|)\,ds, \qquad t \in J.$$

Denoting by $r(t)$ the right-hand side of the above inequality, we have

$$|x'(t)| \leq r(t), \qquad t \in J,$$
$$r(0) = M^*\|\phi\| + M\{|y_0| + c_1\|\phi\| + c_2\} + c_1\|\phi\| + c_2 + M^*c_2T + MTN,$$

and

$$\begin{aligned} r'(t) &\leq c_1v'(t) + M^*c_1v(t) + Mp(t)\Omega(v(t) + r(t)), && t \in J, \\ &\leq c_1\{Mc_1v(t) + MTp(t)\Omega(v(t) + r(t))\} + M^*c_1v(t) + Mp(t)\Omega(v(t) + r(t)), && t \in J. \end{aligned}$$

Let

$$w(t) = v(t) + r(t), \qquad t \in J.$$

Then

$$w(0) = c$$

and

$$\begin{aligned} w'(t) &= v'(t) + r'(t) \\ &\leq c_1[Mc_1 + M + M^*]v(t) + M(c_1T + T + 1)p(t)\Omega(v(t) + r(t)) \\ &\leq c_1[Mc_1 + M + M^*]w(t) + M(c_1T + T+)p(t)\Omega(w(t)), \qquad t \in J. \end{aligned}$$

This implies

$$\int_{w(0)}^{w(t)} \frac{ds}{s + \Omega(s)} \leq \int_0^T m(s)\,ds < \int_c^\infty \frac{ds}{s + \Omega(s)}, \qquad t \in J.$$

This inequality implies that there is a constant K such that

$$w(t) = v(t) + r(t) \leq K, \qquad t \in J.$$

Then

$$|x(t)| \leq v(t) \leq K, \qquad t \in J,$$
$$|x'(t)| \leq r(t) \leq K, \qquad t \in J,$$

and hence,

$$\|x\|^* = \max\{\|x\|_r, \|x'\|_0\} \leq K,$$

where K depends only on T and on the functions m and Ω.

We shall now prove that the operator $F : Z \to Z$ defined by

$$(Fx)(t) = \phi(t), \qquad -r \leq t \leq 0,$$

$$(Fx)(t) = C(t)\phi(0) + S(t)[y_0 - g(0,\phi)] + \int_0^t C(t-s)g(s,x_s)\,ds + \int_0^t S(t-s)B\tilde{W}^{-1} \times \left[x_1 - C(T)\phi(0) - S(T)[y_0 - g(0,\phi)] - \int_0^T C(T-\tau)g(\tau,x_\tau)\,d\tau \times - \int_0^T S(T-\tau)f(\tau,x_\tau,x'(\tau))\,d\tau\right](s)\,ds + \int_0^t S(t-s)f(s,x_s,x'(s))\,ds, \qquad t \in J,$$

is a completely continuous operator.

Let $B_k = \{x \in Z,\ \|x\|^* \leq k\}$ for some $k \geq 1$. We first show that F maps B_k into an equicontinuous family. Let $x \in B_k$ and $t_1, t_2 \in J$. Then if $0 < t_1 < t_2 \leq T$,

$$
\begin{aligned}
&|(Fx)(t_1) - (Fx)(t_2)| \\
&\quad \leq |[C(t_1) - C(t_2)]\,\phi(0)| + |[S(t_1) - S(t_2)]\,[y_0 - g(0,\phi)]| \\
&\qquad + \left|\int_0^{t_1} [C(t_1 - s) - C(t_2 - s)]\, g(s, x_s)\, ds\right| + \left|\int_{t_1}^{t_2} C(t_2 - s)\, g(s, x_s)\, ds\right| \\
&\qquad + \left|\int_0^{t_1} [S(t_1 - s) - S(t_2 - s)]\, B\tilde{W}^{-1}\left[x_1 - C(T)\phi(0) - S(T)\,[y_0 - g(0,\phi)]\right.\right. \\
&\qquad - \left.\left.\int_0^T C(T-\tau) g(\tau, x_\tau)\, d\tau - \int_0^T S(T-\tau) f(\tau, x_\tau, x'(\tau))\, d\tau\right](s)\, ds\right| \\
&\qquad + \left|\int_{t_1}^{t_2} S(t_2 - s) B\tilde{W}^{-1}\left[x_1 - C(T)\phi(0) - S(T)[y_0 - g(0,\phi)]\right.\right. \\
&\qquad - \left.\left.\int_0^T S(T-\tau) f(\tau, x_\tau, x'(\tau))\, d\tau\right](s)\, ds\right| \\
&\qquad + \left|\int_0^{t_1} [S(t_1 - s) - S(t_2 - s)]\, f(s, x_s, x'(s))\, ds\right| \\
&\qquad + \left|\int_{t_1}^{t_2} S(t_2 - s)\, f(s, x_s, x'(s))\, ds\right| \\
&\quad \leq |C(t_1) - C(t_2)|\,\|\phi\| + |S(t_1) - S(t_2)|\,\{|y_0| + c_1\|\phi\| + c_2\} \qquad (4) \\
&\qquad + \int_0^{t_1} |C(t_1 - s) - C(t_2 - s)|\,\{c_1\|x_s\| + c_2\}\, ds \\
&\qquad + \int_{t_1}^{t_2} |C(t_2 - s)|\,\{c_1\|x_s\| + c_2\}\, ds \\
&\qquad + \int_0^{t_1} |S(t_1 - s) - S(t_2 - s)|\,|B|\left|\tilde{W}^{-1}\right|\Big[|x_1| + M\|\phi\| \\
&\qquad + MT\{|y_0| + c_1\|\phi\| + c_2\} + M\int_0^T \{c_1\|x_\tau\| + c_2\}\, d\tau + MT\int_0^T \alpha_k(\tau)\, d\tau\Big]\, ds \\
&\qquad + \int_{t_1}^{t_2} |S(t_2 - s)|\ |B|\ \left|\tilde{W}^{-1}\right|\Big[|x_1| + M\|\phi\| + MT\{|y_0| + c_1\|\phi\| + c_2\} \\
&\qquad + M\int_0^T \{c_1\|x_\tau\| + c_2\}\, d\tau + MT\int_0^T \alpha_k(\tau)\, d\tau\Big]\, ds \\
&\qquad + \int_0^{t_1} |S(t_1 - s) - S(t_2 - s)|\alpha_k(s)\, ds \\
&\qquad + \int_{t_1}^{t_2} |S(t_2 - s)|\alpha_k(s)\, ds,
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
&|(Fx)'(t_1) - (Fx)'(t_2)| \\
&\quad \leq |A\,(S(t_1) - S(t_2))\,\phi(0)| + |[C(t_1) - C(t_2)]\,[y_0 - g(0,\phi)]| \\
&\qquad + |g(t_1, x_{t_1}) - g(t_2, x_{t_2})| + \left|\int_0^{t_1} A\,(S(t_1 - s) - S(t_2 - s))\, g(s, x_s)\, ds\right| \qquad (5) \\
&\qquad + \left|\int_{t_1}^{t_2} AS(t_2 - s)\, g(s, x_s)\, ds\right|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\int_0^{t_1}\left[C(t_1-s)-C(t_2-s)\right]B\tilde{W}^{-1}\left[x_1-C(T)\phi(0)-S(T)\left[y_0-g(0,\phi)\right]\right.\right.\\
&\left.\left.-\int_0^T C(T-\tau)g(\tau,x_\tau)\,d\tau-\int_0^T S(T-\tau)f\left(\tau,x_\tau,x'(\tau)\right)d\tau\right](s)\,ds\right|\\
&+\left|\int_{t_1}^{t_2}C(t_2-s)\,B\tilde{W}^{-1}\left[x_1-C(T)\phi(0)-S(T)\left[y_0-g(0,\phi)\right]\right.\right.\\
&\left.\left.-\int_0^T C(T-\tau)g(\tau,x_\tau)\,d\tau-\int_0^T S(T-\tau)f\left(\tau,x_\tau,x'(\tau)\right)d\tau\right](s)\,ds\right|\\
&+\left|\int_0^{t_1}\left[C(t_1-s)-C(t_2-s)\right]f\left(s,x_s,x'(s)\right)ds\right|\\
&+\left|\int_{t_1}^{t_2}C(t_2-s)\,f\left(s,x_s,x'(s)\right)ds\right|\\
\le{}&\left|A\left(S(t_1)-S(t_2)\right)\right|\|\phi\|+\left|\left[C(t_1)-C(t_2)\right]\right|\left\{|y_0|+c_1\|\phi\|+c_2\right\}\\
&+\left|g(t_1,x_{t_1})-g(t_2,x_{t_2})\right|+\int_0^{t_1}\left|A\left(S(t_1-s)-S(t_2-s)\right)\right|\left\{c_1\|x_s\|+c_2\right\}ds \qquad (5)(\text{cont.})\\
&+\int_{t_1}^{t_2}\left|AS(t_2-s)\right|\left\{c_1\|x_s\|+c_2\right\}ds\\
&+\int_0^{t_1}\left|C(t_1-s)-C(t_2-s)\right||B|\left|\tilde{W}^{-1}\right|\left[|x_1|+M\|\phi\|\right.\\
&\left.+MT\left\{|y_0|+c_1\|\phi\|+c_2\right\}+M\int_0^T\left\{c_1\|x_\tau\|+c_2\right\}d\tau+MT\int_0^T\alpha_k(\tau)\,d\tau\right]ds\\
&+\int_{t_1}^{t_2}\left|C(t_2-s)\right||B|\left|\tilde{W}^{-1}\right|\left[|x_1|+M\|\phi\|+MT\left\{\|y_0\|+c_1\|\phi\|+c_2\right\}\right.\\
&\left.+M\int_0^T\left\{c_1\|x_\tau\|+c_2\right\}d\tau+MT\int_0^T\alpha_k(\tau)\,d\tau\right]ds\\
&+\int_0^{t_1}\left|C(t_1-s)-C(t_2-s)\right|\alpha_k(s)\,ds+\int_{t_1}^{t_2}\left|C(t_2-s)\right|\alpha_k(s)\,ds.
\end{aligned}
$$

The right-hand sides of (4) and (5) are independent of $y \in B_k$ and tend to zero as $t_2 - t_1 \to 0$, since $C(t), S(t)$ are uniformly continuous for $t \in J$. The compactness of $C(t), S(t)$ for $t > 0$ implies the continuity in the uniform operator topology. The compactness of $S(t)$ follows from that of $C(t)$.

Thus, F maps B_k into an equicontinuous family of functions. It is easy to see that the family FB_k is uniformly bounded.

Next we show $\overline{FB_k}$ is compact. Since we have shown FB_k is an equicontinuous collection, it suffices by the Arzela-Ascoli theorem to show that F maps B_k into a precompact set in X.

Let $0 < t \le T$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $x \in B_k$, we define

$$
\begin{aligned}
(F_\epsilon x)(t) &= C(t)\phi(0)+S(t)[y_0-g(0,\phi)]+\int_0^{t-\epsilon}C(t-s)g(s,x_s)\,ds\\
&+\int_0^{t-\epsilon}S(t-s)B\tilde{W}^{-1}\left[x_1-C(T)\phi(0)-S(T)[y_0-g(0,\phi)]-\int_0^T C(T-\tau)g(s,x_\tau)\,d\tau\right.\\
&\left.-\int_0^T S(T-\tau)f\left(\tau,x_\tau,x'(\tau)\right)d\tau\right](s)\,ds+\int_0^{t-\epsilon}S(t-s)f\left(s,x_s,x'(s)\right)ds, \qquad t\in J.
\end{aligned}
$$

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for each $t \in J$, and since

$$|f(t, x_{nt}, x_n'(t)) - f(t, x_t, x'(t))| \leq 2\alpha_q(t),$$

we have by dominated convergence theorem,

$$\begin{aligned}
\|Fx_n - Fx\| &= \sup_{t\in J} \left| \int_0^t C(t-s)[g(s, x_{ns}) - g(s, x_s)]\, ds \right. \\
&\quad - \int_0^t S(t-s)B\tilde{W}^{-1} \left[\int_0^T C(T-\tau)[g(\tau, x_{n\tau}) - g(\tau, x_\tau)]\, d\tau \right. \\
&\quad \left. + \int_0^T S(T-\tau)\left[f(\tau, x_{n\tau}, x_n'(\tau)) - f(\tau, x_\tau, x'(\tau))\right] d\tau \right](s)\, ds \\
&\quad \left. + \int_0^t S(t-s)\left[f(s, x_{ns}, x_n'(s)) - f(s, x_s, x'(s))\right] ds \right| \\
&\leq \int_0^T |C(t-s)[g(s, x_{ns}) - g(s, x_s)]|\, ds \\
&\quad - \int_0^T \left| S(t-s)B\tilde{W}^{-1} \left[\int_0^T C(T-\tau)[g(\tau, x_{n\tau}) - g(\tau, x_\tau)]\, d\tau \right.\right. \\
&\quad \left.\left. + \int_0^T S(T-\tau)\left[f(\tau, x_{n\tau}, x_n'(\tau)) - f(\tau, x_\tau, x'(\tau))\right] d\tau \right] \right| ds \\
&\quad + \int_0^T |S(t-s)\left[f(s, x_{ns}, x_n'(s)) - f(s, x_s, x'(s))\right] ds| \to 0
\end{aligned}$$

and

$$\begin{aligned}
\|(Fx_n)' - (Fx)'\| &= \sup_{t\in J} \left| [g(t, x_{nt}) - g(t, x_t)] + \int_0^t AS(t-s)\left[g(s, x_{ns}) - g(s, x_s)\right] ds \right. \\
&\quad - \int_0^t C(t-s)B\tilde{W}^{-1} \left[\int_0^T C(T-\tau)[g(\tau, x_{n\tau}) - g(\tau, x_\tau)]\, d\tau \right. \\
&\quad \left. + \int_0^T S(T-\tau)\left[f(\tau, x_{n\tau}, x_n'(\tau)) - f(\tau, x_\tau, x'(\tau))\right] d\tau \right](s)\, ds \\
&\quad \left. + \int_0^t C(t-s)\left[f(s, x_{ns}, x_n'(s)) - f(s, x_s, x'(s))\right] ds \right| \\
&\leq \int_0^T |g(t, x_{nt}) - g(t, x_t)|\, ds + \int_0^t |AS(t-s)\left[g(s, x_{ns}) - g(s, x_s)\right]|\, ds \\
&\quad - \int_0^T \left| C(t-s)B\tilde{W}^{-1} \left[\int_0^T C(T-\tau)[g(\tau, x_{n\tau}) - g(\tau, x_\tau)]\, d\tau \right.\right. \\
&\quad \left.\left. + \int_0^T S(T-\tau)\left[f(\tau, x_{n\tau}, x_n'(\tau)) - f(\tau, x_\tau, x'(\tau))\right] d\tau \right] \right| ds \\
&\quad + \int_0^T |C(t-s)\left[f(s, x_{ns}, x_n'(s)) - f(s, x_s, x'(s))\right] ds| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus, F is continuous. This completes the proof that F is completely continuous.

Finally, the set $\xi(F) = \{x \in Z : x = \lambda F x, \lambda \in (0,1)\}$ is bounded, as we proved in the first step. Consequently, by Leray-Schauder alternative, the operator F has a fixed point in Z. This means that any fixed point of F is a mild solution of (1) on J satisfying $(Fx)(t) = x(t)$. Thus, system (1) is controllable on J.

4. EXAMPLE

Consider the following partial delay differential equation:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial t}(y,t)-\eta(t,z(y,t-r))\right)&=z_{yy}(y,t)+\sigma\left(t,z(y,t-r),\frac{\partial z}{\partial t}(y,t)\right)+\mu(y,t),\\ z(0,t)=z(\pi,t)&=0,\qquad \text{for } t>0,\\ z(y,t)&=\phi(y,t),\qquad \text{for } -r\le t\le 0,\\ \frac{\partial z}{\partial t}(y,0)&=z_0(y),\quad t\in J=[0,T],\qquad \text{for } 0<y<\pi,\end{aligned}\tag{6}$$

where ϕ is continuous, $\eta : J\times(0,\pi)\to(0,\pi)$ is continuous and strongly measurable, $\sigma : J\times(0,\pi)\times(0,\pi)\to(0,\pi)$ is continuous and strongly measurable, and $\mu : (0,\pi)\times J\to(0,\pi)$ is continuous in t.

Let $X=L^2[0,\pi]$ and let $A : X\to X$ be defined by

$$Aw=w'',\qquad w\in D(A),$$

where $D(A)=\{w\in X : w, w' \text{ are absolutely continuous, } w''\in X,\ w(0)=w(\pi)=0\}$. Then

$$Aw=\sum_{n=1}^{\infty}-n^2(w,w_n)w_n,\qquad w\in D(A),$$

where $w_n(s)=\sqrt{2/\pi}\sin ns$, $n=1,2,3,\dots$ is the orthogonal set of eigenfunctions of A.

It can be easily shown that A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t\in R$, in X and is given by [9]

$$C(t)w=\sum_{n=1}^{\infty}\cos nt(w,w_n)w_n,\qquad w\in X.$$

The associated sine family is given by

$$S(t)w=\sum_{n=1}^{\infty}\frac{1}{n}\sin nt(w,w_n)w_n,\qquad w\in X.$$

Let $g : J\times C\to X$ be defined by

$$g(t,\phi)(y)=\eta(t,\phi(y,-r)),\qquad \phi\in C,\quad y\in(0,\pi).$$

Also there exist positive constants c_1 and c_2 such that

$$\|\eta(t,\phi)\|\le c_1\|\phi\|+c_2.$$

Let $f : J\times C\times X\to X$ be defined by

$$f(t,\phi,w)(y)=\sigma(t,\phi(y,-r),w(y)),\qquad \phi\in C,\quad w\in X,\quad y\in(0,\pi).$$

Further, the function σ satisfies the following condition.

There exists a continuous function $p : J\to[0,\infty)$ such that

$$\|\sigma(t,\phi,w)\|\le p(t)\Omega(\|\phi\|+|w|),\qquad t\in J,\quad \phi\in C,\quad w\in X,$$

where $\Omega : [0,\infty)\to(0,\infty)$ is a continuous nondecreasing function such that

$$\int_0^T p(s)\,ds<\int_c^{\infty}\frac{ds}{s+\Omega(s)},$$

and c is a known constant. Let $B : U \subset J \to X$ be defined by

$$(Bu)(t)(y) = \mu(y,t), \qquad y \in (0,\pi),$$

such that it satisfies condition (H_4). Hence, by the above theorem, system (6) is controllable on J.

REMARK. CONSTRUCTION OF $\tilde{W}^{-1}$. (See [4].) Let

$$Y = \frac{L^2[J,U]}{\ker W}.$$

Since $\ker W$ is closed, Y is a Banach space under the norm

$$\|[u]\|_Y = \inf_{u\in[u]} \|u\|_{L^2[J,U]} = \inf_{W\hat{u}=0} \|u+\hat{u}\|_{L^2[J,U]},$$

where $[u]$ are the equivalence classes of u.

Define $\tilde{W} : Y \to X$ by

$$\tilde{W}[u] = Wu, \qquad u \in [u].$$

Now $\tilde{W}$ is one-to-one and

$$\left\|\tilde{W}[u]\right\|_X \le \|W\| \, \|[u]\|_Y.$$

We claim that V = Range W is a Banach space with the norm

$$\|v\|_V = \left\|\tilde{W}^{-1}v\right\|_Y.$$

This norm is equivalent to the graph norm on $D(\tilde{W}^{-1})$ = Range W, $\tilde{W}$ is bounded and since $D(\tilde{W}) = Y$ is closed, $\tilde{W}^{-1}$ is closed, and so the above norm makes Range $W = V$ a Banach space.

Moreover,

$$\begin{aligned}\|Wu\|_V = \left\|\tilde{W}^{-1}Wu\right\|_Y &= \left\|\tilde{W}^{-1}\tilde{W}[u]\right\| \\ &= \|[u]\| = \inf_{u\in[u]} \|u\| \le \|u\|,\end{aligned}$$

so

$$W \in \pounds\ \left(L^2[J,U],V\right).$$

Since $L^2[J,U]$ is reflexive and $\ker W$ is weakly closed, the infimum in the definition of the norm on Y is attained. For any $v \in V$, we can therefore choose a control $u \in L^2[J,U]$ such that $u = \tilde{W}^{-1}v$.

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