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On the Gibbs Phenomenon for Wavelet Expansions

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It is shown that a Gibbs phenomenon occurs in the wavelet expansion of a function with a jump discontinuity at 0 for a wide class of wavelets. Additional results are provided on the asymptotic behavior of the Gibbs splines and on methods to remove the Gibbs phenomenon.

1. INTRODUCTION

If S_n denotes the nth partial sum of the Fourier expansion of the 2-periodic function defined by

$$F(x)=\begin{cases}-1 & \text{for} \quad -1\leqslant x<0,\\ 1 & \text{for} \quad 0\leqslant x<1,\end{cases} \tag{1.1}$$

then, of course, $S_n(x)\to 1$ for all $0<x<1$. However, there are sequences x_n of positive numbers converging to 0 such that $S_n(x_n)$ converges to a number greater than 1. Indeed

$$\lim_{n\to\infty} S_n(a/n)=\frac{2}{\pi}\int_0^{\pi a}\frac{\sin t}{t}\,dt$$

so that

$$\lim_{n\to\infty} S_n(1/n)=\frac{2}{\pi}\int_0^{\pi}\frac{\sin t}{t}\,dt=1.17898\ldots>1,$$

e.g., see [7, p. 43]. This fact was pointed out by Gibbs [8] in 1899; for the history of this observation see [3]. A similar phenomenon exists in a vicinity of any jump of a piecewise smooth periodic function F.

Recently, the question has arisen as to whether there exists a Gibbs phenomenon for orthogonal wavelet expansions, see [11], [14], [15], and

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[16]. As a simple illustrative example, let us look at the Shannon scaling function

$$\phi(x)=\frac{\sin(\pi x)}{\pi x}. \tag{1.2}$$

The corresponding multiresolution subspaces V_m, $m\in\mathbf{Z}$, of $L^2(\mathbf{R})$ consist of those (entire) functions in $L^2(\mathbf{R})$ whose Fourier transforms vanish outside the interval $(-2^m\pi, 2^m\pi)$. For a given f in $L^2(\mathbf{R})$, the orthogonal projections $Q_m f$ of f onto the spaces V_m converge to f in the $L^2(\mathbf{R})$ norm as $m\to\infty$. They form partial sums of the wavelet expansion of f with respect to the wavelet induced by ϕ (see Section 2), and they are given by

$$(Q_m f)(x)=\int_{-\infty}^{\infty}\frac{\sin(2^m\pi(x-y))}{\pi(x-y)}f(y)\,dy.$$

Let f be defined by

$$f(x)=\begin{cases}-1 & \text{for} \quad -1<x<0,\\ 1 & \text{for} \quad 0<x<1,\\ 0 & \text{otherwise.}\end{cases} \tag{1.3}$$

Then we obtain

$$(Q_m f)(2^{-m}a)=\int_{-a}^{a}\frac{\sin(\pi x)}{\pi x}\,dx-\int_{-a+2^m}^{a+2^m}\frac{\sin(\pi x)}{\pi x}\,dx.$$

Thus

$$\lim_{m\to\infty}(Q_m f)(2^{-m}a)=\frac{2}{\pi}\int_0^{\pi a}\frac{\sin t}{t}\,dt.$$

This shows that there exists a Gibbs phenomenon in the sense that $(Q_m f)(2^{-m}a)$ converges to a number greater than 1 for certain positive a, e.g., for $a=1$. This is just a Gibbs phenomenon for the Fourier transform, analogous to the standard one for Fourier series.

It is known that there is no Gibbs phenomenon for the Haar wavelet. However, for all other studied classes of wavelets a Gibbs phenomenon was found. We now report on some of these results.

Richards [14] studied the Gibbs phenomenon for the expansion into spline functions without directly referring to multiresolution theory. Let $T_n^{[k]}$ be the 2-periodic spline of degree $k-1$ having knots j/n, $j\in\mathbf{Z}$, that best approximates the 2-periodic function (1.1) in the norm of $L^2[-1,1]$. In [14] it is shown that the sequence $T_n^{[k]}(x/n)$, $n=1, 2, \ldots$, converges locally uniformly to a spline function $S^{[k]}$ of degree $k-1$ with knots at the integers. Let us call this cardinal spline $S^{[k]}$ the Gibbs spline of degree $k-1$. It is the best approximation in the L^2 norm of the function defined by $h(x)=1$ if $x\geqslant 0$ and $h(x)=-1$ if $x<0$ by a cardinal spline of degree $k-1$. If there is $x>0$ such that $S^{[k]}(x)>1$, then there exists a Gibbs phenomenon in the spline approximation of F. In [14] the Gibbs splines $S^{[k]}$, $k\leqslant 8$, are computed numerically. It is shown that an overshoot is always observed in these cases. Then the behavior of $S^{[k]}$ as $k\to\infty$ is investigated. It is conjectured that

$$\lim_{k\to\infty} S^{[k]}(x)=\frac{2}{\pi}\int_0^{\pi x}\frac{\sin t}{t}\,dt$$

locally uniformly in x. The conjecture is proved for odd degree $k-1$ and $x\in\mathbf{Z}$. The Gibbs splines $S^{[k]}$ will also appear in Section 4 of this paper in connection with the best $L^2(\mathbf{R})$ approximation in multiresolution spaces generated by spline functions. We will give a complete proof of the conjecture of Richards, and we will also prove that, for every k, there exists $x>0$ such that $S^{[k]}(x)>1$. Our approach demonstrates the usefulness of multiresolution theory in spline approximation theory.

Kelly [11] studied the Gibbs phenomenon for expansions associated with a multiresolution analysis of closed linear subspaces V_m, $m\in\mathbf{Z}$ of $L^2(\mathbf{R})$; see Section 2. The orthogonal projection $Q_m f$ of $f\in L^2(\mathbf{R})$ onto V_m is the best approximation of f in the norm of $L^2(\mathbf{R})$ by a function in V_m. If $m\to\infty$, then $Q_m f$ converges to f in the norm of $L^2(\mathbf{R})$. Kelly gave a necessary and sufficient condition for the existence of the Gibbs phenomenon in this situation in terms of a Gibbs function. This Gibbs function generalizes the Gibbs splines $S^{[k]}$ of [14]. The Gibbs function can be expressed in terms of the scaling function of the underlying multiresolution analysis. We restate and reprove Kelly's result in Section 3. This result is then used in [11] to show that there exists the Gibbs phenomenon for the expansion associated with the Daubechies wavelets with compact support. These wavelets were introduced in [5]. Kelly also considered the Gibbs phenomenon at points different from zero and obtained approximate values for the overshoot by numerical calculations.

Using Kelly's general result, Shim [15] showed that there also exists the Gibbs phenomenon for the expansion into Meyer-type wavelets. Moreover, Shim showed that the Gibbs phenomenon can be avoided if appropriate summation methods are used. A similar observation is known from the theory of Fourier series: there is no Gibbs phenomenon if Fourier series are summed by the Fejér method (Cesàro summability). Some of these results are presented in Section 5.

It should be noticed that all of the mentioned results on the existence of the Gibbs phenomenon for certain families of wavelets use special properties of the wavelets under consideration that are not shared by other classes of wavelets. It is natural to ask whether we can derive the existence of the Gibbs phenomenon for wavelet expansions directly from the defining properties of wavelets. This is in fact possible. We prove in Section 3: if the scaling function is continuously differentiable and of sufficient decay at $\pm\infty$, then there is a Gibbs phenomenon in the associated wavelet expansion.

2. Wavelet Expansions

The presentation in this paper is essentially self-contained so that knowledge of wavelet theory is not required to understand it. Good references for orthogonal wavelet theory are Chui [4, Ch. 5], Daubechies [6, Ch. 5] and Walter [16, Ch. 3].

Let ϕ be a scaling function, i.e., $\phi: \mathbf{R} \to \mathbf{R}$ is a square integrable function having the properties:

1. the functions $\phi(x-n)$, $n \in \mathbf{Z}$, form an orthonormal system in $L^2(\mathbf{R})$;
2. the multiresolution subspaces V_m, $m \in \mathbf{Z}$, of $L^2(\mathbf{R})$, defined as the closed linear spans of the orthonormal systems $\phi_{m,n}(x) = 2^{m/2}\phi(2^m x - n)$, $n \in \mathbf{Z}$, are nested: $\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$
3. the union of the spaces V_m, $m \in \mathbf{Z}$, is dense in $L^2(\mathbf{R})$.

It should be noted that we consider only real-valued scaling functions and that we do not make a priori assumptions on their smoothness or on their decay at $\pm\infty$. By the first property, the orthogonal projection $Q_m f$ of $f \in L^2(\mathbf{R})$ onto V_m is given by

$$Q_m f = \sum_{n \in \mathbf{Z}} (f, \phi_{m,n})\, \phi_{m,n}, \tag{2.1}$$

where $(\cdot, \cdot)$ denotes the scalar product in $L^2(\mathbf{R})$. Because of the second and third property, the sequence $Q_m f$ converges to f in the $L^2(\mathbf{R})$ norm as $m \to \infty$ for every $f \in L^2(\mathbf{R})$. This sequence plays the role of the sequence of partial sums of Fourier series in the classical theory.

Indeed, the quantity $Q_m f$ is a partial sum of the wavelet expansion associated with the given scaling function. Let ψ be a corresponding (mother) wavelet, i.e., a function in V_1 such that the system $\psi(x-n)$, $n \in \mathbf{Z}$, forms an orthonormal basis of the orthogonal complement of V_0 within V_1. Then the system $\psi_{m,n}(x) = 2^{m/2}\psi(2^m x - n)$, $m, n \in \mathbf{Z}$, is an orthonormal basis of $L^2(\mathbf{R})$: every function $f \in L^2(\mathbf{R})$ admits the $L^2(\mathbf{R})$-convergent wavelet expansion

$$f = \sum_{m,\, n \in \mathbf{Z}} (f, \psi_{m,n})\, \psi_{m,n}. \tag{2.2}$$

Now $Q_m f$ is the partial sum

$$Q_m f = \sum_{n} \sum_{k < m} (f, \psi_{k,n})\, \psi_{k,n}.$$

However, as in many papers on wavelet theory, we will not use wavelets directly.

If it is possible to interchange sum and integral in (2.1), we can write Q_m as an integral operator

$$(Q_m f)(x) = \int_{-\infty}^{\infty} 2^m q(2^m x, 2^m y)\, f(y)\, dy, \tag{2.3}$$

where the kernel $q(x, y)$ is defined by

$$q(x, y) = \sum_{n \in \mathbf{Z}} \phi(x-n)\, \phi(y-n) \qquad \text{for} \quad x, y \in \mathbf{R}. \tag{2.4}$$

We collect some properties of q in the following lemma (compare with [13, (6.2) on p. 33]).

LEMMA 2.1. *Let ϕ be a continuous scaling function satisfying*

$$|\phi(x)| \leqslant K(1+|x|)^{-\beta} \qquad \textit{for} \quad x \in \mathbf{R} \tag{2.5}$$

with constants K and $\beta > 1$. Then

(a) *the kernel $q(x, y)$ is continuous and satisfies the estimate*

$$|q(x, y)| \leqslant L(1+|x-y|)^{-\beta} \qquad \textit{for all} \quad x, y \in \mathbf{R},$$

where L is a constant;

(b) *every function in V_m is continuous;*

(c) *equation* (2.3) *holds for every $f \in L^2(\mathbf{R})$, $m \in \mathbf{Z}$ and $x \in \mathbf{R}$.*

Proof. (a) By assumption, the defining series (2.4) for q converges locally uniformly. Thus continuity of ϕ implies continuity of q. For the proof of the estimate, we assume without loss of generality that $|x+y| \leqslant 1$ and $x \geqslant y$ because $q(x+1, y+1) = q(x, y) = q(y, x)$. If $n \geqslant 0$ then

$$|x-n| = \left|\frac{x+y}{2} + \frac{x-y}{2} - n\right| \geqslant \left|\frac{x-y}{2} - n\right| - \frac{1}{2},$$

and

$$|y-n| = \left|\frac{x+y}{2} - \frac{x-y}{2} - n\right| \geqslant \frac{x-y}{2} + n - \frac{1}{2} \geqslant \frac{x-y}{2} - \frac{1}{2}.$$

Thus

$$(1+|x-n|)(1+|y-n|) \geqslant \tfrac{1}{4}(1+x-y)(1+|x-y-2n|).$$

The same result holds for $n<0$ if we replace n by $-n$ on the right hand side. Using (2.4) and (2.5) we have shown that

$$|q(x, y)| \leqslant K^2 4^{\beta}(1+x-y)^{-\beta} \times \left(\sum_{n \geqslant 0} (1+|x-y-2n|)^{-\beta} + \sum_{n<0} (1+|x-y+2n|)^{-\beta}\right).$$

Since $\beta > 1$ the sum $\sum_{n \in \mathbf{Z}} (1+|t-2n|)^{-\beta}$ is bounded by a constant independent of $t \in \mathbf{R}$. Therefore the above inequality implies (a).

(b) This follows from the fact that the expansion $f = \sum_n (f, \phi_{m,n}) \phi_{m,n}$ of every $f \in V_m$ is locally uniformly convergent.

(c) Without loss of generality we assume that $m=0$. Since

$$\sum_n \int_{-\infty}^{\infty} |\phi(x-n)\,\phi(y-n)\,f(y)|\,dy \leqslant \left(\int_{-\infty}^{\infty} |f(y)|^2\,dy\right)^{1/2} \sum_n |\phi(x-n)| < \infty$$

we are permitted to interchange sum and integral in (2.1) yielding (2.3) for almost all x. This equation then holds for all x because both sides represent continuous functions of x. This is true for the left hand side by part (b), and it holds for the right hand side by a well known theorem on the continuous parameter dependence of Lebesgue integrals in combination with the estimate given in part (a). ▮

We will also need the following well known result. We include a proof for the interest of completeness and because the method of proof (making use of the second property of scaling functions) will also be useful to establish an important lemma in the next section.

LEMMA 2.2. *If* ϕ *is a continuous scaling function satisfying* (2.5) *with* $\beta > 1$, *then*

$$\int_{-\infty}^{\infty} q(x, y)\, dy = 1 \qquad \text{for all} \quad x \in \mathbf{R}. \tag{2.6}$$

Proof. Let $d = k/2^n$, $k \in \mathbf{Z}$, $n \in \mathbf{N}$, be a dyadic number. Then $\phi(2^{-m}x + d)$ belongs to V_0 for all $m \geqslant n$. In fact, the second property of scaling functions shows that $\phi(2^{-m}x)$ is in V_0, and then $\phi(2^{-m}x + d) = \phi(2^{-m}(x + 2^{m-n}k))$ is in V_0, too. Thus, by Lemma 2.1(c) and definition of Q_0,

$$\phi(2^{-m}x + d) = \int_{-\infty}^{\infty} q(x, y)\, \phi(2^{-m}y + d)\, dy.$$

Since $q(x, y)$ is integrable with respect to y by Lemma 2.1(a) and ϕ is bounded, we obtain for $m \to \infty$ by the Lebesgue dominated convergence theorem

$$\phi(d) = \int_{-\infty}^{\infty} q(x, y)\, \phi(d)\, dy.$$

Since there is a dyadic number d such that $\phi(d) \neq 0$ this yields the desired result. ▮

3. THE GIBBS PHENOMENON FOR WAVELET EXPANSIONS

We first need a precise definition of what we mean by the Gibbs phenomenon for wavelets.

DEFINITION 3.1. Let $f: \mathbf{R} \to \mathbf{R}$ be a square integrable bounded function with a jump discontinuity at 0, i.e., the limits $f(0-) = \lim_{0 > x \to 0} f(x)$ and $f(0+) = \lim_{0 < x \to 0} f(x)$ exist and are different. We say that the wavelet expansion of f with respect to a given scaling function ϕ shows a Gibbs phenomenon at the right hand side of 0 if there is a sequence $0 < x_m \to 0$ such that $(Q_m f)(x_m)$ converges to a number greater than $f(0+)$ if $f(0+) > f(0-)$ or to a number less than $f(0+)$ if $f(0+) < f(0-)$ as $m \to \infty$. Similarly, we speak of a Gibbs phenomenon at the left hand side of 0 if there is a sequence $0 > x_m \to 0$ such that $(Q_m f)(x_m)$ converges to a number greater than $f(0-)$ if $f(0-) > f(0+)$ or to a number less than $f(0-)$ if $f(0-) < f(0+)$ as $m \to \infty$.

We remark that this definition refers only to the orthogonal projections Q_m onto the multiresolution spaces V_m but not directly to the scaling function ϕ. We also remark that in the above definition we tacitly assumed that

$Q_m f$ is a continuous function. Without such an assumption, $Q_m f$ could be redefined arbitrarily on a null set without changing it as an element of $L^2(\mathbf{R})$, and the sequence $(Q_m f)(x_m)$ would be meaningless.

We further remark that $(Q_m f)(x-d)=(Q_m g)(x)$ if $g(x)=f(x-d)$ for every dyadic number $d=k/2^n$, $k\in\mathbf{Z}$, $n\in\mathbf{N}$ and $m\geqslant n$. This follows from the fact that $h(x-2^{-m})$ is in V_m whenever h is in V_m. Therefore, the wavelet expansion of f shows a Gibbs phenomenon at $x=d$ (which is defined in an obvious way) if and only if the wavelet expansion of g shows a Gibbs phenomenon at $x=0$.

Let f be a function as in Definition 3.1, and let x_m be any sequence of positive numbers converging to 0. Then, under the assumption of Lemma 2.1,

$$(Q_m f)(x_m)=\int_{-\infty}^{\infty} 2^m q(2^m x_m, 2^m y)\, f(y)\, dy=\int_{-\infty}^{\infty} q(2^m x_m, t)\, f(2^{-m}t)\, dt.$$

Define a function g by $g(x)=f(x)-f(0-)$ if $x<0$ and $g(x)=f(x)-f(0+)$ if $x\geqslant 0$, and set

$$\varepsilon_m=\int_{-\infty}^{\infty} q(2^m x_m, t)\, g(2^{-m}t)\, dt.$$

Then, by Lemma 2.1(a),

$$\begin{aligned}|\varepsilon_m| &\leqslant \int_{-\infty}^{\infty} \frac{L}{(1+|2^m x_m-t|)^\beta}\,|g(2^{-m}t)|\, dt\\ &=\int_{-\infty}^{\infty} \frac{L}{(1+|s|)^\beta}\,|g(2^{-m}s+x_m)|\, ds.\end{aligned}$$

By the Lebesgue dominated convergence theorem, the second integral converges to 0 as $m\to\infty$ because g is bounded and $g(x)\to 0$ as $0\neq x\to 0$. Hence there is a sequence ε_m converging to 0 such that

$$(Q_m f)(x_m)=\varepsilon_m+f(0-)\int_{-\infty}^{0} q(2^m x_m, t)\, dt+f(0+)\int_{0}^{\infty} q(2^m x_m, t)\, dt.$$

Using Lemma 2.2 we can write this as

$$(Q_m f)(x_m)=\varepsilon_m+f(0-)+(f(0+)-f(0-))\int_{0}^{\infty} q(2^m x_m, t)\, dt. \quad (3.1)$$

We use this identity to reprove the following theorem established in [11].

THEOREM 3.2. *Let ϕ be a continuous scaling function satisfying* (2.5) *with* $\beta > 1$. *Then the following statements are equivalent*:

(a) *there is a Gibbs phenomenon at the right hand side of* 0 *for the wavelet expansion of at least one bounded square integrable function* $f\colon \mathbf{R} \to \mathbf{R}$ *with a jump discontinuity at* 0;

(b) *there is a Gibbs phenomenon at the right hand side of* 0 *for the wavelet expansion of all such* f;

(c) *there is an* $a > 0$ *such that* $\int_0^\infty q(a, t)\, dt > 1$.

Proof. (c) implies (c): Let $a > 0$ be such that $w := \int_0^\infty q(a, t)\, dt > 1$. Let f be a function as in Definition 3.1, and set $x_m = 2^{-m}a$. Then, by (3.1), $(Q_m f)(x_m) \to f(0-)(1-w) + f(0+)\, w$ as $m \to \infty$. Since $w > 1$, the limit is greater than $f(0+)$ if $f(0+) > f(0-)$ and less than $f(0+)$ in the other case. Thus (b) holds.

(b) trivially implies (a).

(a) implies (c): Let f be a function as given by (a), and let $0 < x_m \to 0$ be such that $(Q_m f)(x_m)$ converges to a number greater than $f(0+)$ if $f(0+) > f(0-)$ or less than $f(0+)$ otherwise. Then (3.1) shows that $\int_0^\infty q(2^m x_m, t)\, dt$ converges to a number greater than 1 as $m \to \infty$. This implies that there is m such that $a := 2^m x_m$ satisfies $\int_0^\infty q(a, t)\, dt > 1$. ∎

Of course, there is a similar result for the Gibbs phenomenon at the left hand side of 0 with (c) replaced by the condition that there is $a < 0$ such that $\int_0^\infty q(a, t)\, dt < 0$ (or $\int_{-\infty}^0 q(a, t)\, dt > 1$). Because of the equivalence of (a) and (b) we can simply speak of a Gibbs phenomenon at the right (left) hand side of zero.

If f is the function defined by (1.3), then we obtain

$$\lim_{m \to \infty} (Q_m f)(2^{-m}x) = \int_{-\infty}^{\infty} q(x, y)\, h(y)\, dy \tag{3.2}$$

where h is defined by

$$h(x) = \begin{cases} 1 & \text{if } x \geq 0; \\ -1 & \text{if } x < 0. \end{cases} \tag{3.3}$$

Let us call this limit the *Gibbs function* associated with the given scaling function. It is a continuous function under the hypotheses of the theorem.

We will also need the difference r between h and the Gibbs function:

$$r(x) = h(x) - \int_{-\infty}^{\infty} q(x, y)\, h(y)\, dy. \tag{3.4}$$

There is a Gibbs phenomenon at the right hand side of 0 if and only if there exists $a > 0$ such that $r(a) < 0$, and at the left hand side of 0 if and only if there exists $a < 0$ such that $r(a) > 0$.

We now use the above theorem to prove existence of the Gibbs phenomenon for wavelet expansions. We first need two more lemmas.

LEMMA 3.3. *Let ϕ be a continuous scaling function satisfying* (2.5) *with $\beta > 1$. Then there is a constant M such that $|r(x)| \leqslant M(1+|x|)^{1-\beta}$ for $x \in \mathbf{R}$. If $\beta > 3/2$ then $r \in L^2(\mathbf{R})$ and r is orthogonal to V_0.*

Proof. By Lemma 2.2, we can write

$$r(x) = \begin{cases} 2\int_{-\infty}^{0} q(x, y)\, dy & \text{if} \quad x \geqslant 0, \\ -2\int_{0}^{\infty} q(x, y)\, dy & \text{if} \quad x < 0. \end{cases}$$

Now Lemma 2.1(a) easily implies the first part of the statement of the lemma. It is then clear that $r \in L^2(\mathbf{R})$ if $\beta > 3/2$. Let f be any integrable function in V_0. We multiply (3.4) by f and integrate to obtain

$$\int_{-\infty}^{\infty} r(x)\, f(x)\, dx = \int_{-\infty}^{\infty} f(x)\, h(x)\, dx - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q(x, y)\, f(x)\, h(y)\, dy\, dx.$$

Using the Fubini theorem for the double integral and $f(y) = \int_{-\infty}^{\infty} q(x, y)\, f(x)\, dx$, we obtain

$$\int_{-\infty}^{\infty} r(x)\, f(x)\, dx = \int_{-\infty}^{\infty} f(x)\, h(x)\, dx - \int_{-\infty}^{\infty} f(y)\, h(y)\, dy = 0.$$

This implies that r is orthogonal to all functions $\phi(x-n)$, $n \in \mathbf{Z}$. It follows that r is orthogonal to V_0. ∎

LEMMA 3.4. *Let ϕ be a continuous and bounded scaling function which is differentiable at a dyadic number d and $\phi'(d) \neq 0$. Let $g \in L^2(\mathbf{R})$ be orthogonal to V_0, and let $xg(x)$ be in $L^1([\mathbf{R}])$. Then $\int_{-\infty}^{\infty} xg(x)\, dx = 0$.*

Proof. The assumptions imply that $g \in L^1(\mathbf{R})$. We first show that $\int_{-\infty}^{\infty} g(x)\, dx = 0$. Let $c = k/2^n$, $k \in \mathbf{Z}$, $n \in \mathbf{N}$, be a dyadic number. Then, as we saw in the proof of Lemma 2.2, the function $\phi(2^{-m}x + c)$ is in V_0 for every $m \geqslant n$. Therefore,

$$\int_{-\infty}^{\infty} \phi(2^{-m}x + c)\, g(x)\, dx = 0 \qquad \text{for} \quad m \geqslant n. \tag{3.5}$$

Letting $m \to \infty$ and using the Lebesgue dominated convergence theorem we find that $\phi(c) \int_{-\infty}^{\infty} g(x)\,dx = 0$. Since this is true for all dyadic numbers c, we obtain $\int_{-\infty}^{\infty} g(x)\,dx = 0$. Using (3.5) with $c = d$ we can write

$$\int_{-\infty}^{\infty} \frac{\phi(2^{-m}x + d) - \phi(d)}{2^{-m}x}\, xg(x)\,dx = 0.$$

Since ϕ is differentiable at d, we obtain for $m \to \infty$ using the dominated convergence theorem again that $\phi'(d) \int_{-\infty}^{\infty} xg(x)\,dx = 0$. Since $\phi'(d) \neq 0$ this implies the desired result. ▮

We can now prove the main theorem of this section.

THEOREM 3.5. *Let ϕ be a continuous scaling function which is differentiable at a dyadic number with nonvanishing derivative there, and which satisfies* (2.5) *with $\beta > 3$. Then the corresponding wavelet expansion shows a Gibbs phenomenon at the right hand side or left hand side of* 0.

Proof. Since $\beta > 3$ Lemma 3.3 shows that the function $g = r$ satisfies the assumptions of Lemma 3.4. Hence

$$\int_{-\infty}^{\infty} xr(x)\,dx = 0.$$

If we would have $r(x) \geqslant 0$ for $x > 0$ and $r(x) \leqslant 0$ for $x < 0$, then this would imply that $r(x) = 0$ almost everywhere. This is impossible because $r - h$ is continuous and h has a jump at $x = 0$. Hence there is $x > 0$ such that $r(x) < 0$ or there is $x < 0$ such that $r(x) > 0$. The remark after (3.4) now implies that there is a Gibbs phenomenon at the right or left hand side of 0. ▮

The inequality $\beta > 3$ appears to enter the assumptions only because of our method of proof. It should be possible to weaken this condition. However, the remaining assumptions cannot be entirely removed because the Haar scaling function (the characteristic function of $[0, 1)$) has to be excluded by the assumptions of the theorem.

The theorem only asserts that there is a Gibbs phenomenon at the right *or* left hand side of zero. If $f(-x) \in V_0$ whenever $f \in V_0$ (e.g., this is true for the scaling functions considered in the next two sections), then, of course, the theorem yields that there is a Gibbs phenomenon on both sides of 0.

4. The Gibbs Phenomenon for Spline Wavelets

In this section we consider wavelets related to spline approximation. The kth order cardinal B-spline $N^{[k]}$ is defined as the k-fold convolution product of the characteristic function of the interval $[0, 1]$ for $k=2, 3, \ldots$ (cf. [4, p. 17]). These functions are not scaling functions in the sense of Section 2 because the first orthogonality condition is not satisfied. The corresponding orthogonalized scaling function $\phi^{[k]}$ (leading to wavelets introduced by Battle [2] and Lemarié [12]) is defined as the function whose Fourier transform is given by

$$\hat{\phi}^{[k]}(\omega)=\left(\frac{1-e^{-i\omega}}{i\omega}\right)^{k}\sigma_{k}(\omega)^{-1/2}, \tag{4.1}$$

where

$$\sigma_k(\omega)=\sin(\omega/2)^{2k}\sum_{n\in\mathbf{Z}}(\omega/2+n\pi)^{-2k}; \tag{4.2}$$

see [4, p. 216]. The space $V_0=V_0^{[k]}$ consists of all square integrable and $k-2$ times continuously differentiable functions that agree with a polynomial function of degree at most $k-1$ on each interval $[n, n+1]$ for $n\in\mathbf{Z}$.

The scaling function $\phi^{[k]}$ satisfies (2.5) for all $\beta>0$. Moreover, $f(-x)\in V_0^{[k]}$ whenever $f\in V_0^{[k]}$. Therefore, Theorem 3.5 yields that there is a Gibbs phenomenon in the corresponding wavelet expansion at both sides of 0 for all $k=2, 3, \ldots$ In fact, in the case of spline wavelets we have the following stronger result whose proof is even simpler than that of Theorem 3.5.

Theorem 4.1. *Let r be defined as in* (3.4) *with respect to the scaling function $\phi^{[k]}$. Then there is $0<x<k$ such that $r(x)<0$, and there is $-k<y<0$ such that $r(y)>0$. Consequently, the corresponding wavelet expansion shows a Gibbs phenomenon at the left and right hand side of* 0.

Proof. The kth order cardinal B-spline $N^{[k]}$ belongs to $V_0^{[k]}$, vanishes outside the interval $(0, k)$ and is positive in this interval. By Lemma 3.3, r is orthogonal to $N^{[k]}$. Hence r changes sign in $(0, k)$ unless r is identically zero in $(0, k)$. The latter is impossible because $r(x)$ is positive for small $x>0$ (the Gibbs function vanishes at 0 by symmetry). If we replace $N^{[k]}$ by $N^{[k]}(x+k)$, we see in the same way that r changes sign in $(-k, 0)$, too. ∎

Without using wavelet theory, the existence of the Gibbs phenomenon was proved for $k=2, \ldots, 8$ and for k sufficiently large by Richards [14].

Let $S^{[k]}$ be the Gibbs function associated with $\phi^{[k]}$, i.e.,

$$S^{[k]}(x)=\int_{-\infty}^{\infty} q^{[k]}(x, y)\, h(y)\, dy, \tag{4.3}$$

where $q^{[k]}$ is the kernel (2.4) corresponding to $\phi^{[k]}$ and h is the function defined by (3.3). Since $T_n^{[k]}=Q_m F$ with $n=2^m$ and the 2-periodic function F defined by (1.1), the function $S^{[k]}$ agrees with the Gibbs spline of order k introduced by Richards [14, p. 338]. Richards [14, p. 335] conjectures the following result.

THEOREM 4.2. *We have*

$$\lim_{k\to\infty} S^{[k]}(x)=\frac{2}{\pi}\int_0^{\pi x}\frac{\sin t}{t}\,dt$$

locally uniformly in x.

The right hand side of this equation is just the Gibbs function for the Shannon scaling function (see Section 1). In [14] the above result is proved for integers x and for even k approaching infinity. The following considerations will lead to a proof of Theorem 4.2. Our first task is to prove some estimates for $\hat{\phi}^{[k]}$.

LEMMA 4.3. *The following estimates hold for all* $\omega\in\mathbf{R}$:

$$|\hat{\phi}^{[k]}(\omega)|\leqslant 1, \tag{4.4}$$

$$|\hat{\phi}^{[k]}(\omega)|\leqslant\left|\frac{\sin(\omega/2)}{\omega/2}\right|^k(\pi/2)^k, \tag{4.5}$$

$$|\hat{\phi}^{[k]}(\omega)|\leqslant\min(1, |\pi/\omega|^k). \tag{4.6}$$

Proof. Because of the first property of scaling functions we know [4, Thm. 3.23] that

$$\sum_n |\hat{\phi}^{[k]}(\omega+2n\pi)|^2=1, \tag{4.7}$$

so the first estimate is clear. By [4, p. 90], we have

$$\sigma_k(\omega)\geqslant\sigma_k(\pi) \qquad \text{for all} \quad \omega\in\mathbf{R}.$$

The definition of σ_k shows immediately that

$$\sigma_k(\pi) \geqslant (\pi/2)^{-2k},$$

so the second estimate follows. The third estimate follows from the first two inequalities by noting that $|\sin(\omega/2)| \leqslant 1$. ▮

LEMMA 4.4. *The 2π-periodic functions*

$$\sum_n |\hat{\phi}^{[k]}(\omega + 2n\pi)|$$

are uniformly bounded with respect to $\omega \in \mathbf{R}$ and $k \geqslant 2$. Moreover,

$$\sum_{n \neq 0} |\hat{\phi}^{[k]}(\omega + 2n\pi)| \to 0 \qquad \text{as} \quad k \to \infty \qquad \text{for} \quad |\omega| < \pi.$$

Proof. Let $-\pi \leqslant \omega \leqslant \pi$. Then, by (4.6),

$$\sum_n |\hat{\phi}^{[k]}(\omega + 2n\pi)| \leqslant 1 + \sum_{n \neq 0} \left| \frac{\pi}{\omega + 2n\pi} \right|^k.$$

Since the basis of the above power is $\leqslant 1$, we obtain

$$\sum_n |\hat{\phi}^{[k]}(\omega + 2n\pi)| \leqslant 1 + \sum_{n \neq 0} \left| \frac{\pi}{\omega + 2n\pi} \right|^2 \leqslant 1 + 2 \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

This proves the first part of the lemma. Let $|\omega/\pi| \leqslant s < 1$. Then, by (4.6),

$$\sum_{n \neq 0} |\hat{\phi}^{[k]}(\omega + 2n\pi)| \leqslant \sum_{n \neq 0} \left| \frac{\pi}{\omega + 2n\pi} \right|^k$$

$$\leqslant 2 \sum_{n=1}^{\infty} (2n - s)^{-k} \to 0 \qquad \text{as} \quad k \to \infty$$

which completes the proof. ▮

LEMMA 4.5. *The sequence of functions* $\exp(i\omega k/2)\, \hat{\phi}^{[k]}(\omega)$ *converges to the characteristic function $\chi_{(-\pi, \pi)}(\omega)$ of the interval $(-\pi, \pi)$ pointwise a.e. and in $L^2(\mathbf{R})$ as $k \to \infty$.*

Proof. If $|\omega| > \pi$, the pointwise convergence follows from (4.6). If $|\omega| < \pi$, then, by (4.7) and Lemma 4.4,

$$0 \leqslant 1 - |\hat{\phi}^{[k]}(\omega)|^2 = \sum_{n \neq 0} |\hat{\phi}^{[k]}(\omega + 2n\pi)|^2$$

$$\leqslant \sum_{n \neq 0} |\hat{\phi}^{[k]}(\omega + 2n\pi)| \to 0 \qquad \text{as} \quad k \to \infty.$$

This implies $|\hat{\phi}^{[k]}(\omega)| \to 1$ as $k \to \infty$. Since $|\hat{\phi}^{[k]}(\omega)| = \exp(i\omega k/2)\, \hat{\phi}^{[k]}(\omega)$, this proves the pointwise convergence for $|\omega| < \pi$. By (4.6), the sequence $|\hat{\phi}^{[k]}(\omega)|$ is uniformly dominated by a function in $L^2(\mathbf{R})$. Thus pointwise convergence implies convergence in the $L^2(\mathbf{R})$ norm. ∎

For generalizations of the above lemma we refer to [1] and [9]. It follows that

$$\phi^{[k]}(x + k/2) \to \frac{\sin(\pi x)}{\pi x} \qquad \text{as} \quad k \to \infty$$

in the $L^2(\mathbf{R})$ norm. We recognize the right hand side as the Shannon scaling function.

We now turn to the investigation of the asymptotic behavior of the kernel function $q^{[k]}$ as $k \to \infty$. The Fourier transform of $q^{[k]}(x, y)$ with respect to y is given by

$$\hat{q}^{[k]}(x, \omega) = \hat{\phi}^{[k]}(\omega) \sum_n \phi^{[k]}(x - n) \exp(-i\omega n).$$

If we use a well known identity related to the Poison summation formula [10, p. 128], we obtain

$$\hat{q}^{[k]}(x, \omega) = \hat{\phi}^{[k]}(\omega) \sum_n \overline{\hat{\phi}^{[k]}(\omega + 2n\pi)} \exp\left(-i(\omega + 2n\pi)\, x\right). \tag{4.8}$$

Applying the following lemma to (4.3), we can rewrite the Gibbs function as

$$S^{[k]}(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Im} \hat{q}^{[k]}(x, \omega)}{\omega}\, d\omega. \tag{4.9}$$

LEMMA 4.6. *Let* $f: \mathbf{R} \to \mathbf{R}$ *be such that* $f \in L^2(\mathbf{R}) \cap L^1(\mathbf{R})$ *and* $xf(x) \in L^1(\mathbf{R})$. *Then*

$$\int_{-\infty}^{\infty} f(x)\, h(x)\, dx = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Im} \hat{f}(\omega)}{\omega}\, d\omega.$$

Proof. We have

$$\frac{\operatorname{Im} \hat{f}(\omega)}{\omega} = -\int_{-\infty}^{\infty} \frac{\sin(x\omega)}{\omega} f(x)\, dx.$$

The assumptions guarantee that $\sin(x\omega) f(x)/\omega$ is integrable over $\omega \in (-a, a)$, $x \in \mathbf{R}$, for every $a > 0$. Therefore, by the Fubini theorem,

$$\int_{-a}^{a} \frac{\operatorname{Im} \hat{f}(\omega)}{\omega} \, d\omega = -\int_{-\infty}^{\infty} \left(\int_{-a}^{a} \frac{\sin(x\omega)}{\omega} \, d\omega \right) f(x) \, dx.$$

Letting $a \to \infty$, the Lebesgue dominated convergence theorem yields the stated result. ∎

LEMMA 4.7. *For every* $x \in \mathbf{R}$, *we have*

$$\hat{q}^{[k]}(x, \omega) \to \chi_{(-\pi, \pi)}(\omega) \exp(-i\omega x) \qquad \textit{as} \quad k \to \infty$$

pointwise a.e. and in the $L^2(\mathbf{R})$ *norm.*

Proof. The pointwise convergence follows easily from (4.8) and Lemmas 4.4 and 4.5. We obtain convergence in the $L^2(\mathbf{R})$ norm because the functions $\hat{q}^{[k]}(x, \cdot)$ are uniformly dominated by a function in $L^2(\mathbf{R})$, see Lemmas 4.3 and 4.4. ∎

As a consequence we find that

$$q^{[k]}(x, y) \to \frac{\sin(\pi(y-x))}{\pi(y-x)} \qquad \text{as} \quad k \to \infty,$$

in the $L^2(\mathbf{R})$ norm for fixed x as functions of y.

We are now in a position to prove Theorem 4.2.

Proof of Theorem 4.2. It follows from (4.6) and Lemma 4.4 that

$$|\hat{q}^{[k]}(x, \omega)| \leqslant L |\pi/\omega|^k$$

where L is a constant independent of x, ω, k. This implies that

$$\int_{|\omega| \geqslant \pi} \frac{\operatorname{Im} \hat{q}^{[k]}(x, \omega)}{\omega} \, d\omega \to 0 \qquad \text{as} \quad k \to \infty$$

uniformly with respect to $x \in \mathbf{R}$. It remains to treat the integral in (4.9) between the limits $-\pi \leqslant \omega \leqslant \pi$. We first consider only the terms with $n \neq 0$ in (4.8). By (4.5), we have for $n \neq 0$

$$|\hat{\phi}^{[k]}(\omega + 2n\pi)| \leqslant \frac{|\sin(\omega/2)|^k}{|\omega/2 + n\pi|^k} \left(\frac{\pi}{2}\right)^k \leqslant \frac{|\omega|}{|\omega/2 + n\pi|^k} \left(\frac{\pi}{2}\right)^k,$$

where we estimated $|\sin(\omega/2)|^k \leqslant |\omega|\,|\sin(\omega/2)|^{k-1} \leqslant |\omega|$. For $|\omega/\pi| \leqslant s \leqslant 1$ we obtain

$$\left|\frac{\hat{\phi}^{[k]}(\omega+2n\pi)}{\omega}\right| \leqslant \frac{1}{(2|n|-s)^k}.$$

This shows that the functions

$$\sum_{n\neq 0}\left|\frac{\hat{\phi}^{[k]}(\omega+2n\pi)}{\omega}\right|, \qquad k\geqslant 2,$$

are uniformly bounded for $\omega \in (-\pi,\pi)$ and converge pointwise to 0 as $k\to\infty$ in this interval. Hence its integrals over $\omega\in(-\pi,\pi)$ converge to 0 as $k\to\infty$. We have thus shown that

$$S^{[k]}(x) - \frac{1}{\pi}\int_{-\pi}^{\pi} |\hat{\phi}^{[k]}(\omega)|^2 \frac{\sin(\omega x)}{\omega}\, d\omega \to 0 \qquad \text{as} \quad k\to\infty \tag{4.10}$$

uniformly with respect to $x\in\mathbf{R}$. We can now write

$$\left|\int_{-\pi}^{\pi} |\hat{\phi}^{[k]}(\omega)|^2 \frac{\sin(\omega x)}{\omega}\, d\omega - \int_{-\pi}^{\pi} \frac{\sin(\omega x)}{\omega}\, d\omega\right|$$

$$\leqslant |x| \int_{-\pi}^{\pi} (1-|\hat{\phi}^{[k]}(w)|^2)\, d\omega \to 0 \tag{4.11}$$

as $k\to\infty$ locally uniformly with respect to x. We just used (4.4) and Lemma 4.5. Now (4.10) and (4.11) imply Theorem 4.2. ∎

5. Approximation with Positive Kernels

In order to avoid the Gibbs phenomenon we can try to approximate a function f by functions in V_m other than the orthogonal projections $Q_m f$. First we note that there is an obvious generalization of some of the ideas in Sections 2 and 3. Let us call a function $p\colon \mathbf{R}^2\to\mathbf{R}$ an *admissible* kernel if it has the following properties:

1. p is continuous;
2. there is a bounded function $\rho\in L^1(\mathbf{R})$ such that $|p(x,y)|\leqslant \rho(x-y)$ for all $x, y\in\mathbf{R}$;
3. $\int_{-\infty}^{\infty} p(x,y)\,dy = 1$ for all $x\in\mathbf{R}$.

The integral operators

$$(P_\lambda f)(x) = \int_{-\infty}^{\infty} \lambda p(\lambda x, \lambda y)\, f(y)\, dy, \qquad \lambda > 0, \tag{5.1}$$

then map functions $f \in L^2(\mathbf{R})$ to continuous functions in $L^2(\mathbf{R})$. For example, to show that $P_1 f \in L^2(\mathbf{R})$ we use the Schwarz inequality to estimate

$$|(P_1 f)(x)|^2 \leqslant \int_{-\infty}^{\infty} |p(x, y)|\, dy \int_{-\infty}^{\infty} |p(x, y)|\, |f(y)|^2\, dy.$$

Thus

$$\int_{-\infty}^{\infty} |(P_1 f)(x)|^2\, dx \leqslant \left(\int_{-\infty}^{\infty} \rho(t)\, dt \right)^2 \int_{-\infty}^{\infty} |f(y)|^2\, dy < \infty.$$

In the notation of Section 2, we have $Q_m = P_\lambda$ for $\lambda = 2^m$ if $p = q$ is derived from a scaling function that satisfies the assumption of Lemma 2.1. Moreover, if $p(x, y) = u(x - y)$ is an admissible kernel of convolution type, then the family $\lambda p(\lambda x, \lambda y)$, $\lambda > 0$, is a summability kernel in the sense of [10, p. 124]. It is easy to show that $(P_\lambda f)(x)$ converges locally uniformly to $f(x)$ for every bounded and continuous function f whenever p is an admissible kernel; compare [10, ex. 10, p. 130].

Analogous to Definition 3.1, we define the Gibbs phenomenon for the approximation of a function f with a jump discontinuity at 0 by $P_\lambda f$ as $\lambda \to \infty$. We verify that Theorem 3.2 remains valid in this case: there is a Gibbs phenomenon at the right hand side of zero if and only if there is $a > 0$ such that $\int_0^\infty p(a, y)\, dy > 1$. In particular, we see that there is no Gibbs phenomenon if the kernel p is positive (i.e., $p(x, y) \geqslant 0$ for all $x, y \in \mathbf{R}$). Indeed, if p is positive then $0 \leqslant \int_0^\infty p(a, y)\, dy \leqslant 1$ for all $a \in \mathbf{R}$. Our goal is to find positive admissible kernels with the additional property that P_{2^m} maps $L^2(\mathbf{R})$ into V_m, $m \in \mathbf{Z}$, for given multiresolution spaces V_m. Of course, it is sufficient to verify this latter property for $m = 0$.

As an example, we first consider the Shannon scaling function (1.2) with corresponding (not admissible and not positive) kernel

$$q(x, y) = \frac{\sin(\pi(x - y))}{\pi(x - y)}.$$

For $\lambda > 0$ and $f \in L^2(\mathbf{R})$, we define

$$f_\lambda(x) = \int_{-\infty}^{\infty} \lambda q(\lambda x, \lambda y)\, f(y)\, dy.$$

f_λ is the orthogonal projection of f onto the subspace of functions whose Fourier transforms vanish outside $(-\lambda\pi, \lambda\pi)$. If $\lambda = 2^m$ then $Q_m f = f_\lambda$. Analogous to the Fejér method of summation of Fourier series, we consider

$$(P_\lambda f)(x) = \frac{1}{\lambda}\int_0^\lambda f_\mu(x)\,d\mu = \frac{1}{\lambda}\int_0^\lambda \int_{-\infty}^{\infty} \mu q(\mu x, \mu y)\, f(y)\, dy\, d\mu.$$

Since $\mu q(\mu x, \mu y) f(y)$ is integrable over $(\mu, y) \in (0, \lambda) \times \mathbf{R}$, the Fubini theorem leads to

$$(P_\lambda f)(x) = \int_{-\infty}^{\infty} \lambda p(\lambda x, \lambda y)\, f(y)\, dy,$$

where

$$p(x, y) = \int_0^1 t q(tx, ty)\, dt.$$

The new kernel is given explicitly by

$$p(x, y) = \int_0^1 \frac{\sin(\pi t(x-y))}{\pi(x-y)}\, dt = 2\left\{\frac{\sin(\pi(x-y)/2)}{\pi(x-y)}\right\}^2. \tag{5.2}$$

We see that P_λ is a convolution operator $P_\lambda f = u_\lambda * f$ where

$$u_\lambda(x) = \lambda u(\lambda x) \qquad \text{and} \qquad u(x) = 2\left\{\frac{\sin(\pi x/2)}{\pi x}\right\}^2. \tag{5.3}$$

THEOREM 5.1. *The function $p(x, y)$ of* (5.2) *is an admissible and positive kernel for the Shannon system in the sense that $P_{2^m} f \in V_m$ for all $f \in L^2(\mathbf{R})$ and $m \in \mathbf{Z}$.*

Proof. Clearly, p is admissible and positive. Let $f \in L^2(\mathbf{R})$. In order to show that $P_{2^m} f \in V_m$ it is sufficient to consider the case $m = 0$. We note that the Fourier transform of $P_1 f = u * f$ is the product of the Fourier transform of f and a function that vanishes outside $[-\pi, \pi]$. Since the Fourier transforms of functions in V_0 are exactly those functions that vanish outside this interval, it follows that $P_1 f \in V_0$. ∎

We now construct a positive summability kernel for wavelets generalizing those introduced by Meyer [13, p. 23]. We say that ϕ is a scaling function of Meyer-type if its Fourier transform $\hat{\phi}$ shares the following properties:

1. $\hat{\phi} \in L^2(\mathbf{R})$ vanishes outside the interval $[-4\pi/3, 4\pi/3]$;
2. $\hat{\phi}(\omega) = 1$ for $\omega \in [-2\pi/3, 2\pi/3]$;
3. $|\hat{\phi}(\omega)|^2 + |\hat{\phi}(\omega + 2\pi)|^2 = 1$ for all $\omega \in [-4\pi/3, -2\pi/3]$;
4. $\hat{\phi}$ is real-valued and even.

It is easy to see that ϕ is a real-valued even (entire) scaling function. Therefore, Theorem 3.5 shows that we always have a Gibbs phenomenon in the wavelet expansion using a wavelet of Meyer-type as soon as we can take $\beta > 3$ in (2.5). However, this assumption is not necessary as was shown in a different way in [15].

THEOREM 5.2. *Let ϕ be a scaling function of Meyer-type, and let*

$$u(x) = \lambda u(\lambda x) \qquad \text{where} \quad u(x) = \frac{1}{4} \phi\left(\frac{x}{4}\right)^2.$$

Then $p(x, y) = u(x - y)$ is an admissible and positive kernel for the given ϕ, i.e., $P_{2^m} f \in V_m$ for all $f \in L^2(\mathbf{R})$ and $m \in \mathbf{Z}$.

Proof. Since $\int_{-\infty}^{\infty} \phi(x)^2 \, dx = 1$, it follows that $\int_{-\infty}^{\infty} u(x) \, dx = 1$ which proves the first part of the statement. Let $f \in L^2(\mathbf{R})$. In order to show that $P_{2^m} f$ belongs to V_m it is again sufficient to consider the case $m = 0$. Now the Fourier transform of $P_1 f = u * f$ is the product of the Fourier transform of u and the Fourier transform of f. Since $\hat{\phi}$ vanishes outside $[-4\pi/3, 4\pi/3]$, the Fourier transform of $\phi(x)^2$ vanishes outside $[-8\pi/3, 8\pi/3]$. Then the Fourier transform of u and thus also the Fourier transform of $P_1 f$ vanishes outside $[-2\pi/3, 2\pi/3]$. This already implies that $P_1 f \in V_0$ as is shown in the following simple lemma. ∎

LEMMA 5.3. *In the Meyer-type case, every function $f \in L^2(\mathbf{R})$ whose Fourier transform vanishes outside $[-2\pi/3, 2\pi/3]$ belongs to V_0.*

Proof. Let f be a function as given in the statement of the lemma. Let F be the 2π-periodic extension of $\hat{f}$. Because of the first two properties of $\hat{\phi}$ for scaling functions of Meyer-type, we have $\hat{f} = \hat{\phi} F$. If we replace F by its Fourier series and apply the inverse Fourier transform, we see that f is a l^2 linear combination of the functions $\phi(x - n)$, $n \in \mathbf{Z}$. Hence $f \in V_0$. ∎

We now show how to find a positive kernel for the spline approximation considered in Section 4. We define

$$p^{[k]}(x, y) = \sum_n N^{[k]}(x - n) \, N^{[k]}(y - n), \tag{5.4}$$

where $N^{[k]}$ is the kth order cardinal B-spline. The series is locally uniformly convergent and thus defines a continuous function.

THEOREM 5.4. *For every* $k = 2, 3, \ldots$, $p^{[k]}(x, y)$ *is an admissible and positive kernel with the property that* $P_{2^m} f$ *belongs to the spline multiresolution space* $V_m^{[k]}$ *for every* $f \in L^2(\mathbf{R})$ *and* $m \in \mathbf{Z}$.

Proof. The second property of an admissible kernel is satisfied trivially because $p^{[k]}(x, y)$ is bounded and vanishes for $|x - y| \geqslant k$. The third property follows from the fact that $\sum_n N^{[k]}(x - n) = 1$ for all x (which implies $\int_{-\infty}^{\infty} N^{[k]}(y)\, dy = 1$); see [4, Thm. 4.3 (vi)]. Thus $p^{[k]}$ is admissible. It is positive because $N^{[k]}(x) \geqslant 0$ for all x. Since $N^{[k]} \in V_0^{[k]}$, we see that the remaining part of the statement is true. ∎

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