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Localization of the Riemann–Roch Character

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We present a K-theoretic approach to the Guillemin–Sternberg conjecture (V. Guillemin and S. Sternberg, *Invent. Math.* **67** (1982), 515–538), about the commutativity of geometric quantization and symplectic reduction, which was proved by E. Meinrenken (*J. Amer. Math. Soc.* **9** (1996), 373–389; *Adv. Math.* **134**, (1998), 240–277) and Tian–Zhang (Y. Tian and W. Zhang, *Invent. Math.* **132** (1998), 229–259). Besides providing a new proof of this conjecture for the full non-

invariant almost complex structure and an abstract moment map.

Key Words: moment map; reduction; geometric quantization; induction; transversally elliptic symbol.

Contents.

1. INTRODUCTION

This article is devoted to the study of the "quantization commutes with reduction" principle of Guillemin–Sternberg [17]. The object of this paper is twofold. The first goal is to give a K-theoretic approach to this problem, which provides a new proof of results obtained by Meinrenken [29], Meinrenken–Sjamaar [30], and Tian–Zhang [35]. The second goal is to define an extension to the *non-symplectic* case.

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In the Kostant–Souriau framework one considers a prequantum line bundle L over a compact symplectic manifold (M, ω): L carries a Hermitian connection ∇^L with curvature form equal to $-\iota\omega$. Suppose now that a compact Lie group G, with Lie algebra $\mathfrak{g}$, acts on $L \to M$, living the data (ω, ∇^L) invariant. Then the G-action on (M, ω) is Hamiltonian with moment map $f_G: M \to \mathfrak{g}^*$ given by the Kostant formula: $\mathscr{L}^L(X) - \nabla^L_{X_M} = \iota\langle f_G, X\rangle$, $X \in \mathfrak{g}$. Here $\mathscr{L}^L(X)$ is the infinitesimal action of X on the section of $L \to M$ and X_M is the vector field on M generated by $X \in \mathfrak{g}$.

Choose now an invariant almost complex structure J on M that is compatible with ω, in the sense that $\omega(-, J-)$ defines a Riemannian metric. It defines a quantization map

$$RR^{G,J}(M, -): K_G(M) \to R(G),$$

from the equivariant K-theory of complex vector bundles over M to the character ring of G. The "quantization commutes with reduction" Theorem tells us how the multiplicities of $RR^{G,J}(M, L)$ behave (see Theorem C).

Here our main goal is to compute the multiplicity of the trivial representation in $RR^{G,J}(M, L)$, when the data (L, J) are not associated to a symplectic form.

We consider a compact manifold M on which a compact Lie group G acts, and which carries a G-invariant almost complex structure J. Let $L \to M$ be a G-equivariant Hermitian line bundle over M, equipped with a Hermitian connection ∇^L on L. This defines a map $f_L: M \to \mathfrak{g}^*$ by the equation

$$\mathscr{L}^L(X) - \nabla^L_{X_M} = \iota\langle f_L, X\rangle, \qquad X \in \mathfrak{g}. \tag{1.1}$$

(see [10, Sect. 7.1]). The map f_L is an *abstract moment map* in the sense of Karshon [20], since f_L is equivariant, and for any $X \in \mathfrak{g}$, the function $\langle f_L, X\rangle$ is locally constant on the submanifold $M^X := \{m \in M, X_M(m) = 0\}$.

If 0 is a regular value of f_L, $\mathscr{Z} := f_L^{-1}(0)$ is a smooth submanifold of M which carries a locally free action of G. We consider the orbifold reduced space $\mathscr{M}_{\mathrm{red}} = \mathscr{Z}/G$ and we denote $\pi: \mathscr{Z} \to \mathscr{M}_{\mathrm{red}}$ the projection. In Lemma 6.9 we show that the almost complex structure J induces an orientation o_{red} on $\mathscr{M}_{\mathrm{red}}$ and a Spin^c structure on $(\mathscr{M}_{\mathrm{red}}, o_{\mathrm{red}})$. Let $\mathscr{Q}(\mathscr{M}_{\mathrm{red}}, -): K_{\mathrm{orb}}(\mathscr{M}_{\mathrm{red}}) \to \mathbb{Z}$ be the quantization map defined by the Spin^c structure and let $L_{\mathrm{red}} \to \mathscr{M}_{\mathrm{red}}$ be the orbifold line bundle induced by L.

We obtain the following "quantization commutes with reduction" theorem.

THEOREM A. *Let $L \to M$ be a G-equivariant Hermitian line bundle over M, equipped with a Hermitian connection ∇^L on L. Let $f_L: M \to \mathfrak{g}^*$ be the corresponding abstract moment map. If 0 is a regular value of f_L, we have*

$$[RR^{G,J}(M, L^{\overset{k}{\otimes}})]^G = \mathcal{Q}(\mathcal{M}_{\rm red}, L_{\rm red}^{\overset{k}{\otimes}}), \qquad k \in \mathbb{N} - \{0\}, \tag{1.2}$$

if any of the following hold:

(i) $G = T$ *is a torus; or*

(ii) $k \in \mathbb{N}$ *is large enough , so that the ball* $\{\xi \in \mathfrak{g}^*, \|\xi\| \leqslant \frac{1}{k}\|\theta\|\}$ *is contained in the set of regular values of* f_L. *Here* $\theta = \sum_{\alpha > 0} \alpha$ *is the sum of the positive roots of* G, *and* $\|\cdot\|$ *is a* G-*invariant Euclidean norm on* $\mathfrak{g}^*$.

Here, for $V \in R(G)$, *we denote* $[V]^G \in \mathbb{Z}$ *the multiplicity of the trivial representation.*

A similar result was proved by Jeffrey–Kirwan [19] in the Hamiltonian setting when one relaxes the condition of positivity of J with respect to the symplectic form. See also [13] for a similar result in the Spin^c setting, when $G = S^1$.

As an example, let us apply Theorem A to the counterexemple due to Vergne which shows that quantization does not always commute with reduction. Let $G = SU(2)$ and let M be the $SU(2)$-coadjoint orbit passing through the unique positive root θ. Thus M is the projective line bundle $\mathbb{CP}^1$ with ω equal to twice the standard Kähler form. The prequantum line bundle is $L = \mathcal{O}(2)$ and $RR^G(M, L^{-1}) = [RR^G(M, L^{-1})]^G = -1$. Since $\mathcal{M}_{\rm red} = \emptyset$ we have $[RR^G(M, L^{-1})]^G \neq \mathcal{Q}(\mathcal{M}_{\rm red}, (L^{-1})_{\rm red})$: the condition (ii) of Theorem A does not hold since θ is not a regular value of the moment map $M \hookrightarrow \mathfrak{g}^*$. But if we take $(L^{-1})^{\overset{k}{\otimes}}$ with $k > 1$ the condition (ii) is satisfied, and thus $[RR^G(M, (L^{-1})^{\overset{k}{\otimes}})]^G = 0$ for $k > 1$. In fact a direct computation shows that $-RR^G(M, (L^{-1})^{\overset{k}{\otimes}})$ is the character of the irreducible $SU(2)$-representation with highest weight $(k-1)\,\theta$ for all $k \geqslant 1$.

The result of Theorem A can be rewritten when J defines an almost complex structure $J_{\rm red}$ on $\mathcal{M}_{\rm red}$. It happens when the following decomposition holds

$$\mathbf{T}M|_{\mathcal{Z}} = \mathbf{T}\mathcal{Z} \oplus J(\mathfrak{g}_{\mathcal{Z}}) \qquad \text{with} \qquad \mathfrak{g}_{\mathcal{Z}} := \{X_{\mathcal{Z}}, X \in \mathfrak{g}\}. \tag{1.3}$$

First we note that (1.3) always holds in the Hamiltonian case when J is compatible with the symplectic form. Condition (1.3) already appears in the works of Jeffrey–Kirwan [19], and Cannas da Silva–Karshon–Tolman [13].

In all this paper we fix a G-invariant scalar product on $\mathfrak{g}^*$ which induces an identification $\mathfrak{g} \simeq \mathfrak{g}^*$. Thus f_G can be considered as a map from M to $\mathfrak{g}$, and we define the endomorphism $\mathcal{D}$ of the bundle $\mathfrak{g} \times \mathcal{Z}$ by: $\mathcal{D}(X) = -df_G(J(X_{\mathcal{Z}}))$, for $X \in \mathfrak{g}$. Condition (1.3) is then equivalent to: $\det \mathcal{D}(z) \neq 0$ for all $z \in \mathcal{Z}$. The endomorphism $\mathcal{D}$ defines a complex structure $J_{\mathcal{D}}$ on

$\mathscr{Z}\times\mathfrak{g}_{\mathbb{C}}$, so the vector bundle $\mathscr{Z}\times\mathfrak{g}_{\mathbb{C}}$ inherits two irreducible complex spinor bundles $\mathscr{Z}\times\bigwedge_{\mathbb{C}}^{\bullet}\mathfrak{g}_{\mathbb{C}}$ and $\mathscr{Z}\times\bigwedge_{J_{\mathscr{D}}}^{\bullet}\mathfrak{g}_{\mathbb{C}}$ related by

$$\textstyle\bigwedge_{J_{\mathscr{D}}}^{\bullet}\mathfrak{g}_{\mathbb{C}}\times\mathscr{Z}=\bigwedge_{\mathbb{C}}^{\bullet}\mathfrak{g}_{\mathbb{C}}\times\mathscr{Z}\otimes\pi^{*}L_{\mathscr{D}},$$

where $L_{\mathscr{D}}\to\mathscr{M}_{\mathrm{red}}$ is a line bundle (see (6.11)). In this case we prove in Proposition 6.12 that (1.2) has the form

$$[RR^{G,J}(M,L^{\overset{k}{\otimes}})]^{G}=\pm RR^{J_{\mathrm{red}}}(\mathscr{M}_{\mathrm{red}},L_{\mathrm{red}}^{\overset{k}{\otimes}}\otimes L_{\mathscr{D}}),\tag{1.4}$$

where $\pm$ is the sign of $\det\mathscr{D}$, and where $RR^{J_{\mathrm{red}}}(\mathscr{M}_{\mathrm{red}},-)$ is the Riemann–Roch character defined by J_{red}.

In this paper, we start from an abstract moment map $f_G\colon M\to\mathfrak{g}^*$, and we extend the result of Theorem A to the f_G-moment bundles, and the f_G-positive bundles. These notions were introduced in the Hamiltonian setting by Meinrenken–Sjamaar [30] and Tian–Zhang [35]. Let us recall the definitions.

Let H be a maximal torus of G with Lie algebra $\mathfrak{h}$.

DEFINITION 1.1. A G-equivariant line bundle over M is called a f_G-moment bundle if for all components F of the fixed-point set M^H the weight of the H-action on $L|_F$ is equal to $f_G(F)$.

It is easy to see that the definition is independent of the choice of the maximal torus. Note that $f_G(F)\in\mathfrak{h}^*=(\mathfrak{g}^*)^H$, since f_G is equivariant. Any Hermitian line bundle L is tautologically a moment bundle relative to the abstract moment map f_L.

For any $\beta\in\mathfrak{g}$, we denote by $\mathbb{T}_\beta$ the torus of G generated by $\exp_G(t.\beta)$, $t\in\mathbb{R}$, and M^β the submanifold of points fixed by $\mathbb{T}_\beta$.

DEFINITION 1.2. A complex G-vector bundle E is called *f_G-positive* if the following hold: for any $m\in M^\beta\cap f_G^{-1}(\beta)$, we have

$$\langle\xi,\beta\rangle\geqslant 0$$

for any weights ξ of the $\mathbb{T}_\beta$-action on E_m. A complex G-vector bundle E is called *f_G-strictly positive* when furthermore the last inequality is strict for any $\beta\neq 0$.

For any *f_G-strictly positive* complex vector bundle E, and any $\beta\in\mathfrak{g}$ such that $M^\beta\cap f_G^{-1}(\beta)\neq\varnothing$, we define $\eta_{E,\beta}=\inf_\xi\langle\xi,\beta\rangle$, where ξ runs over the set of weights for the $\mathbb{T}_\beta$-action on the fibers of each complex vector bundle $E|_{\mathscr{Z}}$, $\mathscr{Z}$ being a connected component of M^β that intersects $f_G^{-1}(\beta)$.

It is not difficult to see that a f_G-moment bundle L is f_G-strictly positive with $\eta_{L,\beta}=\|\beta\|^2$, for any $\beta\in\mathfrak{g}$ such that $M^\beta\cap f_G^{-1}(\beta)\neq\varnothing$ (see Lemma 7.9). The bundle $M\times\mathbb{C}\to M$ is the trivial example of f_G-positive complex vector bundle over M.

Let $\mathfrak{h}_+$ be the choice of some positive Weyl chamber in $\mathfrak{h}$. We prove in Lemma 6.3 that the set $\mathscr{B}_G := \{\beta \in \mathfrak{h}_+, M^\beta \cap f_G^{-1}(\beta) \neq \emptyset\}$ is finite.

THEOREM B. *Let $f_G\colon M \to \mathfrak{g}^*$ be an abstract moment map with 0 as regular value. Let E be a f_G-strictly positive G-complex vector bundle over M (see Def.* 1.2*). We have*

$$[RR^{G,J}(M, E^{\overset{k}{\otimes}})]^G = \mathscr{Q}(\mathscr{M}_{\text{red}}, E_{\text{red}}^{\overset{k}{\otimes}}), \qquad k \in \mathbb{N} - \{0\}, \tag{1.5}$$

if any of the following hold:

(i) *$G = T$ is a torus; or*

(ii) *k is large enough, so that $k.\eta_{E,\beta} > \sum_{\alpha>0} \langle \alpha, \beta\rangle$, for any $\beta \in \mathscr{B}_G - \{0\}$; here the sum $\sum_{\alpha>0}$ is taken over the positive roots of G.*

Moreover if (1.3) *holds,* (1.5) *becomes*

$$[RR^{G,J}(M, E^{\overset{k}{\otimes}})]^G = \pm RR^{J_{\text{red}}}(\mathscr{M}_{\text{red}}, E_{\text{red}}^{\overset{k}{\otimes}} \otimes L_{\mathscr{D}}).$$

Let us explain why Theorem B applied to a G-hermitian line bundle L with the abstract moment map $f_G = f_L$ implies Theorem A. It is sufficient to prove that condition (ii) of Theorem A implies condition (ii) of Theorem B. The curvature of (L, ∇^L) is $(\nabla^L)^2 = -\iota\, \omega^L$, where ω^L is a differential 2-form on M. From the equivariant Bianchi formula (see Proposition 7.4 in [10]) we get $\langle df_L, X\rangle = -\omega^L(X_M, -)$ for any $X \in \mathfrak{g}$. So, for any $\beta \in \mathscr{B}_G - \{0\}$, and $m \in M^\beta \cap f_L^{-1}(\beta)$, the last equality gives $\langle df_L|_m, \beta\rangle = 0$, hence β is a critical value of f_L. Suppose now that $k \in \mathbb{N}$ is large enough so that the ball $\{\xi \in \mathfrak{g}^*, \|\xi\| \leqslant \frac{1}{k}\|\theta\|\}$ is included in the set of regular values of f_L. This gives first $\|\beta\| > \frac{1}{k}\|\theta\|$ and then $\eta_{L,\beta} = \|\beta\|^2 > \frac{1}{k}\langle \theta, \beta\rangle$, for any $\beta \in \mathscr{B}_G - \{0\}$.

In the last section of this paper, we restrict ourselves to the Hamiltonian case. In this situation, the abstract moment map f_G and the almost complex structure J are related by means of a G-invariant symplectic 2-form ω :

- f_G is the moment map associated to a Hamiltonian action of G over (M, ω): $d\langle f_G, X\rangle = -\omega(X_M, -)$, for $X \in \mathfrak{g}$, and
- the data (ω, J) are *compatible*: $(v, w) \to \omega(v, Jw)$ is a Riemannian metric on M.

When 0 is a regular value of f_G, the compatible data (ω, J) induce compatible data $(\omega_{\text{red}}, J_{\text{red}})$ on $\mathscr{M}_{\text{red}}$. We have then a map $RR^{J_{\text{red}}}(\mathscr{M}_{\text{red}}, -)$. If 0 is not a regular value of f_G, we consider elements a in the principal face

τ of the Weyl chamber (see Sect. 7.4). For generic elements $a \in \tau \cap f_G(M)$, the set $\mathcal{M}_a := f_G^{-1}(G \cdot a)/G$ is a symplectic orbifold and one can consider the quantization map $RR^{J_a}(\mathcal{M}_a, -)$ relative to the choice of compatible almost complex structure J_a.

In this situation, we recover the results of [29, 30, 35].

THEOREM C (Meinrenken, Meinrenken–Sjamaar, Tian–Zhang). *Let f_G be the moment map associated to a Hamiltonian action of G over (M, ω), and let J be a ω-compatible almost complex structure. Let $E \to M$ be a G-vector bundle.*

If $0 \notin f_G(M)$ and E is f_G-strictly positive, we have $[RR^{G,J}(M, E)]^G = 0$.

If $0 \in f_G(M)$ then:

(i) *If 0 is a regular value, we have $[RR^{G,J}(M, E)]^G = RR^{J_{\mathrm{red}}}(\mathcal{M}_{\mathrm{red}}, E_{\mathrm{red}})$, if E is f_G-positive.*

(ii) *If 0 is not a regular value of f_G and $E = L$ is a f_G-moment bundle, we have $[RR^{G,J}(M, L)]^G = RR^{J_a}(\mathcal{M}_a, L_a)$, for every generic value a of $\tau \cap f_G(M)$ sufficiently close to 0. Here L_a is the orbifold line bundle $L|_{f_G^{-1}(G\cdot a)}/G$.*

We now turn to an introduction of our method. We associate to the abstract moment map $f_G\colon M \to \mathfrak{g}$ the vector field

$$\mathcal{H}^G_m = [f_G(m)]_M . m, \qquad m \in M,$$

and we denote by C^{f_G} the set where $\mathcal{H}^G$ vanishes. There are two important cases. First, when the map f_G is constant, equal to an element γ in the center of $\mathfrak{g}$, the set C^{f_G} corresponds to the submanifold M^γ. Second, when f_G is the moment map associated with a Hamiltonian action of G over M. In this situation, Witten [39] introduces the vector field $\mathcal{H}^G$ to propose, in the context of equivariant cohomology, a localization on the set of critical points of the function $\|f_G\|^2$: here $\mathcal{H}^G$ is the Hamiltonian vector field of $\frac{-1}{2}\|f_G\|^2$, hence $\mathcal{H}^G_m = 0 \leftrightarrow d(\|f_G\|^2)_m = 0$. This idea has been developed by the author in [31, 32].

Using a deformation argument in the context of transversally elliptic operator introduced by Atiyah [1] and Vergne [38], we prove in Section 4 that the map[1] RR^G can be localized near C^{f_G}. More precisely, we have the finite decomposition $C^{f_G} = \bigcup_{\beta \in \mathcal{B}_G} C^G_\beta$ with $C^G_\beta = G(M^\beta \cap f_G^{-1}(\beta))$, and

$$RR^G(M, E) = \sum_{\beta \in \mathcal{B}_G} RR^G_\beta(M, E). \tag{1.6}$$

[1] When the almost complex strcture J is understood, we denote by RR^G the quantization map.

Each term $RR^G_\beta(M,E)$ is a generalized character of G that only depends on the behavior of the data M, E, J, f_G near the subset C^G_β. In fact, $RR^G_\beta(M,E)$ is the index of a transversally elliptic operator defined in an open neighborhood of C^G_β.

Our proof of Theorems B and C is in two steps. First we compute the term $RR^G_0(M,E)$ which is the Riemann–Roch character localized near $f_G^{-1}(0)$. After, we prove that $[RR^G_\beta(M,E)]^G=0$ for every $\beta\neq 0$. For this purpose, the analysis of the localized Riemann–Roch characters $RR^G_\beta(M,-)\colon K_G(M)\to R^{-\infty}(G)$ is divided in three cases[2] :

Case 1. $\beta=0$.

Case 2. $\beta\neq 0$. and $G_\beta=G$.

Case 3. $G_\beta\neq G$.

We work out Case 1 in Section 6.2. We compute the generalized character $RR^G_0(M,E)$ when 0 is a regular value of f_G. We prove in particular that the multiplicity of the trivial representation in $RR^G_0(M,E)$ is $\mathscr{Q}(\mathscr{M}_{\mathrm{red}}, E_{\mathrm{red}})$. This last quantity is equal to $\pm RR^{J_{\mathrm{red}}}(\mathscr{M}_{\mathrm{red}}, E_{\mathrm{red}}\otimes L_{\mathscr{D}})$ when (1.3) holds.

Case 2 is studied in Section 5 for the particular situation where f_G is constant, equal to a G-invariant element $\beta\in\mathfrak{g}$. Then $C^{f_G}=C^G_\beta=M^\beta$, and (1.6) becomes $RR^G(M,E)=RR^G_\beta(M,E)$. We prove then a localization formula (see (1.7)) in the spirit of the Atiyah–Segal–Singer formula in equivariant K-theory [3, 34]. Let us sketch out the result.

The normal bundle $\mathscr{N}$ of M^β in M inherits a canonical complex structure $J_{\mathscr{N}}$ on the fibers. We denote by $\overline{\mathscr{N}}\to M^\beta$ the complex vector bundle with the opposite complex structure. The torus $\mathbb{T}_\beta$ is included in the center of G, so the bundle $\overline{\mathscr{N}}$ and the virtual bundle $\wedge^\bullet_{\mathbb{C}}\overline{\mathscr{N}}:=\wedge^{\mathrm{even}}_{\mathbb{C}}\overline{\mathscr{N}}\xrightarrow{0}\wedge^{\mathrm{odd}}_{\mathbb{C}}\overline{\mathscr{N}}$ carry a $G\times\mathbb{T}_\beta$-action: they can be considered as elements of $K_{G\times\mathbb{T}_\beta}(M^\beta)=K_G(M^\beta)\otimes R(\mathbb{T}_\beta)$. Let $K_G(M^\beta)\hat{\otimes}R(\mathbb{T}_\beta)$ be the vector space formed by the infinite formal sums $\sum_a E_a\, h^a$ taken over the set of weights of $\mathbb{T}_\beta$, where $E_a\in K_G(M^\beta)$ for every a. The Riemann–Roch character RR^G can be extended to a map $RR^{G\times\mathbb{T}_\beta}$ that satisfies the commutative diagram

$$\begin{array}{ccc} K_G(M^\beta) & \xrightarrow{\quad RR^G\quad} & R(G) \\ \big\downarrow & & \big\downarrow k \\ K_G(M^\beta)\hat{\otimes}R(\mathbb{T}_\beta) & \xrightarrow{RR^{G\times\mathbb{T}_\beta}} & R(G)\hat{\otimes}R(\mathbb{T}_\beta). \end{array}$$

[2] G_β is the stabilizer of β in G.

The arrow $k\colon R(G) \to R(G) \hat{\otimes} R(\mathbb{T}_\beta)$ is the canonical map defined by $k(\phi)(g, h) := \phi(gh)$. We shall notice that $[k(\phi)]^{G\times\mathbb{T}_\beta} = [\phi]^G$.

In Section 5, we define an inverse, denoted by $[\wedge_{\mathbb{C}}^{\bullet} \overline{\mathcal{N}}\,]_\beta^{-1}$, of $\wedge_{\mathbb{C}}^{\bullet} \overline{\mathcal{N}}$ in $K_G(M^\beta) \hat{\otimes} R(\mathbb{T}_\beta)$ which is polarized by β. It means that $[\wedge_{\mathbb{C}}^{\bullet} \overline{\mathcal{N}}\,]_\beta^{-1} = \sum_a N_a h^a$ with $N_a \neq 0$ only if $\langle a, \beta\rangle \geqslant 0$. We can state now our localization formula as the following equality in $R(G) \hat{\otimes} R(\mathbb{T}_\beta)$:

$$RR^G(M, E) = RR^{G\times \mathbb{T}_\beta}(M^\beta, E_{|M^\beta} \otimes [\wedge_{\mathbb{C}}^{\bullet} \overline{\mathcal{N}}\,]_\beta^{-1}), \tag{1.7}$$

for every $E \in K_G(M)$.

In Section 6.3 we work out Case 2 for the general situation. The map $RR_\beta^G(M^\beta, -)$ is the Riemann–Roch character on the G-manifold M^β, localized near $M^\beta \cap f_G^{-1}(\beta)$, and we extend it to a map $RR_\beta^{G\times\mathbb{T}_\beta}(M^\beta, -)$: $K_G(M^\beta) \hat{\otimes} R(\mathbb{T}_\beta) \to R^{-\infty}(G) \hat{\otimes} R(\mathbb{T}_\beta)$. We prove then the following localization formula

$$RR_\beta^G(M, E) = RR_\beta^{G\times\mathbb{T}_\beta}(M^\beta, E_{|M^\beta} \otimes [\wedge_{\mathbb{C}}^{\bullet} \overline{\mathcal{N}}\,]_\beta^{-1}), \tag{1.8}$$

as an equality in $R^{-\infty}(G) \hat{\otimes} R(\mathbb{T}_\beta)$. With (1.8) in hand, we see easily that $[RR_\beta^G(M, E)]^G = 0$ if the vector bundle E is f_G-strictly positive.

Section 6.4 is devoted to Case 3. The abstract moment map $f_G\colon M \to \mathfrak{g}$ for the G-action on M induces abstract moment maps $f_{G'}\colon M \to \mathfrak{g}'$ for every closed subgroup G' of G. For every $\beta \in \mathcal{B}_G$, we consider the Riemann–Roch characters $RR_\beta^G(M, -)$, $RR_\beta^{G_\beta}(M, -)$, and $RR_\beta^H(M, -)$ localized respectively on $G(M^\beta \cap f_G^{-1}(\beta))$, $M^\beta \cap f_G^{-1}(\beta)$, and $M^\beta \cap f_H^{-1}(\beta)$. The major result of Section 6.4 is the induction formulas proved in Theorem 6.16 and Corollary 6.17, between these three characters. I will explain briefly this result.

Let W be the Weyl group associated to (G, H). The choice of a Weyl chamber $\mathfrak{h}^+$ in $\mathfrak{h}$ determines a complex structure on the real vector space $\mathfrak{g}/\mathfrak{h}$. Our induction formulas make a crucial use of the holomorphic induction map $\mathrm{Hol}_H^G\colon R(H) \to R(G)$ (see (9.92) in Appendix B). Recall that $\mathrm{Hol}_H^G(h^\lambda)$ is, for any weight λ, equal either to zero or to the character of an irreducible representation of G (times ± 1). In Theorem 6.16 we prove the following relation between $RR_\beta^G(M, -)$ and $RR_\beta^H(M, -)$

$$\begin{aligned} RR_\beta^G(M, E) &= \frac{1}{|W_\beta|} \mathrm{Hol}_H^G\Big(\sum_{w\in W} w.RR_\beta^H(M, E)\Big) \\ &= \frac{1}{|W_\beta|} \mathrm{Hol}_H^G(RR_\beta^H(M, E) \wedge_{\mathbb{C}}^{\bullet} \overline{\mathfrak{g}/\mathfrak{h}}), \end{aligned} \tag{1.9}$$

where W_β is the stabilizer of β in W. In Corollary 6.17 we get the other relation:

$$RR^G_\beta(M,E) = \mathrm{Hol}^G_{G_\beta}\big(RR^{G_\beta}_\beta(M,E)\wedge^\bullet_{\mathbb{C}}\overline{\mathfrak{g}/\mathfrak{g}_\beta}\big). \tag{1.10}$$

Let us compare (1.9), with the Weyl integration formula[3]: for any $\phi\in R(G)$ we have $\phi=\mathrm{Hol}^G_H(\phi_{|H})=\mathrm{Hol}^G_H(\phi^+_{|H}\wedge^\bullet_{\mathbb{C}}\overline{\mathfrak{g}/\mathfrak{h}})$, where $\phi_{|H}$ is the restriction of ϕ to H, and $\phi^+_{|H}=\sum_\lambda m(\lambda)\,h^\lambda$ is the unique element in $R(H)\otimes\mathbb{Q}$ such that $\sum_{w\in W} w.\phi^+_{|H}=\phi_{|H}$ and $m(\lambda)\neq 0$ only if $\lambda\in\mathfrak{h}^+$. In (1.9), the W-invariant element $\frac{1}{|W_\beta|}\sum_{w\in W} w.RR^H_\beta(M,E)$ plays the role of the restriction to H of the character $\phi=RR^G_\beta(M,E)$, and $\frac{1}{|W_\beta|}RR^H_\beta(M,E)$ plays the role of $\phi^+_{|H}$.

Since β is a G_β-invariant element, (1.10) reduces the analysis of Case 3 to the one of Case 2. From the result proved in Case 2, we have $[RR^{G_\beta}_\beta(M^\beta,E)]^{G_\beta}=0$ if the vector bundle E is f_G-strictly positive. But this does not implies in general that $[RR^G_\beta(M,E)]^G=0$. We have to take the tensor product of E (so that $E^{\overset{k}{\otimes}}$ becomes more and more f_{G_β}-strictly positive) to see that $[RR^G_\beta(M,E^{\overset{k}{\otimes}})]^G=0$, when $\eta_{E^{\overset{k}{\otimes}},\beta}=k.\eta_{E,\beta}>\sum_{\alpha>0}\langle\alpha,\beta\rangle$.

In the Hamiltonian setting considered in Section 7, our strategy is the same, but at each step we obtain considerable refinements that are the principal ingredients of the proof of Theorem C.

Case 1. When 0 is a regular value of f_G, we show that the Spin^c structure on $\mathscr{M}_{\mathrm{red}}$ is defined by J_{red}, hence $\mathscr{Q}(\mathscr{M}_{\mathrm{red}},-)=RR^{J_{\mathrm{red}}}(\mathscr{M}_{\mathrm{red}},-)$. When 0 is not a regular value of f_G, we use the "shifting trick" to compute the G-invariant part of $RR^G_0(M,E)$ (see Sect. 7.4).

Case 2. For any G-invariant element $\beta\in\mathscr{B}_G$ with $\beta\neq 0$, we prove that the inverse $[\wedge^\bullet_{\mathbb{C}}\overline{\mathscr{N}}]^{-1}_\beta$ is of the form $\sum_a N_a\,h^a$ with $N_a\neq 0$ only if $\langle a,\beta\rangle>0$ (in general we have only $\langle a,\beta\rangle\geqslant 0$).

Case 3. For $\beta\in\mathscr{B}_G$ with $G_\beta\neq G$, we consider the open face σ of the Weyl chamber which contains β, and the corresponding symplectic slice $\mathscr{Y}_\sigma$ which is a G_β-symplectic submanifold of M. The localized Riemann–Roch characters $RR^G_\beta(M,E)$ and $RR^{G_\beta}_\beta(\mathscr{Y}_\sigma,-)$ are related by the following induction formula

$$RR^G_\beta(M,E)=\mathrm{Hol}^G_{G_\beta}\big(RR^{G_\beta}_\beta(\mathscr{Y}_\sigma,E|_{\mathscr{Y}_\sigma})\big).$$

[3] See Remark 9.2.

Notation

Throughout the paper G will denote a compact, connected Lie group, and $\mathfrak{g}$ its Lie algebra. We let H be a maximal torus in G, and $\mathfrak{h}$ be its Lie algebra. The integral lattice $\Lambda \subset \mathfrak{h}$ is defined as the kernel of $\exp: \mathfrak{h} \to H$, and the real weight lattice $\Lambda^* \subset \mathfrak{h}^*$ is defined by: $\Lambda^* := \hom(\Lambda, 2\pi\mathbb{Z})$. Every $\lambda \in \Lambda^*$ defines a 1-dimensional H-representation, denoted $\mathbb{C}_\lambda$, where $h = \exp X$ acts by $h^\lambda := e^{\imath\langle\lambda, X\rangle}$. We let W be the Weyl group of (G, H), and we fix the positive Weyl chambers $\mathfrak{h}_+ \subset \mathfrak{h}$ and $\mathfrak{h}^*_+ \subset \mathfrak{h}^*$. For any dominant weight $\lambda \in \Lambda^*_+ := \Lambda^* \cap \mathfrak{h}^*_+$, we denote by V_λ the G-irreducible representation with highest weight λ, and χ^G_λ its character. We denote by $R(G)$ (resp. $R(H)$) the ring of characters of finite-dimensional G-representations (resp. H-representations). We denote by $R^{-\infty}(G)$ (resp. $R^{-\infty}(H)$) the set of generalized characters of G (resp. H). An element $\chi \in R^{-\infty}(G)$ is of the form $\chi = \sum_{\lambda \in \Lambda^*_+} m_\lambda \chi^G_\lambda$, where $\lambda \mapsto m_\lambda$, $\Lambda^*_+ \to \mathbb{Z}$ has at most polynomial growth. In the same way, an element $\chi \in R^{-\infty}(H)$ is of the form $\chi = \sum_{\lambda \in \Lambda^*} m_\lambda h^\lambda$, where $\lambda \mapsto m_\lambda$, $\Lambda^* \to \mathbb{Z}$ has at most polynomial growth.

Some additional notation will be introduced later :

G_γ: stabilizer subgroup of $\gamma \in \mathfrak{g}$
$\mathbb{T}_\beta$: torus generated by $\beta \in \mathfrak{g}$
M^γ: submanifold of points fixed by $\gamma \in \mathfrak{g}$
$\mathbf{T}M$: tangent bundle of M
$\mathbf{T}_G M$: set of tangent vectors orthogonal to the G-orbits in M
$\mathscr{C}^{-\infty}(G)^G$: set of generalized functions on G, invariant by conjugation
$\mathrm{Ind}^G_{G_\gamma}: \mathscr{C}^{-\infty}(G_\gamma)^{G_\gamma} \to \mathscr{C}^{-\infty}(G)^G$: induction map
$\mathrm{Hol}^G_{G_\gamma}: R(G_\gamma) \to R(G)$: holomorphic induction map
$RR^G_\beta(M, -)$: Riemann–Roch character localized on $G.(M^\beta \cap f_G^{-1}(\beta))$
$\mathrm{Char}(\sigma)$: characteristic set of the symbol σ
$\mathrm{Thom}_G(M, J)$: Thom symbol
$\mathrm{Thom}^\gamma_G(M)$: Thom symbol localized near M^γ
$\mathrm{Thom}^f_{G,\beta}(M)$: Thom symbol localized near $G.(M^\beta \cap f_G^{-1}(\beta))$.

2. QUANTIZATION OF COMPACT MANIFOLDS

Let M be a compact manifold provided with an action of a compact connected Lie group G. A G-invariant almost complex structure J on M defines a map $RR^{G,J}(M, -): K_G(M) \to R(G)$ from the equivariant K-theory of complex vector bundles over M to the character ring of G.

Let us recall the definition of this map. The almost complex structure on M gives the decomposition $\bigwedge \mathbf{T}^*M \otimes \mathbb{C} = \bigoplus_{i,j} \bigwedge^{i,j} \mathbf{T}^*M$ of the bundle of differential forms. Using Hermitian structure in the tangent bundle $\mathbf{T}M$ of M, and in the fibers of E, we define a twisted Dirac operator

$$\mathscr{D}_E^+ : \mathscr{A}^{0,\,\mathrm{even}}(M, E) \to \mathscr{A}^{0,\,\mathrm{odd}}(M, E),$$

where $\mathscr{A}^{i,j}(M, E) := \Gamma(M, \wedge^{i,j}\mathbf{T}^*M \otimes_{\mathbb{C}} E)$ is the space of E-valued forms of type (i, j). The Riemann–Roch character $RR^{G,J}(M, E)$ is defined as the index of the elliptic operator $\mathscr{D}_E^+$:

$$RR^{G,J}(M, E) = [\mathrm{Ker}\, \mathscr{D}_E^+] - [\mathrm{Coker}\, \mathscr{D}_E^+].$$

In fact, the virtual character $RR^{G,J}(M, E)$ is independent of the choice of the Hermitian metrics on the vector bundles $\mathbf{T}M$ and E.

If M is a compact complex analytic manifold, and E is an holomorphic complex vector bundle, we have $RR^{G,J}(M, E) = \sum_{q=0}^{q=\dim M} (-1)^q [\mathscr{H}^q(M, \mathscr{O}(E))]$, where $\mathscr{H}^q(M, \mathscr{O}(E))$ is the qth cohomology group of the sheaf $\mathscr{O}(E)$ of the holomorphic sections of E over M.

In this paper, we shall use an equivalent definition of the map $RR^{G,J}$. We associate to an invariant almost complex structure J the symbol $\mathrm{Thom}_G(M, J) \in K_G(\mathbf{T}M)$ defined as follows. Consider a Riemannian structure q on M such that the endomorphism J is orthogonal relatively to q, and let h be the following Hermitian structure on $\mathbf{T}M$: $h(v, w) = q(v, w) - \imath q(Jv, w)$ for $v, w \in \mathbf{T}M$. Let $p\colon \mathbf{T}M \to M$ be the canonical projection. The symbol $\mathrm{Thom}_G(M, J)\colon p^*(\wedge_{\mathbb{C}}^{\mathrm{even}} \mathbf{T}M) \to p^*(\wedge_{\mathbb{C}}^{\mathrm{odd}} \mathbf{T}M)$ is equal, at $(x, v) \in \mathbf{T}M$, to the Clifford map

$$Cl_x(v)\colon p^*(\wedge_{\mathbb{C}}^{\mathrm{even}} \mathbf{T}M)|_{(x,v)} \to p^*(\wedge_{\mathbb{C}}^{\mathrm{odd}} \mathbf{T}M)|_{(x,v)}, \tag{2.1}$$

where $Cl_x(v).w = v \wedge w - c_h(v).w$ for $w \in \wedge_{\mathbb{C}}^{\bullet} \mathbf{T}_xM$. Here $c_h(v)\colon \wedge_{\mathbb{C}}^{\bullet} \mathbf{T}_xM \to \wedge^{\bullet-1} \mathbf{T}_xM$ denotes the contraction map relatively to h: for $w \in \mathbf{T}_xM$ we have $c_h(v).w = h(w, v)$. Here $(\mathbf{T}M, J)$ is considered as a complex vector bundle over M.

The symbol $\mathrm{Thom}_G(M, J)$ determines the Bott–Thom isomorphism $\mathrm{Thom}_J\colon K_G(M) \to K_G(\mathbf{T}M)$ by $\mathrm{Thom}_J(E) := \mathrm{Thom}_G(M, J) \otimes p^*(E)$, $E \in K_G(M)$. To make the notation clearer, $\mathrm{Thom}_J(E)$ is the symbol $\sigma^E\colon p^*(\wedge_{\mathbb{C}}^{\mathrm{even}} \mathbf{T}M \otimes E) \to p^*(\wedge_{\mathbb{C}}^{\mathrm{odd}} \mathbf{T}M \otimes E)$ with

$$\sigma^E(x, v) := Cl_x(v) \otimes Id_{E_x}, \qquad (x, v) \in \mathbf{T}M, \tag{2.2}$$

where E_x is the fiber of E at $x \in M$.

Consider the index map $\mathrm{Index}_M^G\colon K_G(\mathbf{T}^*M) \to R(G)$ where $\mathbf{T}^*M$ is the cotangent bundle of M. Using a G-invariant auxiliary metric on $\mathbf{T}M$, we can identify the vector bundle $\mathbf{T}^*M$ and $\mathbf{T}M$, and produce an "index" map

$\text{Index}^G_M \colon K_G(\mathbf{T}M) \to R(G)$. We verify easily that this map is independent of the choice of the metric on $\mathbf{T}M$.

LEMMA 2.1. *We have the commutative diagram*

$$\begin{array}{ccc} K_G(M) & \xrightarrow{\text{Thom}_J} & K_G(\mathbf{T}M) \\ & \searrow^{RR^{G,J}} & \downarrow \text{Index}^G_M \\ & & R(G). \end{array} \tag{2.3}$$

Proof. If we use the natural identification $(\bigwedge^{0,1}\mathbf{T}^*M, \imath) \cong (\mathbf{T}M, J)$ of complex vector bundles over M, we see that the principal symbol of the operator $\mathscr{D}^+_E$ is equal to σ^E (see [14]). ▮

We will conclude with the following Lemma. Let J^0, J^1 be two G-invariant almost complex structures on M, and let RR^{G,J^0}, RR^{G,J^1} be the respective quantization maps.

LEMMA 2.2. *The maps RR^{G,J^0} and RR^{G,J^1} are identical in the following cases:*

(i) *There exists a G-invariant Section $A \in \Gamma(M, \text{End}(\mathbf{T}M))$, homotopic to the identity* in $\Gamma(M, \text{End}(\mathbf{T}M))^G$ such that A_x is invertible, and $A_x . J^0_x = J^1_x . A_x$ for every $x \in M$.

(ii) There exists an homotopy $J^t, t \in [0, 1]$ of G-invariant almost complex structures between J^0 and J^1.

Proof of (i). Take a Riemannian structure q^1 on M such that $J^1 \in O(q^1)$ and define another Riemannian structure q^0 by $q^0(v, w) = q^1(Av, Aw)$ so that $J^0 \in O(q^0)$. The Section A defines a bundle unitary map $\underline{A}$: $(\mathbf{T}M, J^0, h^0) \to (\mathbf{T}M, J^1, h^1)$, $(x, v) \to (x, A_x . v)$, where $h^l(.,.) := q^l(.,.) - \imath q^l(J^l.,.)$, $l = 0, 1$. This gives an isomorphism $A^\wedge_x \colon \bigwedge_{J^0}\mathbf{T}_xM \to \bigwedge_{J^1}\mathbf{T}_xM$ such that the following diagram is commutative

$$\begin{array}{ccc} \bigwedge_{J^0}\mathbf{T}_xM & \xrightarrow{Cl_x(v)} & \bigwedge_{J^0}\mathbf{T}_xM \\ \downarrow A^\wedge_x & & \downarrow A^\wedge_x \\ \bigwedge_{J^1}\mathbf{T}_xM & \xrightarrow{Cl_x(A_x.v)} & \bigwedge_{J^1}\mathbf{T}_xM. \end{array}$$

Then $A^\wedge$ induces an isomorphism between the symbols $\text{Thom}_G(M, J^0)$ and $\underline{A}^*(\text{Thom}_G(M, J^1))$: $(x, v) \to \text{Thom}_G(M, J^1)(x, A_x . v)$. Here $\underline{A}^* \colon K_G(\mathbf{T}M) \to K_G(\mathbf{T}M)$ is the map induced by the isomorphism $\underline{A}$. Thus the complexes $\text{Thom}_G(M, J^0)$ and $\underline{A}^*(\text{Thom}_G(M, J^1))$ define the same class in $K_G(\mathbf{T}M)$. Since A is homotopic to the identity, we have $\underline{A}^* = \text{Identity}$. We have proved that $\text{Thom}_G(M, J^0) = \text{Thom}_G(M, J^1)$ in $K_G(\mathbf{T}M)$, and by Lemma 2.1 this shows that $RR^{G,J^0} = RR^{G,J^1}$.

Proof of (ii). We construct A as in (i). Take first $A^{1,0} := Id - J^1 J^0$ and remark that $A^{1,0}.J^0 = J^1.A^{1,0}$. Here we consider the homotopy $A_u^{1,0} := Id - uJ^1 J^0$, $u \in [0, 1]$. If $-J^1 J^0$ is close to Id, for example $|Id + J^1 J^0| \leqslant 1/2$, the bundle map $A_u^{1,0}$ will be invertible for every $u \in [0, 1]$. Then we can conclude with Point (i). In general we use the homotopy J^t, $t \in [0, 1]$. First, we decompose the interval $[0, 1]$ in $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1$ and we consider the maps $A^{t_{l+1}, t_l} := Id - J^{t_{l+1}} J^{t_l}$, with the corresponding homotopy $A_u^{t_{l+1}, t_l}$, $u \in [0, 1]$, for $l = 0, ..., k-1$. Because $-J^{t_{l+1}} J^{t_l} \to Id$ when $t \to t'$, the bundle maps $A_u^{t_{l+1}, t_l}$ are invertible for all $u \in [0, 1]$ if $t_{l+1} - t_l$ is small enough. Then we take the G-equivariant bundle map $A := \Pi_{l=0}^{k-1} A^{t_{l+1}, t_l}$ with the homotopy $A_u := \Pi_{l=0}^{k-1} A_u^{t_{l+1}, t_l}$, $u \in [0, 1]$. We have $A.J^0 = J^1.A$ and A_u is invertible for every $u \in [0, 1]$, hence we conclude with the point (i). ∎

3. TRANSVERSALLY ELLIPTIC SYMBOLS

We give here a brief review of the material we need in the next sections. The references are [1, 11, 12, 38].

Let M be a *compact* manifold provided with a G-action. Like in the previous section, we identify the tangent bundle $\mathbf{T}M$ and the cotangent bundle $\mathbf{T}^*M$ via a G-invariant metric $(.,.)_M$ on $\mathbf{T}M$. For any $X \in \mathfrak{g}$, we denote by X_M the following vector field: for $m \in M$, $X_M(m) := \frac{d}{dt} \exp(-tX).m|_{t=0}$.

If E^0, E^1 are G-equivariant vector bundles over M, a morphism $\sigma \in \Gamma(\mathbf{T}M, \hom(p^*E^0, p^*E^1))$ of G-equivariant complex vector bundles will be called a symbol. The subset of all $(x, v) \in \mathbf{T}M$ where $\sigma(x, v)$: $E_x^0 \to E_x^1$ is not invertible will be called the characteristic set of σ, and denoted $\mathrm{Char}(\sigma)$.

We denote by $\mathbf{T}_G M$ the following subset of $\mathbf{T}M$:

$$\mathbf{T}_G M = \{(x, v) \in \mathbf{T}M, (v, X_M(m))_M = 0 \text{ for all } X \in \mathfrak{g}\}.$$

A G-equivariant symbol σ will be called *elliptic* if σ is invertible outside a compact subset of $\mathbf{T}M$ ($\mathrm{Char}(\sigma)$ is compact), and it will be called *transversally elliptic* if the restriction of σ to $\mathbf{T}_G M$ is invertible outside a compact subset of $\mathbf{T}_G M$ ($\mathrm{Char}(\sigma) \cap \mathbf{T}_G M$ is compact). An elliptic symbol σ defines an element of $K_G(\mathbf{T}M)$, and the index of σ is a virtual finite dimensional representation of G [3, 4, 5, 6]. A transversally elliptic symbol σ defines an element of $K_G(\mathbf{T}_G M)$, and the index of σ is defined (see [1] for the analytic index and [11, 12] for the cohomological one) and is a trace class virtual representation of G. Remark that any elliptic symbol of $\mathbf{T}M$ is transversally elliptic, hence we have a restriction map $K_G(\mathbf{T}M) \to K_G(\mathbf{T}_G M)$, which makes the following diagram

$$\begin{array}{ccc} K_G(\mathbf{T}M) & \longrightarrow & K_G(\mathbf{T}_G M) \\ \Big\downarrow \mathrm{Index}_M^G & & \Big\downarrow \mathrm{Index}_M^G \\ R(G) & \longrightarrow & R^{-\infty}(G). \end{array} \tag{3.1}$$

commutative.

3.1. *Index map on non-compact manifolds.* Let U be a non-compact G-manifold. Lemma 3.6 and Theorem 3.7 of [1] tell us that for any open G-embedding $j\colon U \hookrightarrow M$ into a compact manifold we have a pushforward map $j_*\colon K_G(\mathbf{T}_G U) \to K_G(\mathbf{T}_G M)$ such that the composition

$$K_G(\mathbf{T}_G U) \xrightarrow{j_*} K_G(\mathbf{T}_G M) \xrightarrow{\mathrm{Index}_M^G} R^{-\infty}(G)$$

is independent of the choice of $j\colon U \hookrightarrow M$.

LEMMA 3.1. *Let U be a G-invariant open subset of a G-manifold $\mathcal{X}$. If U is relatively compact, there exists an open G-embedding $j\colon U \hookrightarrow M$ into a compact G-manifold.*

Proof. Here we follow the proof given by Boutet de Monvel in [9]. Let $\chi \in \mathscr{C}^\infty(\mathcal{X})^G$ be a function with compact support, such that $0 \leqslant \chi \leqslant 1$ and $\chi = 1$ on U. Let $q\colon \mathcal{X} \times \mathbb{R} \to \mathbb{R}$ be the function defined by $q(m, t) = \chi(m) - t^2$. The interval $(-\infty, 1]$ is the image of q, and the fibers $q^{-1}(\varepsilon)$ are compact for every $\varepsilon > 0$. According to Sard's Theorem there exists a regular value $0 < \varepsilon_0 < 1$ of q. The set $q^{-1}(\varepsilon_0)$ is then a compact G-invariant submanifold of $\mathcal{X} \times \mathbb{R}$, and $j\colon U \to q^{-1}(\varepsilon_0)$, $m \mapsto (m, \sqrt{1 - \varepsilon_0})$ is an open embedding. ∎

COROLLARY 3.2. *The index map* $\mathrm{Index}_U^G\colon K_G(\mathbf{T}_G U) \to R^{-\infty}(G)$ *is defined when U is a G-invariant relatively compact open subset of a G-manifold.*

3.2. *Excision lemma.* Let $j\colon U \hookrightarrow M$ be the inclusion map of a G-invariant open subset on a compact manifold, and let $j_*\colon K_G(\mathbf{T}_G U) \to K_G(\mathbf{T}_G M)$ be the pushforward map. We have two index maps Index_M^G, and Index_U^G such that $\mathrm{Index}_M^G \circ j_* = \mathrm{Index}_U^G$. Suppose that σ is a transversally elliptic symbol on $\mathbf{T}M$ with characteristic set contained in $\mathbf{T}M|_U$. Then, the restriction $\sigma|_U$ of σ to $\mathbf{T}U$ is a transversally elliptic symbol on $\mathbf{T}U$, and

$$j_*(\sigma|_U) = \sigma \quad \text{in} \quad K_G(\mathbf{T}_G M). \tag{3.2}$$

In particular, it gives $\mathrm{Index}_M^G(\sigma) = \mathrm{Index}_U^G(\sigma|_U)$.

3.3. *Locally free action.* Let G and H be compact Lie groups and let M be a *compact* $G\times H$ manifold.

In a first place, we suppose that G acts freely on M, and we denote by $\pi\colon M\to M/G$ the principal fibration. Note that the map π is H-equivariant. In this situation we have $\mathbf{T}_{G\times H}M\cong\pi^*(\mathbf{T}_H(M/G))$, and thus an isomorphism

$$\pi^*\colon K_H(\mathbf{T}_H(M/G))\to K_{G\times H}(\mathbf{T}_{G\times H}M). \tag{3.3}$$

We rephrase now Theorem 3.1 of Atiyah in [1].

For each irreducible G-representation V_μ, we associate the complex vector bundle $\underline{V}_\mu := M\times_H V_\mu$ on M/G and denote by $\underline{V}^*_\mu$ its dual. The group H acts trivially on V_μ, this makes $\underline{V}^*_\mu$ a H-vector bundle.

THEOREM 3.3 (Atiyah). *If $\sigma\in K_H(\mathbf{T}_H(M/G))$, then we have the following equality in $R^{-\infty}(G\times H)$*

$$\mathrm{Index}_M^{G\times H}(\pi^*\sigma)=\sum_{\mu\in\Lambda^*_+}\mathrm{Index}^H_{M/G}(\sigma\otimes\underline{V}^*_\mu).V_\mu. \tag{3.4}$$

In particular the G-invariant part of $\mathrm{Index}_M^{G\times H}(\pi^*\sigma)$ *is* $\mathrm{Index}^H_{M/G}(\sigma)$.

A classical example is when $M=G$, $G=G_r$ acts by right multiplications on G, and $G=G_l$ acts by left multiplications on G. Then the zero map $\sigma_0\colon G\times\mathbb{C}\to G\times\{0\}$ defines a $G_r\times G_l$-transversally elliptic symbol associated to the zero differential operator $\mathscr{C}^\infty(G)\to 0$. This symbol is equal to the pullback of $\mathbb{C}\in K_{G_r}(\mathbf{T}_{G_r}\{\text{point}\})\cong R(G_r)$. In this case $\mathrm{Index}_G^{G_r\times G_l}(\sigma_0)$ is equal to $L^2(G)$, the L^2-index of the zero operator on $\mathscr{C}^\infty(G)$. The G_r-vector bundle $\underline{V}^*_\mu\to\{\text{point}\}$ is just the vector space V^*_μ with the canonical action of G_r. Finally, (3.4) is the Peter-Weyl decomposition of $L^2(G)$ in $R^{-\infty}(G_r\times G_l)$: $L^2(G)=\sum_{\mu\in\Lambda^*_+}V^*_\mu\otimes V_\mu$.

We suppose now that G acts locally freely on M. The quotient $\mathscr{X}:=M/G$ is an orbifold, a space with finite-quotient singularities. One considers on $\mathscr{X}$ the H-equivariant *proper* orbifold vector bundles and the corresponding $R(H)$-module $K_{orb,H}(\mathscr{X})$ [21]. In the same way we consider the H-equivariant proper elliptic symbols on the orbifold $\mathbf{T}\mathscr{X}$ and the corresponding $R(H)$-module $K_{orb,H}(\mathbf{T}\mathscr{X})$. The principal fibration $\pi\colon M\to\mathscr{X}$ induces isomorphisms $K_{orb,H}(\mathscr{X})\simeq K_{G\times H}(M)$ and $K_{orb,H}(\mathbf{T}\mathscr{X})\simeq K_{G\times H}(\mathbf{T}_H M)$ that we both denote by π^*. The index map

$$\mathrm{Index}^H_{\mathscr{X}}\colon K_{orb,H}(\mathbf{T}\mathscr{X})\to R(H) \tag{3.5}$$

is defined by the following equation: for any $\sigma\in K_{orb,H}(\mathbf{T}\mathscr{X})$, $\mathrm{Index}^H_{\mathscr{X}}(\sigma) := [\mathrm{Index}_M^{G\times H}(\pi^*\sigma)]^G$.

We are particularly interested in the case where the bundle $\mathbf{T}_G M \to M$ carries a $G \times H$-equivariant almost complex structure J. Taking the quotient by G, it defines a H-equivariant almost complex structure $J_{\mathscr{X}}$ on the orbifold tangent bundle $\mathbf{T}\mathscr{X} \to \mathscr{X}$. Like in the smooth case, we have the Thom symbol $\mathrm{Thom}_H(\mathscr{X}, J_{\mathscr{X}}) \in K_{orb, H}(\mathbf{T}\mathscr{X})$ and a Riemann–Roch character RR^H: $K_{orb, H}(\mathscr{X}) \to R(H)$ related as in Lemma 2.1.

3.4. *Induction.* Let $i: H \hookrightarrow G$ be a closed subgroup with Lie algebra $\mathfrak{h}$, and $\mathscr{Y}$ be a H-manifold (as in Corollary 3.2). We have two principal bundles $\pi_1: G \times \mathscr{Y} \to \mathscr{Y}$ for the G-action, and $\pi_2: G \times \mathscr{Y} \to \mathscr{X} := G \times_H \mathscr{Y}$ for the diagonal H-action. The map $i_*: K_H(\mathbf{T}_H\mathscr{Y}) \to K_G(\mathbf{T}_G\mathscr{X})$ is well defined by the following commutative diagram

$$\begin{array}{ccc} K_H(\mathbf{T}_H\mathscr{Y}) & \xrightarrow{\pi_1^*} & K_{G\times H}(\mathbf{T}_{G\times H}(G\times\mathscr{Y})) \\ & \searrow^{i_*} & \downarrow (\pi_2^*)^{-1} \\ & & K_G(\mathbf{T}_G\mathscr{X}), \end{array} \tag{3.6}$$

since π_1^* and π_2^* are isomorphisms.

Let us show how to compute $i_*(\sigma)$, for an H-transversally elliptic symbol $\sigma \in \Gamma(\mathbf{T}Y, \hom(E^0, E^1))$, where E^0, E^1 are H-equivariant vector bundles over $\mathbf{T}\mathscr{Y}$. First we note[4] that $\mathbf{T}(G \times_H \mathscr{Y}) \cong G \times_H (\mathfrak{g}/\mathfrak{h} \oplus \mathbf{T}\mathscr{Y})$, and $\mathbf{T}_G(G \times_H \mathscr{Y}) \cong G \times_H (\mathbf{T}_H\mathscr{Y})$. So we extend trivially σ to $\mathfrak{g}/\mathfrak{h} \oplus \mathbf{T}\mathscr{Y}$, and we define $i_*(\sigma) \in \Gamma(G \times_H (\mathfrak{g}/\mathfrak{h} \oplus \mathbf{T}\mathscr{Y}), \hom(G \times_H E^0, G \times_H E^1))$ by $i_*(\sigma)([g; \xi, x, v]) := \sigma(x, v)$ for $g \in G$, $\xi \in \mathfrak{g}/\mathfrak{h}$ and $(x, v) \in \mathbf{T}\mathscr{Y}$.

To express the G-index of $i_*(\sigma)$ in terms of the H-index of σ, we need the induction map

$$\mathrm{Ind}_H^G: \mathscr{C}^{-\infty}(H)^H \to \mathscr{C}^{-\infty}(G)^G, \tag{3.7}$$

where $\mathscr{C}^{-\infty}(H)$ is the set of generalized functions on H, and the H and G invariants are taken with the conjugation action. The map Ind_H^G is defined as follows: for $\phi \in \mathscr{C}^{-\infty}(H)^H$, we have $\int_G \mathrm{Ind}_H^G(\phi)(g)\, f(g)\, dg = \mathrm{cst} \int_H \phi(h)\, f|_H(h)\, dh$, for every $f \in \mathscr{C}^{\infty}(G)^G$, where $\mathrm{cst} = \mathrm{vol}(G, dg)/\mathrm{vol}(H, dh)$.

We can now recall Theorem 4.1 of Atiyah in [1].

THEOREM 3.4. *Let $i: H \to G$ be the inclusion of a closed subgroup, let $\mathscr{Y}$ be a H-manifold satisfying the hypothesis of Corollary 3.2, and set $\mathscr{X} = G \times_H \mathscr{Y}$. Then we have the commutative diagram*

[4] These identities come from the following $G \times H$-equivariant isomorphism of vector bundles over $G \times \mathscr{Y}$: $\mathbf{T}_H(G \times \mathscr{Y}) \to G \times (\mathfrak{g}/\mathfrak{h} \oplus \mathbf{T}\mathscr{Y})$, $(g, m; \frac{d}{dt}|_{t=0}(g.e^{tX}) + v_m) \mapsto (g, m; pr_{\mathfrak{g}/\mathfrak{h}}(X) + v_m)$. Here $pr_{\mathfrak{g}/\mathfrak{h}} : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the orthogonal projection.

$$\begin{array}{ccc} K_H(\mathbf{T}_H\mathscr{Y}) & \xrightarrow{i*} & K_G(\mathbf{T}_G\mathscr{X}) \\ \Big\downarrow \scriptstyle{\mathrm{Index}^H_{\mathscr{Y}}} & & \Big\downarrow \scriptstyle{\mathrm{Index}^G_{\mathscr{X}}} \\ \mathscr{C}^{-\infty}(H)^H & \xrightarrow[\mathrm{Ind}^G_H]{} & \mathscr{C}^{-\infty}(G)^G. \end{array}$$

3.5. *Reduction.* Let us recall a multiplicative property of the index for the product of manifold. Let a compact Lie group G acts on two manifolds $\mathscr{X}$ and $\mathscr{Y}$, and assume that another compact Lie group H acts on $\mathscr{Y}$ commuting with the action of G. The external product of complexes on $\mathbf{T}\mathscr{X}$ and $\mathbf{T}\mathscr{Y}$ induces a multiplication (see [1] and [38], Section 2):

$$\begin{gathered} K_G(\mathbf{T}\mathscr{X}) \times K_{G\times H}(\mathbf{T}\mathscr{Y}) \to K_{G\times H}(\mathbf{T}(\mathscr{X}\times\mathscr{Y})) \\ (\sigma_1, \sigma_2) \mapsto \sigma_1 \odot \sigma_2. \end{gathered} \tag{3.8}$$

Let us recall the definition of this external product. Let $E^{\pm}$, $F^{\pm}$ be $G\times H$-equivariant Hermitian vector bundles over $\mathscr{X}$ and $\mathscr{Y}$ respectively, and let $\sigma_1 \colon E^+ \to E^-$, $\sigma_2 \colon F^+ \to F^-$ be $G\times H$-equivariant symbols. We consider the $G\times H$-equivariant symbol

$$\sigma_1 \odot \sigma_2 : E^+ \otimes F^+ \oplus E^- \otimes F^- \to E^- \otimes F^+ \oplus E^+ \otimes F^-$$

defined by

$$\sigma_1 \odot \sigma_2 = \begin{pmatrix} \sigma_1 \otimes I & -I \otimes \sigma_2^* \\ I \otimes \sigma_2 & \sigma_1^* \otimes I \end{pmatrix}. \tag{3.9}$$

We see that the set $\mathrm{Char}(\sigma_1 \odot \sigma_2) \subset \mathbf{T}\mathscr{X} \times \mathbf{T}\mathscr{Y}$ is equal to $\mathrm{Char}(\sigma_1) \times \mathrm{Char}(\sigma_2)$. This exterior product defines the $R(G)$-module structure on $K_G(\mathbf{T}\mathscr{X})$, by taking $\mathscr{Y} = \mathit{point}$ and $H = \{e\}$. If we take $\mathscr{X} = \mathscr{Y}$ and $H = \{e\}$, the product on $K_G(\mathbf{T}\mathscr{X})$ is defined by

$$\sigma_1 \,\tilde{\odot}\, \sigma_2 := s_{\mathscr{X}}^*(\sigma_1 \odot \sigma_2), \tag{3.10}$$

where $s_{\mathscr{X}} \colon \mathbf{T}\mathscr{X} \to \mathbf{T}\mathscr{X} \times \mathbf{T}\mathscr{X}$ is the diagonal map.

In the transversally elliptic case we need to be careful in the definition of the exterior product, since $\mathbf{T}_{G\times H}(\mathscr{X}\times\mathscr{Y}) \neq \mathbf{T}_G\mathscr{X} \times \mathbf{T}_H\mathscr{Y}$.

DEFINITION 3.5. Let σ be a H-transversally elliptic symbol on $\mathbf{T}\mathscr{Y}$. This symbol is called *H-transversally good* if the characteristic set of σ intersects $\mathbf{T}_H\mathscr{Y}$ in a compact subset of $\mathscr{Y}$.

Recall Lemma 3.4 and Theorem 3.5 of Atiyah in [1]. Let σ_1 be a G-transversally elliptic symbol on $\mathbf{T}\mathcal{X}$, and σ_2 be a H-transversally elliptic symbol on $\mathbf{T}\mathcal{Y}$ that is G-equivariant. Suppose furthermore that σ_2 is *H-transversally good*, then the product $\sigma_1 \odot \sigma_2$ is $G\times H$-transversally elliptic. Since every class of $K_{G\times H}(\mathbf{T}_H\mathcal{Y})$ can be represented by an *H-transversally good* elliptic symbol, we have a multiplication

$$\begin{aligned} K_G(\mathbf{T}_G\mathcal{X}) \times K_{G\times H}(\mathbf{T}_H\mathcal{Y}) &\to K_{G\times H}(\mathbf{T}_{G\times H}(\mathcal{X}\times\mathcal{Y})) \\ (\sigma_1, \sigma_2) &\mapsto \sigma_1 \odot \sigma_2. \end{aligned} \tag{3.11}$$

Suppose now that the manifolds $\mathcal{X}$ and $\mathcal{Y}$ satisfy the condition of Corollary 3.2. So, the index maps $\mathrm{Index}^G_{\mathcal{X}}$, $\mathrm{Index}^{G\times H}_{\mathcal{Y}}$, and $\mathrm{Index}^{G\times H}_{\mathcal{X}\times\mathcal{Y}}$ are well defined. According to Theorem 3.5 of [1], we know that

$$\mathrm{Index}^{G\times H}_{\mathcal{X}\times\mathcal{Y}}(\sigma_1 \odot \sigma_2) = \mathrm{Index}^G_{\mathcal{X}}(\sigma_1)\cdot \mathrm{Index}^{G\times H}_{\mathcal{Y}}(\sigma_2) \quad \text{in} \quad R^{-\infty}(G\times H),$$

$$\text{for any } \sigma_1 \in K_G(\mathbf{T}_G\mathcal{X}) \text{ and } \sigma_2 \in K_{G\times H}(\mathbf{T}_H(\mathcal{X}\times H)). \tag{3.12}$$

In the rest of this Section we suppose that the subgroup $H\subset G$ is the stabilizer of an element $\gamma\in\mathfrak{g}$. The manifold G/H carries a G-invariant complex structure J_γ defined by the element γ: at $e\in G/H$, the map $J_\gamma(e)$ equals $ad(\gamma).(\sqrt{-ad(\gamma)^2})^{-1}$ on $\mathbf{T}_e(G/H)=\mathfrak{g}/\mathfrak{h}$.

We recall now the definition of the map $r^{\gamma}_{G,H}\colon K_G(\mathbf{T}_G\mathcal{X})\to K_H(\mathbf{T}_H\mathcal{X})$ introduced by Atiyah in [1]. We consider the manifold $\mathcal{X}\times G$ with two actions of $G\times H$: for $(g,h)\in G\times H$ and $(x,a)\in\mathcal{X}\times G$, we have $(g,h).(x,a) := (g.x, gah^{-1})$ on $\mathcal{X}\overset{1}{\times} G$, and we have $(g,h).(x,a) := (h.x, gah^{-1})$ on $\mathcal{X}\overset{2}{\times} G$.

The map $\Theta\colon \mathcal{X}\overset{2}{\times} G\to\mathcal{X}\overset{1}{\times} G$, $(x,a)\mapsto(a.x,a)$ is $G\times H$-equivariant, and induces $\Theta^*\colon K_{G\times H}(\mathbf{T}_{G\times H}(\mathcal{X}\overset{1}{\times} G))\to K_{G\times H}(\mathbf{T}_{G\times H}(\mathcal{X}\overset{2}{\times} G))$. The G-action is free on $\mathcal{X}\overset{2}{\times} G$, so the quotient map $\pi\colon\mathcal{X}\overset{2}{\times} G\to\mathcal{X}$ induces an isomorphism $\pi^*\colon K_H(\mathbf{T}_H\mathcal{X})\to K_{G\times H}(\mathbf{T}_{G\times H}(\mathcal{X}\overset{2}{\times} G))$. We denote by $\sigma^{\gamma}_{\mathfrak{g}/\mathfrak{h}}\in K_{G\times H}(\mathbf{T}_H G)$ the pullback of the Thom class $\mathrm{Thom}_G(G/H, J_\gamma)\in K_G(\mathbf{T}(G/H))$, via the quotient map $G\to G/H$.

Consider the manifold $\mathcal{Y}=G$ with the action of $G\times H$ defined by $(g,h).a = gah^{-1}$ for $a\in G$, and $(g,h)\in G\times H$. Since the symbol $\sigma^{\gamma}_{\mathfrak{g}/\mathfrak{h}}$ is H-transversally good on $\mathbf{T}G$, the product by $\sigma^{\gamma}_{\mathfrak{g}/\mathfrak{h}}$ induces, by (3.11), the map

$$\begin{aligned} K_G(\mathbf{T}_G\mathcal{X}) &\to K_{G\times H}(T_{G\times H}(\mathcal{X}\overset{1}{\times} G)) \\ \sigma &\mapsto \sigma\odot\sigma^{\gamma}_{\mathfrak{g}/\mathfrak{h}}. \end{aligned}$$

DEFINITION 3.6 (Atiyah). Let H the stabilizer of $\gamma \in \mathfrak{g}$ in G. The map $r^{\gamma}_{G,H}\colon K_G(\mathbf{T}_G\mathscr{X}) \to K_H(\mathbf{T}_H\mathscr{X})$ is defined for every $\sigma \in K_G(\mathbf{T}_G\mathscr{X})$ by

$$r^{\gamma}_{G,H}(\sigma) := (\pi^*)^{-1} \circ \Theta^*(\sigma \odot \sigma^{\gamma}_{\mathfrak{g}/\mathfrak{h}}).$$

Theorem 4.2 in [1] tells us that the following diagram is commutative

$$\begin{array}{ccc} K_G(\mathbf{T}_G\mathscr{X}) & \xrightarrow{r^{\gamma}_{G,H}} & K_H(\mathbf{T}_H\mathscr{X}) \\ \Big\downarrow \mathrm{Index}^G_{\mathscr{X}} & & \Big\downarrow \mathrm{Index}^H_{\mathscr{X}} \\ \mathscr{C}^{-\infty}(G)^G & \xleftarrow{\mathrm{Ind}^G_H} & \mathscr{C}^{-\infty}(H)^H. \end{array} \tag{3.13}$$

We show now a more explicit description of the map $r^{\gamma}_{G,H}$. Consider the moment map

$$\mu_G\colon \mathbf{T}^*\mathscr{X} \to \mathfrak{g}^*$$

for the (canonical) Hamiltonian action of G on the symplectic manifold $\mathbf{T}^*\mathscr{X}$. If we identify $\mathbf{T}\mathscr{X}$ with $\mathbf{T}^*\mathscr{X}$ via a G-invariant metric, and $\mathfrak{g}$ with $\mathfrak{g}^*$ via a G-invariant scalar product, the "moment map" is a map $\mu_G\colon \mathbf{T}\mathscr{X} \to \mathfrak{g}$ defined as follows. If $E^1, \ldots, E^l$ is an orthonormal basis of $\mathfrak{g}$, we have $\mu_G(x, v) = \sum_i (E^i_M(x), v)_M E^i$ for $(x, v) \in \mathbf{T}\mathscr{X}$. The moment map admits the decomposition $\mu_G = \mu_H + \mu_{G/H}$, relative to the H-invariant orthogonal decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$. It is important to note that $\mathbf{T}_G\mathscr{X} = \mu_G^{-1}(0)$, $\mathbf{T}_H\mathscr{X} = \mu_H^{-1}(0)$, and $\mathbf{T}_G\mathscr{X} = \mathbf{T}_H\mathscr{X} \cap \mu_{G/H}^{-1}(0)$.

The real vector space $\mathfrak{g}/\mathfrak{h}$ is endowed with the complex structure defined by γ. Consider over $\mathbf{T}\mathscr{X}$ the H-equivariant symbol

$$\begin{aligned} \sigma^{\mathscr{X}}_{G,H}\colon \mathbf{T}\mathscr{X} \times \textstyle\bigwedge^{\mathrm{even}}_{\mathbb{C}} \mathfrak{g}/\mathfrak{h} &\to \mathbf{T}\mathscr{X} \times \textstyle\bigwedge^{\mathrm{odd}}_{\mathbb{C}} \mathfrak{g}/\mathfrak{h} \\ (x, v; w) &\to (x, v; w'), \end{aligned}$$

with $w' = Cl(\mu_{G/H}(x, v)).w$. Here $\mathfrak{h}^{\perp} \simeq \mathfrak{g}/\mathfrak{h}$, and $Cl(X)\colon \bigwedge_{\mathbb{C}} \mathfrak{g}/\mathfrak{h} \to \bigwedge_{\mathbb{C}} \mathfrak{g}/\mathfrak{h}$, $X \in \mathfrak{g}/\mathfrak{h}$, denotes the Clifford action. This symbol has $\mu_{G/H}^{-1}(0)$ for characteristic set. For any symbol σ over $\mathbf{T}\mathscr{X}$, with characteristic set $\mathrm{Char}(\sigma)$, the product $\sigma \,\tilde{\odot}\, \sigma^{\mathscr{X}}_{G,H}$, defined at (3.10), is a symbol over $\mathbf{T}\mathscr{X}$ with characteristic set $\mathrm{Char}(\sigma \,\tilde{\odot}\, \sigma^{\mathscr{X}}_{G,H}) = \mathrm{Char}(\sigma) \cap \mu_{G/H}^{-1}(0)$. Then, if σ is a G-transversally elliptic symbol over $\mathbf{T}\mathscr{X}$, the product $\sigma \,\tilde{\odot}\, \sigma^{\mathscr{X}}_{G,H}$ is a H-transversally elliptic symbol.

PROPOSITION 3.7. *The map $r^{\gamma}_{G,H}\colon K_G(\mathbf{T}_G\mathscr{X}) \to K_H(\mathbf{T}_H\mathscr{X})$ has the following equivalent definition: for every $\sigma \in K_G(\mathbf{T}_G\mathscr{X})$*

$$r^{\gamma}_{G,H}(\sigma) = \sigma \,\tilde{\odot}\, \sigma^{\mathscr{X}}_{G,H} \quad \text{in} \quad K_H(\mathbf{T}_H\mathscr{X}).$$

Proof. We have to show that for every $\sigma \in K_G(\mathbf{T}_G\mathscr{X})$, $\sigma \tilde{\odot} \sigma^{\mathscr{X}}_{G,H} = (\pi^*)^{-1} \circ \Theta^*(\sigma \odot \sigma^{\gamma}_{\mathfrak{g}/\mathfrak{h}})$ in $K_H(\mathbf{T}_H\mathscr{X})$. Let $p_G: \mathbf{T}G \to G$ and $p_{\mathscr{X}}: \mathbf{T}\mathscr{X} \to \mathscr{X}$ be the canonical projections. The symbol $\sigma^{\gamma}_{\mathfrak{g}/\mathfrak{h}}: p^*_G(G \times \wedge^{\text{even}}_{\mathbb{C}} \mathfrak{g}/\mathfrak{h}) \to p^*_G(G \times \wedge^{\text{odd}}_{\mathbb{C}} \mathfrak{g}/\mathfrak{h})$ is defined by $\sigma^{\gamma}_{\mathfrak{g}/\mathfrak{h}}(a, Z) = Cl(Z_{\mathfrak{g}/\mathfrak{h}})$ for $(a, Z) \in \mathbf{T}G \simeq G \times \mathfrak{g}$, where $Z_{\mathfrak{g}/\mathfrak{h}}$ is the $\mathfrak{g}/\mathfrak{h}$-component of $Z \in \mathfrak{g}$.

Consider $\sigma: p^*_{\mathscr{X}}E_0 \to p^*_{\mathscr{X}}E_1$, a G-transversally elliptic symbol on $\mathbf{T}\mathscr{X}$, where E_0, E_1 are G-complex vector bundles over $\mathscr{X}$. The product $\sigma \odot \sigma^{\gamma}_{\mathfrak{g}/\mathfrak{h}}$ acts on the bundles $p^*_{\mathscr{X}}E_{\bullet} \otimes p^*_G(G \times \wedge^{\bullet}_{\mathbb{C}} \mathfrak{g}/\mathfrak{h})$ at $(x, v; a, Z) \in \mathbf{T}(\mathscr{X} \times G)$ by

$$\sigma(x, v) \odot Cl(Z_{\mathfrak{g}/\mathfrak{h}}).$$

The pullback $\sigma_o := \Theta^*(\sigma \odot \sigma_{\mathfrak{g}/\mathfrak{h}})$ acts on the bundle $G \times (p^*_{\mathscr{X}}E_{\bullet} \otimes \wedge^{\bullet}_{\mathbb{C}} \mathfrak{g}/\mathfrak{h})$ (here we identify $\mathbf{T}(\mathscr{X} \times G)$ with $G \times (\mathfrak{g} \oplus \mathbf{T}\mathscr{X})$). At $(x, v; a, Z) \in \mathbf{T}(\mathscr{X} \times G)$ we have

$$\sigma_o(x, v; a, Z) = \sigma \odot \sigma^{\gamma}_{\mathfrak{g}/\mathfrak{h}}(a.x, v'; a, Z'), \qquad \text{with}$$

$(v', Z') = ([\mathbf{T}_{(x,a)}\Theta]^*)^{-1}(v, Z)$. Here $\mathbf{T}_{(x,a)}\Theta: \mathbf{T}_{(x,a)}(\mathscr{X} \times G) \to \mathbf{T}_{(a.x,a)}(\mathscr{X} \times G)$ is the tangent map of Θ at (x, a), and $[\mathbf{T}_{(x,a)}\Theta]^*: \mathbf{T}_{(a.x,a)}(\mathscr{X} \times G) \to \mathbf{T}_{(x,a)}(\mathscr{X} \times G)$ its transpose. A small computation shows that $Z' = Z + \mu_G(v)$ and $v' = a.v$. Finally, we get

$$\sigma_o(x, v; a, Z) = \sigma(a.x, a.v) \odot Cl(Z_{\mathfrak{g}/\mathfrak{h}} + \mu_{G/H}(v)).$$

Hence, the symbol $(\pi^*)^{-1}(\sigma_o)$ acts on the bundle $p^*_{\mathscr{X}}E_{\bullet} \otimes \wedge^{\bullet}_{\mathbb{C}} \mathfrak{g}/\mathfrak{h}$ by

$$(\pi^*)^{-1}(\sigma_o)(x, v) = \sigma(x, v) \odot Cl(\mu_{G/H}(v)). \qquad \blacksquare$$

For any G-invariant function $\phi \in \mathscr{C}^{\infty}(G)^G$, the Weyl integration formula can be written[5]

$$\phi = \mathrm{Ind}^G_H(\phi_{|H} \wedge^{\bullet}_{\mathbb{C}} \mathfrak{g}/\mathfrak{h}) \text{ in } \mathscr{C}^{-\infty}(G)^G. \tag{3.14}$$

where $\phi_{|H} \in \mathscr{C}^{\infty}(H)^H$ is the restriction to $H = G_{\gamma}$. Equality (3.14) remains true for any $\phi \in \mathscr{C}^{-\infty}(G)^G$ that admits a restriction to H.

LEMMA 3.8. *Let σ be a G-transversally elliptic symbol. Suppose furthermore that σ is H-transversally elliptic. This symbol defines two classes $\sigma \in K_G(\mathbf{T}_G\mathscr{X})$ and $\sigma_{|H} \in K_H(\mathbf{T}_H\mathscr{X})$ with the relation*[6] *$r^{\gamma}_{G,H}(\sigma) = \sigma_{|H} \otimes \wedge^{\bullet}_{\mathbb{C}} \mathfrak{g}/\mathfrak{h}$. Hence for the generalized character $\mathrm{Index}^G_{\mathscr{X}}(\sigma) \in R^{-\infty}(G)$ we have a "Weyl integration" formula*

$$\mathrm{Index}^G_{\mathscr{X}}(\sigma) = \mathrm{Ind}^G_H(\mathrm{Index}^H_{\mathscr{X}}(\sigma_{|H}) \wedge^{\bullet}_{\mathbb{C}} \mathfrak{g}/\mathfrak{h}). \tag{3.15}$$

[5] See Remark 9.2.

[6] Here we note $\sigma_{|H} \otimes \wedge^{\bullet}_{\mathbb{C}} \mathfrak{g}/\mathfrak{h}$ for the difference $\sigma_{|H} \otimes \wedge^{\text{even}}_{\mathbb{C}} \mathfrak{g}/\mathfrak{h} - \sigma_{|H} \otimes \wedge^{\text{odd}}_{\mathbb{C}} \mathfrak{g}/\mathfrak{h}$.

Proof. If σ is H-transversally elliptic, the symbol $(x, v) \to \sigma(x, v) \odot Cl(\mu_{G/H}(v))$ is homotopic to $(x, v) \to \sigma(x, v) \odot Cl(0)$ in $K_H(\mathbf{T}_H\mathscr{X})$. Hence $\sigma_{|H} \odot \sigma^{\mathscr{X}}_{G,H} = \sigma_{|H} \otimes \wedge^{\bullet}_{\mathbb{C}}\, \mathfrak{g}/\mathfrak{h}$ in $K_H(\mathbf{T}_H\mathscr{X})$. (3.15) follows from the diagram (3.13). ∎

COROLLARY 3.9. *Let σ be a G-transversally elliptic symbol which furthermore is H-transversally elliptic, and let $\phi \in \mathscr{C}^{-\infty}(G)^G$ which admits a restriction to H. We have*

$$\phi = \mathrm{Index}^G_{\mathscr{X}}(\sigma) \leftrightarrow \phi_{|H} = \mathrm{Index}^H_{\mathscr{X}}(\sigma_{|H}).$$

In fact, if we come back to the definition of the analytic index given by Atiyah [1], one can show the following stronger result. If σ be a G-transversally elliptic symbol which is also H-transversally elliptic, then $\mathrm{Index}^G_{\mathscr{X}}(\sigma) \in \mathscr{C}^{-\infty}(G)^G$ admits a *restriction* to H equal to $\mathrm{Index}^H_{\mathscr{X}}(\sigma_{|H}) \in \mathscr{C}^{-\infty}(H)^H$.

4. LOCALIZATION—THE GENERAL PROCEDURE

We recall briefly the notations. Let (M, J, G) be a compact G-manifold provided with a G-invariant almost complex structure. We denote by $RR^{G,J}: K_G(M) \to R(G)$ (or simply RR^G), the corresponding quantization map. We choose a G-invariant Riemannian metric $(.,.)_M$ on M. We define in this section a general procedure to localize the quantization map through the use of a G-equivariant vector field λ. This idea of localization goes back, when G is a circle group, to Atiyah [1] (see Lecture 6) and Vergne [38] (see part II).

We denote by $\Phi_\lambda: M \to \mathfrak{g}^*$ the map defined by $\langle \Phi_\lambda(m), X\rangle := (\lambda_m, X_M|_m)_M$ for $X \in \mathfrak{g}$. We denote by $\sigma^E(m, v)$, $(m, v) \in \mathbf{T}M$ the elliptic symbol associated to $\mathrm{Thom}_G(M) \otimes p^*(E)$ for $E \in K_G(M)$ (see Section 2).

Let σ^E_1 be the following G-equivariant elliptic symbol

$$\sigma^E_1(m, v) := \sigma^E(m, v - \lambda_m), \qquad (m, v) \in \mathbf{T}M. \tag{4.1}$$

The symbol σ^E_1 is obviously homotopic to σ^E, so they define the same class in $K_G(\mathbf{T}M)$. The characteristic set $\mathrm{Char}(\sigma^E)$ is $M \subset \mathbf{T}M$, but we see easily that $\mathrm{Char}(\sigma^E_1)$ is equal to the graph of the vector field λ, and

$$\mathrm{Char}(\sigma^E_1) \cap \mathbf{T}_G M = \{(m, \lambda_m) \in \mathbf{T}M, m \in \{\Phi_\lambda = 0\}\}.$$

We will now decompose the elliptic symbol σ_1^E in $K_G(\mathbf{T}_G M)$ near

$$C_\lambda := \{\Phi_\lambda = 0\}.$$

If a G-invariant subset C is a union of *connected components* of C_λ there exists a G-invariant open neighborhood $\mathcal{U}^c \subset M$ of C such that $\mathcal{U}^c \cap C_\lambda = C$ and $\partial\mathcal{U}^c \cap C_\lambda = \emptyset$. We associate to the subset C the symbol $\sigma_C^E := \sigma_1^E|_{\mathcal{U}^c} \in K_G(\mathbf{T}_G\mathcal{U}^c)$ which is the restriction of σ_1^E to $\mathbf{T}\mathcal{U}^c$. It is well defined since $\mathrm{Char}(\sigma_1^E|_{\mathcal{U}^c}) \cap \mathbf{T}_G\mathcal{U}^c = \{(m, \lambda_m) \in \mathbf{T}M, m \in C\}$ is compact.

PROPOSITION 4.1. *Let $C^a, a \in A$, be a finite collection of disjoint G-invariant subsets of C_λ, each of them being a union of connected components of C_λ, and let $\sigma_{C^a}^E \in K_G(\mathbf{T}_G\mathcal{U}^a)$ be the localized symbols. If $C_\lambda = \bigcup_a C^a$, we have*

$$\sigma^E = \sum_{a\in A} i_*^a(\sigma_{C^a}^E) \quad in \quad K_G(\mathbf{T}_G M),$$

where $i^a: \mathcal{U}^a \hookrightarrow M$ is the inclusion and $i_*^a: K_G(\mathbf{T}_G\mathcal{U}^a) \to K_G(\mathbf{T}_G M)$ is the corresponding direct image.

Proof. This is a consequence of the property of excision (see Sect. 3.2). We consider disjoint neighborhoods $\mathcal{U}^a$ of C^a, and take $i: \mathcal{U} = \bigcup_a \mathcal{U}^a \hookrightarrow M$. Let $\chi_a \in \mathscr{C}^\infty(M)^G$ be a test function (i.e., $0 \leqslant \chi_a \leqslant 1$) with compact support on $\mathcal{U}^a$ such that $\chi_a(m) \neq 0$ if $m \in C^a$. Then the function $\chi := \sum_a \chi_a$ is a G-invariant test function with support in $\mathcal{U}$ such that χ never vanishes on C_λ.

Using the G-equivariant symbol $\sigma_\chi^E(m, v) := \sigma^E(m, \chi(m)\, v - \lambda_m)$, $(m, v) \in \mathbf{T}M$, we prove the following :

(i) the symbol σ_χ^E is G-transversally elliptic and $\mathrm{Char}(\sigma_\chi^E) \subset \mathbf{T}M|_{\mathcal{U}}$,

(ii) the symbols σ_χ^E and σ_1^E are equal in $K_G(\mathbf{T}_G M)$, and

(iii) the restrictions $\sigma_\chi^E|_{\mathcal{U}}$ and $\sigma_1^E|_{\mathcal{U}}$ are equal in $K_G(\mathbf{T}_G\mathcal{U})$.

With Point (i) we can apply the excision property to σ_χ^E, hence $\sigma_\chi^E = i_*(\sigma_\chi^E|_{\mathcal{U}})$. By (ii) and (iii), the last equality gives $\sigma_1^E = i_*(\sigma_1^E|_{\mathcal{U}}) = \sum_a i_*^a(\sigma_{C^a}^E)$.

Proof of (i). The point (m, v) belongs to $\mathrm{Char}(\sigma_\chi^E)$ if and only if $\chi(m)\, v = \lambda_m (*)$. If m is not included in $\mathcal{U}$, we have $\chi(m) = 0$ and the equality $(*)$ becomes $\lambda_m = 0$. But $\{\lambda = 0\} \subset C_\lambda \subset \mathcal{U}$, thus $\mathrm{Char}(\sigma_\chi^E) \subset \mathbf{T}M|_{\mathcal{U}}$. The point (m, v) belongs to $\mathrm{Char}(\sigma_\chi^E) \cap \mathbf{T}_G M$ if and only if $\chi(m)\, v = \lambda_m$ and v is orthogonal to the G-orbit in m. This imposes $m \in C_\lambda$, and finally we see that $\mathrm{Char}(\sigma_\chi^E) \cap \mathbf{T}_G M \simeq C_\lambda$ is compact because the function χ never vanishes on C_λ.

Proof of (ii). We consider the symbols σ_t^E, $t \in [0, 1]$ defined by

$$\sigma_t^E(m, v) = \sigma^E(m, (t + (1-t)\, \chi(m))\, v - \lambda_m).$$

We see as above that σ_t^E is an homotopy of G-transversally elliptic symbols on $\mathbf{T}M$.

Proof of (iii). Here we use the homotopy $\sigma_t^E|_{\mathscr{U}}$, $t \in [0, 1]$. ∎

Because $RR^G(M, E) = \mathrm{Index}_M^G(\sigma^E) \in R(G)$, we obtain from Proposition 4.1 the following decomposition

$$RR^G(M, E) = \sum_{a \in A} \mathrm{Index}_{\mathscr{U}^a}^G(\sigma_{C^a}^E) \quad \text{in} \quad R^{-\infty}(G). \tag{4.2}$$

The rest of this article is devoted to the description, in some particular cases, of the Riemann–Roch character localized near C^a:

$$\begin{aligned} RR_{C^a}^G(M, -) \colon K_G(M) &\to R^{-\infty}(G) \\ E &\mapsto \mathrm{Index}_{\mathscr{U}^a}^G(\sigma_{C^a}^E). \end{aligned} \tag{4.3}$$

5. LOCALIZATION ON M^β

Let (M, J, G) be a compact G-manifold provided with a G-invariant almost complex structure. Let β be an element in the *center* of the Lie algebra of G, and consider the G-invariant vector field $\lambda := \beta_M$ generated by the infinitesimal action of β. In this case we have obviously

$$\{\Phi_{\beta_M} = 0\} = \{\beta_M = 0\} = M^\beta.$$

In this section, we compute the localization of the quantization map on the submanifold M^β following the technique explained in Section 4. We first need to understand the case of a vector space.

The principal results of this section, i.e., Proposition 5.4 and Theorem 5.8, were obtained by Vergne [38, Part II], in the Spin case for an action of the circle group.

5.1. *Action on a vector space.* Let (V, q, J) be a real vector space equipped with a complex structure J and an Euclidean metric q such that

$J \in O(q)$. Suppose that a compact Lie group G acts on (V, q, J) in a unitary way, and that there exists β in the center of $\mathfrak{g}$ such that

$$V^{\beta} = \{0\}.$$

We denote by $\mathbb{T}_{\beta}$ the torus generated by $\exp(t.\beta), t \in \mathbb{R}$, and $\mathfrak{t}_{\beta}$ its Lie algebra.

The complex $\mathrm{Thom}_G(V, J)$ does not define an element in $K_G(\mathbf{T}V)$ because its characteristic set is V.

DEFINITION 5.1. Let $\mathrm{Thom}_G^{\beta}(V) \in K_G(\mathbf{T}_G V)$ be the G-transversally[7] elliptic complex defined by

$$\mathrm{Thom}_G^{\beta}(V)(x, v) := \mathrm{Thom}_G(V)(x, v - \beta_V(x)) \qquad \text{for} \quad (x, v) \in \mathbf{T}V.$$

Before computing the index of $\mathrm{Thom}_G^{\beta}(V)$ explicitly, we compare it with the pushforward $j_!(\mathbb{C}) \in K_G(\mathbf{T}V)$ where $j: \{0\} \hookrightarrow V$ is the inclusion and $\mathbb{C} \to \{0\}$ is the trivial line bundle. Recall that $\mathrm{Index}_V^G(j_!(\mathbb{C})) = 1$.

We denote by $\overline{V}$ the real vector space V endowed with the complex structure $-J$, and $\bigwedge_{\mathbb{C}}^{\bullet} \overline{V} := \bigwedge_{\mathbb{C}}^{\mathrm{even}} \overline{V} - \bigwedge_{\mathbb{C}}^{\mathrm{odd}} \overline{V}$ the corresponding element in $R(G)$.

LEMMA 5.2. *We have* $\bigwedge_{\mathbb{C}}^{\bullet} \overline{V} . \mathrm{Thom}_G^{\beta}(V) = j_!(\mathbb{C})$ *in* $K_G(\mathbf{T}_G V)$, *hence*

$$\textstyle\bigwedge_{\mathbb{C}}^{\bullet} \overline{V}. \mathrm{Index}_V^G(\mathrm{Thom}_G^{\beta}(V)) = 1 \quad \textit{in} \quad R^{-\infty}(G).$$

Proof. The class $j_!(\mathbb{C})$ is represented by the symbol $\sigma_o : \mathbf{T}V \times \bigwedge_{\mathbb{C}}^{\mathrm{even}} (V \otimes \mathbb{C}) \to \mathbf{T}V \times \bigwedge_{\mathbb{C}}^{\mathrm{odd}} (V \otimes \mathbb{C})$, $(x, v, w) \mapsto (x, v, Cl(x + \imath v).w)$. If we use the following isomorphism of complex G-vector spaces

$$\begin{aligned} V \otimes \mathbb{C} &\to V \oplus \overline{V} \\ x + \imath v &\mapsto (v - J(x), v + J(x)), \end{aligned}$$

we can write $\sigma_o = \sigma_- \odot \sigma_+$, where the symbols[8] $\sigma_{\pm}$ act on $\mathbf{T}V \times \bigwedge_{\mathbb{C}}^{\bullet} V_{\pm}$ through the Clifford maps $\sigma_{\pm}(x, v) = Cl(v \mp J(x))$. Finally we see that the following G-transversally elliptic symbols on $\mathbf{T}V$ are homotopic

[7] One can verify that $\mathrm{Char}(\mathrm{Thom}_G^{\beta}(V)) \cap \mathbf{T}_G V = \{(0, 0)\}$.
[8] $V_+ = V$ and $V_- = \overline{V}$.

$$
\begin{aligned}
Cl(v+J(x)) &\odot Cl(v-J(x))\\
Cl(v+J(x)) &\odot Cl(v-\beta_V(x))\\
Cl(0) &\odot Cl(v-\beta_V(x)).
\end{aligned}
$$

The lemma is proved since $(x, v) \to Cl(0) \odot Cl(v-\beta_V(x))$ represents the class $\wedge_{\mathbb{C}}^{\bullet} \overline{V} \,.\, \mathrm{Thom}_G^{\beta}(V)$ in $K_G(\mathbf{T}_G V)$. ∎

We compute now the index of $\mathrm{Thom}_G^{\beta}(V)$. For $\alpha \in \mathfrak{t}_\beta^*$, we define the G-invariant subspaces[9] $V(\alpha) := \{v \in V,\ \rho(\exp X)(v) = e^{\imath\langle \alpha, X\rangle} \cdot v,\ \forall X \in \mathfrak{t}_\beta\}$, and $(V \otimes \mathbb{C})(\alpha) := \{v \in V \otimes \mathbb{C},\ \rho(\exp X)(v) = e^{\imath\langle \alpha, X\rangle} v,\ \forall X \in \mathfrak{t}_\beta\}$.

An element $\alpha \in \mathfrak{t}_\beta^*$, is called a weight for the action of $\mathbb{T}_\beta$ on (V, J) (resp. on $V \otimes \mathbb{C}$) if $V(\alpha) \neq 0$ (resp. $(V \otimes \mathbb{C})(\alpha) \neq 0$). We denote by $\Delta(\mathbb{T}_\beta, V)$ (resp. $\Delta(\mathbb{T}_\beta, V \otimes \mathbb{C})$) the set of weights for the action of $\mathbb{T}_\beta$ on V (resp. $V \otimes \mathbb{C}$). We shall note that $\Delta(\mathbb{T}_\beta, V \otimes \mathbb{C}) = \Delta(\mathbb{T}_\beta, V) \cup -\Delta(\mathbb{T}_\beta, V)$.

DEFINITION 5.3. We denote by $V^{+,\beta}$ the following G-stable subspace of V

$$
V^{+,\beta} := \sum_{\alpha \in \Delta_+(\mathbb{T}_\beta, V)} V(\alpha),
$$

where $\Delta_+(\mathbb{T}_\beta, V) = \{\alpha \in \Delta(\mathbb{T}_\beta, V),\ \langle \alpha, \beta\rangle > 0\}$. In the same way, we denote by $(V \otimes \mathbb{C})^{+,\beta}$ the following G-stable subspace of $V \otimes \mathbb{C}$: $(V \otimes \mathbb{C})^{+,\beta} := \sum_{\alpha \in \Delta_+(\mathbb{T}_\beta, V \otimes \mathbb{C})} (V \otimes \mathbb{C})(\alpha)$, where $\Delta_+(\mathbb{T}_\beta, V \otimes \mathbb{C}) = \{\alpha \in \Delta(\mathbb{T}_\beta, V \otimes \mathbb{C}),\ \langle \alpha, \beta\rangle > 0\}$.

For any representation W of G, we denote by $\det W$ the representation $\wedge_{\mathbb{C}}^{\max} W$. In the same way, if $W \to M$ is a G complex vector bundle we denote by $\det W$ the corresponding line bundle.

PROPOSITION 5.4. *We have the following equality in $R^{-\infty}(G)$:*

$$
\mathrm{Index}_V^G(\mathrm{Thom}_G^{\beta}(V)) = (-1)^{\dim_{\mathbb{C}} V^{+,\beta}} \det V^{+,\beta} \otimes \sum_{k \in \mathbb{N}} S^k((V \otimes \mathbb{C})^{+,\beta}),
$$

where $S^k((V \otimes \mathbb{C})^{+,\beta})$ is the kth symmetric product over $\mathbb{C}$ of $(V \otimes \mathbb{C})^{+,\beta}$.

Proposition 5.4 and Lemma 5.2 give the two important properties of the generalized function $\chi := \mathrm{Index}_G^V(\mathrm{Thom}_G^{\beta}(V))$. First χ is an inverse, in $R^{-\infty}(G)$, of the function $g \in G \to \det_V^{\mathbb{C}}(1-g^{-1})$ which is the trace of the

[9] We denote by $z \cdot v := x.v + y.J(v)$, $z = x + \imath y \in \mathbb{C}$, the action of $\mathbb{C}$ on the complex vector space (V, J), and $zw = v \otimes zz'$, $w = v \otimes z' \in V \otimes \mathbb{C}$ the canonical action of $\mathbb{C}$ on $V \otimes \mathbb{C}$.

(virtual) representation $\bigwedge_{\mathbb{C}}^{\bullet} \overline{V}$. Second, the decomposition of χ into irreducible characters of G is of the form $\chi = \sum_\lambda m_\lambda \chi_\lambda^G$ with $m_\lambda \neq 0 \Rightarrow \langle \lambda, \beta \rangle \geqslant 0$.

Definition 5.5. For any $R(G)$-module A, we denote by $A \hat{\otimes} R(\mathbb{T}_\beta)$, the $R(G) \otimes R(\mathbb{T}_\beta)$-module formed by the infinite formal sums $\sum_\alpha E_\alpha h^\alpha$ taken over the set of weights of $\mathbb{T}_\beta$, where $E_\alpha \in A$ for every α.

We denote by $[\bigwedge_{\mathbb{C}}^{\bullet} \overline{V}]_\beta^{-1}$ the infinite sum $(-1)^r \det V^{+,\beta} \otimes \sum_{k \in \mathbb{N}} S^k((V \otimes \mathbb{C})^{+,\beta})$, with $r = \dim_{\mathbb{C}} V^{+,\beta}$. It can be considered either as an element of $R^{-\infty}(G)$, $R(G) \hat{\otimes} R(\mathbb{T}_\beta)$, or $R^{-\infty}(\mathbb{T}_\beta)$.

Let $\mathscr{V} \to \mathscr{X}$ be a G-complex vector bundle such that $\mathscr{V}^\beta = \mathscr{X}$. The torus $\mathbb{T}_\beta$ acts on the fibers of $\mathscr{V} \to \mathscr{X}$, so we can polarize the $\mathbb{T}_\beta$-weights and define the vector bundles $\mathscr{V}^{+,\beta}$ and $(\mathscr{V} \otimes \mathbb{C})^{+,\beta}$. In this case, the infinite sum $[\bigwedge_{\mathbb{C}}^{\bullet} \overline{\mathscr{V}}]_\beta^{-1} := (-1)^{\dim_{\mathbb{C}} \mathscr{V}^{+,\beta}} \det \mathscr{V}^{+,\beta} \otimes \sum_{k \in \mathbb{N}} S^k((\mathscr{V} \otimes \mathbb{C})^{+,\beta})$ is an inverse of $\bigwedge_{\mathbb{C}}^{\bullet} \overline{\mathscr{V}}$ in $K_G(\mathscr{X}) \hat{\otimes} R(\mathbb{T}_\beta)$.

The rest of this Section is devoted to the proof of Proposition 5.4. The case $V^{+,\beta} = V$ or $V^{+,\beta} = \{0\}$ is considered by Atiyah [1] (see Lecture 6) and Vergne [38] (see Lemma 6, Part II).

Let H be a maximal torus of G containing $\mathbb{T}_\beta$. The symbol $\mathrm{Thom}_G^\beta(V)$ is also H-transversally elliptic and let $\mathrm{Thom}_H^\beta(V)$ be the corresponding class in $K_H(\mathbf{T}_H V)$. Following Corollary 3.9, we can reduce the proof of Proposition 5.4 to the case where the group G is equal to the torus H.

Proof of Proposition 5.4 *for a torus action.* We first recall the index theorem proved by Atiyah in Lecture 6 of [1]. Let $\mathbb{T}_m$ the circle group act on $\mathbb{C}$ with the representation t^m, $m > 0$. We have two classes $\mathrm{Thom}_{\mathbb{T}_m}^{\pm}(\mathbb{C}) \in K_{\mathbb{T}_m}(\mathbf{T}_{\mathbb{T}_m}(\mathbb{C}))$ that correspond respectively to $\beta = \pm\imath \in Lie(S^1)$. Atiyah denotes these elements $\overline{\partial}^{\pm}$.

Lemma 5.6 (Atiyah). *We have, for $m > 0$, the following equalities in $R^{-\infty}(\mathbb{T}_m)$:*

$$\mathrm{Index}_{\mathbb{C}}^{\mathbb{T}_m}(\mathrm{Thom}_{\mathbb{T}_m}^{+}(\mathbb{C})) = \left[\frac{1}{1-t^{-m}}\right]^{+} = -t^m . \sum_{k \in \mathbb{N}} (t^m)^k$$

$$\mathrm{Index}_{\mathbb{C}}^{\mathbb{T}_m}(\mathrm{Thom}_{\mathbb{T}_m}^{-}(\mathbb{C})) = \left[\frac{1}{1-t^{-m}}\right]^{-} = \sum_{k \in \mathbb{N}} (t^{-m})^k.$$

Here we follow the notation of Atiyah: $[\frac{1}{1-t^{-m}}]^{+}$ and $[\frac{1}{1-t^{-m}}]^{-}$ are the Laurent expansions of the meromorphic function $t \in \mathbb{C} \to \frac{1}{1-t^{-m}}$ around $t = 0$ and $t = \infty$ respectively.

From this lemma we can compute the index of $\mathrm{Thom}^{\pm}_{\mathbb{T}_m}(\mathbb{C})$ when $m<0$. Suppose $m<0$ and consider the morphism $\kappa\colon \mathbb{T}_m \to \mathbb{T}_{|m|}$, $t \to t^{-1}$. Using the induced morphism $\kappa^*\colon K_{\mathbb{T}_{|m|}}(\mathbf{T}_{\mathbb{T}_{|m|}}(\mathbb{C})) \to K_{\mathbb{T}_m}(\mathbf{T}_{\mathbb{T}_m}(\mathbb{C}))$, we see that $\kappa^*(\mathrm{Thom}^{\pm}_{\mathbb{T}_{|m|}}(\mathbb{C})) = \mathrm{Thom}^{\mp}_{\mathbb{T}_m}(\mathbb{C})$. This gives $\mathrm{Index}^{\mathbb{T}_m}_{\mathbb{C}}(\mathrm{Thom}^{+}_{\mathbb{T}_m}(\mathbb{C})) = \kappa^*(\sum_{k\in\mathbb{N}} (t^{-|m|})^k) = \sum_{k\in\mathbb{N}} (t^{-m})^k$ and $\mathrm{Index}^{\mathbb{T}_m}_{\mathbb{C}}(\mathrm{Thom}^{-}_{\mathbb{T}_m}(\mathbb{C})) = \kappa^*(-t^{|m|}.\sum_{k\in\mathbb{N}}(t^{|m|})^k) = -t^m \sum_{k\in\mathbb{N}}(t^m)^k$.

We can summarize these different cases as follows.

Lemma 5.7. *Let $\mathbb{T}_\alpha$ the circle group act on $\mathbb{C}$ with the representation $t \to t^\alpha$ for $\alpha \in \mathbb{Z}\setminus\{0\}$. Let $\beta \in Lie(\mathbb{T}_\alpha) \simeq \mathbb{R}$ a non-zero element. We have the following equalities in $R^{-\infty}(\mathbb{T}_\alpha)$:*

$$\mathrm{Index}^{\mathbb{T}_\alpha}_{\mathbb{C}}(\mathrm{Thom}^{\beta}_{\mathbb{T}_\alpha}(\mathbb{C}))(t) = \left[\frac{1}{1-u^{-1}}\right]^{\varepsilon}_{u=t^\alpha},$$

where ε is the sign of $\langle\alpha, \beta\rangle$.

We decompose now the vector space V into an orthogonal sum $V = \bigoplus_{i\in I} \mathbb{C}_{\alpha_i}$, where $\mathbb{C}_{\alpha_i}$ is a H-stable subspace of dimension 1 over $\mathbb{C}$ equipped with the representation $t\in H \to t^{\alpha_i} \in \mathbb{C}$. Here the set I parametrizes the weights for the action of H on V, counted with their multiplicities. Consider the circle group $\mathbb{T}_i$ with the trivial action on $\bigoplus_{k\neq i} \mathbb{C}_{\alpha_k}$ and with the canonical action on $\mathbb{C}_{\alpha_i}$. We consider V equipped with the action of $H\times \Pi_k \mathbb{T}_k$. The symbol $\mathrm{Thom}^{\beta}_H(V)$ is $H\times\Pi_k\mathbb{T}_k$-equivariant and is either H-transversally elliptic, $H\times\Pi_k\mathbb{T}_k$-transversally elliptic (we denote by σ_B the corresponding class), or $\Pi_k\mathbb{T}_k$-transversally elliptic (we denote by σ_A the corresponding class). We have the following canonical morphisms :

$$\begin{array}{ccccc} K_H(\mathbf{T}_H V) & \leftarrow & K_{H\times\Pi_k\mathbb{T}_k}(\mathbf{T}_H V) & \to & K_{H\times\Pi_k\mathbb{T}_k}(\mathbf{T}_{H\times\Pi_k\mathbb{T}_k} V) \\ \mathrm{Thom}^{\beta}_H(V) & \leftarrow & \sigma_{B_1} & \to & \sigma_B, \\ K_{H\times\Pi_k\mathbb{T}_k}(\mathbf{T}_{H\times\Pi_k\mathbb{T}_k} V) & \leftarrow & K_{H\times\Pi_k\mathbb{T}_k}(\mathbf{T}_{\Pi_k\mathbb{T}_k} V) & \to & K_{\Pi_k\mathbb{T}_k}(\mathbf{T}_{\Pi_k\mathbb{T}_k} V) \\ \sigma_B & \leftarrow & \sigma_{B_2} & \to & \sigma_A. \end{array} \tag{5.1}$$

We consider the following characters:

— $\phi(t) \in R^{-\infty}(H)$ the H-index of $\mathrm{Thom}^{\beta}_H(V)$,

— $\phi_B(t, t_1, \ldots, t_l) \in R^{-\infty}(H\times\Pi_k\mathbb{T}_k)$ the $H\times\Pi_k\mathbb{T}_k$-index of σ_B (the same for σ_{B_1} and σ_{B_2}).

— $\phi_A(t_1, \ldots, t_l) \in R^{-\infty}(\Pi_k\mathbb{T}_k)$ the $\Pi_k\mathbb{T}_k$-index of σ_A.

They satisfy the relations

(i) $\phi(t) = \phi_B(t, 1, \ldots, 1)$ and $\phi_B(1, t_1, \ldots, t_l) = \phi_A(t_1, \ldots, t_l)$.

(ii) $\phi_B(tu, t_1u^{-\alpha_1}, \ldots, t_lu^{-\alpha_1}) = \phi_B(t, t_1, \ldots, t_l)$, for all $u\in H$.

Point (i) is a consequence of the morphisms (5.1). Point (ii) follows from the fact that the elements $(u, u^{-\alpha_1}, \ldots, u^{-\alpha_l})$, $u \in H$ act trivially on V.

The symbol σ_A can be expressed through the map

$$K_{\mathbb{T}_1}(\mathbf{T}_{\mathbb{T}_1}\mathbb{C}_{\alpha_1}) \times K_{\mathbb{T}_2}(\mathbf{T}_{\mathbb{T}_2}\mathbb{C}_{\alpha_2}) \times \cdots \times K_{\mathbb{T}_l}(\mathbf{T}_{\mathbb{T}_l}\mathbb{C}_{\alpha_l}) \to K_{\Pi_k \mathbb{T}_k}(\mathbf{T}_{\Pi_k \mathbb{T}_k} V)$$
$$(\sigma_1, \sigma_2, \ldots, \sigma_l) \mapsto \sigma_1 \odot \sigma_2 \odot \cdots \odot \sigma_l.$$

Here we have $\sigma_A = \bigodot_{k=1}^{l} \mathrm{Thom}_{\mathbb{T}_k}^{\varepsilon_k}(\mathbb{C}_{\alpha_k})$ in $K_{\Pi_k \mathbb{T}_k}(\mathbf{T}_{\Pi_k \mathbb{T}_k} V)$, where ε_k is the sign of $\langle \alpha_k, \beta \rangle$. Finally, we get

$$\phi(u) = \phi_B(u, 1, \ldots, 1) = \phi_B(1, u^{\alpha_1}, \ldots, u^{\alpha_1})$$
$$= \phi_A(u^{\alpha_1}, \ldots, u^{\alpha_1}) = \Pi_k \left[\frac{1}{1 - t^{-1}} \right]_{t = u^{\alpha_k}}^{\varepsilon_k}.$$

To finish the proof, it suffices to note that the following identification of H-vector spaces holds: $V^{+,\beta} \simeq \bigoplus_{\varepsilon_k > 0} \mathbb{C}_{\alpha_k}$ and $(V \otimes \mathbb{C})^{+,\beta} \simeq \bigoplus_k \mathbb{C}_{\varepsilon_k \alpha_k}$. ▮

5.1. *Localization of the quantization map on* M^β. Let $\beta \neq 0$ be a G-invariant element of $\mathfrak{g}$. The localization formula that we prove for the Riemann–Roch character $RR^G(M, -)$ will hold in[10] $\hat{R}(G) := \hom_{\mathbb{Z}}(R(G), \mathbb{Z})$.

Let $\mathcal{N}$ be the normal bundle of M^β in M. For $m \in M^\beta$, we have the decomposition $\mathbf{T}_m M = \mathbf{T}_m M^\beta \oplus \mathcal{N}|_m$. The linear action of β on $T_m M$ precises this decomposition. The map $\mathcal{L}^M(\beta)\colon \mathbf{T}_m M \to \mathbf{T}_m M$ commutes with the map J and satisfies $\mathbf{T}_m M^\beta = \ker(\mathcal{L}^M(\beta))$. Here we take $\mathcal{N}|_m := \mathrm{Image}(\mathcal{L}^M(\beta))$. Then the almost complex structure J induces a G-invariant almost complex structure J_β on M^β, and a complex structure $J_{\mathcal{N}}$ on the fibers of $\mathcal{N} \to M^\beta$. We have then a quantization map $RR^G(M^\beta, -)\colon K_G(M^\beta) \to R(G)$. The torus $\mathbb{T}_\beta$ acts linearly on the fibers of the complex vector bundle $\mathcal{N}$. Thus we associate the polarized complex G-vector bundles $\mathcal{N}^{+,\beta}$ and $(\mathcal{N} \otimes \mathbb{C})^{+,\beta}$ (see Definition 5.5).

THEOREM 5.8. *For every $E \in K_G(M)$, we have the following equality in $\hat{R}(G)$:*

$$RR^G(M, E) = (-1)^{r_{\mathcal{N}}} \sum_{k \in \mathbb{N}} RR^G(M^\beta, E|_{M^\beta} \otimes \det \mathcal{N}^{+,\beta} \otimes S^k((\mathcal{N} \otimes \mathbb{C})^{+,\beta}),$$

where $r_{\mathcal{N}}$ is the locally constant function on M^β equal to the complex rank of $\mathcal{N}^{+,\beta}$.

Before proving this result let us rewrite this localization formula in a more synthetic way. The $G \times \mathbb{T}_\beta$-Riemann–Roch character $RR^{G \times \mathbb{T}_\beta}(M^\beta, -)$ is

[10] An element of $\hat{R}(G)$ is simply a formal sum $\sum_\lambda m_\lambda \chi_\lambda^G$ with $m_\lambda \in \mathbb{Z}$ for all λ.

extended canonically to a map from $K_G(M^\beta)\,\hat{\otimes}\,R(\mathbb{T}_\beta)$ to $R(G)\,\hat{\otimes}\,R(\mathbb{T}_\beta)$ (see Definition 5.5). Note that the surjective morphism $G\times\mathbb{T}_\beta\to G, (g,t)\mapsto g.t$ induces maps $R(G)\to R(G)\otimes R(\mathbb{T}_\beta)$, $K_G(M)\to K_{G\times\mathbb{T}_\beta}(M)$, both denoted k, with the tautological relation $k(RR^G(M,E))=RR^{G\times\mathbb{T}_\beta}(M,k(E))$. To simplify, we will omit the morphism k in our notations.

Let $\bar{\mathcal{N}}$ be the normal bundle $\mathcal{N}$ with the opposite complex structure. With the convention of Definition 5.5 the element $\bigwedge_{\mathbb{C}}^{\bullet}\bar{\mathcal{N}}\in K_{G\times\mathbb{T}_\beta}(M^\beta)\simeq K_G(M^\beta)\otimes R(\mathbb{T}_\beta)$ admits a polarized inverse $[\bigwedge_{\mathbb{C}}^{\bullet}\bar{\mathcal{N}}]_\beta^{-1}\in K_G(M^\beta)\,\hat{\otimes}\,R(\mathbb{T}_\beta)$. Finally the result of Theorem 5.8 can be written as the following equality in $R(G)\,\hat{\otimes}\,R(\mathbb{T}_\beta)$:

$$RR^G(M,E)=RR^{G\times\mathbb{T}_\beta}(M^\beta, E|_{M^\beta}\otimes[\textstyle\bigwedge_{\mathbb{C}}^{\bullet}\bar{\mathcal{N}}]_\beta^{-1}). \tag{5.2}$$

Note that Theorem 5.8 gives a proof of some rigidity properties [7, 30]. Let H be a maximal torus of G. Following Meinrenken and Sjamaar, a G-equivariant complex vector bundle $E\to M$ is called *rigid* if the action of H on $E|_{M^H}$ is trivial. Take $\beta\in\mathfrak{h}$ such that $M^\beta=M^H$, and apply Theorem 5.8, with β and $-\beta$, to $RR^H(M,E)$, with E rigid.

If we take $+\beta$, Theorem 5.8 shows that $h\in H\to RR^H(M,E)(h)$ is of the form $h\in H\to\sum_{a\in\hat{H}}n_a h^a$ with $n_a\neq 0\Rightarrow\langle a,\beta\rangle\geqslant 0$. (see Lemma 9.4). If we take $-\beta$, we find $RR^H(M,E)(h)=\sum_{a\in\hat{H}}n_a h^a$, with $n_a\neq 0\Rightarrow -\langle a,\beta\rangle\geqslant 0$. Comparing the two results, and using the genericity of β, we see that $RR^H(M,E)$ is a *constant* function on H, hence $RR^G(M,E)$ is then a constant function on G. We can now rewrite the equation of Theorem 5.8, where we keep on the right hand side the *constant* terms:

$$RR^G(M,E)=\sum_{F\subset M^{H,+}}RR(F,E|_F). \tag{5.3}$$

Here the summation is taken over all connected components F of M^H such that $\mathcal{N}_F^{+,\beta}=0$ (i.e., we have $\langle\xi,\beta\rangle<0$ for all weights ξ of the H-action on the normal bundle $\mathcal{N}_F$ of F).

Proof of Theorem 5.8. Let $\mathcal{U}$ be a G-invariant tubular neighborhood[11] of M^β in M. We know from Section 4 that $RR^G(M,E)=\mathrm{Index}^G_{\mathcal{U}}(\mathrm{Thom}^\beta_G(M,J)\otimes E|_{\mathcal{U}})$ where

$$\mathrm{Thom}^\beta_G(M,J)(m,w):=\mathrm{Thom}_G(M,J)(m,w-\beta_{\mathcal{N}}(m)),\qquad (m,w)\in\mathbf{T}\mathcal{U}.$$

Let $\phi\colon\mathcal{V}\to\mathcal{U}$ be G-invariant diffeomorphism with a G-invariant neighborhood $\mathcal{V}$ of M^β in the normal bundle $\mathcal{N}$. We denote by $\mathrm{Thom}^\beta_G(\mathcal{V},J)$

[11] To simplify the notation, we keep the notation M^β even if we work in fact on a connected component of the submanifold M^β.

the symbol $\phi^*(\mathrm{Thom}^{\beta}_{G}(M, J))$. Here we still denote by J the almost complex structure transported on $\mathcal{V}$ via the diffeomorphism $\mathcal{U} \simeq \mathcal{V}$.

Let $p\colon \mathcal{N} \to M^{\beta}$ be the canonical projection. The choice of a G-invariant connection on $\mathcal{N}$ induces an isomorphism of G-vector bundles over $\mathcal{N}$:

$$\begin{aligned} \mathbf{T}\mathcal{N} &\xrightarrow{\sim} p^*(\mathbf{T}M^{\beta} \oplus \mathcal{N}) \\ w &\mapsto \mathbf{T}p(w) \oplus (w)^V \end{aligned} \tag{5.4}$$

Here $w \to (w)^V$, $\mathbf{T}\mathcal{N} \to p^*\mathcal{N}$ is the projection that associates to a tangent vector its *vertical* part (see [10, Sect. 7] or [31, Sect. 4.1]). The map $\tilde{J} := p^*(J_{\beta} \oplus J_{\mathcal{N}})$ defines an almost complex structure on the manifold $\mathcal{N}$ that is constant over the fibers of p. With this new almost complex structure $\tilde{J}$ we construct the G-transversally elliptic symbol over $\mathcal{N}$

$$\mathrm{Thom}^{\beta}_{G}(\mathcal{N})(n, w) = \mathrm{Thom}_{G}(\mathcal{N}, \tilde{J})(n, w - \beta_{\mathcal{N}}(n)), \qquad (n, w) \in \mathbf{T}\mathcal{N}.$$

We denote by $i\colon \mathcal{V} \to \mathcal{N}$ the inclusion map, and $i_*\colon K_G(\mathbf{T}_G\mathcal{V}) \to K_G(\mathbf{T}_G\mathcal{N})$ the induced map.

Lemma 5.9. *We have*

$$i_*(\mathrm{Thom}^{\beta}_{G}(\mathcal{V}, J)) = \mathrm{Thom}^{\beta}_{G}(\mathcal{N}) \quad in \quad K_G(\mathbf{T}\mathcal{N}).$$

Proof. We proceed as in Lemma 2.2. The complex structure J_n, $n \in \mathcal{V}$ and $\tilde{J}_n$, $n \in \mathcal{N}$ are equal on M^{β}, and are related by the homotopy $J^t_{(x,v)} := J_{(x,t.v)}$, $u \in [0, 1]$ for $n = (x, v) \in \mathcal{V}$. Then, as in Lemma 2.2, we can construct an invertible bundle map $A \in \Gamma(\mathcal{V}, \mathrm{End}(\mathbf{T}\mathcal{V}))^G$, which is homotopic to the identity and such that $A.J = \tilde{J}.A$ on $\mathcal{V}$. We conclude as in Lemma 2.2 that the symbols $\mathrm{Thom}^{\beta}_{G}(\mathcal{V}, J)$ and $\mathrm{Thom}^{\beta}_{G}(\mathcal{N})|\mathcal{V}$ are equal in $K_G(\mathbf{T}\mathcal{V})$. Then the Lemma follows from the excision property. ▮

Since $E \simeq p^*(E|_{M^{\beta}})$, for any G-complex vector bundle E over $\mathcal{N}$, the former lemma tells us that $RR^G(M, E) = \mathrm{Index}^G_{\mathcal{N}}(\mathrm{Thom}^{\beta}_{G}(\mathcal{N}) \otimes p^*(E|_{M^{\beta}}))$.

We consider now the Hermitian vector bundle $\mathcal{N} \to M^{\beta}$ with the action of $G \times \mathbb{T}_{\beta}$. First we use the decomposition $\mathcal{N} = \bigoplus_{\alpha} \mathcal{N}^{\alpha}$ relatively to the unitary action of $\mathbb{T}_{\beta}$ on the fibers of $\mathcal{N}$. Let N^{α} be an Hermitian vector space of dimension equal to the rank of $\mathcal{N}^{\alpha}$, equipped with the representation $t \to t^{\alpha}$ of $\mathbb{T}_{\beta}$. Let U be the group of $\mathbb{T}_{\beta}$*-equivariant unitary maps* of the vector space $N := \bigoplus_{\alpha} N^{\alpha}$, and let R be the $\mathbb{T}_{\beta}$-equivariant unitary frame of $(\mathcal{N}, J_{\mathcal{N}})$ framed on N. Note that R is provided with a $U \times G$-action and a trivial action of $\mathbb{T}_{\beta}$: for $x \in M^{\beta}$, any element of $R|_x$ is a $\mathbb{T}_{\beta}$-equivariant unitary map from N to $\mathcal{N}|_x$. The manifold $\mathcal{N}$ is isomorphic to $R \times_U N$, where G acts on R and $\mathbb{T}_{\beta}$ acts on N.

We denote by $\mathrm{Thom}^{\beta}_{G\times\mathbb{T}_\beta}(\mathcal{N})$ the $G\times\mathbb{T}_\beta$ canonical extension of $\mathrm{Thom}^{\beta}_{G}(\mathcal{N})$. It can be considered as a G, $G\times\mathbb{T}_\beta$, or $\mathbb{T}_\beta$-transversally elliptic symbol. Here we consider $\mathrm{Thom}^{\beta}_{G\times\mathbb{T}_\beta}(\mathcal{N})$ as an element of $K_{G\times\mathbb{T}_\beta}(\mathbf{T}_{\mathbb{T}_\beta}(R\times_U N))$. Recall that we have two isomorphisms

$$\pi_N^*\colon K_{G\times\mathbb{T}_\beta}(\mathbf{T}_{\mathbb{T}_\beta}(R\times_U N)) \xrightarrow{\sim} K_{G\times\mathbb{T}_\beta\times U}(\mathbf{T}_{\mathbb{T}_\beta\times U}(R\times N)), \tag{5.5}$$

$$\pi^*\colon K_G(\mathbf{T}M^\beta) \xrightarrow{\sim} K_{G\times U}(\mathbf{T}_U R), \tag{5.6}$$

where $\pi_N\colon R\times N\to R\times_U N\simeq\mathcal{N}$ and $\pi\colon R\to R/U\simeq M^\beta$ are the quotient maps relative to the free U-action. Following (3.11), we have a product

$$K_{G\times U}(\mathbf{T}_U R)\times K_{\mathbb{T}_\beta\times U}(\mathbf{T}_{\mathbb{T}_\beta}N)\to K_{G\times\mathbb{T}_\beta\times U}(\mathbf{T}_{\mathbb{T}_\beta\times U}(R\times N)). \tag{5.7}$$

The following Thom classes

— $\mathrm{Thom}^{\beta}_{G\times\mathbb{T}_\beta}(\mathcal{N})\in K_{G\times\mathbb{T}_\beta}(\mathbf{T}_{\mathbb{T}_\beta}(R\times_U N))$,
— $\mathrm{Thom}^{\beta}_{\mathbb{T}_\beta\times U}(N)\in K_{\mathbb{T}_\beta\times U}(\mathbf{T}_{\mathbb{T}_\beta}N)$, and
— $\mathrm{Thom}_G(M^\beta)\in K_G(\mathbf{T}M^\beta)$

are related by the following equality in $K_{G\times\mathbb{T}_\beta\times U}(\mathbf{T}_{\mathbb{T}_\beta\times U}(R\times N))$:

$$\pi_N^*\,\mathrm{Thom}^{\beta}_{G\times\mathbb{T}_\beta}(\mathcal{N}) = (\pi^*\,\mathrm{Thom}_G(M^\beta))\odot\mathrm{Thom}^{\beta}_{\mathbb{T}_\beta\times U}(N). \tag{5.8}$$

We will justify (5.8) later. Every $E\in K_G(M)$, when restrict to M^β, admit the decomposition $E|_{M^\beta}=\sum_{a\in\widehat{\mathbb{T}_\beta}}E^a\otimes\mathbb{C}_a$ in $K_{G\times\mathbb{T}_\beta}(M^\beta)\simeq K_G(M^\beta)\otimes R(\mathbb{T}_\beta)$. Multiplication of (5.8) by E gives

$$\begin{aligned}&\pi_N^*(\mathrm{Thom}^{\beta}_{G\times\mathbb{T}_\beta}(\mathcal{N})\otimes E|_{M^\beta})\\ &\quad=\sum_{a\in\widehat{\mathbb{T}_\beta}}\pi^*(\mathrm{Thom}_G(M^\beta)\otimes E^a)\odot(\mathrm{Thom}^{\beta}_{\mathbb{T}_\beta\times U}(N)\otimes\mathbb{C}_a).\end{aligned}$$

Following (3.12) and Theorem (3.3), the last equality gives, after taking the index and the U-invariant,

$$\begin{aligned}&RR^{G\times\mathbb{T}_\beta}(M,E)\\ &\quad=\sum_a\Big[\sum_{i\in\hat{U}}RR^G(M^\beta,E^a\otimes\underline{W}_i^*)\cdot W_i\cdot\mathrm{Index}^{\mathbb{T}_\beta\times U}(\mathrm{Thom}^{\beta}_{\mathbb{T}_\beta\times U}(N))\cdot\mathbb{C}_a\Big]^U.\end{aligned} \tag{5.9}$$

Here we used that $RR^{G\times\mathbb{T}_\beta}(M,E)$ is equal to the U-invariant part of $\mathrm{Index}^{G\times\mathbb{T}_\beta\times U}(\pi_N^*(\mathrm{Thom}^{\beta}_{G\times\mathbb{T}_\beta}(\mathcal{N})\otimes E|_{M^\beta}))$, and the index of $\pi^*(\mathrm{Thom}_G(M^\beta)\otimes E^a)$ is equal to $\sum_{i\in\hat{U}}RR^G(M^\beta,E^a\otimes\underline{W}_i^*).W_i$.

Now we observe that for any $L \in R(U)$, the U-invariant part of $\sum_{i \in \hat{U}} RR^G(M^\beta, E|_{M^\beta} \otimes \underline{W}_i^*).W_i \otimes L$ is equal to $RR^G(M^\beta, E|_{M^\beta} \otimes \underline{L})$ with $\underline{L} = R \times_U L$. With the computation of $\mathrm{Index}^{\mathbb{T}_\beta \times U}(\mathrm{Thom}^\beta_{\mathbb{T}_\beta \times U}(N))$ given in Proposition 5.4 we obtain finally

$$RR^{G \times \mathrm{T}_\beta}(M, E)$$
$$= (-1)^{r_{\mathcal{N}}} \sum_{k \in \mathbb{N}} RR^{G \times \mathrm{T}_\beta}(M^\beta, E|_{M^\beta} \otimes \det \mathcal{N}^{+,\beta} \otimes S^k((\mathcal{N} \otimes \mathbb{C})^{+,\beta}))$$

which implies the equality of Theorem 5.8.

We give now an explanation for (5.8), which is a direct consequence of the fact that the almost complex structure $\tilde{J}$ admits the decomposition $\tilde{J} = p^*(J_\beta \oplus J_{\mathcal{N}})$. Hence $\bigwedge_{\mathbb{C}}^{\bullet} \mathbf{T}_n \mathcal{N}$ equipped with the map $Cl_n(v - \beta_{\mathcal{N}}(n))$, $v \in \mathbf{T}_n \mathcal{N}$ is isomorphic to $\bigwedge_{\mathbb{C}}^{\bullet} \mathbf{T}_x M^\beta \otimes \bigwedge_{\mathbb{C}}^{\bullet} \mathcal{N}|_x$ equipped with $Cl_x(v_1) \odot Cl_x(v_2 - \beta_{\mathcal{N}}(n))$ where $x = p(n)$, and the vector $v \in \mathbf{T}_n \mathcal{N}$ is decomposed, following the isomorphism (5.4), in $v = v_1 + v_2$ with $v_1 \in \mathbf{T}_x M^\beta$ and $v_2 \in \mathcal{N}|_x$. Note that the vector $w = \beta_{\mathcal{N}}(n) \in \mathbf{T}_n \mathcal{N}$ is vertical, i.e., $w = (w)^V$. ∎

6. LOCALIZATION VIA AN ABSTRACT MOMENT MAP

Let (M, J, G) be a compact G-manifold provided with a G-invariant almost complex structure. We denote by $RR^G \colon K_G(M) \to R(G)$ the quantization map. Here we suppose that the G-manifold is equipped with an *abstract moment map* [15, 20].

DEFINITION 6.1. A smooth map $f_G \colon M \to \mathfrak{g}^*$ is called an abstract moment map if

(i) the map f_G is equivariant for the action of the group G, and

(ii) [12]for every connected Lie subgroup $K \subset G$ with Lie algebra $\mathfrak{k}$, the induced map $f_K \colon M \to \mathfrak{k}^*$ is locally constant on the submanifold M^K of fixed points for the K-action (the map f_K is the composition of f_G with the projection $\mathfrak{g}^* \to \mathfrak{k}^*$).

The terminology "moment map" is usually used when we work in the case of a Hamiltonian action. More precisely, when the manifold is equipped with a symplectic 2-form ω which is G-invariant, a *moment map* $\Phi \colon M \to \mathfrak{g}^*$ relative to ω is a G-equivariant map satisfying $d\langle \Phi, X \rangle = -\omega(X_M, -)$, $X \in \mathfrak{g}$.

[12] Condition (ii) is equivalent to the following: for every $X \in \mathfrak{g}$, the function $\langle f_G, X \rangle$ is locally constant on M^X.

For the rest of this paper we make the choice of a G-invariant scalar product over $\mathfrak{g}^*$. This defines an identification $\mathfrak{g}^* \simeq \mathfrak{g}$, and we work with a given abstract moment map $f_G: M \to \mathfrak{g}$.

DEFINITION 6.2. Let $\mathscr{H}^G$ be the G-invariant vector field over M defined by

$$\mathscr{H}^G_m := (f_G(m)_M)_m, \qquad \forall m \in M.$$

The aim of this section is to compute the localization, as in Section 4, with the G-invariant vector field $\mathscr{H}^G$. We know that the Riemann–Roch character is localized near the set $\{\Phi_{\mathscr{H}^G} = 0\}$, but we see that $\{\Phi_{\mathscr{H}^G} = 0\} = \{\mathscr{H}^G = 0\}$. We will denote by C^{f_G} this set. Let H be a maximal torus of G, with Lie algebra $\mathfrak{h}$, and let $\mathfrak{h}_+$ be a Weyl chamber in $\mathfrak{h}$.

LEMMA 6.3. *There exists a finite subset $\mathscr{B}_G \subset \mathfrak{h}_+$, such that*

$$C^{f_G} = \bigcup_{\beta \in \mathscr{B}_G} C^G_\beta, \qquad \textit{with} \quad C^G_\beta = G.(M^\beta \cap f_G^{-1}(\beta)).$$

Proof. We first observe that $\mathscr{H}^G_m = 0$ if and only if $f_G(m) = \beta'$ and $\beta'_M|_m = 0$, that is $m \in M^{\beta'} \cap f_G^{-1}(\beta')$, for some $\beta' \in \mathfrak{g}$. For every $\beta' \in \mathfrak{g}$, there exists $\beta \in \mathfrak{h}_+$, with $\beta' = g.\beta$ for some $g \in G$. Hence $M^{\beta'} \cap f_G^{-1}(\beta') = g.(M^\beta \cap f_G^{-1}(\beta))$. We have shown that $C^{f_G} = \bigcup_{\beta \in \mathfrak{h}_+} C^G_\beta$, and we need to prove that the set $\mathscr{B}_G := \{\beta \in \mathfrak{h}_+, M^\beta \cap f_G^{-1}(\beta) \neq \emptyset\}$ is finite. Consider the set $\{H_1, ..., H_l\}$ of stabilizers for the action of the torus H on the compact manifold M. For each $\beta \in \mathfrak{h}$ we denote by $\mathbb{T}_\beta$ the subtorus of H generated by $\exp(t.\beta)$, $t \in \mathbb{R}$, and we observe that

$$\begin{aligned} M^\beta \cap f_G^{-1}(\beta) \neq \emptyset &\Leftrightarrow \exists H_i \text{ such that } \mathbb{T}_\beta \subset H_i \text{ and } M^{H_i} \cap f_G^{-1}(\beta) \neq \emptyset \\ &\Leftrightarrow \exists H_i \text{ such that } \beta \in f_G(M^{H_i}) \cap Lie(H_i). \end{aligned}$$

But $f_G(M^{H_i}) \cap Lie(H_i) \subset f_{H_i}(M^{H_i})$ is a finite set after Definition 6.1. The proof is now completed. ∎

DEFINITION 6.4. Let $\mathrm{Thom}^f_{G,[\beta]}(M) \in K_G(\mathbf{T}_G \mathscr{U}^{G,\beta})$ defined by

$$\mathrm{Thom}^f_{G,[\beta]}(M)(x, v) := \mathrm{Thom}_G(M)(x, v - \mathscr{H}^G_x), \qquad \text{for} \quad (x, v) \in \mathbf{T}\mathscr{U}^{G,\beta}.$$

Here $i^{G,\beta}: \mathscr{U}^{G,\beta} \hookrightarrow M$ is any G-invariant neighborhood of C^G_β such that $\overline{\mathscr{U}^{G,\beta}} \cap C^{f_G} = C^G_\beta$.

DEFINITION 6.5. For every $\beta \in \mathcal{B}_G$, we denote by $RR^G_\beta(M, -)\colon K_G(M) \to R^{-\infty}(G)$ the localized Riemann–Roch character near C^G_β, defined as in (4.3), by

$$RR^G_\beta(M, E) = \mathrm{Index}^G_{\mathcal{U}^{G,\beta}}(\mathrm{Thom}^f_{G,[\beta]}(M) \otimes E_{|\mathcal{U}^{G,\beta}}),$$

for $E \in K_G(M)$. Note that the map $RR^G_\beta(M, -)$ is well defined on a *non-compact* manifold M when the abstract moment map is proper, since we can take $\mathcal{U}^{G,\beta}$ relatively compact and the index map $\mathrm{Index}^G_{\mathcal{U}^{G,\beta}}$ is then defined (see Corollary 3.2).

According to Proposition 4.1, we have the partition $RR^G(M, -) = \sum_{\beta \in \mathcal{B}_G} RR^G_\beta(M, -)$, and the rest of this article is devoted to the analysis of the maps $RR^G_\beta(M, -)$, $\beta \in \mathcal{B}_G$.

In Sections 6.3 and 6.4 we prove that $[RR^G_\beta(M, E)]^G = 0$, when E is f_G*-strictly positive* with $\eta_{E,\beta} > \langle \theta, \beta \rangle$ (see Def. 1.2 for the notion of f_G*-positivity*). The next two Sections are devoted to the computation of $RR^G_0(M, -)$ when 0 is a regular value of the abstract moment map f_G.

5.1. *Induced* Spin^c *structures.* In this Section we first review the notion of Spin^c-structures (see [25, 14, 33]). After we show that the almost complex structure J on M induces a Spin^c-structure on $\mathcal{M}_{\mathrm{red}}$.

The group Spin_n is the connected double cover of the group SO_n. Let $\eta\colon \mathrm{Spin}_n \to \mathrm{SO}_n$ be the covering map, and let ε be the element who generates the kernel. The group Spin^c_n is the quotient $\mathrm{Spin}_n \times_{\mathbb{Z}_2} \mathrm{U}_1$, where $\mathbb{Z}_2$ acts by $(\varepsilon, -1)$. There are two canonical group homomorphisms

$$\eta\colon \mathrm{Spin}^c_n \to \mathrm{SO}_n\,, \qquad \mathrm{Det}\colon \mathrm{Spin}^c_n \to \mathrm{U}_1$$

such that $\eta^c = (\eta, \mathrm{Det})\colon \mathrm{Spin}^c_n \to \mathrm{SO}_n \times \mathrm{U}_1$ is a double covering map.

Let $p\colon E \to M$ be an oriented Euclidean vector bundle of rank n, and let $\mathrm{P}_{\mathrm{SO}}(E)$ be its bundle of oriented orthonormal frames. A Spin^c-structure on E is a Spin^c_n-principal bundle $\mathrm{P}_{\mathrm{Spin}^c}(E) \to M$, together with a Spin^c-equivariant map $\mathrm{P}_{\mathrm{Spin}^c}(E) \to \mathrm{P}_{\mathrm{SO}}(E)$. The line bundle $\mathbb{L} := \mathrm{P}_{\mathrm{Spin}^c}(E) \times_{\mathrm{Det}} \mathbb{C}$ is called the determinant line bundle associated to $\mathrm{P}_{\mathrm{Spin}^c}(E)$. We have then a double covering map[13]

$$\eta^c_E\colon \mathrm{P}_{\mathrm{Spin}^c}(E) \to \mathrm{P}_{\mathrm{SO}}(E) \times \mathrm{P}_{\mathrm{U}}(\mathbb{L}), \tag{6.1}$$

where $\mathrm{P}_{\mathrm{U}}(\mathbb{L}) := \mathrm{P}_{\mathrm{Spin}^c}(E) \times_{\mathrm{Det}} \mathrm{U}_1$ is the associated U_1-principal bundle over M.

A Spin^c-structure on an oriented Riemannian manifold is a Spin^c-structure on its tangent bundle. If a group K acts on the bundle E, preserving the orientation and the Euclidean structure, we defines a

[13] If P, Q are principal bundle over M respectively for the groups G and H, we denote simply by $P \times Q$ their fibering product over M which is a $G \times H$ principal bundle over M.

K-equivariant Spin^c-structure by requiring $\mathrm{P}_{\mathrm{Spin}^c}(E)$ to be a K-equivariant principal bundle, and (6.1) to be $(K \times \mathrm{Spin}_n^c)$-equivariant.

We assume now that E is of even rank $n = 2m$. Let Δ_{2m} be the irreducible complex Spin representation of Spin_{2m}^c. Recall that $\Delta_{2m} = \Delta_{2m}^+ \oplus \Delta_{2m}^-$ inherits a canonical Clifford action $\mathbf{c}\colon \mathbb{R}^{2m} \to \mathrm{End}_{\mathbb{C}}(\Delta_{2m})$ which is Spin_{2m}^c-equivariant, and which interchanges the graduation: $\mathbf{c}(v) : \Delta_{2m}^{\pm} \to \Delta_{2m}^{\mp}$, for every $v \in \mathbb{R}^{2m}$. Let

$$\mathscr{S}(E) := \mathrm{P}_{\mathrm{Spin}^c}(E) \times_{\mathrm{Spin}_{2m}^c} \Delta_{2m} \tag{6.2}$$

be the irreducible complex spinor bundle over $E \to M$. The orientation on the fibers of E defines a graduation $\mathscr{S}(E) := \mathscr{S}(E)^+ \oplus \mathscr{S}(E)^-$. Let $\overline{E}$ be the bundle E with opposite orientation. A Spin^c structure on E induces a Spin^c-structure on $\overline{E}$, with the same determinant line bundle, and such that $\mathscr{S}(\overline{E})^{\pm} = \mathscr{S}(E)^{\mp}$.

More generally, we associated to an Euclidean vector bundle $p\colon E \to M$ its Clifford bundle $\mathrm{Cl}(E) \to M$. A complex vector bundle $\mathscr{S} \to M$ is called a complex spinor bundle over $E \to M$ if it is a left-$\mathrm{Cl}(E)$-module; moreover $\mathscr{S}$ is called irreducible if $\mathrm{Cl}(E) \otimes \mathbb{C} \simeq \mathrm{End}_{\mathbb{C}}(\mathscr{S})$. In fact the notion of Spin^c-structure (in terms of principal bundle) on a Euclidean bundle $E \to M$ is equivalent to the existence of an irreducible complex spinor bundle over $E \to M$ [33].

Since $E = \mathrm{P}_{\mathrm{Spin}^c}(E) \times_{\mathrm{Spin}_{2m}^c} \mathbb{R}^{2m}$, the bundle $p^*\mathscr{S}(E)$ is isomorphic to $\mathrm{P}_{\mathrm{Spin}^c}(E) \times_{\mathrm{Spin}_{2m}^c} (\mathbb{R}^{2m} \oplus \Delta_{2m})$.

Definition 6.6. Let S-Thom(E): $p^*\mathscr{S}(E)^+ \to p^*\mathscr{S}(E)^-$ be the symbol defined by

$$\mathrm{P}_{\mathrm{Spin}^c}(E) \times_{\mathrm{Spin}_{2m}^c} (\mathbb{R}^{2m} \oplus \Delta_{2m}^+) \to \mathrm{P}_{\mathrm{Spin}^c}(E) \times_{\mathrm{Spin}_{2m}^c} (\mathbb{R}^{2m} \oplus \Delta_{2m}^-)$$
$$[p; v, w] \mapsto [p, v, \mathbf{c}(v)\, w].$$

When E is the tangent bundle of a manifold M, the symbol S-Thom(E) is denoted by S-Thom(M). If a group K acts equivariantly on the Spin^c-stucture, we denote by S-Thom$_K(E)$ the equivariant symbol.

The characteristic set of S-Thom(E) is $M \simeq \{$ zero section of $E\}$, hence it defines a class in $K(E)$ if M is compact. When $E = \mathbf{T}M$, the symbol S-Thom(M) corresponds to the *principal symbol* of the Spin^c Dirac operator associated to the Spin^c-structure [14]. When M is compact, we define a quantization map $\mathscr{Q}(M, -)\colon K(M) \to \mathbb{Z}$ by the relation $\mathscr{Q}(\mathrm{M}, V) := \mathrm{Index}_{\mathrm{M}}(\text{S-Thom}(\mathrm{M}) \otimes V)$: $\mathscr{Q}(M, V)$ is the index of the Spin^c Dirac operator on M twisted by V.

These notions extend to the orbifold case. Let M be a manifold with a locally free action of a compact Lie group G. The quotient $\mathcal{X} := M/G$ is an orbifold, a space with finite quotient singularities. A Spin^c structure on $\mathcal{X}$ is by definition a G-equivariant Spin^c structure on the bundle $\mathbf{T}_G M \to M$; where $\mathbf{T}_G M$ is identified with the pullback of $\mathbf{T}\mathcal{X}$ via the quotient map $\pi\colon M \to \mathcal{X}$. We define in the same way $\text{S-Thom}(\mathcal{X}) \in K_{\text{orb}}(\mathbf{T}\mathcal{X})$, such that $\pi^*\text{S-Thom}(\mathcal{X}) = \text{S-Thom}_G(\mathbf{T}_G M)$. The pullback by π induces an isomophism $\pi^*\colon K_{\text{orb}}(\mathbf{T}\mathcal{X}) \simeq K_G(\mathbf{T}_G M)$. The quantization map $\mathcal{Q}(\mathcal{X}, -)$ is defined by: $\mathcal{Q}(\mathcal{X}, \mathcal{E}) = \text{Index}_{\mathcal{X}}(\text{S-Thom}(\mathcal{X}) \otimes \mathcal{E})$.

LEMMA 6.7. *Let $E \to M$ be an oriented G-bundle. Let g_0, g_1 be two G-invariant metric on the fibers of E, and suppose that (E, g_0) admits an equivariant* Spin^c*-stucture denoted by* $\mathrm{P}_{\mathrm{Spin}^c}(E, g_0)$. *The trivial homotopy $g_t = (1-t).g_0 + t.g_1$ between the metrics, induces an equivariant homotopy between the principal bundles* $\mathrm{P}_{\mathrm{SO}}(E, g_0)$, $\mathrm{P}_{\mathrm{SO}}(E, g_1)$ *which can be lift to an equivariant homotopy between* $\mathrm{P}_{\mathrm{Spin}^c}(E, g_0)$ *and a* Spin^c*-bundle over (E, g_1). When the base M is compact, the corresponding symbols* $\text{S-Thom}_G(E, g_0)$ *and* $\text{S-Thom}_G(E, g_1)$ *define the same class in $K_G(E)$.*

Proof. Let $\mathcal{S}$ be the irreducible complex spinor bundle associated to $\mathrm{P}_{\mathrm{Spin}^c}(E, g_0)$. We denote by $\mathbf{c}_0\colon \mathrm{Cl}(E, g_0) \to \mathrm{End}_{\mathbb{C}}(\mathcal{S})$ the corresponding Clifford action. Let A_t be the unique g_0-symmetric endomorphism of E such that $g_t(v, w) = g_0(A_t(v), A_t(w))$. The composition $\mathbf{c}_0 \circ A_t$ is then a Clifford action of (E, g_t) on $\mathcal{S}$. It defines a Spin^c-structure on the bundle (E, g_t) which is homotopic to $\mathrm{P}_{\mathrm{Spin}^c}(E, g_0)$. ∎

Consider now the case of a *complex* vector bundle $E \to M$, of complex rank m. The orientation on the fibers of E is given by the complex structure J. Let $\mathrm{P}_{\mathrm{U}}(E)$ be the bundle of unitary frames on E. We have a morphism $\mathrm{j} : \mathrm{U}_m \to \mathrm{Spin}^c_{2m}$ which makes the diagram[14]

$$\begin{array}{ccc} \mathrm{U}_m & \xrightarrow{\ \mathrm{j}\ } & \mathrm{Spin}^c_{2m} \\ & \searrow^{\mathrm{i}\times\det} & \downarrow \eta^c \\ & & \mathrm{SO}_{2m} \times \mathrm{U}_1. \end{array} \tag{6.3}$$

commutative [25]. Then

$$\mathrm{P}_{\mathrm{Spin}^c}(E) := \mathrm{Spin}^c_{2m} \times_{\mathrm{j}} \mathrm{P}_{\mathrm{U}}(E) \tag{6.4}$$

defines a Spin^c-structure over E, with bundle of irreducible spinors $\mathcal{S}(E) = \bigwedge^\bullet_{\mathbb{C}} E$ and determinant line bundle equal to $\det_{\mathbb{C}} E$.

[14] Here i: $\mathrm{U}_m \hookrightarrow \mathrm{SO}_{2m}$ is the canonical inclusion map.

Remark 6.8. Let M be a manifold equipped with an almost complex structure J. The symbol S-Thom(M) defined by the Spin^c-structure (6.4), and the Thom symbol Thom(M, J) defined in Section 2 coincide.

Consider our case of interest, where M is a compact G-manifold equipped with an equivariant almost complex structure J and with an abstract moment map $f_G: M \to \mathfrak{g}^*$. Here we assume that 0 is a regular value of f_G: $\mathscr{Z} := f_G^{-1}(0)$ is a smooth submanifold of M with a locally free action of G. Let $\mathscr{M}_{\text{red}} := \mathscr{Z}/G$ be the corresponding "reduced" space, and let $\pi: \mathscr{Z} \to \mathscr{M}_{\text{red}}$ be the projection map. On $\mathscr{Z}$ we have an exact sequence $0 \to \mathbf{T}\mathscr{Z} \to \mathbf{T}M|_{\mathscr{Z}} \xrightarrow{df_G} \mathfrak{g}^* \times \mathscr{Z} \to 0$, and $\mathbf{T}\mathscr{Z} = \mathbf{T}_G\mathscr{Z} \oplus \mathfrak{g}_{\mathscr{Z}}$ where $\mathfrak{g}_{\mathscr{Z}} \simeq \mathfrak{g} \times \mathscr{Z}$ denotes the trivial bundle corresponding to the subspace of $\mathbf{T}\mathscr{Z}$ formed by the vector field generated by the infinitesimal action of $\mathfrak{g}$. So $\mathbf{T}M|_{\mathscr{Z}}$ admits the decomposition

$$\mathbf{T}M|_{\mathscr{Z}} = \mathbf{T}_G\mathscr{Z} \oplus \mathfrak{g}_{\mathscr{Z}} \oplus \mathfrak{g}^* \times \mathscr{Z}. \tag{6.5}$$

The bundle $\pi^*(\mathbf{T}\mathscr{M}_{\text{red}})$ is identified with $\mathbf{T}_G\mathscr{Z}$. Thus the decomposition (6.5) can be rewritten

$$\mathbf{T}M|_{\mathscr{Z}} = \pi^*(\mathbf{T}\mathscr{M}_{\text{red}}) \oplus \mathfrak{g}_{\mathbb{C}} \times \mathscr{Z}. \tag{6.6}$$

with the convention $\mathfrak{g}_{\mathscr{Z}} = (\mathfrak{g} \otimes i\mathbb{R}) \times \mathscr{Z}$ and $\mathfrak{g}^* \times \mathscr{Z} = (\mathfrak{g} \otimes \mathbb{R}) \times \mathscr{Z}$.

LEMMA 6.9. *The data* (J, f_G) *induce :*

- *an orientation* o_{red} *on* $\mathscr{M}_{\text{red}}$,
- *a* Spin^c-*structure* Q_{red} *on* $(\mathscr{M}_{\text{red}}, o_{\text{red}})$.

Moreover, the irreducible complex spinor bundle $\bigwedge_J^{\bullet} \mathbf{T}M$, *when restricted to* $\mathscr{Z}$, *defines a complex spinor bundle over* $\pi^*(\mathbf{T}\mathscr{M}_{\text{red}}) \oplus \mathfrak{g}_{\mathbb{C}} \times \mathscr{Z}$ *that is homotopic to* $\pi^*\mathscr{S}(\mathscr{M}_{\text{red}}) \otimes \bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{\mathbb{C}} \times \mathscr{Z}$.

Proof. Since $\mathfrak{g}_{\mathbb{C}} \times \mathscr{Z}$ is canonically oriented by the complex multiplication by i, the orientation $o(J)$ on M determines an orientation $o(\mathscr{M}_{\text{red}})$ on $\mathbf{T}\mathscr{M}_{\text{red}}$ such that $o(J) = o(\mathscr{M}_{\text{red}})\, o(\imath)$.

Let g_0 be the Riemannian metric on $\mathbf{T}M|_{\mathscr{Z}}$ equal to the restriction to $\mathscr{Z}$ of the Riemannian metric on M (which is taken compatible with J). If P is the Spin^c-structure on M determined by J (see 6.4), the restriction $\mathrm{P}|_{\mathscr{Z}}$ is then a Spin^c-structure on $(\mathbf{T}M|_{\mathscr{Z}}, g_0)$. Let g_1 be a G-invariant metric on the bundle $\mathbf{T}M|_{\mathscr{Z}}$ which makes (6.6) an orthogonal sum, and which is constant on the the trivial bundle $\mathfrak{g}_{\mathbb{C}} \times \mathscr{Z}$. We know from Lemma 6.7 that the Spin^c-structure $\mathrm{P}|_{\mathscr{Z}}$ on $(\mathbf{T}M|_{\mathscr{Z}}, g_0)$ is homotopic to Spin^c-structure P_1 on $(\mathbf{T}M|_{\mathscr{Z}}, g_1)$ (both are G-equivariant).

The $SO_{2k} \times U_l$-principal bundle $P_{SO}(\pi^*(T\mathcal{M}_{red})) \times P_U(\mathfrak{g}_{\mathbb{C}} \times \mathcal{Z})$ is a reduction[15] of the SO_{2n} principal bundle $P_{SO}(\pi^*(T\mathcal{M}_{red}) \oplus \mathfrak{g}_{\mathbb{C}} \times \mathcal{Z})$, thus we have the commutative diagram

$$\begin{array}{ccc} Q & \longrightarrow & P_{SO}(\pi^*(T\mathcal{M}_{red})) \times P_U(\mathfrak{g}_{\mathbb{C}} \times \mathcal{Z}) \times P_U(\mathbb{L}|_{\mathcal{Z}}) \\ \downarrow & & \downarrow \\ P_1 & \longrightarrow & P_{SO}(\pi^*(T\mathcal{M}_{red}) \oplus \mathfrak{g}_{\mathbb{C}} \times \mathcal{Z}) \times P_U(\mathbb{L}|_{\mathcal{Z}}), \end{array} \tag{6.7}$$

where $\mathbb{L} = \det_{\mathbb{C}}(TM, J)$. Here Q is a $(\eta^c)^{-1}(SO_{2k} \times U_l) \simeq Spin^c_{2k} \times U_l$-principal bundle. Finally we see that $Q_{red} = Q/(U_l \times G)$ is a $Spin^c$ structure on $\mathcal{M}_{red}$ with determinant line bundle $\mathbb{L}_{red} = \det_{\mathbb{C}}(TM|_{\mathcal{Z}})/G$.

The irreducible complex spinor bundle $\bigwedge_J^{\bullet} TM$, when restricted to $\mathcal{Z}$, is homotopic to $\mathcal{S}' = P_1 \times_{Spin^c_{2n}} \Delta_{2m}$. Using (6.7) we get

$$\begin{aligned} \mathcal{S}' &= Q \times_{(Spin^c_{2k} \times U_l)} (\Delta_{2k} \otimes \textstyle\bigwedge^{\bullet} \mathbb{C}^l) \\ &= ((Q/U_l) \times_{Spin^c_{2k}} \Delta_{2k}) \otimes ((Q/Spin^c_{2k}) \times_{U_l} \textstyle\bigwedge^{\bullet} \mathbb{C}^l) \\ &= \pi^* \mathcal{S}(\mathcal{M}_{red}) \otimes (\textstyle\bigwedge^{\bullet} \mathfrak{g}_{\mathbb{C}}) \times \mathcal{Z}. \end{aligned}$$

Here we have used the identifications $Q/Spin^c_{2k} = P_U(\mathfrak{g}_{\mathbb{C}} \times \mathcal{Z})$ and $P_U(\mathfrak{g}_{\mathbb{C}} \times \mathcal{Z}) \times_{U_l} \bigwedge^{\bullet} \mathbb{C}^l = (\bigwedge^{\bullet} \mathfrak{g}_{\mathbb{C}}) \times \mathcal{Z}$. ∎

We shall consider the particular case where J defines an almost complex structure on $\mathcal{M}_{red}$. It happens when the following decomposition holds

$$TM|_{\mathcal{Z}} = T\mathcal{Z} \oplus J(\mathfrak{g}_{\mathcal{Z}}). \tag{6.8}$$

With (6.8), $TM|_{\mathcal{Z}}$ decomposes in $TM|_{\mathcal{Z}} = \pi^*(T\mathcal{M}_{red}) \oplus \mathfrak{g}_{\mathcal{Z}} \oplus J(\mathfrak{g}_{\mathcal{Z}})$: let us denote by $pr: TM|_{\mathcal{Z}} \to \pi^*(T\mathcal{M}_{red})$ the corresponding projection. Since $\mathfrak{g}_{\mathcal{Z}} \oplus J(\mathfrak{g}_{\mathcal{Z}})$ is invariant by J, the endomorphism $J_{red} := pr \circ J$ is a G-invariant almost complex structure on $\pi^*(T\mathcal{M}_{red})$.

Using the identification $\mathfrak{g} \simeq \mathfrak{g}^*$, one considers the endomorphism $\mathcal{D}$ of the trivial bundle $\mathfrak{g} \times \mathcal{Z}$ defined by

$$\mathcal{D}(X) = -df_G(J(X_{\mathcal{Z}})), \qquad \text{for} \quad X \in \mathfrak{g}. \tag{6.9}$$

Condition (6.8) is then equivalent to : $\det \mathcal{D}(z) \neq 0$ for all $z \in \mathcal{Z}$. We shall use the normalized map $\mathcal{D}({}^t\mathcal{D}\mathcal{D})^{-1/2}$, which is an orthogonal map for the fixed Euclidean structure on $\mathfrak{g}$ (to simplify we keep the same notation $\mathcal{D}$ for it). Let $J_{\mathcal{D}}$ be the complex structure on the trivial bundle $\mathfrak{g}_{\mathbb{C}} \times \mathcal{Z}$ defined by the following matrix

$$J_{\mathcal{D}} := \begin{pmatrix} 0 & -\mathcal{D} \\ \mathcal{D}^{-1} & 0 \end{pmatrix}.$$

[15] Here $2n = \dim M$, $2k = \dim \mathcal{M}_{red}$ and $l = \dim(\mathfrak{g})$, so $n = k + l$.

LEMMA 6.10. *Suppose that the decomposition* (6.8) *holds. On*[16] $\mathbf{T}M|_{\mathscr{Z}} = \pi^*(\mathbf{T}\mathscr{M}_{\mathrm{red}}) \oplus \mathfrak{g}_{\mathbb{C}} \times \mathscr{Z}$ *the almost complex structure* J *is homotopic to* $J_{\mathrm{red}} \times J_{\mathscr{D}}$. *Hence the irreducible complex spinor bundle* $\bigwedge_J^{\bullet} \mathbf{T}M$, *when restricted to* $\mathscr{Z}$, *defines a complex spinor bundle over* $\pi^*(\mathbf{T}\mathscr{M}_{\mathrm{red}}) \oplus \mathfrak{g}_{\mathbb{C}} \times \mathscr{Z}$ *which is homotopic to* $\pi^*(\bigwedge_{J_{\mathrm{red}}}^{\bullet} \mathbf{T}\mathscr{M}_{\mathrm{red}}) \otimes \bigwedge_{J_{\mathscr{D}}}^{\bullet} \mathfrak{g}_{\mathbb{C}} \times \mathscr{Z}$.

Proof. Trough the decomposition $\mathbf{T}M|_{\mathscr{Z}} = \pi^*(\mathbf{T}\mathscr{M}_{\mathrm{red}}) \oplus \mathfrak{g}_{\mathscr{Z}} \oplus J(\mathfrak{g}_{\mathscr{Z}})$, the map J is described by the matrix

$$\begin{pmatrix} J_{\mathrm{red}} & 0 \\ A & \imath \end{pmatrix},$$

hence J is homotopic to

$$J' = \begin{pmatrix} J_{\mathrm{red}} & 0 \\ 0 & \imath \end{pmatrix}.$$

In the decomposition (6.6), J' has the following matrix

$$\begin{pmatrix} J_{\mathrm{red}} & B \\ 0 & C \end{pmatrix},$$

with $C \in \mathrm{End}(\mathfrak{g}_{\mathbb{C}} \times \mathscr{Z})$ of the form

$$\begin{pmatrix} -\mathscr{D} b \mathscr{D}^{-1} & -\mathscr{D} \\ b^2 \mathscr{D}^{-1} + \mathscr{D}^{-1} & b \end{pmatrix}.$$

Hence J' is tied to $J_{\mathrm{red}} \times J_{\mathscr{D}}$ through the homotopies $t \to t\,B$ and $t \to t\,b$, $0 \leqslant t \leqslant 1$. ▮

6.2. *The map* RR_0^G. The map $RR_0^G(M, -)\colon K_G(M) \to R^{-\infty}(G)$ is the Riemann–Roch character localized near $C_0^G = f_G^{-1}(0)$ (see Definition 6.5). In particular, $RR_0^G(M, -)$ is the zero map if 0 does not belong to $f_G(M)$. In this Section, we assume that $0 \in f_G(M)$ is a regular value of f_G. We have proved in the past Section that J induces an orientation $o(\mathscr{M}_{\mathrm{red}})$ on the reduced space $\mathscr{M}_{\mathrm{red}}$ together with a Spin$^{\mathrm{c}}$-structure on $(\mathscr{M}_{\mathrm{red}}, o(\mathscr{M}_{\mathrm{red}}))$. Let S-Thom$(\mathscr{M}_{\mathrm{red}})$ be the elliptic symbol defined by this Spin$^{\mathrm{c}}$-structure and let $\mathscr{Q}(\mathscr{M}_{\mathrm{red}}, -)$ be the corresponding quantization map.

PROPOSITION 6.11. *For every G-equivariant vector bundle* $E \to M$, *we have*

$$RR_0^G(M, E) = \sum_{\mu \in \Lambda_+^*} \mathscr{Q}(\mathscr{M}_{\mathrm{red}}, E_{\mathrm{red}} \otimes \underline{V_\mu^*}).V_\mu \quad \textit{in} \quad R^{-\infty}(G), \tag{6.10}$$

[16] Here we use the decomposition (6.6) of $\mathbf{T}M|_{\mathscr{Z}}$.

Here $E_{\mathrm{red}} = E/G$ *is the orbifold vector bundle on* $\mathcal{M}_{\mathrm{red}}$ *induced by* E, *and* $\underline{V_\mu} = \mathcal{Z} \times_G V_\mu$. *In particular, the G-invariant part of* $RR_0^G(M, E)$ *is equal to* $\mathcal{Q}(\mathcal{M}_{\mathrm{red}}, E_{\mathrm{red}}) \in \mathbb{Z}$.

Equality (6.10) is obtained by Vergne [38, Part II] in the case of a Hamiltonian action of the circle group on a compact symplectic manifold.

Suppose now that the decomposition (6.8) holds. The trivial bundle $\mathfrak{g}_{\mathbb{C}} \times \mathcal{Z}$ has two irreducible complex spinor bundles $\bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{\mathbb{C}} \times \mathcal{Z}$ and $\bigwedge_{J_{\mathcal{D}}}^{\bullet} \mathfrak{g}_{\mathbb{C}} \times \mathcal{Z}$. Thus

$$\textstyle\bigwedge_{J_{\mathcal{D}}}^{\bullet} \mathfrak{g}_{\mathbb{C}} \times \mathcal{Z} = \bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{\mathbb{C}} \times \mathcal{Z} \otimes \pi^* L_{\mathcal{D}} \tag{6.11}$$

where $\pi^* L_{\mathcal{D}} \to \mathcal{Z}$ is the line bundle equal to $\mathrm{Hom}_{Cl_{\mathbb{C}}}(\bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{\mathbb{C}} \times \mathcal{Z}, \bigwedge_{J_{\mathcal{D}}}^{\bullet} \mathfrak{g}_{\mathbb{C}} \times \mathcal{Z})$: at $z \in \mathcal{Z}$, $\pi^* L_{\mathcal{D}}|_z$ is the complex vector space of linear maps $\bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{\mathbb{C}} \to \bigwedge_{J_{\mathcal{D}}(z)}^{\bullet} \mathfrak{g}_{\mathbb{C}}$ commuting with the Clifford actions (see [33]). Note that $\bigwedge_{J_{\mathcal{D}}}^{\pm} \mathfrak{g}_{\mathbb{C}} \times \mathcal{Z} = \bigwedge_{\mathbb{C}}^{\pm} \mathfrak{g}_{\mathbb{C}} \times \mathcal{Z} \otimes \pi^* L_{\mathcal{D}}$ if the orientation of $J_{\mathcal{D}}$ coincide with those defined by $\imath$ (i.e., $\det \mathcal{D} > 0$). If $\det \mathcal{D} < 0$, we have $\bigwedge_{J_{\mathcal{D}}}^{\pm} \mathfrak{g}_{\mathbb{C}} \times \mathcal{Z} = \bigwedge_{\mathbb{C}}^{\mp} \mathfrak{g}_{\mathbb{C}} \times \mathcal{Z} \otimes \pi^* L_{\mathcal{D}}$.

PROPOSITION 6.12. *Suppose that the decomposition* (6.8) *holds, and let* $RR^{J_{\mathrm{red}}}(\mathcal{M}_{\mathrm{red}}, -)$ *be the quantization map given by* J_{red}. *For every G-equivariant vector bundle* $E \to M$, *we have*

$$[RR_0^G(M, E)]^G = \pm\, RR^{J_{\mathrm{red}}}(\mathcal{M}_{\mathrm{red}}, E_{\mathrm{red}} \otimes L_{\mathcal{D}}), \tag{6.12}$$

where $\pm$ *is the sign of* $\det \mathcal{D}$.

Proof of Proposition 6.11. Following Definition 6.5, the map $RR_0^G(M, -)$ is defined by $\mathrm{Thom}^f_{G,[0]}(M) \in K_G(\mathbf{T}_G \mathcal{U}^{G,0})$, where $\mathcal{U}^{G,0}$ is a (small) neighborhood of $\mathcal{Z}$ in M. Since 0 is a regular value of f_G, $\mathcal{U}^{G,0}$ is diffeomorphic to $\mathcal{Z} \times \mathfrak{g}^*$, and the moment map is equal to the projection $f\colon \mathcal{Z} \times \mathfrak{g}^* \to \mathfrak{g}^*$ in a neighborhood of $\mathcal{Z}$ in $\mathcal{Z} \times \mathfrak{g}^*$. We denote by $\sigma_{\mathcal{Z}} \in K_G(\mathbf{T}_G(\mathcal{Z} \times \mathfrak{g}^*))$ the symbol corresponding to $\mathrm{Thom}^f_{G,[0]}(M)$ through the diffeomorphism $\mathcal{U}^{G,0} \cong \mathcal{Z} \times \mathfrak{g}^*$. Let $\mathrm{Index}^G_{\mathcal{Z} \times \mathfrak{g}^*}\colon K_G(\mathbf{T}_G(\mathcal{Z} \times \mathfrak{g}^*)) \to R^{-\infty}(G)$ be the index map on $\mathcal{Z} \times \mathfrak{g}^*$. The map $RR_0^G(M, -)$ is defined by $RR_0^G(M, E) = \mathrm{Index}^G_{\mathcal{Z} \times \mathfrak{g}^*}(\sigma_{\mathcal{Z}} \otimes E_{|\mathcal{Z}})$.

Following Atiyah [1, Theorem 4.3], the inclusion map $j\colon \mathcal{Z} \hookrightarrow \mathcal{Z} \times \mathfrak{g}^*$ induces an $R(G)$-module morphism $j_!\colon K_G(\mathbf{T}_G \mathcal{Z}) \to K_G(\mathbf{T}_G(\mathcal{Z} \times \mathfrak{g}^*))$, with the commutative diagram

$$\begin{array}{ccc} K_G(\mathbf{T}_G\mathscr{Z}) & \xrightarrow{j_!} & K_G(\mathbf{T}_G(\mathscr{Z}\times\mathfrak{g}^*)) \\ & \searrow^{\mathrm{Index}^G_{\mathscr{Z}}} & \downarrow{\scriptstyle \mathrm{Index}^G_{\mathscr{Z}\times\mathfrak{g}^*}} \\ & & R^{-\infty}(G) \end{array} . \tag{6.13}$$

More generally, the map $i_!\colon K_G(\mathbf{T}_G\mathscr{Z}) \to K_G(\mathbf{T}_G\mathscr{Y})$ is defined by Atiyah for any embedding $i\colon \mathscr{Z} \hookrightarrow \mathscr{Y}$ of G-manifolds with $\mathscr{Z}$ compact.

Consider now the case where i is the zero-section of a G-vector bundle $\mathscr{E} \to \mathscr{Z}$. In general the map $i_!$ is *not* an isomorphism. If furthermore the G-action is *locally free* over $\mathscr{Z}$, then $\mathbf{T}_G\mathscr{Z}$, $\mathbf{T}_G\mathscr{E}$ are respectively subbundles of $\mathbf{T}\mathscr{Z} \to \mathscr{Z}$, $\mathbf{T}\mathscr{E} \to \mathscr{E}$, and the projection $\mathbf{T}_G\mathscr{E} \to \mathbf{T}_G\mathscr{Z}$ is a vector bundle isomorphic to $s^*(\mathbf{T}\mathscr{E})$ (where $s\colon \mathbf{T}_G\mathscr{Z} \hookrightarrow \mathbf{T}\mathscr{Z}$ is the inclusion). Hence the vector bundle $\mathbf{T}_G\mathscr{E} \to \mathbf{T}_G\mathscr{Z}$ inherits a complex structure over the fibers (coming from the complex vector bundle $\mathbf{T}\mathscr{E} \to \mathbf{T}\mathscr{Z}$). In this situation, the map $i_!\colon K_G(\mathbf{T}_G\mathscr{Z}) \to K_G(\mathbf{T}_G\mathscr{E})$ is the Thom isomorphism.

In the case of the (trivial) vector bundle $\mathscr{Z}\times\mathfrak{g}^* \to \mathscr{Z}$, the map $j_!\colon K_G(\mathbf{T}_G\mathscr{Z}) \to K_G(\mathbf{T}_G(\mathscr{Z}\times\mathfrak{g}^*))$ is then an *isomorphism*. Take $\tilde{\sigma}_{\mathscr{Z}} = (j_!)^{-1}$ $(\sigma_{\mathscr{Z}})$, and from the commutative diagram (6.13) we have $RR^G_0(M, E) = \mathrm{Index}^G_{\mathscr{Z}}(\tilde{\sigma}_{\mathscr{Z}} \otimes E|_{\mathscr{Z}})$. From Theorem 3.3 we get

$$\mathrm{Index}^G_{\mathscr{Z}}(\tilde{\sigma}_{\mathscr{Z}} \otimes E|_{\mathscr{Z}}) = \sum_{\mu\in\Lambda^*_+} \mathrm{Index}_{\mathscr{M}_{\mathrm{red}}}(\sigma^{\mathrm{red}} \otimes E_{\mathrm{red}} \otimes \underline{V_\mu}^*).V_\mu,$$

where $\sigma^{\mathrm{red}} \in K_{\mathrm{orb}}(\mathbf{T}\mathscr{M}_{\mathrm{red}})$ corresponds to $\tilde{\sigma}_{\mathscr{Z}} = (j_!)^{-1}(\sigma_{\mathscr{Z}})$ through the isomorphism $\pi^*\colon K_{\mathrm{orb}}(\mathbf{T}\mathscr{M}_{\mathrm{red}}) \to K_G(\mathbf{T}_G\mathscr{Z})$. Proposition 6.12 follows immediately from

LEMMA 6.13. *We have*

$$j_! \circ (\pi)^*(\text{S-Thom}(\mathscr{M}_{\mathrm{red}})) = \sigma_{\mathscr{Z}}$$

in $K_G(\mathbf{T}_G(\mathscr{Z}\times\mathfrak{g}^*))$.

Proof. Let $\mathscr{S}(M)$ the irreducible spinor bundle defined by the almost complex structure J. Let $\tilde{J}$ be the almost complex structure on $\mathscr{Z}\times\mathfrak{g}^*$, equal to J on $\mathscr{Z}$, and which is constant on the fibers of the projection $\mathscr{Z}\times\mathfrak{g}^* \to \mathscr{Z}$. Since the almost complex structures J and $\tilde{J}$ are *homotopic* near $\mathscr{Z}$, the complex $\sigma_{\mathscr{Z}}$ can be defined on $\mathscr{Z}\times\mathfrak{g}^*$ with $\tilde{J}$: we take $\mathscr{S}(M)|_{\mathscr{Z}}\times\mathfrak{g}^*$ for bundle of spinors over $\mathscr{Z}\times\mathfrak{g}^*$. Following (6.6) and (6.5), for $(z, \xi) \in \mathscr{Z}\times\mathfrak{g}^*$ a vector $v \in \mathbf{T}_{(z,\xi)}(\mathscr{Z}\times\mathfrak{g}^*)$ decomposes into $v = v_1 + X + \imath Y$, where $v_1 \in \pi^*(\mathbf{T}M_\xi)$, and $X + \imath Y \in \mathfrak{g}_{\mathbb{C}}$. The map $\sigma_{\mathscr{Z}}(z, \xi; v)$ acts on $\mathscr{S}(M)_z$ by the Clifford action pushed by the vector field[17] $\mathscr{H}^G(z, \xi) = \imath\,\xi$:

$$\sigma_{\mathscr{Z}}(z, \xi; v) = \mathrm{Cl}_z(v_1 + X + \imath(Y - \xi)).$$

[17] The tangent vector $\mathscr{H}^G(z, \xi) \in \mathfrak{g}_{\mathscr{Z}}|_z$ is equal to $\imath\xi \in \mathfrak{g}_{\mathbb{C}} \times \mathscr{Z}$.

Using now Lemma 6.9, we see that $\sigma_{\mathscr{X}}$ is homotopic to the symbol $\sigma'_{\mathscr{X}}$, which acts on the product $(\pi^*\mathscr{S}(\mathscr{M}_{\text{red}}) \otimes \bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{\mathbb{C}} \times \mathscr{X}) \times \mathfrak{g}^*$ by

$$\sigma'_{\mathscr{X}}(z, \xi; v) = \mathrm{Cl}_z(v_1) \odot \mathrm{Cl}(X + \imath(Y - \xi)).$$

Now we see that the map $\mathrm{Cl}_z(v_1) \odot \mathrm{Cl}(X + \imath(Y - \xi))$ is homotopic, as a G-transversally elliptic symbol, to $\mathrm{Cl}_z(v_1) \odot \mathrm{Cl}(\xi + \imath X)$. The K-theory class of this former symbol is equal to $(\pi)^*$ (S-Thom($\mathscr{M}_{\text{red}}$)) $\odot\, k_!(\mathbb{C})$ (where k: $\{0\} \hookrightarrow \mathfrak{g}^*$) which is the symbol map of $j_! \circ (\pi)^*$ (S-Thom($\mathscr{M}_{\text{red}}$)) (see the construction of the map $j_!$ in [1] [Lecture 4]). We have shown that $\mathrm{j}_! \circ (\pi)^*$ (S-Thom($\mathscr{M}_{\text{red}}$))$=\sigma_{\mathscr{X}}$ in $K_G(\mathbf{T}_G(\mathscr{X} \times \mathfrak{g}^*))$. ∎

Proof of Proposition 6.12. Here the proof is similar to the former proof but we use Lemma 6.10 instead of Lemma 6.9. One as to show that

$$j_! \circ (\pi)^* \,(\text{S-Thom}(\mathscr{M}_{\text{red}}) \otimes L_{\mathscr{D}}) = \pm \sigma_{\mathscr{X}}$$

in $K_G(\mathbf{T}_G(\mathscr{X} \times \mathfrak{g}^*))$, where $\pm$ is the sign of $\det \mathscr{D}$. By Lemma 6.10, we see as before that $\sigma_{\mathscr{X}}$ is homotopic to the product

$$\mathrm{Cl}_z(v_1) \odot \mathrm{Cl}_{J_{\mathscr{D}}}(\xi + \imath X) \tag{6.14}$$

acting on $(\bigwedge_{J_{\text{red}}}^{\bullet} \pi^*(\mathbf{T}\mathscr{M}_{\text{red}}) \otimes \bigwedge_{J_{\mathscr{D}}}^{\bullet} \mathfrak{g}_{\mathbb{C}} \times \mathscr{X}) \times \mathfrak{g}^*$. Now we use the isomorphism of irreducible complex spinor bundles (6.11) where we have two different orientations $o(J_{\mathscr{D}})$ and $o(\imath)$ on $\mathfrak{g}_{\mathbb{C}} \times \mathscr{X}$: $o(J_{\mathscr{D}}) = \pm o(\imath)$ where $\pm$ is the sign of $\det \mathscr{D}$. Hence the transversally elliptic symbol (6.14) is equal to

$$\pm \mathrm{Cl}_z(v_1) \odot \mathrm{Cl}(\xi + \imath X) \odot \mathrm{Id}_{L_{\mathscr{D}}}$$

acting on $(\bigwedge_{J_{\text{red}}}^{\bullet} \pi^*(\mathbf{T}\mathscr{M}_{\text{red}}) \otimes \bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{\mathbb{C}} \times \mathscr{X} \otimes L_{\mathscr{D}}) \times \mathfrak{g}^*$. ∎

6.3 *The map* RR_β^G *when* $G_\beta = G$. When $\beta \in \mathscr{B}_G - \{0\}$ is in the center of $\mathfrak{g}$, the map $RR_\beta^G(M, -)$ is the Riemann–Roch character localized near $M^\beta \cap f_G^{-1}(\beta)$. In this Section we prove that $[RR_\beta^G(M, E)]^G = 0$ if E is a f_G-strictly positive complex vector bundle.

The almost complex structure J and the abstract moment map f_G: $M \to \mathfrak{g}$ restrict on M^β to an almost complex structure J_β and a abstract moment map $f_G|_{M^\beta}$. The set $M^\beta \cap f_G^{-1}(\beta) = (f_G|_{M^\beta})^{-1}(\beta)$ is a component of the critical set of $C^{f_G|_{M^\beta}}$, and we denote by $RR_\beta^G(M^\beta, -)$: $K_G(M^\beta) \to R^{-\infty}(G)$ the Riemann–Roch character on M^β localized near the component $(f_G|_{M^\beta})^{-1}(\beta)$ (see Definition 6.5).

Here we proceed as in Section 5. Let $p\colon \mathcal{N} \to M^\beta$ be the normal bundle of M^β in M. The torus $\mathbb{T}_\beta \hookrightarrow G$ acts linearly on the fibers of the complex vector bundle $\mathcal{N}$, thus we associate, as in Theorem 5.8, the polarized complex G-vector bundles $\mathcal{N}^{+,\beta}$ and $(\mathcal{N} \otimes \mathbb{C})^{+,\beta}$.

PROPOSITION 6.14. *For every $E \in K_G(M)$, we have the following equality in $\hat{R}(G)$:*

$$RR^G_\beta(M, E) = (-1)^{r_{\mathcal{N}}} \sum_{k \in \mathbb{N}} RR^G_\beta(M^\beta, E|_{M^\beta} \otimes \det \mathcal{N}^{+,\beta} \otimes S^k((\mathcal{N} \otimes \mathbb{C})^{+,\beta}),$$

where $r_{\mathcal{N}}$ is the locally constant function on M^β equal to the complex rank of $\mathcal{N}^{+,\beta}$.

Consider the $G \times \mathbb{T}_\beta$-Riemann–Roch character $RR^{G\times\mathbb{T}_\beta}_\beta(M^\beta, -)$ localized near $M^\beta \cap f_G^{-1}(\beta)$. It can be extended trivially to a map, still denoted by $RR^{G\times\mathbb{T}_\beta}_\beta(M^\beta, -)$, from $K_G(M^\beta) \hat{\otimes} R(\mathbb{T}_\beta)$ to $R^{-\infty}(G) \hat{\otimes} R(\mathbb{T}_\beta)$. Following Definition 5.5 the element $\bigwedge^\bullet_{\mathbb{C}} \overline{\mathcal{N}} \in K_{G\times\mathbb{T}_\beta}(M^\beta) \simeq K_G(M^\beta) \otimes R(\mathbb{T}_\beta)$ admits a polarized inverse $[\bigwedge^\bullet_{\mathbb{C}} \overline{\mathcal{N}}]^{-1}_\beta \in K_G(M^\beta) \hat{\otimes} R(\mathbb{T}_\beta)$. Finally the result of Proposition 6.14 can be written as the following equality in $R^{-\infty}(G) \hat{\otimes} R(\mathbb{T}_\beta)$:

$$RR^G_\beta(M, E) = RR^{G\times\mathbb{T}_\beta}_\beta(M^\beta, E|_{M^\beta} \otimes [\textstyle\bigwedge^\bullet_{\mathbb{C}} \overline{\mathcal{N}}]^{-1}_\beta). \tag{6.15}$$

Consider the decomposition of $RR^G_\beta(M, E) = \sum_\lambda m_{\beta,\lambda}(E)\, \chi^G_\lambda$ in irreducible characters χ^G_λ, $\lambda \in \Lambda^*_+$. Let E be a f_G-strictly positive complex vector bundle over M, and let $\eta_{E,\beta} > 0$ be the constant defined in Definition 1.2. If $\mathcal{Z}$ is a connected component of M^β which intersects $f_G^{-1}(\beta)$, every weight a of the $\mathbb{T}_\beta$-action on the fibers of the complex vector bundle $E^{\overset{k}{\otimes}}|_{\mathcal{Z}} \otimes \det \mathcal{N}^{+,\beta} \otimes S^k((\mathcal{N} \otimes \mathbb{C})^{+,\beta})$ satisfy $\langle a, \beta\rangle \geqslant k.\eta_{E,\beta}$. Lemma 9.4 and Corollary 9.5, applied to this situation, show that

$$m_{\beta,\lambda}(E^{\overset{k}{\otimes}}) \neq 0 \;\Rightarrow\; \langle \lambda, \beta\rangle \geqslant k.\eta_{E,\beta}. \tag{6.16}$$

In particular $[RR^G_\beta(M, E)]^G = m_{\beta,0}(E) = 0$, so we have proved the

COROLLARY 6.15. *Let E be a f_G-strictly positive* complex vector bundle over M (see Def. 1.2). For any $\beta \in \mathcal{B}_G - \{0\}$, with $G_\beta = G$, the G-invariant part of $RR^G_\beta(M, E)$ is equal to 0.

Proof of Proposition 6.14. Here we proceed as in the proof of Theorem 5.8. The almost complex structure J induces an almost complex structure J_β on M^β and a complex structure $J_{\mathcal{N}}$ on the fibers of $\mathcal{N}$. The $G \times \mathbb{T}_\beta$-vector bundle $p\colon \mathcal{N} \to M^\beta$ is isomorphic to $R \times_U N \to M^\beta = R/U$, where R is the $\mathbb{T}_\beta$-equivariant unitary frame of $(\mathcal{N}, J_{\mathcal{N}})$ framed on N.

Let $\mathscr{U}^{G,\beta}$ be a neighborhood of C^G_β in M, and consider the G-transversally elliptic symbol $\mathrm{Thom}^f_{G,[\beta]}(M) \in K_G(\mathbf{T}_G\mathscr{U}^{G,\beta})$ introduced in Definition 6.4. Here we choose $\mathscr{U}^{G,\beta}$ diffeomorphic to an open subset of $\mathscr{N}$ of the form $\mathscr{V} := \{n = (x, v) \in \mathscr{N}, x \in \mathscr{U}$ and $|v| < \varepsilon\}$, where $\mathscr{U}$ is a neighborhood of $(f_G|_{M^\beta})^{-1}(\beta)$ in M^β. The moment map f_G, the vector field $\mathscr{H}^G$, and $\mathrm{Thom}^f_{G,[\beta]}(M)$ are transported by this diffeomorphism to $\mathscr{V}$ (we keep the same symbol for these elements).

We define now the homogeneous vector field $\widetilde{\mathscr{H}}^G$ on $\mathscr{N}$ by

$$\widetilde{\mathscr{H}}^G_n := (f_G(p(n)))_{\mathscr{N}}(n), n \in \mathscr{N}. \tag{6.17}$$

Using the isomorphism $\mathbf{T}\mathscr{N} \simeq p^*(\mathbf{T}M^\beta \oplus \mathscr{N})$ (see (5.4)) the manifold $\mathscr{N}$ is endowed with the almost complex structure $\widetilde{J} := p^*(J_\beta \oplus J_{\mathscr{N}})$. With the data $(\widetilde{J}, \widetilde{\mathscr{H}}^G)$, we construct the following G-transversally elliptic symbol over $\mathscr{N}$:

$$\mathrm{Thom}^f_{G,[\beta]}(\mathscr{N})(n, w) := \mathrm{Thom}_G(\mathscr{N}, \widetilde{J})(n, w - \widetilde{\mathscr{H}}^G_n), \qquad \text{for} \quad (n, w) \in \mathbf{T}\mathscr{N}. \tag{6.18}$$

Let us now verify that

$$\mathrm{Thom}^f_{G,[\beta]}(M) = \mathrm{Thom}^f_{G,[\beta]}(\mathscr{N}) \quad \text{in} \quad K_G(\mathbf{T}_G\mathscr{V}).$$

The invariance of the Thom class after the modification of the almost complex structure is carried out in Lemma 5.9: the class of $\mathrm{Thom}^f_{G,[\beta]}(M)$ is equal in $K_G(\mathbf{T}_G\mathscr{V})$ to the class of the symbol

$$\sigma_1(n, w) := \mathrm{Thom}_G(\mathscr{N}, \widetilde{J})(n, w - \mathscr{H}^G_n), \qquad (n, w) \in \mathbf{T}\mathscr{V}.$$

Using now the family of vectors field $\mathscr{H}^G_t(n) := (f_G(x, t.v))_{\mathscr{V}}(n)$, $t \in [0, 1]$, $n = (x, v) \in \mathscr{V}$, we construct the homotopy

$$\sigma_t(n, w) := \mathrm{Thom}_H(\mathscr{N}, \widetilde{J})(n, w - \mathscr{H}^G_t(n)), \qquad (n, w) \in \mathbf{T}\mathscr{V}$$

of G-transversally elliptic symbol between σ_1 and $\mathrm{Thom}^f_{G,[\beta]}(\mathscr{N})$ (one easily verifies that $\mathrm{Char}(\sigma_t) \cap \mathbf{T}_G\mathscr{V} = C^G_\beta$ for every $t \in [0, 1]$). Finally, we have shown that $\mathrm{Thom}^f_{G,[\beta]}(\mathscr{N}) = \mathrm{Thom}^f_{G,[\beta]}(M)$ in $K_G(\mathbf{T}_G\mathscr{V})$, thus

$$RR^G_\beta(E) = \mathrm{Index}^G_{\mathscr{N}}(\mathrm{Thom}^f_{G,[\beta]}(\mathscr{N}) \otimes p^*(E_{|M^\beta}))$$

for every $E \in K_G(M)$.

Now we proceed as follows. For every $(n, w) \in \mathbf{T}\mathscr{V}$, the Clifford action $\mathrm{Thom}^f_{G,[\beta]}(\mathscr{N})(n, w) = Cl_n(w - \widetilde{\mathscr{H}}^G_n)$ on $\bigwedge^{\bullet}_{\mathbb{C}} \mathbf{T}_n\mathscr{V}$ is equal to the exterior product

$$Cl_x(w_1 - [\widetilde{\mathscr{H}}^G_n]_1) \odot Cl_x(w_2 - [\widetilde{\mathscr{H}}^G_n]_2) \tag{6.19}$$

acting on $\bigwedge^{\bullet}_{\mathbb{C}} \mathbf{T}_x M^\beta \otimes \bigwedge^{\bullet}_{\mathbb{C}} \mathscr{N}|_x$, where $x = p(n)$. Here $w \to w_1$, $\mathbf{T}_n\mathscr{V} \to \mathbf{T}_x M^\beta$ is the tangent map $\mathbf{T}p|_n$, and $w \to w_2 = [w]^V$, $T_n\mathscr{V} \to \mathscr{N}|_x$ is the "vertical" map. We see that $[\widetilde{\mathscr{H}}^G_n]_1 = \mathscr{H}^G_x$ is the vector field on M^β generated by the moment map $f_G|_{M^\beta}$ (see Definition 6.2).

Suppose that the exterior product (6.19) can be modified in

$$Cl_x(w_1 - \mathscr{H}^G_x) \odot Cl_x(w_2 - \beta_{\mathscr{N}}|_n), \tag{6.20}$$

without changing the K-theoretic class. This will prove a modified version of (5.8) in $K_{G\times\mathbb{T}_\beta\times U}(\mathbf{T}_{G\times\mathbb{T}_\beta\times U}(R\times N))$:

$$\pi_N^* \,\mathrm{Thom}^f_{G,[\beta]}(\mathscr{N}) = \pi^* \,\mathrm{Thom}^f_{G,[\beta]}(M^\beta) \odot \mathrm{Thom}^\beta_{\mathbb{T}_\beta\times U}(N), \tag{6.21}$$

where $\pi_N\colon R\times N \to R\times_U N = \mathscr{N}$, $\pi\colon R \to R/U = M^\beta$ are the quotient maps relative to the free U-action, and $\odot$ is the product

$$K_{G\times U}(\mathbf{T}_{G\times U}R) \times K_{\mathbb{T}_\beta\times U}(\mathbf{T}_{\mathbb{T}_\beta}N) \to K_{G\times\mathbb{T}_\beta\times U}(\mathbf{T}_{G\times\mathbb{T}_\beta\times U}(R\times N)). \tag{6.22}$$

The symbols $\mathrm{Thom}^f_{G,[\beta]}(\mathscr{N})$, $\mathrm{Thom}^f_{G,[\beta]}(M^\beta)$ and $\mathrm{Thom}^\beta_{\mathbb{T}_\beta\times U}(N)$ belong respectively to $K_{G\times\mathbb{T}_\beta}(\mathbf{T}_{G\times\mathbb{T}_\beta}(R\times_U N))$, $K_G(\mathbf{T}_G(R/U))$, and $K_{\mathbb{T}_\beta\times U}(\mathbf{T}_{\mathbb{T}_\beta\times U}N)$. Proposition 6.14 follows after taking the index, and the U-invariants, in (6.21).

Finally we explain why the change of $[\widetilde{\mathscr{H}}^G_n]_2$ in $\beta_{\mathscr{N}}|_n$ can be done in (6.19) without changing the class of $\mathrm{Thom}^f_{G,[\beta]}(\mathscr{N})$.

Let $\mu^{\mathscr{N}}\colon \mathfrak{g} \to \Gamma(M^\beta, \mathrm{End}(\mathscr{N}))$ be the "moment" relative to the choice of a connection on $\mathscr{N} \to M^\beta$ (see Definition 7.5 in [10]). Then, for every $X \in \mathfrak{g}$ we have

$$[X_{\mathscr{N}}(x, v)]^V = -\mu^{\mathscr{N}}(X)|_x . v, \qquad (x, v) \in \mathscr{N}$$

(see Proposition 7.6 in [10]). When $X = \beta$, the vector field $\beta_{\mathscr{N}}$ is vertical, hence we have $\mu^{\mathscr{N}}(\beta)|_x . v = \mathscr{L}^{\mathscr{N}}(\beta)|_x . v = -\beta_{\mathscr{N}}(x, v)$, where $\mathscr{L}^{\mathscr{N}}(\beta)$ is the infinitesimal action of β on the fiber of $\mathscr{N} \to M^\beta$. We have also $[\widetilde{\mathscr{H}}^G_n]_2 = -\mu^{\mathscr{N}}(f_G(x))|_x . v$, for every $n = (x, v) \in \mathscr{N}$.

Note that the quadratic form $v \in \mathscr{N}_x \to |\mathscr{L}^{\mathscr{N}}(\beta)|_x . v|^2$ is positive definite for $x \in M^\beta$. Hence, for every $X \in \mathfrak{g}$ close enough to β, the quadratic form $v \in \mathscr{N}_x \to (\mu^{\mathscr{N}}(\beta)|_x . v, \mu^{\mathscr{N}}(X)|_x . v)$ is positive definite for $x \in M^\beta$.

Consider now the homotopy

$$\sigma^t(n,w) := Cl_x(w_1 - \mathcal{H}^G_x) \odot Cl_x(w_2 - t.[\widetilde{\mathcal{H}}^G_n]_2 - (1-t).\beta_{\mathcal{N}}|_n), \qquad (n,v)\in\mathcal{V} \quad t\in[0,1].$$

We see that $(n,w)\in \mathrm{Char}(\sigma^t)\cap \mathbf{T}_G\mathcal{V}$ if and only if

(i) $w_1 = \mathcal{H}^G_x$, $w_2 = t[\widetilde{\mathcal{H}}^G_n]_2 + (1-t)\,\beta_{\mathcal{N}}(n)$, and

(ii) $(w_1, X_{M^\beta}(x)) + (w_2, [X_{\mathcal{N}}(x,v)]^V) = 0$ for all $X\in\mathfrak{g}$.

Take now $X = f_G(x)$ in (ii). Using (i), we get

$$|\mathcal{H}^G_x|^2 + t.|\mu^{\mathcal{N}}(f_G(x))|_x.v|^2 + (1-t).\Sigma(x,v) = 0, \tag{6.23}$$

with $\Sigma(x,v) := (\mu^{\mathcal{N}}(\beta)|_x.v, \mu^{\mathcal{N}}(f_G(x))|_x.v)$.

If $x\in M^\beta$ is sufficiently close to $(f_G|_{M^\beta})^{-1}(\beta)$, the term $\Sigma(x,v)$ is positive for all $v\in\mathcal{N}_x$. In this case, (6.23) gives $\mathcal{H}^G_x = 0$ and $\Sigma(x,v)=0$, which insures that $x\in C^G_\beta$ and $v=0$.

We have proved that $\mathrm{Char}(\sigma^t)\cap\mathbf{T}_G\mathcal{V} = C^G_\beta$ for every $t\in[0,1]$ if $\mathcal{V}$ is "small" enough. Hence σ^t is an homotopy of G-transversally elliptic symbols over $\mathbf{T}\mathcal{V}$ between the exterior products (6.19) and (6.20). ∎

6.2. *Induction formula.* This section is concerned by an induction formula which compare the map $RR^G_\beta(M,-)$ with the similar localized Riemann–Roch characters defined for the maximal torus, and the stabilizer G_β. The idea of this induction comes from a previous paper of the author [32] where a similar induction formula in the context of equivariant cohomology was proved.

Consider the restriction $f_H\colon M\to\mathfrak{h}$ of the moment map f_G to the maximal torus H. In this situation we use the vector field $\mathcal{H}^H|_m = f_H(m)_M|_m$, $m\in M$ to decompose the map $RR^H(M,-)\colon K_H(M)\to R(H)$ near the set $C^{f_H} = \{\mathcal{H}^H = 0\}$. From Lemma 6.3 there exists a finite subset $\mathcal{B}_H\subset\mathfrak{h}$, such that $C^{f_H} = \bigcup_{\beta\in\mathcal{B}_H} C^H_\beta$, with $C^H_\beta = M^\beta\cap f_H^{-1}(\beta)$. As in Definition 6.5, we define for every $\beta\in\mathcal{B}_H$, the map $RR^H_\beta(M,-)\colon K_H(M)\to R^{-\infty}(H)$ which is the Riemann–Roch character localized near C^H_β.

Let W be the Weyl group of (G,H). Note that $\mathcal{B}_H$ is a W-stable subset of $\mathfrak{h}$, and that $\mathcal{B}_G\subset\mathcal{B}_H\cap\mathfrak{h}_+$.

THEOREM 6.16. *We have, for every $\beta\in\mathcal{B}_G$, the following induction formula between $RR^G_\beta(M,-)$ and $RR^H_\beta(M,-)$. For every $E\in K_G(M)$, we have*[18]

[18] See Eqs. (9.2) and (9.4) in Appendix B for the definition of the holomorphic induction maps Hol^G_H and $\mathrm{Hol}^G_{G_\beta}$.

$$RR^G_\beta(M,E) = \frac{1}{|W_\beta|}\,\mathrm{Hol}^G_H\big(RR^H_\beta(M,E)\wedge^\bullet_{\mathbb{C}}\overline{\mathfrak{g}/\mathfrak{h}}\big)$$
$$= \frac{1}{|W_\beta|}\sum_{w\in W}\mathrm{Hol}^G_H\big(w.RR^H_\beta(M,E)\big)$$
$$= \sum_{\beta'\in W.\beta}\mathrm{Hol}^G_H\big(RR^H_{\beta'}(M,E)\big)$$

where W_β is the stabilizer of β in W.

We can use the previous induction formula between G and H index maps to produce an induction formula between G and G_β index maps. Consider the restriction $f_{G_\beta}\colon M\to\mathfrak{g}_\beta$ of the moment map to the stabiliser G_β of β in G. Let $RR^{G_\beta}_\beta(M,-)$ be the Riemann–Roch character localized near $C^{G_\beta}_\beta = M^\beta\cap f_G^{-1}(\beta)$.[19]

COROLLARY 6.17. *For every $\beta\in\mathscr{B}_G$ and every $E\in K_G(M)$, we have*

$$RR^G_\beta(M,E) = \mathrm{Hol}^G_{G_\beta}\big(RR^{G_\beta}_\beta(M,E)\wedge^\bullet_{\mathbb{C}}\overline{\mathfrak{g}/\mathfrak{g}_\beta}\big)\quad \text{in}\quad R^{-\infty}(G).$$

Proof of the Corollary. It comes immediately by applying the induction formula of Theorem 6.16 to the couples (G,H) and (G_β,H).

COROLLARY 6.18. *Let E be a f_G-strictly positive* complex vector bundle over M (see Def. 1.2). We have $[RR^G_\beta(M,E^{\overset{k}{\otimes}})]^G=0$, if $k.\eta_{E,\beta}>\langle\theta,\beta\rangle$. Here $\theta=\sum_{\alpha>0}\alpha$ is the sum of the positive roots of G, and $\eta_{E,\beta}$ is the strictly positive constant defined in Definition 1.2.

Proof of Corollary 6.18. Let us first write the decomposition[20] $RR^{G_\beta}_\beta(M,E^{\overset{k}{\otimes}}) =\sum_{\lambda\in\Lambda^+_\beta} m_{\lambda,\beta}(E^{\overset{k}{\otimes}})\,\chi^{G_\beta}_\lambda$, in irreducible character of G_β. We know from (6.16) that $m_{\lambda,\beta}(E^{\overset{k}{\otimes}})\neq 0\Rightarrow\langle\lambda,\beta\rangle\geqslant k.\eta_{E,\beta}$. Each irreducible character $\chi^{G_\beta}_\lambda$ is equal to $\mathrm{Hol}^{G_\beta}_H(h^\lambda)$, so from Corollary 6.17 we have $RR^G_\beta(M,E^{\overset{k}{\otimes}})=\mathrm{Hol}^G_H\big((\sum_\lambda m_{\lambda,\beta}(E^{\overset{k}{\otimes}})\,h^\lambda)\,\Pi_{\alpha\in\Delta(\mathfrak{g}/\mathfrak{g}_\beta)}(1-h^{-\alpha})\big)$, where $\Delta(\mathfrak{g}/\mathfrak{g}_\beta)$ is the set of H-weight on $\mathfrak{g}/\mathfrak{g}_\beta$.[21] Finally , we see that $RR^G_\beta(M,E^{\overset{k}{\otimes}})$ is a sum of terms of

[19] Note that $M^\beta\cap f_{G_\beta}^{-1}(\beta)=M^\beta\cap f_G^{-1}(\beta)$ because $f_{G_\beta}=f_G$ on M^β.

[20] We choose a set $\Lambda^*_{+,\beta}$ of dominant weight for G_β that contains the set Λ^*_+ of dominant weight for G.

[21] The complex structure on $\mathfrak{g}/\mathfrak{g}_\beta$ is defined by β, so that $\langle\alpha,\beta\rangle>0$ for all $\alpha\in\Delta(\mathfrak{g}/\mathfrak{g}_\beta)$.

the form $m_{\lambda,\beta}(\overset{k}{\otimes}E)\,\mathrm{Hol}_H^G(h^{\lambda-\alpha_I})$ where $\alpha_I=\sum_{\alpha\in I}\alpha$ and I is a subset of $\Delta(\mathfrak{g}/\mathfrak{g}_\beta)$.

We know from Appendix B that $\mathrm{Hol}_H^G(h^{\lambda'})$ is either 0 or the character of an irreducible representation; in particular $\mathrm{Hol}_H^G(h^{\lambda'})$ is equal to ± 1 only if $\langle\lambda', X\rangle\leqslant 0$ for every $X\in\mathfrak{h}_+$ (see Remark 9.3). So $[RR_\beta^G(M,\overset{k}{\otimes}E)]^G\neq 0$ only if there exists a weight λ such that $m_{\lambda,\beta}(\overset{k}{\otimes}E)\neq 0$ and $\mathrm{Hol}_H^G(h^{\lambda-\alpha_I})=\pm 1$. The first condition imposes $\langle\lambda,\beta\rangle\geqslant k.\eta_{E,\beta}$ and the second gives $\langle\lambda,\beta\rangle\leqslant\langle\alpha_I,\beta\rangle$, and combining the two we end with $k.\eta_{E,\beta}\leqslant\langle\alpha_I,\beta\rangle\leqslant\sum_{\alpha\in\Delta(\mathfrak{g}/\mathfrak{g}_\beta)}\langle\alpha,\beta\rangle=\langle\theta,\beta\rangle$. We have proved that $[RR_\beta^G(M,\overset{k}{\otimes}E)]^G=0$ if $k.\eta_{E,\beta}>\langle\theta,\beta\rangle$. ∎

Proof of Theorem 6.16. The first two equalities of the Theorem can be deduced from the third one, that is $RR_\beta^G(M,E)=\sum_{\beta'\in W.\beta}\mathrm{Hol}_H^G(RR_{\beta'}^H(M,E))$. First, it is easy to see that $RR_{w.\beta}^H(M,E)=w.RR_\beta^H(M,E)$ for every $w\in W$ and $\beta\in\mathscr{B}_H$. Then the relation $\mathrm{Hol}_H^G(\phi\wedge_{\mathbb{C}}^{\bullet}\overline{\mathfrak{g}/\mathfrak{h}})=\sum_{w\in W}\mathrm{Hol}_H^G(w.\phi)$, which is true for every $\phi\in R^{-\infty}(H)$ (see Remark 9.2), gives the first equality of the Theorem.

The map $RR_\beta^G(M,-)$ is defined through the symbol $\mathrm{Thom}^f_{G,[\beta]}(M)\in K_G(\mathbf{T}_G\mathscr{U}^{G,\beta})$, where $i^{G,\beta}\colon\mathscr{U}^{G,\beta}\to M$ is any G-invariant neighborhood of C_β^G such that $\overline{\mathscr{U}^{G,\beta}}\cap C^{f_G}=C_\beta^G$ (see Definition 6.4). We define in the same way the localized Thom complex $\mathrm{Thom}^f_{H,[\beta]}(M)\in K_H(\mathbf{T}_H\mathscr{U}^{H,\beta})$.

For notational convenience, we will note in the same way the direct image of $\mathrm{Thom}^f_{G,[\beta]}(M)$ (resp. $\mathrm{Thom}^f_{H,[\beta]}(M)$) in $K_G(\mathbf{T}_GM)$ (resp. $K_H(\mathbf{T}_HM)$) via $i_*^{G,\beta}\colon K_G(\mathbf{T}_G\mathscr{U}^{G,\beta})\to K_G(\mathbf{T}_GM)$ (resp. $i_*^{H,\beta}\colon K_H(\mathbf{T}_H\mathscr{U}^{H,\beta})\to K_H(\mathbf{T}_HM)$).

Then we have $RR_\beta^G(M,E)=\mathrm{Index}_M^G(\mathrm{Thom}^f_{G,[\beta]}(M)\otimes E)$ for $E\in K_G(M)$. The Weyl group acts on $K_H(\mathbf{T}_HM)$ and we remark that $w.\mathrm{Thom}^f_{H,[\beta]}(M)=\mathrm{Thom}^f_{H,[w.\beta]}(M)$ for every $\beta\in\mathscr{B}_H$, and $w\in W$. After taking the index we see that $RR_{w.\beta}^H(M,E)=w.RR_\beta^H(M,E)$ for every G-vector bundle E.

Consider the map $r_{G,H}^\gamma\colon K_G(\mathbf{T}_GM)\to K_H(\mathbf{T}_HM)$ defined with $\gamma\in\mathfrak{h}$ in the interior of the Weyl chamber, so that $G_\gamma=H$ (see Sect. 3.5). The third equality of the Theorem is an immediate consequence of the next Lemma.

LEMMA 6.19. *We have*

$$r_{G,H}^\gamma(\mathrm{Thom}^f_{G,[\beta]}(M))=\sum_{\beta'\in W.\beta}\mathrm{Thom}^f_{H,[\beta']}(M)\otimes\wedge_{\mathbb{C}}^{\bullet}\mathfrak{g}/\mathfrak{h}\quad\text{in}\quad K_H(\mathbf{T}_HM).$$

Proof of Lemma 6.19. Consider a G-invariant open neighborhood $\mathcal{U}^{G,\beta}$ of C^G_β such that $\mathcal{U}^{G,\beta} \cap C^{f_G} = C^G_\beta$. We know from Proposition 3.7 that the class $r^\gamma_{G,H}(\mathrm{Thom}^f_{G,[\beta]}(M))$ is represented by the restriction to $\mathbf{T}\mathcal{U}^{G,\beta}$ of the symbol

$$\sigma_I(m,v) = Cl_m(v - \mathcal{H}^G_m) \odot Cl(\mu_{G/H}(v)), \qquad (m,v) \in \mathbf{T}M.$$

Here $\mu_{G/H}\colon \mathbf{T}M \to \mathfrak{g}/\mathfrak{h}$ is the $\mathfrak{g}/\mathfrak{h}$ part of the Hamiltonian moment map $\mu_G\colon \mathbf{T}M \to \mathfrak{g}$. Let $f_{G/H}\colon M \to \mathfrak{g}/\mathfrak{h}$ (resp. $f_H\colon M \to \mathfrak{h}$) be the $\mathfrak{g}/\mathfrak{h}$-part (resp. the $\mathfrak{h}$-part) of the moment map f_G. We will use in our proof the relation

$$(\mu_{G/H}(\mathcal{H}^G), f_{G/H})_{\mathfrak{g}} = \|\mathcal{H}^G\|^2_M - (\mathcal{H}^G, \mathcal{H}^H)_M. \tag{6.24}$$

Consider the family of H-equivariant symbols σ_θ, $\theta \in [0,1]$ defined on $\mathbf{T}M$ by

$$\sigma_\theta(m,v) = Cl_m(v - \mathcal{H}^G_m) \odot Cl(\theta\mu_{G/H}(v) + (1-\theta)\, f_{G/H}(m)), \qquad (m,v) \in \mathbf{T}M.$$

We see that $(m,v) \in \mathrm{Char}(\sigma_\theta) \leftrightarrow v = \mathcal{H}^G_m$ and $\theta\mu_{G/H}(\mathcal{H}^G_m) + (1-\theta)\, f_{G/H}(m) = 0$. Combining (6.24) with the fact that the vector field $\mathcal{H}^H$ belongs to the H-orbits, we see that $\mathrm{Char}(\sigma_\theta) \cap \mathbf{T}_H M \subset \{\mathcal{H}^G = 0\}$, for every $\theta \in [0,1]$. By this way we have proved that $\sigma_I|_{\mathcal{U}^{G,\beta}}$ is homotopic to the H-transversally elliptic symbol $\sigma_{II}|_{\mathcal{U}^{G,\beta}}$ where

$$\sigma_{II}(m,v) = Cl_m(v - \mathcal{H}^G_m) \odot Cl(f_{G/H}(m)), \qquad (m,v) \in \mathbf{T}M.$$

We transform now σ_{II} via the following homotopy of H-transversally elliptic symbols

$$\sigma^u(m,v) := Cl_m(v - \mathcal{H}^H_m - u.\mathcal{H}^{G/H}_m) \odot Cl(f_{G/H}(m)), \qquad (m,v) \in \mathbf{T}M,$$

for $u \in [0,1]$. Here $\mathrm{Char}(\sigma^u) \cap \mathbf{T}_H M = \{\mathcal{H}^G = 0\} \cap \{f_{G/H} = 0\}$ for all $u \in [0,1]$, hence $\sigma_{II}|_{\mathcal{U}^{G,\beta}}$ is homotopic to the H-transversally elliptic symbol $\sigma_{III}|_{\mathcal{U}^{G,\beta}}$ where

$$\sigma_{III}(m,v) = Cl_m(v - \mathcal{H}^H_m) \odot Cl(f_{G/H}(m)), \qquad (m,v) \in \mathbf{T}M.$$

At this stage we have proved that $\sigma_I|_{\mathcal{U}^{G,\beta}} = \sigma_{III}|_{\mathcal{U}^{G,\beta}}$ in $K_H(\mathbf{T}_H\mathcal{U}^{G,\beta})$. Note that

$$\begin{aligned}\mathrm{Char}(\sigma_{III}|_{\mathcal{U}^{G,\beta}}) \cap \mathbf{T}_H\mathcal{U}^{G,\beta} &= G.(M^\beta \cap f_G^{-1}(\beta)) \cap \{f_{G/H} = 0\}\\ &= W.(M^\beta \cap f_G^{-1}(\beta)),\end{aligned}$$

because $G.\beta \cap \mathfrak{h} = W.\beta$. Let $i: \mathcal{U}^{G,\beta} \hookrightarrow \mathcal{U}$ be a H-invariant neighborhood of $W.(M^\beta \cap f_H^{-1}(\beta))$ such that $\overline{\mathcal{U}} \cap \{\mathcal{H}^H = 0\} = W.(M^\beta \cap f_H^{-1}(\beta))$. The symbol $\sigma_{III}|_{\mathcal{U}}$ is H-transversally elliptic and

$$i_*(\sigma_{III}|_{\mathcal{U}}) = \sigma_{III}|_{\mathcal{U}^{G,\beta}} = \sigma_I|_{\mathcal{U}^{G,\beta}} \quad \text{in} \quad K_H(\mathbf{T}_H\mathcal{U}^{G,\beta}). \tag{6.25}$$

As in the proof of Proposition 4.1, (6.25) is an immediate consequence of the excision property.

The symbol $(m, v) \to Cl_m(v - \mathcal{H}^H_m)$ is H-transversally elliptic on $\mathbf{T}\mathcal{U}$, and equal (by definition) to $\sum_{\beta' \in W.\beta} \mathrm{Thom}^f_{H,[\beta']}(M)$. Hence $\sigma_{III}|_{\mathcal{U}}$ is homotopic, in $K_H(\mathbf{T}_H\mathcal{U})$, to $(m, v) \to Cl_x(v - \mathcal{H}^H_m) \odot 0_{\mathfrak{g}/\mathfrak{h}}$, where $0_{\mathfrak{g}/\mathfrak{h}}$ is the zero map from $\wedge^{\mathrm{even}}_{\mathbb{C}} \mathfrak{g}/\mathfrak{h}$ to $\wedge^{\mathrm{odd}}_{\mathbb{C}} \mathfrak{g}/\mathfrak{h}$. Finally we have shown that $\sigma_{III}|_{\mathcal{U}} = \sum_{\beta' \in W.\beta} \mathrm{Thom}^f_{H,[\beta']}(M) \otimes \wedge^{\bullet}_{\mathbb{C}} \mathfrak{g}/\mathfrak{h}$ in $K_H(\mathbf{T}_H\mathcal{U})$, and then (6.25) finishes the proof. ∎

7. THE HAMILTONIAN CASE

In this section, we assume that (M, ω) is a compact symplectic manifold with a Hamiltonian action of a compact connected Lie group G. The corresponding moment map $\mu_G: M \to \mathfrak{g}^*$ is defined by

$$d\langle \mu_G, X \rangle = -\omega(X_M, -), \qquad \forall X \in \mathfrak{g}. \tag{7.1}$$

The symplectic 2-form ω insures the existence of a G-invariant almost complex structure J *compatible* with ω, i.e, such that:

$$(v, w) \to \omega_x(v, J_x w), \qquad v, w \in \mathbf{T}_x M$$

is symmetric and positive definite for all $x \in M$. We fix once and for all a G-invariant *compatible* almost complex structure J, and we denote by $(-,-)_M := \omega(-, J-)$ the corresponding Riemannian metric. Let $RR^G(M, -)$ be the quantization map defined with the *compatible* almost complex structures J. Since two compatible almost complex structure are homotopic [27], the map $RR^G(M, -)$ does not depend of this choice (see Lemma 2.2).

Here the vector field $\mathcal{H}^G$ is the Hamiltonian vector field of the function[22] $\frac{-1}{2}\|\mu_G\|^2: M \to \mathbb{R}$, and $\{\mathcal{H}^G = 0\}$ is the set of critical points of $\|\mu_G\|^2$. We know from the beginning of Section 6 that we have the decomposition $RR^G(M, -) = \sum_{\beta \in \mathcal{B}_G} RR^G_\beta(M, -)$, where $RR^G_\beta(M, -): K_G(M) \to R^{-\infty}(G)$ is the Riemann–Roch character localized near the critical set $C^G_\beta = G(M^\beta \cap \mu_G^{-1}(\beta))$. In this section we prove the following theorem for the μ_G*-positive* vector bundles (see Def. 1.2).

[22] Equality 7.1 gives $\frac{-1}{2} d\,\|\mu_G\|^2 = \omega(\mathcal{H}^G, -)$

THEOREM 7.1. *Let $E \to M$ be a G-equivariant vector bundle over M. For all $\beta \in \mathscr{B}_G - \{0\}$, the G-invariant part of $RR^G_\beta(M, E)$ is equal to 0 if E is μ_G-positive and $\mu_G^{-1}(0) \neq \emptyset$, or if E is μ_G-strictly positive. If 0 is a regular value of μ_G, the G-invariant part of $RR^G_0(M, E)$ is equal to $RR(\mathscr{M}_{\text{red}}, E_{\text{red}})$.*

In Section 7.4, we consider the general case where 0 is not necessarily a regular value of μ_G, and $E = L$ a moment bundle for μ_G (see Def. 1.1). With our K-theoritic approach we recover the following

THEOREM 7.2 (Meinrenken–Sjamaar). *Let $L \to M$ be a μ_G-moment bundle, and let τ be the principal face of M. The G-invariant part of $RR^G(M, L)$ is equal to $RR(\mathscr{M}_a, L_a)$ for every generic value of $\tau \cap \mu_G(M)$ sufficiently close to 0 (see Sect. 7.4 for the notations).*

7.1 *The map RR^G_0.* We assume that 0 is a regular value of μ_G. The orbifold space $\mathscr{M}_{\text{red}} := \mu_G^{-1}(0)/G$ inherits a symplectic structure ω_{red}. Let $\mathscr{D}(X) = -d\mu_G(J(X_M))$ be the endomorphism of the trivial bundle $\mu_G^{-1}(0) \times \mathfrak{g}$ defined in (6.9). The compatibility of J with ω gives

$$(\mathscr{D}(X), X) = \omega(X_M, J(X_M))_M = \|X_M\|^2,$$

thus decomposition (6.8) holds. A small check shows that the induced almost complex structure J_{red} on $\mathscr{M}_{\text{red}}$ is compatible with ω_{red}. Moreover $t \mapsto t\mathscr{D} + (1-t)\, Id$ is an homotopy of invertible maps between $\mathscr{D}$ and the identity, hence the line bundle $L_{\mathscr{D}} \to \mathscr{M}_{\text{red}}$ defined in (6.11) is trivial. The map RR^G_0 is determined by the Proposition 6.12; in particular

$$[RR^G_0(M, E)]^G = RR^{J_{\text{red}}}(\mathscr{M}_{\text{red}}, E_{\text{red}}),$$

for any $E \in K_G(M)$.

7.2 *The map RR^G_β when $G_\beta = G$.* When $\beta \in \mathscr{B}_G - \{0\}$ is in the center of $\mathfrak{g}$, we proved in Corollary 6.15, that the G-invariant part of $RR^G_\beta(M, E)$ is equal to 0 when E is μ_G-*strictly* positive. In the Hamiltonian case we extend this result for the μ_G-positive bundles.

LEMMA 7.3. *Let $(\mathscr{X}, \omega)$ be a connected symplectic manifold with a G-action, and a proper moment map $\mu\colon \mathscr{X} \to \mathfrak{g}$. Let J be a G-invariant almost complex structure on $\mathscr{X}$ compatible with ω. Let β be a G-invariant element in a Weyl chamber $\mathfrak{h}_+$ of the Lie group G, such that $\mathscr{X}^\beta \cap \mu^{-1}(\beta) \neq \emptyset$. Let $\mathscr{N}^{+,\beta}$ be the polarized normal bundle of $\mathscr{X}^\beta$ in $\mathscr{X}$ (see Def. 5.5 and Theorem 5.8).*

If $\mathcal{N}^{+,\beta}=0$, *we have*

$$\mu(\mathcal{X})\cap\mathfrak{h}_+\subset\{X\in\mathfrak{h}_+,\ (X,\beta)\geqslant\|\beta\|^2\},$$

implying in particular that $\|\beta\|^2$ *is the minimal value of* $\|\mu\|^2$ *on* $\mathcal{X}$.

Proof of the Lemma. Let $\mathcal{Z}$ be a connected component of $\mathcal{X}^\beta$ that intersects $\mu^{-1}(\beta)$, and consider the set of weights $\{\alpha_i,\ i\in I\}$ for the action of $\mathbb{T}_\beta$ on the fibers of the vector bundle $\mathcal{N}\to\mathcal{Z}$. We have then the following description of the function (μ,β) in the neighborhood of $\mathcal{Z}$. For $v\in\mathcal{N}_x$, with the decomposition $v=\bigoplus_i v_i$, we have for $|v|$ small enough $(\mu,\beta)_{(x,v)}=|\beta|^2-\frac{1}{2}\sum_{i\in I}\langle\alpha_i,\beta\rangle\,|v_i|^2$. If $\langle\alpha_i,\beta\rangle<0$ for every $i\in I$, we have

$$(\mu,\beta)\geqslant\|\beta\|^2 \text{ in a neighborhood } \mathcal{V} \text{ of } \mathcal{Z}. \tag{7.2}$$

As $\mu^{-1}(\beta)$ is connected and intersect $\mathcal{Z}$, the last inequality imposes $\mu^{-1}(\beta)\subset\mathcal{Z}$. Take $X\in\mu(\mathcal{X})\cap\mathfrak{h}_+$, and consider $\mathcal{K}:=\mu^{-1}([X,\beta])$. From the convexity theorem [2, 16, 23, 26], the set $\mathcal{K}$ is connected. Then $\mathcal{V}\cap\mathcal{K}$ contains, but is not equal to $\mu^{-1}(\beta)$: there exists $m\in\mathcal{V}\cap\mathcal{K}$ with $\mu(m)\in[X,\beta)$. So $\mu(m)=\beta+t(X-\beta)$ with $t>0$, and $(\mu(m),\beta)\geqslant\|\beta\|^2$. This two conditions imply that $(X,\beta)\geqslant\|\beta\|^2$. ∎

LEMMA 7.4. *Let* $\beta\in\mathcal{B}_G-\{0\}$ *be a G-invariant element such that* $\|\beta\|^2$ *is not the minimal value of* $\|\mu_G\|^2$ *on M. Then for every* μ_G*-positive vector bundle E over M we have the decomposition* $RR^G_\beta(M,E)=\sum_\lambda m_{\beta,\lambda}(E)\,\chi^G_\lambda$ *in irreducible characters with*

$$m_{\beta,\lambda}(E)\neq 0\ \Rightarrow\ \langle\lambda,\beta\rangle>0.$$

In particular, if $\mu_G^{-1}(0)$ *is not empty, the G-invariant part of* $RR^G_\beta(M,E)$ *is equal to* 0 *for every G-invariant* $\beta\in\mathcal{B}_G-\{0\}$. *The result remains when M is non-compact, and the moment map* μ_G *is proper.*

Proof. Recall the localization formula on M^β obtained in Proposition 6.14. For every complex G-vector bundle E over M, we have the following equality in $\hat{R}(G)$

$$RR^G_\beta(M,E)=(-1)^{r_\mathcal{N}}\sum_{k\in\mathbb{N}}RR^G_\beta(M^\beta,E|_{M^\beta}\otimes\det\mathcal{N}^{+,\beta}\otimes S^k((\mathcal{N}\otimes\mathbb{C})^{+,\beta}). \tag{7.3}$$

Suppose that M is non-compact and that the moment map μ_G is proper as a map from a G-invariant open neighborhood of $\mu_G^{-1}(\beta)$ in M to a G-invariant open neighborhood of β in $\mathfrak{g}$. Each terms of (7.3) are well defined and the equality remains valid in this case (It is not difficult to extend the proof given in Section 6.3 to this situation).

If $\|\beta\|^2$ is not the minimal value of $\|\mu_G\|^2$, we know from Lemma 7.3 that the vector bundle $\mathcal{N}^{+,\beta}$ is not trivial over each connected component $\mathcal{Z}$ of M^β that intersects $\mu^{-1}(\beta)$. Then every $\mathbb{T}_\beta$-weight a on the fibers of the complex vector bundle $E|_{\mathcal{Z}} \otimes \det \mathcal{N}^{+,\beta} \otimes S^k((\mathcal{N} \otimes \mathbb{C})^{+,\beta}$ satisfies $\langle a, \beta\rangle > 0$. Lemma 9.4 and Corollary 9.5, applied to this situation, show that $RR_\beta^G(M, E) = \sum_\lambda m_{\beta,\lambda}(E)\, \chi_\lambda^G$ with $m_{\beta,\lambda}(E) \neq 0$ only if $\langle \lambda, \beta\rangle > 0$. ▮

7.3 *The map RR_β^G when $G_\beta \neq G$.* Let σ be the unique open face of $\mathfrak{h}_+$ which contains β. The stabilizer subgroup G_ξ does not depend on the choice of $\xi \in \sigma$, and is denoted by G_σ. Let $\mathfrak{g}_\sigma$ be the Lie algebra of G_σ, and let U_σ the G_σ-invariant open subset of $\mathfrak{g}_\sigma$ defined by $U_\sigma = G_\sigma \cdot \{y \in \mathfrak{h}_+ \mid G_y \subset G_\sigma\}$.

The symplectic cross-section Theorem [18, 26] asserts that the pre-image $\mathcal{Y}_\sigma = \mu_G^{-1}(U_\sigma)$ is a symplectic submanifold of M provided with a Hamiltonian action of G_σ. We denote by ω_σ the symplectic 2-form on $\mathcal{Y}_\sigma$, and $\mu_\sigma \colon \mathcal{Y}_\sigma \to \mathfrak{g}_\sigma$ the moment map. Let J_σ be a G_σ-invariant almost complex structure on $\mathcal{Y}_\sigma$, which is compatible with ω_σ. The vector field $\mathcal{H}^\sigma$ on $\mathcal{Y}_\sigma$ generated by μ_σ vanishes on $C_\beta^\sigma := \mu_\sigma^{-1}(\beta) \cap (\mathcal{Y}_\sigma)^\beta = \mu_G^{-1}(\beta) \cap M^\beta$ (see Definition 6.2). We denote by[23]

$$RR_\beta^{G_\sigma}(\mathcal{Y}_\sigma, -)\colon \tilde{K}_{G_\sigma}(\mathcal{Y}_\sigma) \to R^{-\infty}(G_\sigma)$$

the Riemann–Roch character on $\mathcal{Y}_\sigma$ localized near the *compact* subset C_β^σ by the vector filed $\mathcal{H}^\sigma$. It is well defined even since μ_σ is a proper map (see Definition 6.5).

THEOREM 7.5. *For every $E \in K_G(M)$, we have*

$$RR_\beta^G(M, E) = \mathrm{Hol}_{G_\sigma}^G\big(RR_\beta^{G_\sigma}(\mathcal{Y}_\sigma, E|_{\mathcal{Y}_\sigma})\big) \quad \text{in} \quad R^{-\infty}(G),$$

COROLLARY 7.6. *Let $\beta \in \mathcal{B}_G$ with $G_\beta \neq G$. If $\mu_G^{-1}(0) \neq \emptyset$, we have $[RR_\beta^G(M, E)]^G = 0$, for every μ_G-positive vector bundle $E \to M$. In general, $[RR_\beta^G(M, E)]^G = 0$, for every μ_G-strictly* positive vector bundle E.

Proof of the Corollary. The moment map μ_σ is proper as a map from a G_σ-invariant open neighborhood of $\mu_\sigma^{-1}(\beta)$ in $\mathcal{Y}_\sigma$ to a G_σ-invariant open neighborhood of β in $\mathfrak{g}_\sigma$. If $0 \in \mu_G(M)$ we see that $t\beta \in \mu_\sigma(\mathcal{Y}_\sigma)$ for any $0 < t < 1$, hence $\|\beta\|^2$ is not the minimal value of $\|\mu_\sigma\|^2$.

Proposition 7.4 can be used for the map $RR_\beta^{G_\sigma}(\mathcal{Y}_\sigma, -)$. For any μ_G-positive vector bundle E, we have $RR_\beta^{G_\sigma}(\mathcal{Y}_\sigma, E|_{\mathcal{Y}_\sigma}) = \sum_\lambda m_{\beta,\lambda}(E)\, \chi_\lambda^{G_\sigma}$ with $m_{\beta,\lambda}(E) \neq 0$ only if $\langle \lambda, \beta\rangle > 0$ (the same holds when $0 \notin \mu_G(M)$ and E is μ_G-*strictly* positive). With the induction formula of Theorem 7.5 we get[24]

[23] For a non-compact G-manifold $\mathcal{X}$, we denote by $\tilde{K}_G(\mathcal{X})$ the equivariant K-theory of $\mathcal{X}$ with non-compact support.

[24] $\mathrm{Hol}_{G_\sigma}^G(\chi_\lambda^{G_\sigma}) = \mathrm{Hol}_H^G(h^\lambda)$ since $\chi_\lambda^{G_\sigma} = \mathrm{Hol}_H^{G_\sigma}(h^\lambda)$.

$RR^G_\beta(M, E) = \sum_\lambda m_{\beta,\lambda}(E)\, \mathrm{Hol}^G_H(h^\lambda)$. But $\mathrm{Hol}^G_H(h^\lambda) = \pm 1$ only if $\langle \lambda, X\rangle \leqslant 0$ for every X in the Weyl chamber (see Remark 9.3). This shows

$$\mathrm{Hol}^G_H(h^\lambda) = \pm 1 \;\Rightarrow \langle \lambda, \beta\rangle \leqslant 0 \;\Rightarrow m_{\beta,\lambda}(E) = 0.$$

We have then proved that $[RR^G_\beta(M, E)]^G = 0$. ∎

Proofs of Theorem 7.5. We propose here two different proofs for this induction formula. Both of them use the same technical remark.

The set $G\cdot\mathcal{Y}_\sigma \cong G\times_{G_\sigma}\mathcal{Y}_\sigma$ is a G-invariant open neighborhood of the critical set C^G_β in M. The symplectic form ω, when restricted to $G\times_{G_\sigma}\mathcal{Y}_\sigma$, can be written in terms of the moment map μ_σ and the symplectic form ω_σ.

$$\omega_{[g,y]}(X+v, Y+w) = -(\mu_\sigma(y), [X, Y]) + \omega_\sigma|_y(v, w), \tag{7.4}$$

where $X, Y \in \mathfrak{g}/\mathfrak{g}_\beta$, and $v, w \in \mathbf{T}_y\mathcal{Y}_\sigma$.[25] With the complex structure J_{G/G_σ} on G/G_σ determined by β, we form the almost complex structure $\tilde{J} := J_{G/G_\sigma}\times J_\sigma$ on $G\times_{G_\sigma}\mathcal{Y}_\sigma$. Equation (7.4) shows that $\tilde{J}$ is compatible with ω in a neighborhood of C^G_β, hence $\tilde{J}$ is homotopic to J in a neighborhood of C^G_β in $G\times_{G_\sigma}\mathcal{Y}_\sigma$.

Remark 7.7. The almost complex structures J and $\tilde{J}$ are homotopic in a neighborhood of C^G_β, so as in Lemma 2.2 we see that the computation of the localized Riemann–Roch character $RR^G_\beta(M, E)$ can be done with $\tilde{J}$ instead of J.

First proof of Theorem 7.5. We will show here that Theorem 7.5 is a consequence of the induction formula proved in Theorem 6.16 and of the localization formula obtained in Proposition 6.14. The induction of Corollary 6.17 shows that $RR^G_\beta(M, E) = \mathrm{Hol}^G_{G_\sigma}(RR^{G_\sigma}_\beta(M, E)\wedge^\bullet \overline{\mathfrak{g}/\mathfrak{g}_\sigma})$. So we have to prove the equality

$$RR^{G_\sigma}_\beta(\mathcal{Y}_\sigma, E|_{\mathcal{Y}_\sigma}) = RR^{G_\sigma}_\beta(M, E)\wedge^\bullet \overline{\mathfrak{g}/\mathfrak{g}_\sigma}. \tag{7.5}$$

First we use the localization formula on both sides of the equality. For the map $RR^{G_\sigma}_\beta(M, -)$ this gives

$$RR^{G_\sigma}_\beta(M, E) = RR^{G_\sigma\times \mathrm{T}_\beta}_\beta(M^\beta, E|_{M^\beta}\otimes[\wedge^\bullet_{\mathbb{C}}\mathcal{N}]^{-1}_\beta), \tag{7.6}$$

and for $RR^{G_\sigma}_\beta(\mathcal{Y}_\sigma, -)$ we have

$$RR^{G_\sigma}_\beta(\mathcal{Y}_\sigma, E|_{\mathcal{Y}_\sigma}) = RR^{G_\sigma\times \mathrm{T}_\beta}_\beta((\mathcal{Y}_\sigma)^\beta, E|_{(\mathcal{Y}_\sigma)^\beta}\otimes[\wedge^\bullet_{\mathbb{C}}\overline{\mathcal{N}'}]^{-1}_\beta). \tag{7.7}$$

Here $\mathcal{N}$ and $\mathcal{N}'$ are respectively the normal bundle of M^β in M, and the normal bundle of $(\mathcal{Y}_\sigma)^\beta$ in $\mathcal{Y}_\sigma$. The complex structures on the fibers of $\mathcal{N}$ and $\mathcal{N}'$ are induced respectively by the almost complex structure $\tilde{J}$, and by the almost complex structure J_σ (see Remark 7.7).

[25] We use here the identification $\mathbf{T}(G\times_{G_\sigma}\mathcal{Y}_\sigma)\cong G\times_{G_\sigma}(\mathfrak{g}/\mathfrak{g}_\sigma\oplus \mathbf{T}\mathcal{Y}_\sigma)$ (see (3.6)).

Now we remark that $(\mathcal{Y}_\sigma)^\beta$ is an open neighborhood of $M^\beta \cap \mu_G^{-1}(\beta)$ in M^β, thus we have $RR_\beta^{G_\sigma}(M^\beta, F) = RR_\beta^{G_\sigma}((\mathcal{Y}_\sigma)^\beta, F|_{(\mathcal{Y}_\sigma)^\beta})$ for any equivariant vector bundle F. So (7.6) and (7.7) shows us that (7.5) is equivalent to

$$RR_\beta^{G_\sigma \times \mathbb{T}_\beta}((\mathcal{Y}_\sigma)^\beta, E|_{(\mathcal{Y}_\sigma)^\beta} \otimes [\wedge_{\mathbb{C}}^\bullet \overline{\mathcal{N}}]_\beta^{-1} \otimes [\wedge^\bullet \overline{\mathfrak{g}/\mathfrak{g}_\sigma}])$$
$$= RR_\beta^{G_\sigma \times \mathbb{T}_\beta}((\mathcal{Y}_\sigma)^\beta, E|_{(\mathcal{Y}_\sigma)^\beta} \otimes [\wedge_{\mathbb{C}}^\bullet \overline{\mathcal{N}'}]_\beta^{-1}), \tag{7.8}$$

where $[\wedge^\bullet \overline{\mathfrak{g}/\mathfrak{g}_\sigma}]$ is the trivial bundle $\wedge^\bullet \overline{\mathfrak{g}/\mathfrak{g}_\sigma} \times (\mathcal{Y}_\sigma)^\beta \to (\mathcal{Y}_\sigma)^\beta$.

To finish the proof, we notice that the normal bundle $\mathcal{N} \to M^\beta$, when restricted to $(\mathcal{Y}_\sigma)^\beta$, can be decomposed as $\mathcal{N}|_{(\mathcal{Y}_\sigma)^\beta} = \mathcal{N}' \oplus [\mathfrak{g}/\mathfrak{g}_\sigma]$. Here $[\mathfrak{g}/\mathfrak{g}_\sigma] \to (\mathcal{Y}_\sigma)^\beta$ is the trivial complex vector bundle defined by $[\mathfrak{g}/\mathfrak{g}_\sigma]_m = \{X_{(\mathcal{Y}_\sigma)^\beta}|_m, X \in \mathfrak{g}/\mathfrak{g}_\sigma\}$ for any $m \in (\mathcal{Y}_\sigma)^\beta$. This decomposition gives first the equality $\wedge_{\mathbb{C}}^\bullet \overline{\mathcal{N}} = \wedge_{\mathbb{C}}^\bullet \overline{\mathcal{N}'} \otimes [\wedge_{\mathbb{C}}^\bullet \overline{\mathfrak{g}/\mathfrak{g}_\sigma}]$ and after[26] $[\wedge_{\mathbb{C}}^\bullet \overline{\mathcal{N}}]_\beta^{-1} = [\wedge_{\mathbb{C}}^\bullet \overline{\mathcal{N}'}]_\beta^{-1} \otimes [\wedge_{\mathbb{C}}^\bullet \overline{\mathfrak{g}/\mathfrak{g}_\sigma}]_\beta^{-1}$, which implies $[\wedge_{\mathbb{C}}^\bullet \overline{\mathcal{N}}]_\beta^{-1} \otimes [\wedge_{\mathbb{C}}^\bullet \overline{\mathfrak{g}/\mathfrak{g}_\sigma}] = [\wedge_{\mathbb{C}}^\bullet \overline{\mathcal{N}'}]_\beta^{-1}$. Equation (7.8) is then proved. ▮

Second proof of Theorem 7.5. A G-invariant neighborhood $\mathcal{U}^{G,\beta}$ of the critical set C_β^G in M can be taken of the form $\mathcal{U}^{G,\beta} = G \times_{G_\sigma} \mathcal{U}^{\sigma,\beta}$ where $\mathcal{U}^{\sigma,\beta}$ a relatively compact G_σ-invariant neighborhood of $\mu_G^{-1}(\beta) \cap M^\beta$ in $\mathcal{Y}_\sigma$ such that $\overline{\mathcal{U}^{\sigma,\beta}} \cap \{\mathcal{H}^\sigma = 0\} = \mu_G^{-1}(\beta) \cap M^\beta$.

The maps $RR_\beta^G(M, -)$ and $RR_\beta^{G_\sigma}(\mathcal{Y}_\sigma, -)$ are respectively defined by the localized Thom symbols $\mathrm{Thom}_{G,[\beta]}^\mu(M) \in K_G(\mathbf{T}_G \mathcal{U}^{G,\beta})$ and $\mathrm{Thom}_{G_\sigma,[\beta]}^\mu(\mathcal{Y}_\sigma) \in K_{G_\sigma}(\mathbf{T}_{G_\sigma} \mathcal{U}^{\sigma,\beta})$ (see Definition 6.4). The inclusion $i: G_\sigma \hookrightarrow G$ induces an isomorphism $i_*: K_{G_\sigma}(\mathbf{T}_{G_\sigma} \mathcal{U}^{\sigma,\beta}) \to K_G(\mathbf{T}_G(G \times_{G_\sigma} \mathcal{U}^{\sigma,\beta}))$ (see Sect. 3.4).

LEMMA 7.8. *We have the equality*

$$i_*(\mathrm{Thom}_{G_\sigma,[\beta]}^\mu(\mathcal{Y}_\sigma) \wedge_{\mathbb{C}}^\bullet \mathfrak{g}/\mathfrak{g}_\sigma) = \mathrm{Thom}_{G,[\beta]}^\mu(M).$$

This lemma, combined with Theorem 3.4, shows that $RR_\beta^G(M, E) = \mathrm{Ind}_{G_\sigma}^G (RR_\beta^{G_\sigma}(\mathcal{Y}_\sigma, E|_{\mathcal{Y}_\sigma}) \wedge_{\mathbb{C}}^\bullet \mathfrak{g}/\mathfrak{g}_\sigma) = \mathrm{Hol}_{G_\sigma}^G(RR_\beta^{G_\sigma}(\mathcal{Y}_\sigma, E|_{\mathcal{Y}_\sigma}))$ for any G-complex vector bundle $E \to M$. The proof of Theorem 7.5 is then completed. ▮

Proof of Lemma 7.8. Through the identification $\mathbf{T}(G \times_{G_\sigma} \mathcal{U}^{\sigma,\beta}) \cong G \times_{G_\sigma} (\mathfrak{g}/\mathfrak{g}_\sigma \oplus \mathbf{T}\mathcal{U}^{\sigma,\beta})$, the vector fields $\mathcal{H}^\sigma$ and $\mathcal{H}^G$ satisfy the relation $\mathcal{H}_{[g,y]}^G \cong \mathcal{H}_y^\sigma$, $[g, y] \in \mathcal{U}^{G,\beta}$. The symbol $\sigma_{[g,y;X+v]}$ of $\mathrm{Thom}_{G,[\beta]}^\mu(M)$ at $[g, y; X+v] \in G \times_{G_\sigma} (\mathfrak{g}/\mathfrak{g}_\sigma \oplus \mathbf{T}\mathcal{U}^{\sigma,\beta})$ acts on $\wedge_{\tilde{J}}^\bullet \mathbf{T}_{[g,y]} \mathcal{U}^{G,\beta} \cong \wedge^\bullet \mathfrak{g}/\mathfrak{g}_\sigma \otimes \wedge_{J_\sigma}^\bullet \mathbf{T}_y \mathcal{U}^{\sigma,\beta}$ as the product

$$\sigma_{[g,y;X+v]} = Cl(X) \odot Cl_y(v - \mathcal{H}_y^\sigma).$$

[26] The product of $[\wedge_{\mathbb{C}}^\bullet \overline{\mathcal{N}'}]_\beta^{-1}$ and $[\wedge_{\mathbb{C}}^\bullet \overline{\mathfrak{g}/\mathfrak{g}_\sigma}]_\beta^{-1}$ is well defined in $\tilde{K}_{G_\sigma}((\mathcal{Y}_\sigma)^\beta) \hat{\otimes} R(\mathbb{T}_\beta)$ since these elements are polarized by β: each of them is a sum over the set of weights of $\mathbb{T}_\beta$ of the form $\sum_\alpha E_\alpha h^\alpha$ such that $E_\alpha \neq 0$ only if $\langle \alpha, \beta \rangle \geqslant 0$, and for any $\delta' > \delta \geqslant 0$ the sum $\sum_{\delta \leqslant \langle \alpha, \beta \rangle \leqslant \delta'} E_\alpha h^\alpha$ is finite (see Definition 5.5).

Now we see that $[g, y; X+v] \to Cl(X) \odot Cl_y(v-\mathscr{H}^\sigma_y)$ is homotopic, as G-transversally elliptic symbol, to $\tilde{\sigma}$: $[g, y; X+v] \to Cl(0) \odot Cl_y(v-\mathscr{H}^\sigma_y)$, and $\tilde{\sigma}$ is, by definition, the image of $\mathrm{Thom}^\mu_{G_\sigma, [\beta]}(\mathscr{Y}_\sigma) \wedge^\bullet_{\mathbb{C}} \mathfrak{g}/\mathfrak{g}_\sigma$ by i_*. The proof of Lemma 7.8 is then completed. ▮

7.4 *The singular case.* In this section, we do not assume that 0 is a regular value of μ_G, and we use the "shifting trick" to compute $[RR^G(M, L)]^G$ in term of reduced manifolds of the type $\mu_G^{-1}(a)/G_a$, for every μ_G-moment bundle L. We know from Theorem 7.1 that $[RR^G(M, L)]^G = 0$ if $0 \notin \mu_G(M)$ since every moment bundle is strictly positive (see Lemma 7.9). So, we assume for the rest of this section that $0 \in \mu_G(M)$.

Let $\mathcal{O}_a$ be the coadjoint orbit through $a \in \mathfrak{g}^*$. It has a canonical symplectic 2-form and the moment map $\mathcal{O}_a \to \mathfrak{g}^*$ for the G-action is the inclusion. We denote by $\overline{\mathcal{O}_a}$ the coadjoint orbit $\mathcal{O}_a$ with the opposite symplectic form. The product $M \times \overline{\mathcal{O}_a}$ is a symplectic manifold with a Hamiltonian moment map

$$\mu_a: M \times \overline{\mathcal{O}_a} \to \mathfrak{g}^*$$
$$(m, \xi) \mapsto \mu_G(m) - \xi.$$

On the symplectic manifold $M \times \overline{\mathcal{O}_a}$ we have a quantization map $RR^G(M \times \overline{\mathcal{O}_a}, -)$ with the following property: for any G-vector bundles E and F over M and $\mathcal{O}_a$ respectively, we have $RR^G(M \times \overline{\mathcal{O}_a}, \pi_a^*(E) \otimes (\pi'_a)^*(F)) = RR^G(M, E) \cdot RR^G(\overline{\mathcal{O}_a}, F)$ in $R(G)$. Here we denote by π_a: $M \times \overline{\mathcal{O}_a} \to M$ the projection to the first factor and π'_a the projection to the second factor. Since $RR^G(\overline{\mathcal{O}_a}, \mathbb{C}) = 1$ we have

$$RR^G(M \times \overline{\mathcal{O}_a}, \pi_a^*(L)) = RR^G(M, L). \tag{7.9}$$

We can now compute $[RR^G(M, L)]^G$ by localizing the character $RR^G(M \times \overline{\mathcal{O}_a}, \pi_a^*(L))$ with the moment map μ_a. We need the following Lemma which was proved by Tian–Zhang [36] for the prequantum line bundles.

LEMMA 7.9. *Let L be a μ_G-moment bundle over M. There exists $\epsilon > 0$ such that for any $|a| < \epsilon$, the vector bundle $\pi_a^*(L)$ is μ_a-positive. For $a = 0$, the bundle $L = \pi_0^*(L)$ is μ_G-strictly positive.*

Let $RR_0^G(M \times \overline{\mathcal{O}_a}, -)$ be the Riemann–Roch character localized near $\mu_a^{-1}(0) \simeq \mu_G^{-1}(\mathcal{O}_a)$. Theorem 7.1, Equality 7.9, and Lemma 7.9 show that

$$[RR^G(M, L)]^G = [RR_0^G(M \times \overline{\mathcal{O}_a}, \pi_a^*(L))]^G, \tag{7.10}$$

for any moment bundle L if $a \in \mu_G(M)$ is close enough to 0.

There exists a unique open face τ of the Weyl chamber $\mathfrak{h}_+$ such that $\mu_G(M) \cap \tau$ is dense in $\mu_G(M) \cap \mathfrak{h}_+$. The face τ is called the principal face of (M, μ_G) [26]. All points in the open face τ have the same connected centralizer G_τ. Let A_τ be the identity component of the center of G_τ and $[G_\tau, G_\tau]$ its semi-simple part. Note that we have an identification between the Lie algebra $\mathfrak{a}_\tau$ of A_τ and the linear span of the face τ. The Principal-cross-section Theorem [26] tells us that $Y_\tau := \mu_G^{-1}(\tau)$ is a symplectic G_τ-manifold, with a trivial action of $[G_\tau, G_\tau]$. So, the restriction of μ_G on $\mathcal{Y}_\tau$ is a moment map $\mu_\tau \colon \mathcal{Y}_\tau \to \mathfrak{a}_\tau$ for the Hamiltonian action of the torus A_τ. We decompose the torus A_τ in a product of two subtorus $A_\tau = A_\tau^1 \times A_\tau^2$ where A_τ^1 is the identity component of the principal stabilizer for the action of A_τ on $\mathcal{Y}_\tau$.

We take now a with value in $\tau \cap \mu_G(M)$. For generic values $a \in \tau \cap \mu_G(M)$, $\mu_G^{-1}(a) = \mu_\tau^{-1}(a)$ is a smooth manifold of M with a locally free action of A_τ^2, hence the quotient $\mathcal{M}_a := \mu_G^{-1}(a)/G_a = \mu_\tau^{-1}(a)/(A_\tau^2)$ is a symplectic orbifold. We denote by $RR(\mathcal{M}_a, -)$ the quantization map defined by the choice of a compatible almost complex structure. If L is a μ_G-moment bundle on M, $L|_{\mathcal{Y}_\tau}$ is a μ_τ-moment bundle: the action of $A_\tau^1[G_\tau, G_\tau]$ on $L|_{\mathcal{Y}_\tau}$ is trivial. Then the quotient $L|_{\mu_\tau^{-1}(a)}/G_a = L|_{\mu_\tau^{-1}(a)}/(A_\tau^2)$ is an orbifold line bundle over $\mathcal{M}_a$ for generic a.

We compare now the Riemann–Roch character $RR_0^{G_\tau}(\mathcal{Y}_\tau, -)$ localized near $\mu_\tau^{-1}(a)$ by the moment map $\mu_\tau - a$ and the Riemann–Roch character $RR_0^G(M \times \overline{\mathcal{O}_a}, -)$ localized near $\mu_a^{-1}(0) = G \cdot (\mu_\tau^{-1}(a) \times \{a\})$. All we need is contained in the following

Proposition 7.10. *Let E be a G-vector bundle over M, and take $a \in \tau$. We have $RR_0^G(M \times \overline{\mathcal{O}_a}, \pi_a^* E) = \mathrm{Ind}_{G_\tau}^G(RR_0^{G_\tau}(\mathcal{Y}_\tau, E|_{\mathcal{Y}_\tau}))$, in particular $[RR_0^G(M \times \overline{\mathcal{O}_a}, \pi_a^* E)]^G = [RR_0^{G_\tau}(\mathcal{Y}_\tau, E|_{\mathcal{Y}_\tau})]^{G_\tau}$.*

If L is a μ_G-moment bundle, the action of $A_\tau^1[G_\tau, G_\tau]$ on $L|_{\mathcal{Y}_\tau}$ is trivial, then $[RR_0^{G_\tau}(\mathcal{Y}_\tau, L|_{\mathcal{Y}_\tau})]^{G_\tau} = [RR_0^{A_\tau^2}(\mathcal{Y}_\tau, L|_{\mathcal{Y}_\tau})]^{A_\tau^2}$. Finally, for every generic value $a \in \tau \cap \mu_G(M)$, the quotient $L_a := L|_{\mu_\tau^{-1}(a)}/A_\tau^2$ is an orbifold line bundle over $\mathcal{M}_a$, so from Section 7.1 we get $[RR_0^{A_\tau^2}(\mathcal{Y}_\tau, L|_{\mathcal{Y}_\tau})]^{A_\tau^2} = RR(\mathcal{M}_a, L_a)$.

With this last equality, Proposition 7.9, and equality (7.10) we have proved the central result of this section

Proposition 7.11. *Suppose that $0 \in \mu_G(M)$. If L is a μ_G-moment bundle, there exist $\epsilon > 0$, such that*

$$[RR^G(M, L)]^G = RR(\mathcal{M}_a, L_a),$$

for every generic value $a \in \tau \cap \mu_G(M)$ with $|a| < \epsilon$.

7.4.1. *Proof of Lemma* 7.9. Let L be a μ_G-moment bundle over M, where $\mu_G\colon M \to \mathfrak{g}^*$ is a Hamiltonian moment map. Recall that the Lie algebra $\mathfrak{g}$ is identified to $\mathfrak{g}^*$ trough an invariant scalar product $(-, -)$. Let H be a maximal torus of G with Lie algebra $\mathfrak{h}$.

LEMMA 7.12. *For $\beta \in \mathfrak{h}$ and $m \in M^\beta \cap \mu_G^{-1}(\gamma)$, the weight α for the action of $\mathbb{T}_\beta$ on L_m satifies $(\alpha, \beta) = (\gamma, \beta)$.*

Proof. Let N be the connected component of M^β containing m, and let m' be a point of N^H. Since N is connected, α is also the weight for the action of $\mathbb{T}_\beta$ on $L_{m'}$, and $\mu_G(m')$ is the weight for the action of H on $L_{m'}$: then $(\alpha, X) = (\mu_G(m'), X)$ for every $X \in Lie(\mathbb{T}_\beta)$. But the map $x \to (\mu_G(x), \beta)$ is constant on N, then $(\gamma, \beta) = (\mu_G(m), \beta) = (\mu_G(m'), \beta) = (\alpha, \beta)$. ∎

The element a is taken in $\mathfrak{h}$. The critical set of the function $\|\mu_a\|^2\colon M \times \mathcal{O}_a \to \mathbb{R}$ admits the following decomposition $\mathrm{Cr}(\|\mu_a\|^2) = G\cdot(\mathrm{Cr}(\|\mu_a\|^2) \cap (M \times \{a\})) = G\cdot((\mathrm{Cr}(\|\mu_{G_a} - a\|^2) \cap \mu_G^{-1}(\mathfrak{g}_a)) \times \{a\})$, where $\mu_{G_a}\colon M \to \mathfrak{g}_a$ is the moment map for the action of G_a. Let $\mathcal{B}_a$ the finite subset of $\mathfrak{h}$ defined by $\mathcal{B}_a = \{\beta \in \mathfrak{h}, M^\beta \cap \mu_G^{-1}(\beta + a) \neq \emptyset\}$. Finally we have the decomposition

$$\mathrm{Cr}(\|\mu_a\|^2) = \bigcup_{\beta \in \mathcal{B}_a} G\cdot(M^\beta \cap \mu_G^{-1}(\beta + a) \times \{a\}).$$

Using Lemma 7.12, we see that $\pi_a^* L$ is μ_a-positive if and only if

$$(\beta + a, \beta) \geqslant 0 \qquad \text{for every} \quad \beta \in \mathcal{B}_a. \tag{7.11}$$

We first see that it is trivially true if $a = 0$: in this case L is strictly positive.

Let $\mu_H\colon M \to \mathfrak{h}$ be the moment map for the maximal torus H. Consider the finite set $\mathcal{B}_{H,a}$ which parameterizes the critical set of $\|\mu_H - a\|^2$: $\mathcal{B}_{H,a} = \{\beta \in \mathfrak{h}, M^\beta \cap \mu_H^{-1}(\beta + a) \neq \emptyset\}$. We have obviously the inclusion $\mathcal{B}_a \subset \mathcal{B}_{H,a}$, so it suffices to show (7.11) for $\mathcal{B}_{H,a}$.

To finish our proof we use now a characterization of the set $\mathcal{B}_{H,a}$ we gave in [31]. There exists a finite collection $\mathcal{B}$ of affine subspaces of $\mathfrak{h}$ such that

$$\mathcal{B}_{H,a} \subset \{P_\Delta(a) - a, \Delta \in \mathcal{B}\}$$

for every $a \in \mathfrak{h}$. Here $P_\Delta\colon \mathfrak{h} \to \mathfrak{h}$ is the orthogonal projection on Δ. It is now easy to compute the sign of $(P_\Delta(a), P_\Delta(a) - a)$ for all $\Delta \in \mathcal{B}$. A simple computation gives $(P_\Delta(a), P_\Delta(a) - a) = |P_\Delta(0)|^2 - (a, P_\Delta(0))$. Hence, either $0 \in \Delta$ and then $(P_\Delta(a), P_\Delta(a) - a)$ is equal to 0 for all $a \in \mathfrak{h}$, or $0 \notin \Delta$ and then $(P_\Delta(a), P_\Delta(a) - a) > 0$ if $|a| < |P_\Delta(0)|$. We can take $\epsilon = \inf_{0 \notin \Delta} |P_\Delta(0)|$ in Lemma 7.9. ∎

7.4.2. *Proof of Proposition* 7.10. Since the point a takes value in τ we identify the coadjoint orbit $\mathcal{O}_a$ with G/G_τ. Let $\mathcal{H}^a$ be the Hamiltonian vector field of the function $\frac{-1}{2}\|\mu_a\|^2\colon M\times G/G_\tau\to\mathbb{R}$. To simplify the notations, $\mathcal{Y}_\tau$ will denote a small neighborhood of $\mu_G^{-1}(a)$ in the symplectic slice $\mu_G^{-1}(\tau)$ such that the open subset $\mathcal{U}:=(G\times_{G_\tau}\mathcal{Y}_\tau)\times G/G_\tau$ is then a neighborhood of $\mu_a^{-1}(0)=G\cdot(\mu_\tau^{-1}(a)\times\{\bar{e}\})$ which satisfies $\overline{\mathcal{U}}\cap\{\mathcal{H}^a=0\}=\mu_a^{-1}(0)$. Following Definition 6.4, the localized Riemann–Roch character $RR_0^G(M\times G/G_\tau,-)$ is computed by means of the Thom class $\mathrm{Thom}^{\mu_a}_{G,[0]}(M\times G/G_\tau)\in K_G(\mathbf{T}_G\mathcal{U})$. On the other hand, the localized Riemann–Roch character $RR_0^{G_\tau}(\mathcal{Y}_\tau,-)$ is computed by means of the Thom class $\mathrm{Thom}^{\mu_\tau-a}_{G_\tau,[0]}(\mathcal{Y}_\tau)\in K_{G_\tau}(\mathbf{T}_{G_\tau}\mathcal{Y}_\tau)$.

Proposition 7.10 will follow from a simple relation between $\mathrm{Thom}^{\mu_a}_{G,[0]}(M\times G/G_\tau)$ and $\mathrm{Thom}^{\mu_\tau-a}_{G_\tau,[0]}(\mathcal{Y}_\tau)$.

First, one considers the isomorphism

$$\begin{aligned}\phi\colon \mathcal{U}&\to\mathcal{U}'\\ ([g;y],[h])&\to[g;[g^{-1}h],y],\end{aligned}\tag{7.12}$$

with $\mathcal{U}':=G\times_{G_\tau}(G/G_\tau\times\mathcal{Y}_\tau)$, and let $\phi^*\colon K_G(\mathbf{T}_G\mathcal{U}')\to K_G(\mathbf{T}_G\mathcal{U})$ be the induced isomorphism. After one consider the inclusion $i\colon G_\tau\hookrightarrow G$ which induces an isomorphism $i_*\colon K_{G_\tau}(\mathbf{T}_{G_\tau}(G/G_\tau\times\mathcal{Y}_\tau))\to K_G(\mathbf{T}_G\mathcal{U}')$ (see Sect. 3.4). Let $j\colon\mathcal{Y}_\tau\hookrightarrow G/G_\tau\times\mathcal{Y}_\tau$ be the G_τ-invariant inclusion map defined by $j(y):=(\bar{e},y)$. We have then a pushforward map $j_!\colon K_{G_\tau}(\mathbf{T}_{G_\tau}\mathcal{Y}_\tau)\to K_{G_\tau}(\mathbf{T}_{G_\tau}(G/G_\tau\times\mathcal{Y}_\tau))$. Finally we have produced a map $\Theta:=\phi^*\circ i_*\circ j_!$ from $K_{G_\tau}(\mathbf{T}_{G_\tau}\mathcal{Y}_\tau)$ to $K_G(\mathbf{T}_G\mathcal{U})$, such that $\mathrm{Index}^G_{\mathcal{U}}(\Theta(\sigma))=\mathrm{Ind}^G_{G_\tau}(\mathrm{Index}^{G_\tau}_{\mathcal{Y}_\tau}(\sigma))$ for every $\sigma\in K_{G_\tau}(\mathbf{T}_{G_\tau}\mathcal{Y}_\tau)$.

Proposition 7.10 is an immediate consequence of the following

LEMMA 7.13. *We have the equality*

$$\Theta(\mathrm{Thom}^{\mu_\tau-a}_{G_\tau,[0]}(\mathcal{Y}_\tau))=\mathrm{Thom}^{\mu_a}_{G,[0]}(M\times G/G_\tau).$$

Proof. Let $\mu'_a:=\mu_a\circ\phi^{-1}$ be the moment map on $\mathcal{U}'$, and let $\mathcal{H}'^{,a}$ be the Hamiltonian vector field of $\frac{-1}{2}\|\mu'_a\|^2$. For the tangent manifold $\mathbf{T}\mathcal{U}'$ we have the decomposition

$$\mathbf{T}\mathcal{U}'\simeq G\times_{G_\tau}(\mathfrak{g}/\mathfrak{g}_\tau\oplus G\times_{G_\tau}\overline{(\mathfrak{g}/\mathfrak{g}_\tau)}\oplus\mathbf{T}\mathcal{Y}_\tau).$$

A small computation gives $\mathcal{H}'^{,a}(m)=pr_{\overline{\mathfrak{g}/\mathfrak{g}_\tau}}(ha)+R(m)+\mathcal{H}^\tau_a(y)+S(m)$ for $m=[g;y,[h]]\in\mathcal{U}'$, where $R(m)\in\mathfrak{g}/\mathfrak{g}_\tau$ and $S(m)\in\mathbf{T}_y\mathcal{Y}_\tau$ vanishes when $m\in G\times_{G_\tau}(\{\bar{e}\}\times\mathcal{Y}_\tau)$, i.e., $[h]=\bar{e}$. Here $\mathcal{H}^\tau_a$ is the Hamiltonian vector field of the function $\frac{-1}{2}\|\mu_\tau-a\|^2\colon\mathcal{Y}_\tau\to\mathbb{R}$.

The transversally elliptic symbol $\sigma_1 := (\phi^{-1})^* (\mathrm{Thom}^{\mu_a}_{G,[0]}(M \times G/G_\tau))$ is equal to the exterior product

$$\sigma_1(m, \xi_1 + \xi_2 + v) \\ = Cl(\xi_1 - pr_{\mathfrak{g}/\mathfrak{g}_\tau}(ha)) \odot Cl(\xi_2 - R(m)) \odot Cl(v - \mathscr{H}^\tau_a - S(m)),$$

with $\xi_1 \in \mathfrak{g}/\mathfrak{g}_\tau$, $\xi_2 \in \overline{\mathfrak{g}/\mathfrak{g}_\tau}$, $v \in \mathbf{T}\mathscr{Y}_\tau$.

Now we simplify the symbol σ_1 without changing its K-theoretic class. Since $\mathrm{Char}(\sigma_1) \cap \mathbf{T}_G\mathscr{U}' = G \times_{G_\tau} (\{\bar{e}\} \times \mathscr{Y}_\tau)$, we can transform σ_1 through the G_τ-invariant diffeomorphism $h = e^X$ from a neighborhood of 0 in $\mathfrak{g}/\mathfrak{g}_\tau$ to a neighborhood of $\bar{e}$ in G/G_τ. This gives $\sigma_2 \in K_G(\mathbf{T}_G(G \times_{G_\tau} (\mathfrak{g}/\mathfrak{g}_\tau \times \mathscr{Y}_\tau)))$ defined by

$$\sigma_2([g, X, y], \xi_1 + \xi_2 + v) \\ = Cl(\xi_1 - pr_{\mathfrak{g}/\mathfrak{g}_\tau}(e^X a)) \odot Cl(\xi_2 - R(m)) \odot Cl(v - \mathscr{H}^\tau_a - S(m)).$$

Now trivial homotopies link σ_2 with the symbol σ_3, where we have removed the terms $R(m)$ and $S(m)$, and where we have replaced $pr_{\mathfrak{g}/\mathfrak{g}_\tau}(e^X a) = [X, a] + o([X, a])$ by the term $[X, a]$:

$$\sigma_3([g, X, y], \xi_1 + \xi_2 + v) = Cl(\xi_1 - [X, a]) \odot Cl(\xi_2) \odot Cl(v - \mathscr{H}^\tau_a).$$

Now, we get $\sigma_3 = i_*(\sigma_4)$ where the symbol $\sigma_4 \in K_{G_\tau}(\mathbf{T}_{G_\tau}(\mathfrak{g}/\mathfrak{g}_\tau \times \mathscr{Y}_\tau))$ is defined by

$$\sigma_4(X, y; \xi_2 + v) = Cl(-[X, a]) \odot Cl(\xi_2) \odot Cl(v - \mathscr{H}^\tau_a).$$

So σ_4 is equal to the exterior product of $(y, v) \to Cl(v - \mathscr{H}^\tau_a)$, which is $\mathrm{Thom}^{\mu_\tau - a}_{G_\tau,[0]}(\mathscr{Y}_\tau)$, with the transversally elliptic symbol on $\mathfrak{g}/\mathfrak{g}_\tau$: $(X, \xi_2) \to Cl(-[X, a]) \odot Cl(\xi_2)$. As in Lemma 5.2, we see that the K-theoretic class of this former symbol is equal to $k_!(\mathbb{C})$ where $k\colon \{0\} \hookrightarrow \mathfrak{g}/\mathfrak{g}_\tau$. This shows that

$$\sigma_4 = k_!(\mathbb{C}) \odot \mathrm{Thom}^{\mu_\tau - a}_{G_\tau,[0]}(\mathscr{Y}_\tau) = j_!(\mathrm{Thom}^{\mu_\tau - a}_{G_\tau,[0]}(\mathscr{Y}_\tau)). \quad \blacksquare$$

8. APPENDIX A: $G = SU(2)$

We restrict our attention to an action of $G = SU(2)$ on a compact manifold M. We suppose that M is endowed with a G-invariant almost complex structure J and an abstract moment map $f\colon M \to \mathfrak{g}$. In this situation, the decomposition $RR^G(M, -) = \sum_{\beta \in \mathscr{B}_G} RR^G_\beta(M, -)$ becomes simple.

Let S^1 be the maximal torus of $SU(2)$, and $f_{S^1}\colon M \to \mathbb{R}$ the induced moment map for the S^1-action. The critical set $\{\mathcal{H}^G = 0\}$ has a particularly simple expression: $\{\mathcal{H}^G = 0\} = f^{-1}(0) \cup G.M_+^{S^1}$, where $M_+^{S^1}$ is the union of the connected components $F \subset M^{S^1}$ with $f_{S^1}(F) > 0$. Note that the critical set $\{\mathcal{H}^{S^1} = 0\}$ is equal to $f_{S^1}^{-1}(0) \cup M^{S^1}$,

The non-symplectic case. Here the induction formula of Theorem 6.16, and Proposition 6.14 gives

$$RR^G(M, E) = RR_0^G(M, E) + \mathrm{Hol}_{S^1}^G(\Theta(E)(t).(1-t^{-2})) \tag{8.1}$$

where $\Theta(E) \in R^{-\infty}(S^1)$ is determined by

$$\Theta(E) = (-1)^{r_{\mathcal{N}}} \sum_{k \in \mathbb{N}} RR^{S^1}(M_+^{S^1}, E|_{M_+^{S^1}} \otimes \det \mathcal{N}^+ \otimes S^k((\mathcal{N} \otimes \mathbb{C})^+)). \tag{8.2}$$

Here $\mathcal{N} \to M_+^{S^1}$ is the normal bundle of $M_+^{S^1}$ in M.

The Hamiltonian case. Here we suppose that (M, ω) is a symplectic manifold, with moment map μ and a ω-compatible almost complex structure J. Let $\mathcal{Y} = \mu^{-1}(\mathbb{R}_{>0})$ be the symplectic slice associated to the interior of the Weyl chamber $\mathbb{R}_{>0} \subset Lie(S^1)$.

The induction formula of Theorem 7.5 gives

$$RR^G(M, E) = RR_0^G(M, E) + \mathrm{Hol}_{S^1}^G(\tilde{\Theta}(E)) \tag{8.3}$$

where $\tilde{\Theta}(E) \in R^{-\infty}(S^1)$ is determined by

$$\tilde{\Theta}(E) = (-1)^{r_{\tilde{\mathcal{N}}}} \sum_{k \in \mathbb{N}} RR^{S^1}(M_+^{S^1}, E|_{M_+^{S^1}} \otimes \det \tilde{\mathcal{N}}^+ \otimes S^k((\tilde{\mathcal{N}} \otimes \mathbb{C})^+)). \tag{8.4}$$

Here $\tilde{\mathcal{N}} \to M_+^{S^1}$ is the normal bundle of $M_+^{S^1}$ in $\mathcal{Y}$.

Recall that the irreducible characters ϕ_n of $G = SU(2)$ are labeled by $\mathbb{Z}_{\geqslant 0}$, and are completely determined by the relation $\phi_n = \mathrm{Hol}_{S^1}^G(t^n)$ in $R(G)$ (See Lemma 9.1). Hence the component $\mathrm{Hol}_{S^1}^G(\Theta(E)(t).(1-t^{-2}))$ of (8.1) does not contain the trivial character ϕ_0 if $\Theta(E) = \sum_{n \in \mathbb{Z}} a_n t^n$ with

$$a_n \neq 0 \Rightarrow n \geqslant 3. \tag{8.5}$$

Equation (8.2) tells us that (8.5) is satisfied if the weights for the action of S^1 in the fibers of the complex vector bundle $E|_{M_+^{S^1}} \otimes \det \mathcal{N}^+$ are all bigger than 3.

The conditions are weaker in the "Hamiltonian" situation. The term $\mathrm{Hol}^{G}_{S^1}(\widetilde{\Theta}(E))$ of (8.3) does not contain the trivial character ϕ_0 if $\widetilde{\Theta}(E)=\sum_{n\in\mathbb{Z}} a_n t^n$ with

$$a_n \neq 0 \Rightarrow n \geqslant 1, \tag{8.6}$$

and this condition is fulfilled if the weights for the action of S^1 in the fibers of the complex vector bundle $E|_{M_+^{S^1}} \otimes \det \widetilde{\mathcal{N}}^+$ are all bigger than 1. Here we have another important difference: the vector bundle $\widetilde{\mathcal{N}}^+ \to M_+^{S^1}$ is not equal to the zero bundle if $0 \in \mu(M)$ (see Lemma 7.3).

We see finally that, in the Hamiltonian case, the condition "*E is μ-positive*" implies

$$0 \in \mu(M) \Rightarrow [RR^G(M, E)]^G = [RR^G_0(M, E)]^G.$$

9. APPENDIX B: INDUCTION MAP AND MULTIPLICITIES

Let G be a compact connected Lie group, with maximal torus H, and $\mathfrak{h}^*_+ \subset \mathfrak{h}^* = (\mathfrak{g}^*)^H$ some choice of positive Weyl chamber. We denote by $\mathfrak{R}_+$ the associated system of positive roots, and we label the irreducible representations of G by the set $\Lambda^*_+ = \Lambda^* \cap \mathfrak{h}^*_+$ of dominant weights. For any weights $\alpha \in \Lambda^*$ we denote by $H \to \mathbb{C}^*$, $h \mapsto h^\alpha$ the corresponding character: $(\exp(X))^\alpha = e^{\imath\langle\alpha, X\rangle}$ for $X \in \mathfrak{h}$.

Let W be the Weyl group of (G, H), and $\mathrm{L}^2(H)$ be the vector space of square integrable complex functions on H. For $f \in \mathrm{L}^2(H)$, we consider $J(f) = \sum_{w\in W} (-1)^w\, w.f$, where $W \to \{1, -1\}$, $w \to (-1)^w$, is the signature operator and $w.f \in \mathrm{L}^2(H)$ is defined by $w.f(h) = f(w^{-1}.h)$, $h \in H$ (see Section 7.4 of [8]). The map $\frac{1}{|W|} J$ is the orthogonal projection from $\mathrm{L}^2(H)$ to the space of W-anti-invariant elements of $\mathrm{L}^2(H)$.

Let $\rho \in \mathfrak{h}^*$ be the half sum of the positive roots. The function $H \to \mathbb{C}^*$, $h \mapsto h^\rho$ is well defined as an element of $\mathrm{L}^2(H)$ (even if ρ is not a weight). The Weyl's character formula can be written in the following way. For any dominant weight $\lambda \in \Lambda^*_+$, the restriction $\chi^G_\lambda|_H$ of the irreducible G-character χ^G_λ satisfies

$$J(h^\rho).\chi^G_\lambda|_H = J(h^{\lambda+\rho}) \quad \text{in} \quad \mathrm{L}^2(H). \tag{9.1}$$

For our purpose we give an expression of the character χ^G_λ through the induction map $\mathrm{Ind}^G_H\colon \mathscr{C}^{-\infty}(H) \to \mathscr{C}^{-\infty}(G)^G$ (see (3.7)). Consider the affine action of the Weyl group on the set of weights: $w \circ \lambda = w.(\lambda+\rho) - \rho$ for $w \in W$ and $\lambda \in \Lambda^*$.

LEMMA 9.1. (1) *For any dominant weight* $\lambda \in \Lambda^*_+$, *the character* χ^G_λ *is determined by the relation* $\chi^G_\lambda = \mathrm{Ind}^G_H(h^\lambda \Pi_{\alpha\in\mathfrak{R}_+}(1-h^\alpha))$ *in* $\mathscr{C}^{-\infty}(G)^G$.

(2) *For* $\lambda \in \Lambda^*$ *and* $w \in W$, *we have* $\mathrm{Ind}^G_H(h^{w\circ\lambda}\Pi_{\alpha\in\mathfrak{R}_+}(1-h^\alpha)) = (-1)^w\,\mathrm{Ind}^G_H(h^\lambda \Pi_{\alpha\in\mathfrak{R}_+}(1-h^\alpha))$.

(3) *For any weight* λ, *the following statements are equivalent:*

(a) $\mathrm{Ind}^G_H(h^\lambda \Pi_{\alpha\in\mathfrak{R}_+}(1-h^\alpha)) = 0$,

(b) $W \circ \lambda \cap \Lambda^*_+ = \varnothing$,

(c) *The element* $\lambda+\rho$ *is not a regular element of* $\mathfrak{h}^*$.

Proof of (1). To prove it, we need the following relations proved in [8, Sect. 7.4]:

(i) $\overline{J(h^\rho)} = h^{-\rho}\Pi_{\alpha\in\mathfrak{R}_+}(1-h^\alpha)$,

(ii) $J(h^\rho).\overline{J(h^\rho)} = \Pi_{\alpha\in\mathfrak{R}}(1-h^\alpha)$.

Let dg and dt be respectively the normalized Haar measures on G and H. For any $f \in \mathscr{C}^\infty(G)^G$ we have

$$\int_G \chi^G_\lambda(g)\, f(g)\, dg = \frac{1}{|W|}\int_H \chi^G_\lambda|_H(h)\, \Pi_{\alpha\in\mathfrak{R}}(1-h^\alpha)\, f|_H(h)\, dh \qquad [1]$$

$$= \frac{1}{|W|}\int_H J(h^{\lambda+\rho})\,\overline{J(h^\rho)}\, f|_H(h)\, dh \qquad [2]$$

$$= \int_H h^{\lambda+\rho}\,\overline{J(h^\rho)}\, f|_H(h)\, dh \qquad [3]$$

$$= \int_H h^\lambda\, \Pi_{\alpha\in\mathfrak{R}_+}(1-h^\alpha)\, f|_H(h)\, dh. \qquad [4]$$

The first equality is the Weyl integration formula. Equality [2] comes from (ii) and (9.1). Since $\frac{1}{|W|}J$ is the orthogonal projection on $\mathrm{L}^2(H)^{W\text{-anti-invariant}}$ and $h \mapsto \overline{J(h^\rho)}\, f|_H(h)$ is W-anti-invariant we obtain the third equality. The equality [4] comes from (i).

Proof of (2). From (i), wee see that $h^{w\circ\lambda}\Pi_{\alpha\in\mathfrak{R}_+}(1-h^\alpha) = h^{w(\lambda+\rho)}\overline{J(h^\rho)} = (-1)^w w^{-1}.(h^{\lambda+\rho}\overline{J(h^\rho)}) = (-1)^w w^{-1}.(h^\lambda \Pi_{\alpha\in\mathfrak{R}_+}(1-h^\alpha))$, hence the relation (2) is proved since Ind^G_H is W-invariant.

Proof of (3). The implication (a) $\Rightarrow$ (b) is an immediate consequence of (1) and (2). Proposition 3 in Section 7.4 of [8] tells us that $\{J(h^{\lambda'+\rho}),\ \lambda' \in \Lambda^*_+\}$ is an orthogonal basis of the Hilbert space $\mathrm{L}^2(H)^{W\text{-anti-invariant}}$. For

$\lambda \in \Lambda^*$ and $\lambda' \in \Lambda^*_+$ we have $\langle J(h^{\lambda+\rho}), J(h^{\lambda'+\rho})\rangle_{L^2} = |W| \langle J(h^{\lambda+\rho}), h^{\lambda'+\rho}\rangle_{L^2} = |W| \sum_{w \in W} (-1)^w \int_T t^{w \circ \lambda - \lambda'} dt$. Thus, the condition $W \circ \lambda \cap \Lambda^*_+ = \varnothing$ is equivalent to $J(h^{\lambda+\rho}) = 0$. But the equality [2] gives $\mathrm{Ind}_H^G(h^\lambda \Pi_{\alpha \in \mathfrak{R}_+} (1-h^\alpha)) = \frac{1}{|W|} \mathrm{Ind}_H^G(J(h^{\lambda+\rho})\, h^{-\rho} \Pi_{\alpha \in \mathfrak{R}_+} (1-h^\alpha))$, hence $J(h^{\lambda+\rho}) = 0$ implies the point (a). We have proved that (b) $\Rightarrow$ (a). Finally we see that $J(h^{\lambda+\rho}) = 0 \leftrightarrow \exists w \in W, w.(\lambda+\rho) = \lambda+\rho \leftrightarrow \lambda+\rho$ is not a regular value of $\mathfrak{h}^*$. We have proved that (b) $\leftrightarrow$ (c). ∎

From the previous Lemma, we see that $v \mapsto \mathrm{Ind}_H^G(v(h)\, \Pi_{\alpha \in \mathfrak{R}_+} (1-h^\alpha))$ is the holomorphic induction map

$$\mathrm{Hol}_H^G \colon R(H) \to R(G). \tag{9.2}$$

We keep the same notation for the extended map $\mathrm{Hol}_H^G \colon R^{-\infty}(H) \to R^{-\infty}(G)$. Note that the choice of a positive Weyl chamber $\mathfrak{h}^*_+$ determines a complex structure on $\mathfrak{g}/\mathfrak{h}$, and $\Pi_{\alpha \in \mathfrak{R}_+} (1-h^\alpha)$ is the trace of the virtual H-representation $\bigwedge^\bullet_{\mathbb{C}} \mathfrak{g}/\mathfrak{h} \in R(H)$. Then the map Hol_H^G will be defined simply by the relation $\mathrm{Hol}_H^G(v) = \mathrm{Ind}_H^G(v \bigwedge^\bullet_{\mathbb{C}} \mathfrak{g}/\mathfrak{h})$.

Remark 9.2. The relations (i) and (ii) used in the proof of the past lemma show that $\sum_{w \in W} w.\Pi_{\alpha>0} (1-h^\alpha) = \sum_{w \in W} w.(\overline{J(h^\rho)\, h^\rho}) = \overline{J(h^\rho)}.J(h^\rho) = \Pi_\alpha (1-h^\alpha)$. In other words $\sum_{w \in W} w.\bigwedge^\bullet_{\mathbb{C}} \mathfrak{g}/\mathfrak{h} = (\bigwedge^\bullet_{\mathbb{R}} \mathfrak{g}/\mathfrak{h}) \otimes \mathbb{C} = \bigwedge^\bullet_{\mathbb{C}} \mathfrak{g}/\mathfrak{h}\ \bigwedge^\bullet_{\mathbb{C}} \overline{\mathfrak{g}/\mathfrak{h}}$ in $R(H)$. These equalities give

$$\mathrm{Ind}_H^G\left(\left(\sum_w w.\phi\right) \textstyle\bigwedge^\bullet_{\mathbb{C}} \mathfrak{g}/\mathfrak{h}\right) = \mathrm{Ind}_H^G(\phi \textstyle\bigwedge^\bullet_{\mathbb{R}} \mathfrak{g}/\mathfrak{h}) \tag{9.3}$$

since Ind_H^G is W-invariant. The Weyl integration formula is usually state as the relation $f = \frac{1}{|W|} \mathrm{Ind}_H^G(f|_H \bigwedge^\bullet_{\mathbb{R}} \mathfrak{g}/\mathfrak{h})$ for any $f \in \mathscr{C}^\infty(G)^G$. But $f|_H$ is W-invariant, so (9.3) gives $\frac{1}{|W|} \mathrm{Ind}_H^G(f|_H \bigwedge^\bullet_{\mathbb{R}} \mathfrak{g}/\mathfrak{h}) = \mathrm{Ind}_H^G(f|_H \bigwedge^\bullet_{\mathbb{C}} \mathfrak{g}/\mathfrak{h})$. Finally, for any $\phi \in R(G)$, the Weyl integration formula is equivalent to the following equality in $R(G)$:

$$\phi = \mathrm{Hol}_H^G(\phi|_H).$$

Remark 9.3. A weight λ satisfies $\mathrm{Hol}_H^G(h^\lambda) = \pm 1$ if and only if $0 \in W \circ \lambda \cap \Lambda^*_+$, that is, $\lambda = -(\rho - w.\rho)$ for some $w \in W$. But a small computation shows that $\rho - w.\rho = \sum_{\alpha>0,\, w^{-1}.\alpha<0} = \alpha$, hence $\langle \rho - w.\rho, X\rangle \geqslant 0$ for any $X \in \mathfrak{h}_+$. Finally the equality $\mathrm{Hol}_H^G(h^\lambda) = \pm 1$ implies that $\langle \lambda, X\rangle \leqslant 0$ for any $X \in \mathfrak{h}_+$.

Consider now the stabiliser G_β of the non-zero element $\beta \in \mathfrak{h}_+$. The subgroup H is also a maximal torus of G_β. The Weyl group W_β of (G_β, H) is identified with $\{w \in W, w.\beta = \beta\}$. We consider a Weyl chamber $\mathfrak{h}^*_{+,\beta} \subset \mathfrak{h}^*$ for G_β that contains the Weyl chamber $\mathfrak{h}^*_+$ of G. The irreducible representations $\chi_\lambda^{G_\beta}$, $\lambda \in \Lambda^*_{+,\beta}$ of G_β are labeled by the set $\Lambda^*_{+,\beta} = \Lambda^* \cap \mathfrak{h}^*_{+,\beta}$ of dominant weights.

We have a unique "holomorphic" induction map $\mathrm{Hol}^G_{G_\beta}\colon R(G_\beta) \to R(G)$ such that $\mathrm{Hol}^G_H = \mathrm{Hol}^G_{G_\beta} \circ \mathrm{Hol}^{G_\beta}_H$. This map is defined precisely by the equation[27]

$$\mathrm{Hol}^G_{G_\beta}(v) = \mathrm{Ind}^G_{G_\beta}(v \wedge^\bullet_{\mathbb{C}} \mathfrak{g}/\mathfrak{g}_\beta), \tag{9.4}$$

for every $v \in R(G_\beta)$.

We finish this appendix with some general remarks about P-transversally elliptic symbols on a compact manifold M, when a subgroup $\mathbb{T}$ in the center of P acts trivially on M.

More precisely, let H be a compact maximal torus in P, $\mathfrak{h}_+$ be a choice of a positive Weyl chamber in the Lie algebra $\mathfrak{h}$ of H, and let $\beta \in \mathfrak{h}_+$ be a nonzero element in the center of the Lie algebra $\mathfrak{p}$ of P.[28] We suppose here that the subtorus $\mathbb{T} \subset H$, which is equal to the closure of $\{\exp(t.\beta), t \in \mathbb{R}\}$, acts trivially on M.

Every P-equivariant complex vector bundle $E \to M$ can be decomposed relatively to the $\mathbb{T}$-action: $E = \bigoplus_{a \in \hat{\mathbb{T}}} E^a \otimes \mathbb{C}_a$, where $E^a := \hom_{\mathbb{T}}(\mathbb{C}_a, E)$[29] is a P-complex vector bundle with a trivial action of $\mathbb{T}$. Then, each P-equivariant symbol $\sigma : p^*(E_1) \to p^*(E_2)$ where E_1, E_2 are P-equivariant complex vector bundles over M, and where $p\colon \mathbf{T}M \to M$ is the canonical projection, admits a finite $P \times \mathbb{T}$-equivariant decomposition

$$\sigma = \sum_{a \in \hat{\mathbb{T}}} \sigma^a \otimes \mathbb{C}_a. \tag{9.5}$$

Here $\sigma^a\colon p^*(E_1^a) \to p^*(E_2^a)$ is a P-equivariant symbol, trivial for the $\mathbb{T}$-action.

Let us consider the inclusion map $i\colon \mathbb{T} \hookrightarrow H$, with the induced maps $i\colon Lie(\mathbb{T}) \to \mathfrak{h}$ at the level of Lie algebra and $i^*\colon \mathfrak{h}^* \to Lie(\mathbb{T})^*$. Note that $i^*(\lambda)$ is a weight for $\mathbb{T}$ if λ is a weight for H.

LEMMA 9.4. *Let M be a P-manifold with the same properties as above. Let $\sigma\colon p^*(E_1) \to p^*(E_2)$ be a P-equivariant transversally elliptic symbol over M and denote by $m_\lambda(\sigma)$, $\lambda \in \Lambda^*_{P,+}$, the multiplicities of its index: $\mathrm{Index}^P_M(\sigma) = \sum_{\lambda \in \Lambda^*_{P,+}} m_\lambda(\sigma)\, \chi^P_\lambda$. Then, if $m_\lambda(\sigma) \neq 0$, the weight $a = i^*(\lambda)$ occurs in the decomposition* (9.5).

[27] We take on $\mathfrak{g}/\mathfrak{g}_\beta$ the complex structure defined by β.
[28] The Lie group P is supposed connected then $\beta \in (\mathfrak{p})^P$.
[29] The torus $\mathbb{T}$ acts on the complex line $\mathbb{C}_a$ with the representation $t \to t^a$.

COROLLARY 9.5. *Suppose that the weights* $a \in \hat{\mathbb{T}}$ *which occur in the decomposition* (9.5) *satisfy* $\langle a, \beta \rangle \geqslant \eta$ *for some fixed* $\eta \in \mathbb{R}$. *Then, for the multiplicities, we get*

$$m_\lambda(\sigma) \neq 0 \;\Rightarrow \langle \lambda, \beta \rangle \geqslant \eta.$$

In particular, $\mathrm{Index}^P_M(\sigma)$ *does not contain the trivial representation when* $\eta > 0$.

Remark 9.6. The previous Lemma and Corollary remain true if M is a P-invariant open subset of a compact P-manifold.

For the Corollary, we have just to notice that[30] $\langle \lambda, \beta \rangle = \langle a, \beta \rangle$ for $a = i^*(\lambda)$. Then, if we have $\langle a, \beta \rangle \geqslant \eta$ for all $\mathbb{T}$-weights occurring in σ, we get $\langle \lambda, \beta \rangle \geqslant \eta$ for every λ such that $m_\lambda(\sigma) \neq 0$.

Proof of Lemma 9.4. Let P' be a Lie subgroup of P such that $r\colon \mathbb{T} \times P' \to P$, $r(t, g) = t.g$, is a finite covering of P. The map r induces $r^*\colon K_P(\mathbf{T}_P M) \to K_{\mathbb{T}\times P'}(\mathbf{T}_{P'} M)$[31] and an injective map $r^*\colon R^{-\infty}(P) \to R^{-\infty}(\mathbb{T} \times P')$, such that $\mathrm{Index}^{\mathbb{T}\times P'}_M(r^*\sigma) = r^*(\mathrm{Index}^P_M(\sigma))$.

The decomposition (9.5) can be read through the identification $K_{\mathbb{T}\times P'}(\mathbf{T}_{P'} M) = K_{P'}(\mathbf{T}_{P'} M) \otimes R(\mathbb{T})$: we have $r^*\sigma = \sum_{a \in \hat{\mathbb{T}}} \sigma^a \otimes \mathbb{C}_a$ with $\sigma^a \in K_{P'}(\mathbf{T}_{P'} M)$. Hence

$$\mathrm{Index}^{\mathbb{T}\times P'}_M(r^*\sigma)(t, g) = \sum_{a \in \hat{\mathbb{T}}} \mathrm{Index}^{P'}_M(\sigma^a)(g).\, t^a, \qquad (t, g) \in \mathbb{T} \times P'. \tag{9.6}$$

The irreducible characters χ^P_λ satisfy $r^*\chi^P_\lambda(t, g) = \chi^P_\lambda|_{P'}(g).\, t^{i^*(\lambda)}$. If we start from the decomposition $\mathrm{Index}^P_M(\sigma) = \sum_{\lambda \in \Lambda^*_{P,+}} m_\lambda(\sigma)\, \chi^P_\lambda$ relative to the irreducible characters of P, we get

$$r^*(\mathrm{Index}^{\mathbb{T}\times P'}_M(\sigma))(t, g) = \sum_{a \in \hat{\mathbb{T}}} \Bigg(\sum_{i^*(\lambda) = a} m_\lambda(\sigma)\, \chi^P_\lambda|_{P'}(g) \Bigg).\, t^a, \tag{9.7}$$

for any $(t, g) \in \mathbb{T} \times P'$. If we compare (9.6) and (9.7), we get $\mathrm{Index}^{P'}_M(\sigma^a) = \sum_{i^*(\lambda) = a} m_\lambda(\sigma)\, \chi^P_\lambda|_{P'}$. The map $r^*\colon R^{-\infty}(P) \to R^{-\infty}(\mathbb{T} \times P')$ is injective, so $\sum_{i^*(\lambda) = a} m_\lambda(\sigma)\, \chi^P_\lambda|_{P'} = 0$ if and only if $m_\lambda(\sigma) = 0$ for every λ satisfying $i^*(\lambda) = a$. Hence if the multiplicity $m_\lambda(\sigma)$ is non zero, the element $a = i^*(\lambda)$ is a weight for the action of $\mathbb{T}$ on $\sigma\colon p^*(E_1) \to p^*(E_2)$. ∎

30 We use the same notations for $\beta \in Lie(\mathbb{T})$ and $i(\beta) \in \mathfrak{h}$.
31 Note that $\mathbf{T}_{P'} M = \mathbf{T}_P M$ because $\mathbb{T}$ acts trivially on M.

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