Broué’s Conjecture Holds for Principal 3-Blocks with Elementary Abelian Defect Group of Order 9

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In representation theory of finite groups, one of the most important problems now is to solve a conjecture given by M. Broué [2, 4]. He conjectures that, for any prime $p$, if a finite group $G$ has an abelian Sylow $p$-subgroup $P$, then the derived categories of the principal $p$-blocks of $G$ and of the normalizer $N_G(P)$ of $P$ in $G$ are equivalent. We prove in this paper that Broué’s conjecture holds for the principal 3-block of an arbitrary finite group $G$ with an elementary abelian Sylow 3-subgroup $P$ of order 9, by using initiated works for the case where $G$ is simple, which are due to Puig, Okuyama, Waki, Miyachi, and the authors. The result depends on the classification of finite simple groups.

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0. INTRODUCTION AND NOTATION

In representation theory of finite groups, one of the most important problems now is to solve a conjecture given by M. Broué [2, 4]. He conjectures

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(0.1) Conjecture (Broué [2, 6.2. Question], [4, 4.9. Conjecture]). Let $G$ be a finite group with a Sylow $p$-subgroup $P$, where $p$ is a prime, and let $H = N_G(P)$, the normalizer of $P$ in $G$. Then, is it true that the derived categories of the principal $p$-blocks of $G$ and $H$ are equivalent provided $P$ is abelian?

There are only several special cases where Broué’s conjecture (0.1) has been checked, see [4, p. 15] and [14, p. 136] (see also [5, 6, 15–17, 19, 23, 26, 27]).

The purpose of this paper is to present that Broué’s conjecture (0.1) is true for an arbitrary finite group $G$ with an elementary abelian Sylow 3-subgroup $P$ of order 9. As a matter of fact, it recently has been checked that Broué’s conjecture (0.1) holds if the above $G$ is, moreover, simple by the works of Puig [26], Okuyama-Waki [25], Okuyama [23], Kunugi [19], Koshitani–Kunugi [15], and Koshitani–Miyachi [16, 17]. Then, it looks quite natural to ask whether Broué’s conjecture holds for an arbitrary finite group $G$ with the same Sylow 3-subgroup $P$, by making use of the classification of finite simple groups. On the other hand, there is a work by Marcus [21] in which he proves that Broué’s conjecture is liftable from a finite group $G$ to a finite group $\tilde{G}$ if $G$ is a normal subgroup of $\tilde{G}$ such that $\tilde{G}/G$ is a $p'$-group under some hypotheses. However, in general it is not trivial to know that the hypotheses in [21] are satisfied. Nevertheless, in the case where $P$ is elementary abelian of order 9, we can show that Broué’s conjecture holds by preparing tools (lemmas) which are a sort of generalization or application of theorems of Dade [10] and Marcus [21].

The strategy of this paper is the following. Assume $G \triangleleft \tilde{G}$, $\tilde{G}/G$ is a $p'$-group, and $P$ is an abelian Sylow $p$-subgroup of $G$. By a theorem of Alperin and Dade [1, 9], Broué’s conjecture (0.1) is liftable from $G$ to a normal subgroup $G \cdot C_{\tilde{G}}(P)$ of $\tilde{G}$. It is easy to know that the quotient group $\tilde{G}/[G \cdot C_{\tilde{G}}(P)]$ is a subquotient group of $N_{\tilde{G}}(P)/C_{\tilde{G}}(P) \hookrightarrow Aut(P)$. So, if $P$ is small, then we might have a chance to lift Broué’s conjecture from $G \cdot C_{\tilde{G}}(P)$ to $\tilde{G}$. Actually, this is the case, though it is not yet easy.

Our main theorem is the following:

(0.2) Theorem. Let $\mathbb{O}$ be a complete discrete valuation ring of rank 1, and let $k$ be its residue field $\mathbb{O}/J(\mathbb{O})$ such that $k$ is an algebraically closed field of characteristic 3. Let $G$ be an arbitrary finite group with an elementary abelian Sylow 3-subgroup $P$ of order 9, and let $H = N_G(P)$, the normalizer of $P$ in $G$. Then, the principal block algebras $B_0(kG)$ and $B_0(kH)$ of $kG$ and $kH$, respectively, are splendidly equivalent, and therefore so are $B_0(\mathbb{O}G)$ and $B_0(\mathbb{O}H)$. 
(0.3) Remark. The main result (0.2) depends on the classification of finite simple groups.

Throughout this paper we use the following notation and terminology. Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). In this paper a \textit{module} always means a finitely generated right module, unless stated otherwise. Let \( G \) and \( H \) be finite groups. Even in the case where \( M \) is a \((kG, kH)\)-bimodule, we say that \( M \) is a (right) \( k\{G \times H\} \)-module via an action \( m \cdot (g, h) = g^{-1} mh \) for \( m \in M, g \in G, \) and \( h \in H \). For the same \( M, \) let \( M' = \text{Hom}_k(M, k) \) be the \( k \)-dual of \( M \), so that \( M' \) is a \((kH, kG)\)-bimodule. We say that \( G \) and \( H \) have the same \textit{\( p \)-local structure} in the sense of Broué (see [4, 6.5. Theorem], [27, p. 342], and [21, p. 374]). We write \( \delta \) (\( G \)) for \( \{(g, g) \in G \times G \mid g \in G\} \). Let \( \tilde{G} = G \), and let \( N \) be a right \( kG \)-module. Then, we write \( N\tilde{g} \) or \( \tilde{N}g \) for the \( \tilde{g} \)-conjugate of \( N \) where \( \tilde{g} \in \tilde{G} \), namely, \( N\tilde{g} = N \) as \( k \)-vector spaces and the action of \( G \) on \( N\tilde{g} \) is defined by \( n \cdot g = n \cdot \tilde{g}^{-1} \tilde{g}g \) for \( n \in N \) and \( g \in G \). For an abelian Sylow \( p \)-subgroup \( P \) of \( G \), let \( E(G) = N_G(P)/C_G(P) \), so that this is the inertial quotient of the principal \( p \)-block of \( G \) (note that \( E(G) \) is different from the largest semi-simple normal subgroup of \( G \)). Let \( n \) be a positive integer. Then, \( C_n \), \( \Sigma_n \), and \( A_n \), respectively, denote the cyclic group of order \( n \), the symmetric group on \( n \) letters, and the alternating group on \( n \) letters. We write \( n! = p^m \) for a nonnegative integer \( m \) if \( p^m \mid n \) and \( p^{m+1} \nmid n \). We denote by \( D_n \), \( Q_n \), and \( S_n \), respectively, the dihedral, the quaternion, and the semi-dihedral groups of order \( n \). For two groups \( G \) and \( H \) we denote by \( G \times H \) a semi-direct product of \( G \) by \( H \), namely \( G \rtimes G \times H \). We denote by \( kG \) the \( kG \)-module of \( k \)-dimension \( 1 \). We write \( B_0(kG) \) for the principal block algebra of the group algebra \( kG \). For a ring \( R \), we denote by \( 1_R \), \( R^* \), \( Z(R) \), and \( J(R) \), respectively, the unit element, the set of all units, the center, and the Jacobson radical of \( R \). We say that two block algebras \( A \) and \( B \) of \( kG \) and \( kH \), respectively, are \textit{Puig (splendidly Morita) equivalent}, following Broué [4, p. 22] (see also [21, 1.6. Theorem]), and that \( A \) and \( B \) are \textit{splendidly equivalent}, following Rickard [27, p. 337].

Let \( P \) be a Sylow \( p \)-subgroup of \( G \), and let \( H \) be a subgroup of \( G \) with \( H \supseteq N_G(P) \). Then, we can define the Green correspondences \( f_\{G \times H\}^{G \times G} : G \times G \to G \times H \) and \( f_\{H \times H\}^{H \times G} : H \times H \to H \times G \) with respect to a \( p \)-subgroup \( \delta(P) \) of \( G \times G \). Then, let \( M(1) \) and \( N(1) \) be \( M(1) = f_\{G \times H\}^{G \times G}(B_0(kG)) \) and \( N(1) = n \cdot H \)(\( G \)) \( = g_\{H \times G\}^{H \times G}(B_0(kH)) \). Note that \( M(1) \) and \( N(1) \) are \( p \)-permutation (trivial source) \( k\{G \times H\} \)- and \( k\{H \times G\} \)-modules with vertex \( \delta(P) \), respectively. We use the notations \( f_\{G \times H\}^{G \times G} \) and \( g_\{H \times G\}^{G \times G} \) in similar ways. For any subgroup \( H \) of \( G \) we denote by \( \text{Alperin–Scott module of } H \) in \( G \) (see [30, Sect. 3, p. 1549] and [22, Chap. 4, Theorem 8.4 and p. 297]), and we denote by \( P_H(M) \) the (relatively) \( H \)-projective cover of a right \( kG \)-module \( M \) (see
Consider $E \cong G/G$ (see Section 9.3.1]). We denote by $A$ (see Sect. 1, p. 1538]). For $kG$-modules $X$ and $Y$ we write $[X, Y]^G$ for $\dim_k [\text{Hom}_k(X, Y)]$. Let $A$ be a finite-dimensional $k$-algebra and let $M$ be an $A$-module. Then, we denote by $P(M)$ the projective cover of $M$, by $\text{Soc}(M)$ the socle of $M$, by $\Omega(M)$ the kernel of the projective cover $P(M) \to M$ of $M$ (namely $\Omega$ is the Heller functor (operator)), by $\text{IBr}(A)$ the set of all non-isomorphic simple $A$-modules, by $\text{mod}-A$ the category of all finitely generated right $A$-modules, and by $\text{mod}$-$A$ the stable module category of finitely generated right $A$-modules (see [14, p. 193, Definition 9.3.1]). We denote by $c_A(S, T)$ or just by $c(S, T)$ the Cartan invariant with respect to simple $A$-modules $S$ and $T$. We write $Y/X$ when $Y$ is a direct summand of $X$. For other notation and terminology, see the book of Nagao and Tsushima [22].

1. ELEMENTARY GROUP THEORY

In this section we give elementary facts on group theory. The proofs are easy. But, they are helpful for obtaining our main result.

(1.1) Notation. We fix a finite group $G$ and assume that $G$ has an abelian Sylow $p$-subgroup $P$.

(1.2) Lemma. (i) The group $E(G)$ is a $p'$-subgroup of $\text{Aut}(P)$. In particular, if $P = C_3 \times C_3$, then $E(G) \cong SD_{16}$.

In the following assume that $L$ is a subgroup of a finite group $\tilde{G}$ such that $p \nmid |\tilde{G} : L|$, $P \in \text{Syl}_p(L)$, $G \triangleleft \tilde{G}$, and that $\tilde{G}/G$ is a $p'$-group (so that $P \in \text{Syl}_p(\tilde{G})$), and let $G_0 = G \cdot C_{\tilde{G}}(P)$.

(ii) A correspondence $\varphi_L : E(L) \to E(\tilde{G})$ given by $\ell \cdot C_L(P) \mapsto \ell \cdot C_{\tilde{G}}(P)$ for $\ell \in N_L(P)$ is a group-monomorphism, and hence we can consider $E(L) \leq E(\tilde{G})$ via $\varphi_L$. Moreover, $\varphi_G(E(G)) \triangleleft E(\tilde{G})$, so that we may consider $E(G) \triangleleft E(\tilde{G})$ via $\varphi_G$.

(iii) $G_0 \triangleleft \tilde{G}$ and $E(G) \cong E(G_0)$.

(iv) $\tilde{G}/G \cong N_{\tilde{G}}(P)/N_G(P)$.

(v) If, moreover, $C_\tilde{G}(P) \subseteq G$, then we have that $C_{\tilde{G}}(P) = C_G(P)$ and $\tilde{G}/G \cong N_{\tilde{G}}(P)/N_G(P) \cong E(\tilde{G})/E(G)$.

(vi) If $C_{\tilde{G}}(P) \subseteq G$ and $|E(\tilde{G})| = |E(G)|$, then we have $\tilde{G} = G$.

(vii) $E(G) = E(G_0)$ and $E(\tilde{G})/E(G_0) \cong \tilde{G}/G_0$. In particular, if $E(\tilde{G}) \cong E(G_0)$, then $\tilde{G} = G_0$.

(viii) If $\tilde{G}/G \cong E(\tilde{G})/E(G) \cong C_q$ for a prime $q$ with $q \neq p$, then $C_{\tilde{G}}(P) \subseteq G$, namely, $G_0 = G$. 

(i) If $\tilde{G}/G \cong C_q$ for a prime $q$ with $q \neq p$ and $E(\tilde{G}) \cong E(G)$, then $\tilde{G} = G_0$.

**Proof.** (i) Clear since $\text{Aut}(C_3 \times C_3) \cong \text{GL}_2(3)$.

(ii) Clearly, $\varphi_L$ is well defined and is a group-homomorphism. If $\varphi_L(\ell \cdot C_L(P)) = 1$ for $\ell \in N_L(P)$, then $\ell \in C_{\tilde{G}}(P)$, so that $\ell \in C_L(P)$. Hence, $\varphi_L$ is a monomorphism.

Take any $x \in E(G)$ and $\tilde{x} \in E(\tilde{G})$. Write $x = g \cdot C_G(P)$ and $\tilde{x} = \tilde{g} \cdot C_{\tilde{G}}(P)$ for some $g \in N_G(P)$ and $\tilde{g} \in N_{\tilde{G}}(P)$. Then, $\tilde{x}^{-1} \cdot \varphi_L(x) \cdot \tilde{x} = \tilde{g}^{-1} g \cdot C_G(P) \in [N_G(P) \cdot C_G(P)]/C_G(P) = \text{Im}(\varphi_L)$ since $G \triangleleft \tilde{G}$. Hence, $\varphi_L(E(G)) \triangleleft E(\tilde{G})$.

(iii) By the Frattini argument, $\tilde{G} = G \cdot N_{\tilde{G}}(P)$. Then, $G_0$ is a subgroup of $\tilde{G}$ since $G \triangleleft \tilde{G}$. Take any $\tilde{g} \in \tilde{G}$ and $g_0 \in G_0$. We can write $\tilde{g} = gn$ and $g_0 = gc$ for some $g, g' \in G, n \in N_G(P)$, and $c \in C_G(P)$. Now $g_0 \tilde{g}^{-1} = \tilde{g}' \tilde{g}^{-1} = \tilde{g}' \cdot g^{-1} \cdot c \cdot g^{-1}$. Moreover, $\tilde{g} \tilde{g}^{-1} = gcn^{-1} g^{-1} \in G \cdot C_{\tilde{G}}(P) \cdot G = G \cdot C_G(P) = G_0$ since $C_G(P) \triangleleft N_{\tilde{G}}(P)$ and $G \triangleleft \tilde{G}$. Hence, $\tilde{g} \tilde{g}^{-1} \in G \cdot G \cdot C_G(P) = G_0$, which shows $G_0 \triangleleft \tilde{G}$.

Now, let $\psi: E(G) \to E(G_n)$ as in (ii). Then, by (ii), $\psi(E(G)) \triangleleft E(G_n)$. Take any $x_0 \in E(G_n)$. Then, we can write $x_0 = g_0 \cdot C_{G_n}(P)$ for $g_0 \in N_{G_n}(P)$. We can write also that $g_0 = gc$ for $g \in G$ and $c \in C_G(P)$, so that $c = g^{-1} g_0 \in G_0$. Hence, $c \in C_{G_0}(P)$. Thus, $x_0 = gc \cdot C_{G_0}(P) = g \cdot C_{G_n}(P)$. Furthermore, $g \in N_{G_n}(P) \cdot C_{G_n}(P) \leq N_{G_n}(P)$, so that $g \in N_{G_n}(P) \cap G = G_n(P)$. Therefore, $x_0 = g \cdot C_{G_n}(P) = \psi(g \cdot C_G(P))$. Hence, $\psi$ is onto. Then, by (ii), $\psi$ is an isomorphism.

(iv) By the Frattini argument, $\tilde{G}/G = G \cdot N_{\tilde{G}}(P)/G \cong N_{\tilde{G}}(P)/[G \cap N_{\tilde{G}}(P)] = N_{G_n}(P)/N_{G_n}(P)$.

(v) By the assumption, $C_{\tilde{G}}(P) = C_G(P)$. Hence, by (iv),

$$\tilde{G}/G \cong N_{\tilde{G}}(P)/N_{G_n}(P) \cong [N_{\tilde{G}}(P)/C_{\tilde{G}}(P)]/[N_{G_n}(P)/C_G(P)] \cong E(\tilde{G})/E(G).$$

(vi) Easy by (v).

(vii) This follows from (iii), (v), and (vi).

(viii) Assume $C_{\tilde{G}}(P) \not\subseteq G$. Then, $G \leq G \cdot C_{\tilde{G}}(P) = G_0$, so that $G_0 = \tilde{G}$ since $|\tilde{G} : G|$ is a prime. Hence, $E(\tilde{G}) = E(G_0) = E(G)$ by (iii), a contradiction.

(ix) If $G_0 = G$, then $C_{\tilde{G}}(P) \not\subseteq G$, so that $\tilde{G}/G \cong E(\tilde{G})/E(G)$ from (v), a contradiction. Hence $G \not\subseteq G_0$, which implies $G_0 = \tilde{G}$ since $|\tilde{G} : G_0|$ is a prime. ■

(1.3) **Lemma (Gaschütz).** Assume that $G$ has a Sylow $p$-subgroup $P$ such that $P \cong C_p \times C_p$ and that $C_p \rightarrow Z(G)$. Then, there is a normal subgroup $L$ of $G$ such that $G = C_p \times L$. 


Proof. By Sylow’s theorem we can assume \( P = P_1 \times P_2 \) such that \( P_1 \cong P_2 \cong C_p \) and \( P_1 \subseteq Z(G) \). Hence \( P_1 \triangleleft G \) and \( P_1 \) is abelian. Namely, \( P \) splits on \( P \cap P_1 \).

Take any prime \( q \neq p \), and any \( Q \in \text{Syl}_p(G) \). Then, \( Q \cap P_1 = 1 \), which means that \( Q \) splits on \( Q \cap P_1 \). Thus, it follows from a theorem of Gaschütz [29, Chap. 2, Theorem 8.6] that \( G \) splits on \( P_1 \). That is, \( G = P_1 \rtimes L \) for a subgroup \( L \) of \( G \). Since \( P_1 \subseteq Z(G) \), \( P_1 \rtimes L = P_1 \times L \).

(1.4) Lemma. Assume that \( G \triangleleft \tilde{G} \), \( \tilde{G}/G \) is a \( p' \)-group, \( G \) is a non-abelian simple group, and \( O_p(\tilde{G}) = 1 \). Then, \( G \triangleleft \tilde{G} \triangleleft \text{Aut}(G) = G\cdot\text{Out}(G) \).

Proof. Since \( G \) is non-abelian simple, \( C_{\tilde{G}}(G) \cap G = Z(G) = 1 \). Hence, there is a direct product \( C_{\tilde{G}}(G) \times G \triangleleft \tilde{G} \). Since \( \tilde{G}/G \) is a \( p' \)-group, \( p \mid |C_{\tilde{G}}(G)| \). Hence, \( C_{\tilde{G}}(G) = 1 \) since \( O_p(\tilde{G}) = 1 \). Therefore, \( \tilde{G} = N_{\tilde{G}}(G)/C_{\tilde{G}}(G) \triangleleft \text{Aut}(G) \).

(1.5) Lemma. Let \( G, \tilde{H}, \) and \( H \) be subgroups of \( \tilde{G} \) such that \( \tilde{G} \cap \tilde{H} \supseteq H \), and let \( M \) be a right \( kH \)-module, where

\[
\begin{align*}
\tilde{G} &\geq \tilde{H} \\
G &\geq H.
\end{align*}
\]

Let \( \tilde{h} \in \tilde{H} \). Then, a map

\[
M \otimes_{kH} kG \cdot \tilde{h} \longrightarrow (M \otimes_{kH} kH \cdot \tilde{h}) \otimes_{k[H^{-1}\tilde{H}]} k[H^{-1}\tilde{G}\tilde{h}]
\]

is an isomorphism of right \( k[H^{-1}\tilde{G}\tilde{h}] \)-modules.

(1.6) Lemma. Let \( P \in \text{Syl}_p(G) \).

(i) \( B_0(kG)_{G\times G} = P_{\delta(P)}(kG_{G\times G}) = \text{Scott}_{G\times G}(\delta(P)) \).

(ii) Let \( H \) be a subgroup of \( G \) with \( H \supseteq N_G(P) \). Then, it holds that \( M(1) = P_{\delta(P)}(kG_{H}) = \text{Scott}_{G\times H}(\delta(P)) \), and that there is a primitive idempotent \( e_1 \) of \( (kG)^H \) such that \( M(1) = kGe_1 \), where \( (kG)^H = C_{kG}(H) \).

Proof. (i) Let \( Y \) be a subgroup of a finite group \( X \). Let \( X = \cup_{i=1}^n Yx_i \) be a coset decomposition of \( Y \) in \( X \), where \( n = |X : Y| \), \( x_i \in X \), and \( x_1 = 1 \). A map \( \varphi: k_Y \uparrow^X = \oplus_{i=1}^n (k_Y \otimes_{kY} x_i) \rightarrow k_X \) given by \( \sum_{i=1}^n \alpha_i \otimes_{kY} x_i \mapsto \sum_{i=1}^n \alpha_i \), \( \alpha_i \in k \), is a \( Y \)-split epimorphism of \( kX \)-modules since a map \( \psi: k_Y \rightarrow k_Y \uparrow^X \) given by \( \alpha \mapsto \alpha \otimes_{kY} 1 \) is a \( kY \)-homomorphism which satisfies \( \varphi \circ \psi = id_k \). Now, take \( X = G \times G \) and \( Y = \delta(P) \).

Then, \( k_{\delta(P)} \uparrow^{G \times G} \rightarrow k_{G \times G} \) is a \( \delta(P) \)-split \( k[G \times G] \)-epimorphism. Let
A = Bθ(kG). Then, A is a trivial source k[G × G]-module with vertex \( \delta(P) \) by a theorem of Green [22, Chap. 5, Theorem 10.8]. Hence, \( A|_{k\langle \mathcal{P} \rangle} \uparrow_{G × G} \). Since \( [k_{G×G}, k_{\mathcal{P} G}] \uparrow_{G×G} = [k_{\mathcal{P} G}, k_{G×G}] \uparrow_{G×G} = 1 \) and since there are a monomorphism \( k_{G×G} \rightarrow A \) and an epimorphism \( \theta: A \rightarrow k_{G×G} \) of \( k[G \times G] \)-modules, \( \theta \) is a \( \delta(P) \)-split \( k[G × G] \)-epimorphism. Hence, \( P_{\mathcal{P} G} \) is a trivial source \( k[G×G] \)-module with vertex \( \theta \) by a standard argument. Since \( A \) is indecomposable as a \( k[G×G] \)-module, we have \( A = P_{\mathcal{P} G} \). The second equality is given by [30, Proposition 3.1].

(ii) By the definition, \( M(1) \) is a unique indecomposable direct summand of \( B_θ(kG) \downarrow_{G×H}^{G×G} \) as a right \( k[G × H] \)-module with a vertex \( \delta(P) \).

On the other hand, by the definition of Alperin–Scott modules, we know that \( \text{Scott}_{G×H}(\delta(P)) \big| [\text{Scott}_{G×G}(\delta(P))] \downarrow_{G×H} \). Therefore, by (i), we get \( M(1) = \text{Scott}_{G×H}(\delta(P)) \) by looking at the vertices. The rest is easy. ∎

2. FINITE GROUPS WITH AN ELEMENTARY ABELIAN SYLOW 3- SUBGROUP OF ORDER 9

In this section we give a list of finite groups \( G \) such that \( G \) has an elementary abelian Sylow 3-subgroup of order 9, \( O_3^s(G) = 1 \) and \( O^3(G) = G \), by making use of the classification of finite simple groups. Namely, we need to use the classification of finite simple groups to prove (2.1), (2.2), and (2.4). The following two propositions, (2.1) and (2.2), were provided by S. Yoshiara.

(2.1) Proposition. Let \( G \) be a finite group with an elementary abelian Sylow 3-subgroup of order 9 such that \( O_3^s(G) = 1 \) and \( O^3(G) = G \). Then, \( G \) is one of (i) or (ii).

(i) \( G = X \times Y \) for finite simple groups \( X \) and \( Y \) such that both of them have cyclic Sylow 3-subgroups of order 3.

(ii) \( G \) is a non-abelian finite simple group with an elementary abelian Sylow 3-subgroup of order 9.

Proof. See [17, (1.1) Proposition]. ∎

(2.2) Proposition. If \( G \) is a non-abelian finite simple group with an elementary abelian Sylow 3-subgroup of order 9, then \( G \) is one of the following nine types:

(i) \( A_6, A_7, A_8, M_{11}, M_{22}, M_{23}, HS \);
(ii) \( \text{PSL}_3(q) \) for a power \( q \) of a prime with \( q \equiv 4 \) or \( 7 \) (mod 9);
(iii) \( \text{PSU}_3(q^2) \) for a power \( q \) of a prime with \( 2 < q \equiv 2 \) or \( 5 \) (mod 9);
(iv) \( \text{PSp}_4(q) \) for a power \( q \) of a prime with \( q \equiv 4 \) or \( 7 \) (mod 9);
Then we get the following:

(v) \(\text{PSp}_4(q)\) for a power \(q\) of a prime with \(2 < q \equiv 2 \text{ or } 5 \pmod{9}\);

(vi) \(\text{PSL}_4(q)\) for a power \(q\) of a prime with \(2 < q \equiv 2 \text{ or } 5 \pmod{9}\);

(vii) \(\text{PSU}_4(q^2)\) for a power \(q\) of a prime with \(q \equiv 4 \text{ or } 7 \pmod{9}\);

(viii) \(\text{PSL}_5(q)\) for a power \(q\) of a prime with \(q \equiv 2 \text{ or } 5 \pmod{9}\);

(ix) \(\text{PSU}_5(q^2)\) for a power \(q\) of a prime with \(q \equiv 4 \text{ or } 7 \pmod{9}\).

\textbf{Proof.} See, e.g., [17, (1.2) Proposition].

(2.3) \textbf{Remark.} Note that in (iii) of \((2.2)\) \(\text{PSU}_3(2^2) \cong (C_3 \times C_3) \times Q_8\), in (v) of \((2.2)\) \(\text{PSp}_4(2) = \text{Sp}_4(2) \cong \Sigma_{16}\), and in (vi) of \((2.2)\) \(\text{PSL}_4(2) = \text{SL}_4(2) \cong A_8\) from [12, II 10.14 Satz], [12, II 9.21 Satz, II 9.22 Hauptsatz], and [12, II 6.14 Satz (5)], respectively.

(2.4) \textbf{PROPOSITION.} Assume that \(G\) is a finite group with elementary abelian Sylow 3-subgroup \(P\) of order 9 such that \(O_3(G) = 1\) and \(O^3(G) = G\). Then we get the following:

(i) If \(G = C_3 \times C_3\), then \(E(G) = 1\).

(ii) If \(G = C_3 \times X\) for a non-abelian simple group \(X\) with \(|X|_3 = 3\), then \(E(G) = C_2\).

(iii) If \(G = X \times Y\) for non-abelian simple groups \(X\) and \(Y\) with \(|X|_3 = |Y|_3 = 3\), then \(E(G) = C_2 \times C_2\).

(iv) If \(G = A_6\) or \(A_7\), then \(E(G) = C_4\).

(v) If \(G = A_8\), then \(E(G) = D_8\).

(vi) If \(G = M_{11}, M_{23}, \text{ or } HS\), then \(E(G) = SD_{16}\).

(vii) If \(G = M_{22}(\cong \text{PSL}_3(4))\) or \(M_{24}\), then \(E(G) = Q_8\).

(viii) If \(G = \text{PSL}_3(q)\) such that \(q \equiv 4 \text{ or } 7 \pmod{9}\), then \(E(G) = Q_8\).

(ix) If \(G = \text{PSU}_3(q^2)\) such that \(2 < q \equiv 2 \text{ or } 5 \pmod{9}\), then \(E(G) = Q_8\).

(x) If \(G = \text{PSp}_4(q)\) such that \(q \equiv 4 \text{ or } 7 \pmod{9}\), then \(E(G) = D_8\).

(xi) If \(G = \text{PSp}_4(q)\) such that \(2 < q \equiv 2 \text{ or } 5 \pmod{9}\), then \(E(G) = D_8\).

(xii) If \(G = \text{PSL}_4(q)\) such that \(q \equiv 2 \text{ or } 5 \pmod{9}\), then \(E(G) = D_8\).

(xiii) If \(G = \text{PSL}_5(q)\) such that \(q \equiv 2 \text{ or } 5 \pmod{9}\), then \(E(G) = D_8\).

(xiv) If \(G = \text{PSU}_4(q)\) or \(\text{PSU}_5(q)\) such that \(q \equiv 4 \text{ or } 7 \pmod{9}\), then \(E(G) = D_8\).
3. LEMMAS

In this section we give several lemmas which are useful for obtaining our main result. Throughout this section we assume that $k$ is an algebraically closed field of characteristic $p > 0$, and that $G$, $\tilde{G}$, $G'$, $\tilde{G}'$, $X$, $\tilde{X}$, $H$, $H'$, $\tilde{H}$, and $\tilde{H}'$ are all finite groups.

(3.1) Lemma (Alperin–Dade). Suppose that $G \triangleleft \tilde{G}$, $\tilde{G}/G$ is a $p'$-group, $P \in \text{Syl}_p(G)$, and $\tilde{G} = G \cdot C_G(P)$. Let $\tilde{e}$ and $e$ be the block idempotents of $B_0(k\tilde{G})$ and $B_0(kG)$, respectively. Then, $B_0(k\tilde{G})$ and $B_0(kG)$ are category-isomorphic, namely, a map $B_0(kG) \rightarrow B_0(k\tilde{G})$ defined by $a \mapsto a\tilde{e}$ for $a \in B_0(kG)$ is a $k$-algebra-isomorphism, so that $e\tilde{e} = \tilde{e}e = \tilde{e}$. Moreover, a right $k[\tilde{G} \times G]$-module $B_0(k\tilde{G}) = \tilde{e}k\tilde{G} = \tilde{e}k\tilde{G}e$ gives a Puig equivalence between $B_0(k\tilde{G})$ and $B_0(kG)$.

Proof. This is proved in [1, Theorems 1 and 2] and [9, Theorem]. Note that $(k\tilde{G}\tilde{e})_{G \times G} \cong kGe$ as right $k[\tilde{G} \times G]$-modules as in [11, Sect. 4] (see also [3, Theorems 0.1 and 0.2] and [21, (5.4)]).

(3.2) Lemma. Suppose that $G \triangleleft \tilde{G}$, $\tilde{G}/G$ is a $p'$-group, and $P \in \text{Syl}_p(G)$ such that $\tilde{G} = G \cdot C_G(P)$. Let $\tilde{H}$ be a subgroup of $\tilde{G}$ with $\tilde{H} \supseteq N_{\tilde{G}}(P)$, and let $H = G \cap \tilde{H}$. Assume that $\tilde{H} = H \cdot C_\tilde{G}(P)$. Assume, moreover, that a Puig equivalence between $B_0(kG)$ and $B_0(k\tilde{H})$ is given by the right $k[\tilde{G} \times H]$-module $M(1)_{(\tilde{G}, H)}$. Then, the right $k[\tilde{G} \times \tilde{H}]$-module $M(1)_{(\tilde{G}, \tilde{H})}$ gives a Puig equivalence between $B_0(k\tilde{G})$ and $B_0(k\tilde{H})$.

Proof. Clearly, $H \triangleleft \tilde{H}$ and $\tilde{G} = G \cdot \tilde{H}$, and

$$\begin{align*}
\tilde{G} = G \cdot C_{\tilde{G}}(P) & = G \cdot \tilde{H} \quad \mapsto \quad H = H \cdot C_{\tilde{H}}(P) \supseteq N_{\tilde{G}}(P) \\
\tilde{G} & \quad \mapsto \quad H = G \cap \tilde{H} \supseteq N_{\tilde{G}}(P).
\end{align*}$$

Let $\tilde{A} = B_0(k\tilde{G})$, $A = B_0(kG)$, $\tilde{B} = B_0(k\tilde{H})$, and $B = B_0(kH)$, and let $\tilde{e}$ and $\tilde{f}$ be the block idempotents of $\tilde{A}$ and $\tilde{B}$, respectively. By (3.1), $\tilde{A}^A_A$ realizes a Puig equivalence between $\tilde{A}$ and $A$, and so does $\tilde{B}^B_B$ between $B$ and $\tilde{B}$. Let $M(1) = M(1)_{(\tilde{G}, H)}$. By the hypothesis, $M(1)$ realizes a Puig equivalence between $A$ and $B$. Let $\tilde{M} = \tilde{A} \otimes_A M(1) \otimes_B \tilde{B}$. Then, $\tilde{M}$ realizes a Puig equivalence between $\tilde{A}$ and $\tilde{B}$. Since $\tilde{M}^G_{\tilde{G} \times H} = M(1)$ by (3.1), $\tilde{M}$ is an indecomposable $(\tilde{A}, \tilde{B})$-bimodule. Since $\tilde{M} = \tilde{e}(M(1)^{\tilde{G} \times \tilde{H}})\tilde{f}$, (1.6)(ii) implies that $[M(1), k_{\tilde{G} \times H}]^{G \times H} = 1$, so that $[\tilde{M}, k_{\tilde{G} \times \tilde{H}}]^{G \times \tilde{H}} \neq 0$. Since $[k_{\delta(P)}, k_{\tilde{G} \times H}]^{\tilde{G} \times \tilde{H}} = [\text{Scott}_{\tilde{G} \times \tilde{H}}(\delta(P)), k_{\tilde{G} \times \tilde{H}}]^{\tilde{G} \times \tilde{H}} = 1$, it follows from a theorem of Krull–Schmidt that $\tilde{M} \cong \text{Scott}_{\tilde{G} \times \tilde{H}}(\delta(P))$. 


(Note that we get this also by [28, Lemma 2].) Hence we get the assertion from (1.6(ii)).

(3.3) Lemma (Alperin–Dade–Marcus). Assume as in (3.1). Moreover, suppose that $H \triangleleft \tilde{H}$, $P \in \text{Syl}_p(G) \cap \text{Syl}_p(H)$, $\tilde{H} = H \cdot C_{\tilde{H}}(P)$, and $G/G \cong \tilde{H}/H$. If $B_0(kG)$ and $B_0(kH)$ are Morita (resp. Puig, derived and splendidly) equivalent, then $B_0(k\tilde{G})$ and $B_0(k\tilde{H})$ are also Morita (resp. Puig, derived and splendidly) equivalent.

Proof. Easy by (3.1) since a Puig equivalence is a Morita, derived, and splendidly equivalent (see [21, (5.4)]).

(3.4) Lemma (Dade). Let $G \triangleleft \tilde{G}$ such that $\tilde{G}/G$ is a $p'$-group, and let $P \in \text{Syl}_p(G)$. Suppose, furthermore, that $\tilde{G}$ has a subgroup $\tilde{H}$ such that $\tilde{H} \supseteq N_{\tilde{G}}(P)$, and let $H = G \cap \tilde{H}$, so that $\tilde{G} = G\tilde{H}$ and $\tilde{G}/G \cong \tilde{H}/H$, and

$$\tilde{G} = G\tilde{H} \to \tilde{H} \supseteq N_{\tilde{G}}(P)$$

$$\to \quad H = G \cap \tilde{H} \supseteq N_G(P).$$

Let $f \downarrow_{H \times H}^{G \times H} : G \times H \to H \times H$ be the Green correspondence with respect to $\delta(P)$, and let $A = B_0(kG)$, $B = B_0(kH)$, and $M(1) = M(1)_{(G,H)}$.

(i) It holds that $f \downarrow_{H \times H}^{G \times H}(M(1)) = B$ as right $k[H \times H]$-modules. Moreover, $B$ is $\delta(\tilde{H})$-invariant, namely $B \cdot (\tilde{h}, \tilde{h}) = B$ as right $k[H \times H]$-modules for any $\tilde{h} \in \tilde{H}$, and $B$ is extendible to a right $k[(H \times H) \cdot \delta(\tilde{H})]$-module.

(ii) It holds that $M(1)$ is $\delta(\tilde{H})$-invariant as a right $k[G \times H]$-module, and that $M(1)$ is extendible to a right $k[(G \times H) \cdot \delta(\tilde{H})]$-module.

Proof. (i) The first assertion is easy by the definition of Green correspondences and the theorem of Krull–Schmidt. We easily know that $B$ is $\delta(\tilde{H})$-invariant since $B$ is the principal block. Now, let $1_b$ be the block idempotent of $B$ in $kH$, and let $h \in H$ and $\tilde{h} \in \tilde{H}$. Then,

$$h1_b \cdot (\tilde{h}, \tilde{h}) = \tilde{h}^{-1}h1_b \tilde{h} = \tilde{h}^{-1}h\tilde{h} \cdot \tilde{h}^{-1}1_b \tilde{h} = \tilde{h}^{-1}h\tilde{h} \cdot 1_b \in kH1_b,$$

since $\tilde{h}^{-1}1_b \tilde{h} = 1_b$ and $H \triangleleft \tilde{H}$. This means that $B$ extends to a $k[(H \times H) \cdot \delta(\tilde{H})]$-module.

(ii) The first assertion is obtained as in (i). Then, $M(1)$ extends to a $k[(G \times H) \cdot \delta(\tilde{H})]$-module by use of (i) and a result of Dade [10, (6.5)Corollary, p. 94].

(3.5) Lemma. Let $\tilde{G}$, $G$, $\tilde{H}$, $H$, $P$, and $M(1)$ be the same as in (3.4). Assume, moreover, that $C_{\tilde{G}}(P) \subseteq G$, and that the right $k[G \times H]$-module $M(1)$ gives a Puig equivalence between $B_0(kG)$ and $B_0(kH)$. Then, $B_0(k\tilde{G})$ and $B_0(k\tilde{H})$ are Puig equivalent.
Broué’s Conjecture for Principal 3-blocks

Proof. Let \( \tilde{e}, e, \tilde{f}, \) and \( f \) be the block idempotents of \( B_0(k\tilde{G}) \), \( B_0(kG) \), \( B_0(kH) \), and \( B_0(kH) \), respectively. Since \( C_\tilde{G}(P) \subseteq \tilde{G} \), it follows from a result of Fong [22, Chap. 5, Theorem 5.16(ii), Lemma 5.14, and Theorem 5.13(ii)] and Brauer’s third main theorem [22, Chap. 5, Theorem 6.1] that \( e = \tilde{e} \) and \( f = \tilde{f} \). Now, by (3.4)(ii), the right \( k[\tilde{G} \times \tilde{H}] \)-module \( M(1) \) is extendible to a right \( k[(\tilde{G} \times \tilde{H}) \cdot \delta(\tilde{H})] \)-module. Hence, we get the assertion from [21, 3.9.Corollary (a)].

\[ (3.6) \text{Lemma.} \quad \text{Let} \ G \triangleleft \tilde{G}, \ G' \triangleleft \tilde{G}', \ P \in \text{Syl}_p(G), \ H = N_G(P), \ \tilde{H} = N_{\tilde{G}}(P), \ H' = N_{\tilde{G}'}(P), \text{and} \ \tilde{H}' = N_{\tilde{G}'}(P). \text{Assume that there are group isomorphisms} \ \tilde{\theta}: \tilde{H}/O_p(\tilde{H}) \to \tilde{H}/O_p(\tilde{H}') \text{and} \ \theta: H/O_p(H) \to H'/O_p(H') \text{(so that we can identify} \ B_0(k\tilde{H}) \text{and} \ B_0(k\tilde{H}'), \text{and} \ B_0(kH) \text{and} \ B_0(kH'), \text{via} \ \tilde{\theta} \text{and} \ \theta, \text{respectively), and} \ G/\tilde{G} \equiv H/H' \equiv H'/H' \equiv \tilde{G}'/G', \text{and these are} \ p \text{-groups. Let} \ M(1) = M(1)_{(H, H')} \text{and} \ N'(1) = N(1)_{(H', G')} = f_{\tilde{H}/O_p(\tilde{H})} \cdot \tilde{G}' \Rightarrow \tilde{G}'/O_p(\tilde{H}') \Rightarrow \tilde{H}/O_p(\tilde{H}'), \text{where} \ f_{\tilde{H}/O_p(\tilde{H})} \equiv \ g_{\tilde{H}/O_p(\tilde{H})} \cdot \tilde{G}' \Rightarrow \tilde{G}'/O_p(\tilde{H}') \Rightarrow \tilde{H}/O_p(\tilde{H}'), \text{and} \ G' \Rightarrow H' \Rightarrow G'. \text{In this sense, let} \ L(1) = g_{\tilde{H}/O_p(\tilde{H})} \cdot \tilde{G}' \Rightarrow \tilde{G}'/O_p(\tilde{H}') \Rightarrow \tilde{H}/O_p(\tilde{H}'), \text{where} \ g_{\tilde{H}/O_p(\tilde{H})} \equiv \ g_{\tilde{H}/O_p(\tilde{H})} \cdot \tilde{G}' \Rightarrow \tilde{G}'/O_p(\tilde{H}') \Rightarrow \tilde{H}/O_p(\tilde{H}'). \text{Let} \ \delta = \{ (\tilde{h}, \tilde{h}') \in \tilde{H} \times \tilde{H}' \mid \tilde{\theta}(\tilde{h} \cdot O_p(\tilde{H})) = \tilde{h}' \cdot O_p(\tilde{H}') \}. \text{Then, we get the following.} \]

(i) \( B_0(kH) \) is \( \delta \)-invariant as a \( k[H \times H'] \)-module, and, moreover, \( B_0(kH) \) is extendible to a \( k[(H \times H') \cdot \delta] \)-module.

(ii) \( L(1) \) is \( \delta \)-invariant as a \( k[G \times G'] \)-module, and, moreover, \( L(1) \) is extendible to a \( k[(G \times G') \cdot \delta] \)-module.

(iii) Assume, furthermore, that \( C_{\tilde{G}}(P) \subseteq \tilde{G}, \ C_{\tilde{G}'}(P) \subseteq \tilde{G}' \) and that the \( k[\tilde{G} \times \tilde{G}'] \)-module \( M(1) \otimes_{B_0(k\tilde{H})} N'(1) \) has a unique non-projective indecomposable direct summand \( \mathcal{M} \) such that \( \mathcal{M} \) realizes a Puig equivalence between \( B_0(kG) \) and \( B_0(kG') \). Then it holds that \( L(1) \cong \mathcal{M} \) as \( k[G \times G'] \)-modules and that \( B_0(k\tilde{G}) \) and \( B_0(k\tilde{G}') \) are Puig equivalent.
Proof. (i) This follows from (3.4)(i) and the definition of \( \delta \).

(ii) By looking at vertices of indecomposable direct summands of \( B_0(kH)_H \otimes_G^H \), and \( B_0(kH) \otimes_{G \times G}^H \), for any \((\hat{h}, \hat{h}') \in \delta\), we know that \( L(1) \) is \( \delta \)-invariant. Now, just as in the proof of (3.4)(ii), it follows that \( L(1) \) extends to a \([G \times G] \cdot \delta \)-module from (i) and a result of Dade [10, (6.5)Corollary, p. 94].

(iii) By considering vertices, we know \( \mathcal{M} \cong L(1) \) as \( kG \times G \)-modules. Just as in the proof of (3.5), it holds that \( e = \tilde{e} \) and \( e' = \tilde{e}' \), where \( e, \tilde{e}, e', \) and \( \tilde{e}' \) are, respectively, block idempotents of \( B_0(kG) \), \( B_0(kG \times G) \), \( B_0(kG') \), \( B_0(kG'') \). Thus, (ii) and [21, 3.9].Corollary (a)] imply that \( B_0(kG) \) and \( B_0(kG') \) are Puig equivalent.  

(3.7) Lemma. Let \( k \) be an algebraically closed field of characteristic 3. Let \( G \) be an arbitrary finite group with a cyclic Sylow 3-subgroup \( P \) of order 3, and let \( H = N_G(P) \). Then, the Green correspondence with respect to \( P \) gives a Puig equivalence between \( B_0(kG) \) and \( B_0(kH) \); that is, \( M(1)_{(G,H)} \) gives a Puig equivalence.

Proof. The proof is easy (see [23, Remark 1.6] and [16, (5.7) Lemma]).

(3.8) Lemma. Let \( k \) be an algebraically closed field of characteristic 3, and assume that \( G \) has a Sylow 3-subgroup \( P \) such that \( P \cong C_3 \times C_3 \). Let \( H = N_G(P) \). Then, \( B_0(kG) \) and \( B_0(kH) \) are stably equivalent of Morita type via \( M(1)_{(G,H)} \).

Proof. Let \( Q \) be any subgroup of \( P \) with \( Q \neq 1 \). If \( Q = P \), then \( C_G(Q) = C_H(Q) \).

Assume \( Q \leq P \), so that \( Q \cong C_3 \). Thus, by (1.3), \( C_G(Q) = Q \times L \) for a subgroup \( L \) of \( C_G(Q) \). Hence, \( C_H(Q) = Q \times (L \cap H) \). Clearly, \( B_0(kC_G(Q)) = kQ \otimes_k B_0(kL) \) and \( B_0(kC_H(Q)) = kQ \otimes_k B_0(k[L \cap H]) \). Therefore, since \( C_3 \in \text{Syl}_3(L) \), we get from (3.7) that \( M(1)_{(C_G(Q), C_H(Q))} \) gives a Puig equivalence between \( B_0(k(C_G(Q))) \) and \( B_0(k(C_H(Q))) \). Thus, by a result of Broué [4, 6.3. Theorem] (see also [27, Theorem 4.1]), a \( (B_0(kG), B_0(kH)) \)-bimodule \( M(1)_{(G,H)} \) realizes a stable equivalence of Morita type between \( B_0(kG) \) and \( B_0(kH) \).

4. BROUÉ’S CONJECTURE FOR NON-SIMPLE GROUPS

In this section we give several propositions where Broué’s conjecture is checked for non-simple finite groups which have the elementary abelian Sylow 3-subgroup of order 9.
(4.1) Notation and Assumption. Throughout this section we use the following notation and terminology. Let $k$ be an algebraically closed field of characteristic 3, let $\overline{G}$ be a finite group with an elementary abelian Sylow 3-subgroup $P$ of order 9, and let $G < \overline{G}$ such that $3 | [\overline{G} : G]$, so that $P \in \text{Syl}_3(\overline{G}) \cap \text{Syl}_3(G)$. Furthermore, let $\overline{H}$ be a subgroup of $\overline{G}$ such that $\overline{H} \supseteq N_{\overline{G}}(P)$, and let $H$ be a subgroup of $\overline{H}$ such that $H \supseteq N_G(P)$. We denote by $M_{10}, M_{11}, M_{21}, M_{22}$, and $M_{23}$ the Mathieu groups. For groups $G$, $G'$, $G''$, $H$, $\overline{H}$, and $H'$ we denote by $A$, $A'$, $A''$, $B$, $B'$, and $B''$, respectively, $B_0(kG)$, $B_0(k\overline{G})$, $B_0(kG')$, $B_0(k\overline{G}')$, $B_0(kH)$, and $B_0(k\overline{H})$. We use the notation $A_{6,2}$ for $i = 1, 2, 3$ and $A_{6,2}$ as in [8, p. 4], the notation $L_3(2)_i$ for $i = 1, 2, 3$ as in [8, p. 23], and the notation $M_{22,2}$ as in [8, p. 39].

(4.2) Proposition. $B_0(kM_{22})$ and $B_0(kM_{10})$ are Puig equivalent via a right $k[M_{22} \times M_{10}]$-module $M(1)_{(M_{22}, M_{10})}$.

Proof. Let $G' = M_{22}$, $G = M_{10} = A_{6,2}$, and $P \in \text{Syl}_3(G') \cap \text{Syl}_3(G)$. Let $H = N_G(P)$, so that $H = N_G(P)$ and $H \supseteq P \times Q_8$. Let $M'(1) = M(1)_{(G', H)}$ and $N(1) = N(1)_{(H, G)}$. Then, we can consider the Green correspondences

$$
G' \xrightarrow{f} H \xrightarrow{g} G.
$$

First, we note that the right $k[G' \times H]$-module $M'(1)$ induces a stable equivalence of Morita type between $A'$ and $B$, and that the right $k[H \times G]$-module $N(1)$ induces a stable equivalence of Morita type between $B$ and $A$ by (3.8) (and actually $P$ is a T.I. set in $M_{10}$ by [8, p. 4]).

Now, we can write $\text{IBr}(A') = \{k_{G'}, 55, 49, 49', 231\}$, $\text{IBr}(A) = \{k_G, 1_G, 4, 4', 6\}$, and $\text{IBr}(B) = \{k_H, 1_H, 1_2, 1_3, 2\}$, where the numbers of simple modules mean the $k$-dimensions. We easily know that

$$
\begin{align*}
&f(1_G) = 1_1, \quad f(4) = 1_2, \quad f(4') = 1_3, \quad f(6) = k_H 2 1_1, \\
&1_3 1_2
\end{align*}
$$

These are precisely the Green correspondents of the simple $A'$-modules by [23, Example 4.5]. It follows from [20, Theorem 2.1(i)] that there is a unique non-projective indecomposable right $k[G' \times G]$-module $\mathcal{M}$ such that $\mathcal{M}[M(1)] \otimes_{A'} N(1)$ as right $k[G' \times G]$-modules. Then, the above shows that, for any simple $A'$-module $S'$, $S' \otimes_{A'} \mathcal{M}$ is a simple right $A$-module. Therefore, by [20, Theorem 2.1(iii)] and [21, 1.6. Theorem], we know that $\mathcal{M}$ realizes a Puig equivalence between $A'$ and $A$. [\]

(4.3) Lemma. Assume that $\overline{G} = M_{10} = A_{6,2}$ and $\overline{H} = N_G(P)$. Then, $B_0(\overline{kG})$ and $B_0(k\overline{H})$ are derived (even splendidly) equivalent.
Proof. Let $\tilde{G}' = M_{22}$, so that $\tilde{H} = N_{\tilde{G}}(P) \cong P \times Q_8$ by the proof of (4.2). It follows from results of Okuyama [23, Example 4.5 and 24] that $\tilde{A}'$ and $\tilde{B}$ are splendidly equivalent. Since $\tilde{A}'$ and $\tilde{A}$ are Puig equivalent by (4.2), we get the assertion.

(4.4) Lemma. $B_0(k[M_{22},2])$ and $B_0(k[M_{10},2])$ are Puig equivalent (note $M_{10}, 2 \cong A_6 \cdot 2^2$ by [8, p. 4]).

Proof. Let $G' = M_{22}$, $\tilde{G}' = M_{22}, G = M_{10}$, and $\tilde{G} = M_{10}$, 2. Note that $N_{G'}(P) = N_{G}(P) = P \times Q_8$, $N_{\tilde{G}}(P) = N_{\tilde{G}}(P) = P \times SD_{16}$ (see [8, pp. 4, 39]), and $\begin{array}{ll} \tilde{A}' & \tilde{G}' = M_{22} 2 \supset \tilde{G} = M_{10} 2 \supset N_{\tilde{G}}(P) \supset N_{G}(P) = P \times SD_{16} \\ \triangledown & \triangledown \end{array}$

By (4.2), $A'$ and $A$ are Puig equivalent via a right $k[G' \times G]$-module $M(1)_{G', G}$. Since $E(G') \cong Q_8$ and $E(\tilde{G}') \cong SD_{16}$, we have $C_{G'}(P) \subseteq G'$ by (1.2)(viii). Therefore, (3.5) implies that $\tilde{A}'$ and $\tilde{A}$ are Puig equivalent.

(4.5) Lemma. Let $G = M_{10}, 2 = A_6 \cdot 2^2$ and $\tilde{H} = N_{\tilde{G}}(P)$. Then, $B_0(k \tilde{G})$ and $B_0(k \tilde{H})$ are derived (even splendidly) equivalent.

Proof. Let $G = M_{10}$ and $H = N_{G}(P)$. Then, we can write $\text{IBr}(\tilde{A}) = \{k_{\tilde{G}}, 1_1, 1_2, 1_3, 1_6, 6_6, 8\}$ and $\text{IBr}(A) = \{k_G, 1_G, 6, 4, 4'\}$ such that $k^{\tilde{G}} = k_{\tilde{G}} \oplus 1_a$, $1^{\tilde{G}} = 1_b \oplus 1_c$, $6^{\tilde{G}} = 6_a \oplus 6_b$, and $4^{\tilde{G}} = 4'^{\tilde{G}} = 8$. Similarly, we can write $\text{IBr}(\tilde{B}) = \{k_{\tilde{B}}, 1_0, 1_1, 1_2, 2'\}$ such that $k^{\tilde{B}} = k_{\tilde{B}} \oplus 1_0$, $1^{\tilde{B}} = 1_1 \oplus 1_2$, $1_2^{\tilde{B}} = 1_3^{\tilde{B}} = 2'$, $2^{\tilde{B}} = 2_a \oplus 2_b$ (note that, in [23, Sect. 4, Case 4], the simple $\tilde{B}$-modules here are represented in different notation, say, 0, 3, 1, 2, 6, 4, 5 for $k_{\tilde{B}}, 1_0, 1_1, 1_2, 2'$, $2_a, 2_b$, respectively). Then, we can consider the Green correspondences $G \overset{j}{\rightarrow} H$, $G \overset{f}{\rightarrow} H$

with respect to $P$. Let $\tilde{M}(1) = M(1)_{(\tilde{G}, \tilde{B})}$. Then, by (3.8), $\tilde{M}(1)$ gives a stable equivalence of Morita type between $\tilde{A}$ and $\tilde{B}$. It follows from the proof of (4.2) that the Loewy structure of $f(S)$ for simple $A$-modules $S$ is

\[
\begin{align*}
f(k_G) &= k_H, & f(1_0) &= 1_1, & f(6) &= k_H \cdot 1_1, \\
f(4) &= 2, & f(4') &= 2. 
\end{align*}
\]
Next, we claim that the Loewy structure of $\tilde{f}(\tilde{S})$ for simple $\tilde{A}$-modules $\tilde{S}$ is
\[
\begin{align*}
\tilde{f}(k_{\tilde{G}}) &= k_{\tilde{H}}, & \tilde{f}(1_a) &= 1_0, & \tilde{f}(1_b) &= 1_{1b}, & \tilde{f}(1_c) &= 1_{1a}, \\
(\ast) & & \tilde{f}(6_a) &= 1_{1a}2_{a}^{b}, & \tilde{f}(6_b) &= k_{\tilde{H}}^{2_{a}^{b}}1_{1b}, & \tilde{f}(8) &= 2_{a}^{2}2_{b}^{2}.
\end{align*}
\]

First, consider $\tilde{f}(6_a)$. Then, $\tilde{f}(6_a)\downarrow_H = 6_a\downarrow_{\tilde{H}} = 6_a\downarrow_{G}\downarrow_H = 6\downarrow_H = f(6)$.
Hence, we may assume that $\tilde{f}(6_a)/\tilde{f}(6_a)\cdot J(\tilde{K}_{\tilde{H}}) \cong 2_a$. On the other hand, it is easy to know the Loewy structure of projective indecomposable $k\tilde{H}$-modules. Hence, by looking at the quiver with relations for $k\tilde{H} = \tilde{B}$ in [23, Sect. 4, Case 4], we get the structure of $\tilde{f}(6_a)$ as desired. Similar for the rest.

Let $G' = HS$ (the Higman–Sims simple group) and $H' = N_G(P')$, where $P' \in \text{Syl}_3(G')$. Then, we may assume $P' = P$ and $H' = \tilde{H}$. We use the notation $k_{G'}, 22, 748, 154_0, 1176, 321, 1253$, which are simple $A'$-modules as in [23, Example 4.8]. Let $f': G' \to H' = \tilde{H}$ be the Green correspondence with respect to $P$. Then we know that, except for the structure of $\tilde{f}(1_b) = 1_{1b} \not\cong f'(748)$, $\tilde{f}(S) = f'(S')$ for each pair $(S, S') \in \{(k_{\tilde{G}}, k_{G'}), (1_a, 22), (1_c, 154_0), (6_a, 1176), (6_b, 321), (8, 1253)\}$. Namely, we get the following deformations of $\tilde{A}$ and $A'$ by $\tilde{M}(1)$ and $M'(1)$, respectively, where $M'(1) = M(1)_{(G', \tilde{B})}$. That is,
Note that in the above table the modules at the rows are represented modulo projectives. Thus, as in [23, Example 4.8 (1)–(4)], take \( I_0^{(1)} = \{2\} \), which is a so-called nice index set in [23, Sect. 2, (III)]. Then, just as done in [23, Example 4.8 (3), (4)], there are a \( k \)-algebra \( \tilde{B}_1 \) and a \( (\tilde{B}, \tilde{B}_1) \)-bimodule \( \tilde{M}_1 \) such that by an Okuyama deformation with respect to the index set \( I_0^{(1)} \), the \( k \)-algebra \( \tilde{B} \) is deformed into a new \( k \)-algebra \( \tilde{B}_1 \), which satisfies the following:

\[
\begin{array}{ccc}
(\tilde{S} \otimes \tilde{A} \tilde{M}(1))_{\tilde{B}} & k & 1_0 \quad 1_{1b} \quad 1_{1a} \\
\text{(modulo proj.)} & & 2_a \quad 2_b \quad 2' \\
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow
\end{array}
\]

Note that, since \( 1_{1b} \not\in \{2\} = I_0^{(1)} \), the Okuyama deformation does not make any change for \( 1_{1b} \) in \( \tilde{B} \).

Next, as in [23, Example 4.8, (5), (6)], take \( I_0^{(2)} = \{2_a, 2_b\} \), which is a nice index set again. Then, just as in [23, Example 4.8, (5), (6)], there are a \( k \)-algebra \( \tilde{B}_2 \) and a \( (\tilde{B}, \tilde{B}_2) \)-bimodule \( \tilde{M}_2 \) such that, by an Okuyama deformation with respect to \( I_0^{(2)} \), the \( k \)-algebra \( \tilde{B}_1 \) is deformed into another
equivalence induced by a 3-permutation bimodule with vertex δ̂ resulting derived equivalence between [23, Sect. 2 (IV)], for any simple θ̂ from the proof of (4.4). Then, by (4.5), A′ and B̂ are splendidly equivalent. Thus, we get the assertion since A and A′ are Puig equivalent by (4.4).

Note that 11a /\{2_a, 2_b\} = I_0^{(2)} does not make any change for 1_{1n} in B̂_1. Therefore, we finally get that

\[ \left( \tilde{S} \otimes \tilde{A} \tilde{M}(1) \otimes \tilde{B} \tilde{M}_1 \right)_{\tilde{B}_1} = \text{(simple)} \oplus \text{(projective)} \]

for any simple \( \tilde{A} \)-module \( \tilde{S} \), as seen in the above table. Hence, as in [23, Sect. 2 (IV)], \( \tilde{A} \) and \( B_2 \) are Morita (even Puig) equivalent. Consequently, \( \tilde{A} \) and \( B_2 \) are derived equivalent. Now, since we start with a stable equivalence induced by a 3-permutation bimodule with vertex δ(P), the resulting derived equivalence between \( \tilde{A} \) and \( B_2 \) is a splendid equivalence by [24].

**Lemma.** Let \( \tilde{G} = M_{22,2} \) and \( \tilde{H} = N_{\tilde{G}}(P) \). Then, \( B_0(k\tilde{G}) \) and \( B_0(k\tilde{H}) \) are derived (even splendidly) equivalent.

**Proof.** Let \( G' = M_{10,2} \cong A_{6,2} \). We know that \( \tilde{H} = N_{\tilde{G}}(P) \cong P \rtimes SD_{16} \) from the proof of (4.4). Then, by (4.5), \( A' \) and \( B_2 \) are splendidly equivalent. Thus, we get the assertion since \( \tilde{A} \) and \( A' \) are Puig equivalent by (4.4).
**Lemma 4.7.** Let $\tilde{G} = A_6.2_2 = \PGL_2(9)$ and $\tilde{H} = N_{\tilde{G}}(P)$. Then, $B_0(k\tilde{G})$ and $B_0(k\tilde{H})$ are derived (even splendidly) equivalent.

**Proof.** Let $G = A_6$ and $H = N_G(P)$. By [8, p. 2], $H = P \rtimes C_4$ and $\tilde{H} = P \rtimes C_8$.

$$\tilde{G} = A_6.2_2 \quad \quad \quad \tilde{H} = P \rtimes C_8$$

We can write $\text{IBr}(\tilde{A}) = \{k_{\tilde{G}}, 1_{\tilde{G}}, 3_a, 3_b, 3_{b_1}, 3_{b_2}, 4_1, 4_2\}$ and $\text{IBr}(A) = \{k_G, 3_a, 3_b, 4\}$ such that $k_{\tilde{G}}^G = k_{\tilde{G}} \oplus 1_{\tilde{G}}, 3_a^G = 3_a \oplus 3_{a_2}, 3_b^G = 3_{b_1} \oplus 3_{b_2}$, and $4^G_1 = 4_1 \oplus 4_2$. Similarly, we can write $\text{IBr}(\tilde{B}) = \{k_{\tilde{H}}, 1_{\tilde{H}}, 1_2, 1_2^\vee, 1_3, 1_3^\vee, 1_4, 1_4^\vee\}$ and $\text{IBr}(B) = \{k_H, 1_H, 1_H^\vee\}$ such that $k_H^\tilde{H} = k_H \oplus 1_{\tilde{H}}, 1_H^\tilde{H} = 1_2 \oplus 1_2^\vee, 1_H^\tilde{H} = 1_3 \oplus 1_4^\vee$. Then, by making the proof of [23, Example 4.2] duplicated, and by [24], we get the assertion as in the proof of $(4.5)$. 

**Lemma 4.8.** Let $\tilde{G} = L_3(4).2_1$ and $\tilde{H} = N_{\tilde{G}}(P)$. Then, $B_0(k\tilde{G})$ and $B_0(k\tilde{H})$ are derived (even splendidly) equivalent.

**Proof.** Let $G = L_3(4), H = N_G(P)$, and $G_0 = G \cdot C_G(P)$. By [8, p. 23], $\tilde{H} = (P \rtimes Q_8) \rtimes C_2$, so that $E(\tilde{G}) \cong Q_8 \cong E(G)$. If $G_0 = G$, then $C_G(P) \cong G$, so that $(1.2)(\nu)$ implies that $\tilde{G}/G \cong E(\tilde{G})/E(G) = 1$, a contradiction. Hence, $G \leq G_0$, so that $\tilde{G} = G_0$. Then, by (3.1), $\tilde{A}$ and $A$ are Puig equivalent. On the other hand, we get from [23, Example 4.6] and [24] that $A$ and $B$ are derived (even splendidly) equivalent. Since $B \cong \tilde{B} \cong k[P \rtimes Q_8]$, we know that $\tilde{A}$ and $\tilde{B}$ are derived (even splendidly) equivalent.

**Lemma 4.9.** Let $\tilde{G} = L_3(4).2_2$ and $\tilde{H} = N_{\tilde{G}}(P)$. Then, $B_0(k\tilde{G})$ and $B_0(k\tilde{H})$ are derived (even splendidly) equivalent.

**Proof.** Let $G = L_3(4)$ and $H = N_G(P)$. Then, $\tilde{H} \cong P \rtimes SD_{16}$ and $H = P \rtimes Q_8$ from [8, p. 23]. Then, we can use the same notation for simple $B$-modules and $\tilde{B}$-modules as in the proof of $(4.5)$. We can write that $\text{IBr}(\tilde{A}) = \{k_{\tilde{G}}, 1_{\tilde{G}}, 15_{1a}, 15_{1b}, 30, 19_a, 19_b\}$ and $\text{IBr}(A) = \{k_G, 15_1, 15_2, 15_3, 19\}$ such that $k_{\tilde{G}}^G = k_{\tilde{G}} \oplus 1_{\tilde{G}}, 15_{1a}^G = 15_{1a} \oplus 15_{1b}, 15_{2a}^G = 15_3 \oplus 1_4^\vee = 30$, and $19_{1a}^G = 19_a \oplus 19_\prime$. Let

$$\tilde{G} \cong \tilde{H}, \quad G \cong H$$
be the Green correspondences with respect to $P$. Note that $\tilde{M}(1)$ gives a stable equivalence of Morita type between $\tilde{A}$ and $\tilde{B}$ by (3.8), where $\tilde{M}(1) = M(1)_{\tilde{G}, \tilde{B}}$. We know the structure of the Green correspondents $f(S)$ for each simple $A$-module $S$, which is calculated by Schneider (see [19, Lemma 6.1]). Then, we can calculate the Loewy structure of $f(\tilde{S})$ for each simple $\tilde{A}$-module $\tilde{S}$, namely,

$\begin{array}{ccccccc}
\tilde{S} & k_{\tilde{G}} & 1_{\tilde{G}} & 19_a & 19_b & 15_{1a} & 15_{1b} & 30 \\
\|f| & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & (1)
\end{array}$

We may assume that $\tilde{B}$ is a basic $k$-algebra. Namely, all simple $\tilde{B}$-modules are of $k$-dimension 1. Now, compare (1) with [23, Example 4.8 (2)]. So, as in the proof of (4.5) and [23, Example 4.8 (2)–(4)], take $I^{(1)}_0 = \{2^\prime\}$, which is a nice index set in [23, Sect. 2 (III)]. We denote by $K^k(\text{proj}-R)$ the homotopy category of bounded complexes of finitely generated projective right $R$-modules for a $k$-algebra $R$. Let $\tilde{B}_1 = \text{End}^{k(\text{proj}-\tilde{B})}(P(I^{(1)}_0))^*$ and let $\tilde{M}_1$ be a $(\tilde{B}, \tilde{B}_1)$-bimodule which gives a stable equivalence of Morita type between $\tilde{B}$ and $\tilde{B}_1$, see [23, Theorem 1.2 and Sect. 2]. Then, we get a unique non-projective indecomposable direct summand of $\tilde{X} \otimes_{\tilde{B}} \tilde{M}_1$ as a right $\tilde{B}_1$-module for a $\tilde{B}$-module $\tilde{X}$ in several cases. That is, we get the table

$\begin{array}{cccccccc}
\tilde{X} & k & 1_{1b} & 1_{1a} & 2_a & 2_b & 2^{a_1} & 2^{b_1} & (2)
\|f^{(1)}_0 = \{2^\prime\} & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & (2)
\end{array}$

We denote by $\tilde{M}(1)_{\tilde{G}, \tilde{B}}$ gives a stable equivalence of Morita type between $\tilde{A}$ and $\tilde{B}$ by (3.8), where $\tilde{M}(1) = M(1)_{\tilde{G}, \tilde{B}}$. We know the structure of the Green correspondents $f(S)$ for each simple $A$-module $S$, which is calculated by Schneider (see [19, Lemma 6.1]). Then, we can calculate the Loewy structure of $f(\tilde{S})$ for each simple $\tilde{A}$-module $\tilde{S}$, namely,
We also claim the table
\[
\begin{array}{cccccccc}
1_{ib} & 2' & 1_{ia} & 2' & 2_a & 2_b \\
2_a & 2_b & 2_a & 2_a & 1_0 & 2' & k \\
2' & 1_{ib} & 2' & 1_{ia} & 2_b & 2_a \\
\end{array}
\]
(3) \[ \| I_b^{(1)} \| = \{ 2' \} \]

Next, we claim that the Cartan matrix \( C_{\tilde{B}_1} \) for \( \tilde{B}_1 \) has the form
\[
C_{\tilde{B}_1} = \begin{array}{cccccc}
k & 1_{ib} & 1_{ia} & 1_0 & 2_a & 2_b & 2' \\
k & 3 & 1 & 2 & 1 & 1 & 1 & 1 & 2 \\
1_{ib} & 1 & 3 & 1 & 2 & 1 & 1 & 2 \\
1_{ia} & 2 & 1 & 3 & 1 & 1 & 1 & 2 \\
1_0 & 1 & 2 & 1 & 3 & 1 & 1 & 2 \\
2_a & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\
2_b & 1 & 1 & 1 & 1 & 2 & 1 & 1 \\
2' & 2 & 2 & 2 & 2 & 1 & 1 & 3 \\
\end{array}
\]
It is well known that

\[
\sum_{n \in \mathbb{Z}} (-1)^n \dim_k [\text{Hom}_{K^n(\text{proj}\tilde{B})}(P(k)^*, P(1_{1b})^*[n])]
\]

\[
= \sum_{r, s \in \mathbb{Z}} (-1)^{r-s} \dim_k [\text{Hom}_{\tilde{B}}( (P(k)^*)^r, (P(1_{1b})^*)^s )].
\]

Then, by using this and the definition of a tilting complex \( P(I_0^{(1)})^* \) in \( \tilde{B} \), we get (5) (see [23, Sect. 1]). Actually, we can know the relations for the quiver in (4) by making use of the relations for the quiver of \( \tilde{B} \) stated in [23, Appendix Case 4] and the definition of \( \tilde{B}_1 \), which is given as an endomorphism algebra of the tilting complex \( P(I_0^{(1)})^* \).

In particular, we know the structure of \( P(2_a) \), which has the form

\[
P(2_a) \text{ in } \tilde{B}_1 = \begin{array}{c|c|c}
2_a & 2' & 2_b \\
\hline
1_{1a} & 1_{1b} & k \\
\hline
2_a & 1_{1a} & 1_{1b}
\end{array}.
\]

Now, take \( I_0^{(2)} = \{2_a, 2_b\} \), let \( \tilde{B}_2 = \text{End}_{K^n(\text{proj}\tilde{B}_1)}(P(I_0^{(2)})^*) \), and let \( \tilde{M}_2 \) be a \((\tilde{B}_1, \tilde{B}_2)\)-bimodule giving a stable equivalence of Morita type between \( \tilde{B}_1 \) and \( \tilde{B}_2 \). Then, just as in [23, Example 4.8 (4)–(6)], we get the following table by making use of (6):

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
\tilde{X} & k & 1_{0} & 1_{1b} & 1_{1a} & 2_a & 2_b & 2' \\
\hline
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
I_0^{(2)} = \{2_a, 2_b\} & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
(\tilde{X} \otimes_{\tilde{B}_1} \tilde{M}_2)_{\tilde{B}_2} & k & 1_{0} & 1_{1b} & 1_{1a} & 2_a & 2_b & 2' \\
\text{(modulo proj.)}
\end{array}
\]
Then, in particular, the Okuyama deformation given in the fifth and sixth columns in (7) implies the first column of the table (8)

\[
\begin{array}{c|c|c|c|c|c}
 & 1_{1b} & 1_{1a} & 2' & 2' & 2' \\
\hline
2_b & k & 1_{1b} & 1_0 & 1_a & 1_b \\
2_a & 2 & 2 & 2 & 2_b & 2_b \\
\end{array}
\]

The second and third columns in (8) are easily obtained by (7).

Now, as before, by using (7) we get the quiver and relations for \( \tilde{B}_2 \). In particular, we know the structures of \( P(1_{1b}) \), \( P(1_{1a}) \), and \( P(2') \) for \( \tilde{B}_2 \), which have the forms

\[
\begin{array}{c|c|c|c|c|c}
 & 1_{1b} & 1_{1a} & 2' & 2' & 2' \\
\hline
1_{1b} & 2' & 1_{1a} & 2' & 2' & 2' \\
2_b & k & 1_0 & 2_a & 2_b & 2_b \\
1_b & 2 & 2 & 2 & 2_b & 2_b \\
\end{array}
\]

(9) \( P(1_{1b}) \) in \( \tilde{B}_2 = k \ 1_{1b} \ 1_0 \ 2_a \), \( P(1_{1a}) \) in \( \tilde{B}_2 = k \ 1_{1a} \ 1_0 \ 2_b \),

(10) \( P(2') \) in \( \tilde{B}_2 = \)

Take another nice index set; namely, let \( I_0^{(3)} = \{1_{1b}, 1_{1a}\} \). Let \( \tilde{B}_3 = \text{End}_{K^0(\text{proj-}\tilde{B}_2)}(P(I_0^{(3)})) \) and let \( \tilde{M}_3 \) be a \( (\tilde{B}_2, \tilde{B}_3) \)-bimodule which gives a stable equivalence of Morita type between \( \tilde{B}_2 \) and \( \tilde{B}_3 \). Then, as in
[23, Example 4.8 (6)-(8)], we obtain the following table by using (9):

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>1_{1b}</th>
<th>1_{1a}</th>
<th>2</th>
<th>2'</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi$</td>
<td>$k$</td>
<td>$1_0$</td>
<td>$2'$</td>
<td>$k$</td>
<td>$1_0$</td>
</tr>
</tbody>
</table>

(11) $\prod I_0^{(3)} = \{1_{1b}, 1_{1a}\}$

For the index set $I'/I_0^{(3)}$ above, it follows from [23, Sect. 1] and the structure of $P(2')$ for $\tilde{B}_3$ in (10) that a direct summand $P(2')^*$ of the tilting complex $P(I_0^{(3)})^*$ in $\tilde{B}_3$ has the form

(13) $P(2')^* : \cdots \to P(1_{1a}) \oplus P(1_{1b}) \to P(2') \to 0 \to \cdots$,

where the term $P(2')$ in (13) is of degree 0. We denote by $(P(2')^*)^r$ the term of degree $r$ in a complex $P(2')^*$ for $r \in \mathbb{Z}$. Let $c(S, T) = \dim_k(S, T)$ for simple $\tilde{B}_3$-modules $S$ and $T$. Then, we get by (13), (9), and (10) that

\[
\sum_{r, s \in \mathbb{Z}} (-1)^{r-s} \dim_k[\text{Hom}_{\tilde{B}_3}((P(2')^*)^r, (P(2')^*)^s)]
\]

\[
= c(2', 2') - (c(2', 1_{1a}) + c(2', 1_{1b})) - c(1_{1a}, 2') - c(1_{1b}, 2')
\]

\[
+ c(1_{1a}, 1_{1a}) + c(1_{1a}, 1_{1b}) + c(1_{1b}, 1_{1a}) + c(1_{1b}, 1_{1b})
\]

\[
= 3.
\]

On the other hand,

\[
c_{\tilde{B}_3}(2', 2') = \dim_k[\text{Hom}_{\tilde{B}_3}(P(2'), P(2'))]
\]

\[
= \dim_k[\text{Hom}_{K^*(\text{proj-}\tilde{B}_3)}(P(2')^*, P(2')^*)].
\]

Hence, as in the proof (5), we get (12).

Next, take $I_0^{(4)} = \{2'\}$, let $\tilde{B}_4 = \text{End}_{K^*(\text{proj-}\tilde{B}_3)}(P(I_0^{(4)}))^*$, and let $\tilde{M}_4$ be a $(\tilde{B}_3, \tilde{B}_4)$-bimodule giving a stable equivalence of Morita type between $\tilde{B}_3$.
and $\tilde{B}_4$. Let $\tilde{X}$ be a $\tilde{B}_4$-module appearing at the last column in (11). Let $e$ be a primitive idempotent of $\tilde{B}_3$ corresponding to a simple $\tilde{B}_3$-module $2'$. Then, we know $(\text{Soc}(\tilde{X}))e = \text{Soc}(\tilde{X}) \cong 2'$. Moreover, it follows from (12) that $\Omega\tilde{X}/2'$ has no composition factors isomorphic to $2'$. That is, $(\Omega\tilde{X})e \cong 2'$. Hence, as we have done many times already, we get by [23, Lemma 2.1 (2)] that

$$
\tilde{X} \xrightarrow{k} 2' \xrightarrow{1_0} 2_e
$$

(14)

Then, we finally know from (1–3), (7), (8), (11), and (14) that

$$(\tilde{S} \otimes \tilde{A} M_1 \otimes \tilde{B}_1 \tilde{M}_2 \otimes \tilde{B}_2 \tilde{M}_3 \otimes \tilde{B}_3 \tilde{M}_4)_{\tilde{B}_4} = \text{(simple) } \oplus \text{ (projective)}$$

for any simple $\tilde{A}$-module $\tilde{S}$. Therefore, again by [20, Theorem 2.1(iii)], we get that $\tilde{A}$ and $\tilde{B}$ are derived equivalent. Thus, just as in the final part of the proof of (4.5), they are actually splendidly equivalent by [24].

(4.10) Lemma. Let $\tilde{G} = L_3(4).2_3$ or $L_3(4).2_2$, and let $\tilde{H} = N_{\tilde{G}}(P)$. Then, $B_0(k\tilde{G})$ and $B_0(k\tilde{H})$ are derived (even splendidly) equivalent.

Proof. Case 1. $\tilde{G} = L_3(4).2_2$. Let $G = L_3(4).2_2$. Then, we know $E(\tilde{G}) \cong E(G) \cong SD_{16}$ since $E(G) \triangleleft E(\tilde{G}) \hookrightarrow SD_{16}$ by (1.2)(i)–(ii) and since $E(G) \cong SD_{16}$ by the proof of (4.9). Hence, by (1.2)(ix), we get $\tilde{G} = G \cdot C_{\tilde{G}}(P)$. Thus, $\tilde{H} = H \cdot C_{\tilde{H}}(P)$. Therefore, we get from (4.9) and (3.3) that $\tilde{A}$ and $\tilde{B}$ are splendidly equivalent.

Case 2. $\tilde{G} = L_3(4).2_3$. Then, $\tilde{H} \cong P \times SD_{16}$ from [8, p. 23]. Hence, just as in (4.9), $\tilde{A}$ and $\tilde{B}$ are splendidly equivalent.

(4.11) Remark. It follows from (3.1) and the proofs of (4.9) and (4.10) that the three $k$-algebras $B_0(k[L_3(4).2_2])$, $B_0(k[L_3(4).2_3])$, and $B_0(k[L_3(4).2_2])$ are all Puig equivalent.
5. PROOF OF THE MAIN RESULT

In this section we give a proof of the main result (0.2), which is a case-by-case analysis and which is divided into several parts.

(5.1) Notation. Throughout this section except for the proof of (0.2), let \( \tilde{G} \) be an arbitrary finite group such that \( O_3(\tilde{G}) = 1 \) and that \( \tilde{G} \) has a Sylow 3-subgroup \( P \cong C_3 \times C_3 \), and let \( \bar{G} = O^3(\tilde{G}) \), so that \( P \in \text{Syl}_3(\tilde{G}) \cap \text{Syl}_3(G) \). Let \( \tilde{H} = \tilde{N}_{\tilde{G}}(P) \) and \( \bar{H} = N_G(P) \). Then, \( \tilde{H}_0 = H \cdot C_{\tilde{G}}(P) = H \cdot C_{\tilde{G}}(P) \). By the Frattini argument, \( \tilde{G} = G \cdot \tilde{H} \).

We assume that \( k \) is an algebraically closed field of characteristic 3.

We say that

\((\star)\) holds for \( \tilde{G} \) if there is a splendid equivalence between \( B_0(k\tilde{G}) \) and \( B_0(k\tilde{H}) \)

and

\((\ast)\) holds for \( \tilde{G} \) if there is a Puig equivalence between \( B_0(k\tilde{G}) \) and \( B_0(k\tilde{H}) \).

Of course, \((\ast)\) implies \((\star)\).

(5.2) Lemma. If \( E(G) = 1 \), then \((\ast)\) holds for \( \tilde{G} \).

Proof. Since \( G \) is 3-nilpotent, \( \tilde{G} \) is 3-solvable of 3-length 1. So, it is well known that \((\ast)\) holds for \( \tilde{G} \) since \( B_0(k\tilde{G}) \) and \( B_0(k\tilde{H}) \) are isomorphic via restriction just as in (3.1) by a result of Isaacs-Smith [13] (see [11, Sect. 4]).

(5.3) Lemma. If \( E(G) = 2 \), then \((\ast)\) holds for \( \tilde{G} \).

Proof. By (2.4), \( G = Q \times L' \), where \( Q \cong C_3 \) and \( L' \) is a non-abelian simple group such that \( C_3 \in \text{Syl}_3(L') \). Then, \( Q = O_3(G) \) since \( G \) is not 3-solvable. Then, \( Q \cdot \text{char} \: G \triangleleft \tilde{G} \), so that \( Q = O_3(\tilde{G}) \). Let \( \tilde{C} = C_{\tilde{G}}(Q) \). Then, \( \tilde{G}/\tilde{C} = N_{\tilde{G}}(Q)/C_{\tilde{G}}(Q) \leftrightarrow \text{Aut}(Q) \cong C_2 \), and hence \(|\tilde{G}/\tilde{C}| = 1 \) or 2.

Case 1. \( \tilde{G} = \tilde{C} \). Since \( C_3 \cong Q \subseteq Z(\tilde{G}) \), we get by (1.3) that \( \tilde{G} = Q \times L \) for some \( L \triangleleft \tilde{G} \). Let \( R \in \text{Syl}_3(L) \) and \( N = N_L(R) \). Then, by (3.7), the right \( k[L \times N]-\)module \( M(1)_{(L,N)}^{(\tilde{H},R)} \) gives a Puig equivalence between \( B_0(kL) \) and \( B_0(kN) \). Hence, \( M(1)_{(\tilde{G},\tilde{H})} \) gives a Puig equivalence between \( B_0(k\tilde{G}) \) and \( B_0(k\tilde{H}) \). Namely \((\ast)\) holds for \( \tilde{G} \).

Case 2. \(|\tilde{G}/\tilde{C}| = 2 \). Again by (1.3), \( \tilde{C} = Q \times L \) for some \( L \triangleleft \tilde{C} \). Let \( \tilde{N} = N_C(P) \). Then, just as in Case 1, \( M(1)_{(\tilde{C},\tilde{N})} \) gives a Puig equivalence between \( B_0(k\tilde{C}) \) and \( B_0(k\tilde{N}) \). Clearly, \( C_{\tilde{G}}(P) \subseteq \tilde{C} \). Therefore, (3.5) implies that \((\ast)\) holds for \( \tilde{G} \).
(5.4) Lemma. If \(|E(G)| = 16\), then \(\star\) holds for \(\tilde{G}\).

Proof. By (2.4), \(G \in \{M_{11}, M_{23}, HS\}\). By [23, Examples 4.7, 4.8, and 4.9] and [24], there is a splendid equivalence between \(B_0(kG)\) and \(B_0(kH)\). Then, by (1.2)(i)–(iii), we know \(E(G) = E(G_0) = E(\tilde{G})\), so that \(\tilde{G} = G_0\) by (1.2)(iv). Now, by (3.3), the splendid equivalence between \(B_0(kG)\) and \(B_0(kH)\) is liftable to that between \(B_0(k\tilde{G})\) and \(B_0(k\tilde{H})\). \(\blacksquare\)

(5.5) Lemma. If \(E(G) \cong C_2 \times C_2\), then \(\star\) holds for \(\tilde{G}\).

Proof. Since \(C_2 \times C_2 \notin SD_{16}\), we get by (1.2)(i)–(ii) that \(E(\tilde{G}) \in \{C_2 \times C_2, D_8\}\). By (2.4), \(G = X \times Y\) for non-abelian simple groups \(X\) and \(Y\) such that \(C_3 \cong Q \in \text{Syl}_3(X)\) and \(C_3 \cong R \in \text{Syl}_3(Y)\). Let \(H_X = N_X(Q)\), \(H_Y = N_Y(R)\). Then, by (3.7), there are Puig equivalences between \(B_0(kX)\) and \(B_0(kH_X)\) and between \(B_0(kY)\) and \(B_0(kH_Y)\), both of which are given by the Green correspondences with respect to \(Q\) and \(R\), respectively. Hence, \(M(1)^{(G,H)}\) gives a Puig equivalence between \(B_0(kG)\) and \(B_0(kH)\). By (3.2), \(M(1)^{(G_0,H_0)}\) gives a Puig equivalence between \(B_0(kG_0)\) and \(B_0(kH_0)\). Hence, by (i), (ii), and (v) of (1.2), we have \(\tilde{G}/G_0 \cong E(\tilde{G})/E(G_0) \cong 1\) or \(C_2\). Hence, by (3.5), \(\star\) holds for \(\tilde{G}\). \(\blacksquare\)

(5.6) Lemma. Assume that \(G\) is one of (ix), (x), (xiii), or (xiv) in the list of (2.4). Then, \(\star\) holds for \(\tilde{G}\).

Proof. First, assume that \(G\) is of type (ix) of (2.4). Consider the diagram

\[
\begin{array}{ccc}
\tilde{G} = G \cdot \tilde{H} & \rightarrow & \tilde{H} = N_{\tilde{G}}(P) \\
\vee & \rightarrow & \vee \\
G_0 = G \cdot C_{\tilde{G}}(P) & \rightarrow & H_0 = N_{G_0}(P) \\
\vee & \rightarrow & \vee \\
G = \text{PSU}_3(q^2) & \rightarrow & H = N_G(P) = P \rtimes Q_8.
\end{array}
\]

Clearly, \(H_0 = H \cdot C_{H_0}(P)\). Let \(M(1) = M(1)^{(G,H)}\) and \(M_0(1) = M(1)^{(G_0,H_0)}\). It follows from [15, (0.2) and its proof] that \(M(1)\) gives a Puig equivalence between \(B_0(kG)\) and \(B_0(kH)\). Hence, by (3.2), \(M_0(1)\) gives a Puig equivalence between \(B_0(kG_0)\) and \(B_0(kH_0)\). Since \(E(G) = Q_8\) by (2.4), we get from (i), (ii), and (v) of (1.2) that \(\tilde{G}/G_0 \cong E(\tilde{G})/E(G_0) = 1\) or \(C_2\). Hence, as in (5.5), (3.5) implies the assertion.

For the cases (x), (xiii), and (xiv) of (2.4), we similarly get the assertion by using [26, Corollaire 3.6], [16, (0.2)Theorem], and [17, (2.7)Corollary] instead of [15]. \(\blacksquare\)

(5.7) Lemma. If \(G \in \{A_6, A_7, A_8, M_{21} \cong \text{PSL}_3(4), M_{22}\}\), then \(\star\) holds for \(\tilde{G}\).
Proof. Case 1. \( G = A_6 \). By (1.4) and [8, p. 4], \( \tilde{G} \in \{ A_6, \Sigma_6, \text{PGL}_2(9), M_{10}, M_{10}.2 = \text{Aut}(\Sigma_6) \} \). If \( G = A_6 \) or \( \Sigma_6 \), then \((\bigstar)\) holds for \( \tilde{G} \) by [23, Examples 4.2 and 4.4] and [24] (see [5, Sect. 8, p. 150]).

If \( \tilde{G} = \text{PGL}_2(9) \), then \((\bigstar)\) holds for \( \tilde{G} \) by (4.7).
If \( \tilde{G} = M_{10} \), then \((\bigstar)\) holds for \( \tilde{G} \) by (4.3).
If \( \tilde{G} = M_{10}.2 = A_6, 2^2 = \text{Aut}(\Sigma_6) \), then \((\bigstar)\) holds for \( \tilde{G} \) by (4.5).

Case 2. \( G = A_7 \) or \( A_8 \). By (1.4) and [8, pp. 10, 22], \( \tilde{G} = A_n \) or \( \Sigma_n \), where \( n = 7 \) or 8. Hence, \((\star)\) holds for \( G \) by [23, Examples 4.1 and 4.3], [24], and [5, Sect. 8, p. 150].

Case 3. \( G = M_{22} \). By (1.4) and [8, p. 23], \( \tilde{G} \in \{ L_3(4), L_3(4).2_1, L_3(4).2_2, L_3(4).2_3, L_3(4).2^2 \} \) (the notation here adapts to that in [8, p. 23]).
(a) \( \tilde{G} = L_3(4) \). Then, \((\bigstar)\) holds for \( \tilde{G} \) by [23, Example 4.6] and [24].
(b) \( \tilde{G} = L_3(4).2_1 \). Then, \((\bigstar)\) holds for \( \tilde{G} \) by (4.8).
(c) \( \tilde{G} = L_3(4).2_2 \). Then, \((\bigstar)\) holds for \( \tilde{G} \) by (4.9).
(d) \( \tilde{G} = L_3(4).2_3 \). Then, \((\bigstar)\) holds for \( \tilde{G} \) by (4.10).
(e) \( \tilde{G} = L_3(4).2^2 \). Then, \((\bigstar)\) holds for \( \tilde{G} \) by (4.10).

Case 4. \( G = M_{22} \). Then, by (1.4) and [8, p. 39], \( \tilde{G} = M_{22} \) or \( M_{22}.2 \).
For \( \tilde{G} = M_{22} \), \((\bigstar)\) holds by [23, Example 4.5] and [24].
Assume, next, that \( \tilde{G} = M_{22}.2 \). In this case the assertion is obtained by (4.6).

5.8 Lemma. Assume that \( G \) is one of (viii), (xi), or (xii) in the list of (2.4). Then, \((\bigstar)\) holds for \( \tilde{G} \).

Proof. Case 1. Assume that \( G \) is of type (viii) of (2.4). By (2.4), \( E(G) \cong Q_8 \), so that \( E(\tilde{G}) \cong Q_8 \) or \( SD_{16} \) by (1.2)(i)–(ii).
(a) \( E(\tilde{G}) \cong Q_8 \). By (1.2)(ii)–(iii), \( E(\tilde{G}) = E(G_0) = E(G) \). Hence, by (1.2)(vii) (or (viii)), \( \tilde{G} = G_0 \), so that \( H = H_0 \). Now, it follows from [19, Theorem 1.2], [23, Example 4.6], and [24] that there exists a splendid equivalence between \( B_0(kG) \) and \( B_0(kH) \). Hence, by (3.3), the splendid equivalence is lifted to that between \( B_0(kG_0) \) and \( B_0(kH_0) \). This means that \((\bigstar)\) holds for \( \tilde{G} \).
(b) \( E(\tilde{G}) \cong SD_{16} \). Let \( G = L_3(4) \) and \( G^\prime = L_3(4) \). We can consider \( P \in \text{Syl}_4(G) \cap \text{Syl}_4(G^\prime) \). Hence, let \( H^\prime = N_G(P) \). We then may consider \( H = H^\prime \) (see [19]). Let \( A = B_0(kG) \), \( A^\prime = B_0(kG^\prime) \), and \( B = B_0(kH) = B_0(kH^\prime) \). Let \( M(1) = M(1)_{(G,H)} \) and \( N^\prime(1) = N^\prime(1)_{(H,G)} \). It follows from [19, Theorem 1.2] that \( A \) and \( A^\prime \) are Puig equivalent via \( \mathcal{W} \), where \( \mathcal{W} \) is a unique non-projective indecomposable direct summand of \( M(1) \otimes_{\mathbb{F}} N^\prime(1) \).
Let $G_0 = G \cdot C_G(P)$, $A_0 = B_0(kG_0)$, $B_0 = B_0(k[N_G(P)])$, and $\mathfrak{M}_0 = A_0 \otimes_A \mathfrak{M}$. Then, (3.1) implies that $\mathfrak{M}_0$ realizes a Puig equivalence between $A_0$ and $A'$. Now, let $\tilde{G}' = L_3(2)$. We know from (1.2)(v) and (1.2)(vii) that $\tilde{G}/G_0 \cong C_2 \cong \tilde{G}'/G'$. Clearly, $B \cong B_0 \cong k[P \times Q_8]$. Therefore, we get from (3.6)(iii) that $\tilde{A}$ and $\tilde{A}'$ are Puig equivalent, where $\tilde{A} = B_0(k\tilde{G})$ and $\tilde{A}' = B_0(k\tilde{G}')$. We have already known that $\tilde{A}'$ and $\tilde{B}'$ are splendidly equivalent by (4.9), where $\tilde{B}' = B_0(k[N_{\tilde{G}}(P)])$. Clearly, $\tilde{B}' \cong \tilde{B} \cong k[P \times SD_{16}]$, where $\tilde{B} = B_0(k[N_{\tilde{G}}(P)])$. Thus, $\tilde{A}$ and $\tilde{B}$ are splendidly equivalent.

Case 2. Assume that $G$ is of type (xi) of (2.4). By (2.4), $E(G) \cong D_8$, so that $E(\tilde{G}) \cong D_8$ or $SD_{16}$ by (1.2)(i)-(ii).

(a) $E(\tilde{G}) \cong D_8$. As in the proof of Case 1(a), $\tilde{G} = G_0$. Then, we get the assertion just as in Case 1(b) by using [25], [23, Example 3.6 and Remark 3.7], and [24].

(b) $E(\tilde{G}) \cong SD_{16}$. We get the assertion as in Case 1(b) by using (4.5) instead of (4.9) (see [23, Example 3.6]).

Case 3. Assume that $G$ is of type (xii) of (2.4). By (2.4), $E(G) \cong D_8$, so that $E(\tilde{G}) \cong D_8$ or $SD_{16}$ by (1.2)(i)-(ii).

(a) $E(\tilde{G}) \cong D_8$. As in the proof of Case 1(a), we get the assertion by [16, (0.3)Theorem], [23, Example 4.3], and [24].

(b) $E(\tilde{G}) \cong SD_{16}$. We obtain the assertion as in Case 1(b) by using the fact that (0.1) is checked for $G = \Sigma_8 \cong A_8 \cong PSL_n(2,2)$ by Chuang in [5, Sect. 8, p. 150] instead of (4.9) (see [16, (0.4)Corollary]).

Proof of Main Theorem (0.2). Let $G$ be an arbitrary finite group such that $C_3 \times C_3 \cong P \in Syl_3(G)$. Since we consider the principal 3-blocks, we may assume $O_3(G) = 1$. Then, (2.4) and (5.2)–(5.8) imply that $B_0(kG)$ and $B_0(kH)$ are splendidly equivalent. Hence, $B_0(\mathfrak{G})$ and $B_0(\mathfrak{H})$ are splendidly equivalent by a result of Rickard [27, Theorem 5.2].

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