Boundary Layers for Viscous Perturbations of Noncharacteristic Quasilinear Hyperbolic Problems

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In this paper, we study viscous perturbations of quasilinear hyperbolic systems in several dimensions as the viscosity goes to zero. The boundary is noncharacteristic for the hyperbolic system. We in particular describe the boundary layer which arises near the boundary and give a sufficient condition for the convergence of the solution to the solution of some mixed hyperbolic problem with some nonlinear maximal dissipative boundary conditions. A counterexample is given when this condition is not satisfied, and the solution blows up as the viscosity goes to 0.

1. INTRODUCTION

In this paper we consider the Cauchy Dirichlet problem for a parabolic system obtained by small viscosity perturbation of a quasilinear hyperbolic system, the boundary being noncharacteristic for the hyperbolic operator. The goal is to describe the behavior of the (smooth and local in time) solution $u^\varepsilon$ of the viscous problem, as $\varepsilon$ goes to 0.

We prove that the solution $u^\varepsilon$ exists on a domain of space time independent of $\varepsilon \in [0, \varepsilon_0]$ and obtain an asymptotic expansion of $u^\varepsilon$ that describes...
very accurately the nonlinear boundary layer forming near the boundary. A consequence is that $u'$ converges, as $\varepsilon$ goes to 0, to the solution $u^0$ of some well posed quasilinear mixed problem with maximal dissipative nonlinear boundary conditions. These results are obtained under an interesting smallness condition on the data. A counterexample is given, showing that this condition cannot be avoided in general, where the solution of the viscous problem blows up and does not converge, as $\varepsilon \to 0$, to the solution of the hyperbolic limit problem suggested by a formal analysis.

In the linear case this problem was studied by Bardos et al. [2], Lions [18], and Rauch [3], however, without boundary layers analysis. Recently, progress has been made in the nonlinear case using boundary layers analysis. The semilinear case was solved by Guès [11], [12]. In the quasilinear case in one space dimension for systems of conservation laws with convex entropy, the problem was solved by Gisclon and Serre [7, 8, 25].

Concerning the case when the boundary is characteristic for the hyperbolic operator, the problem was solved in the linear case by Bardos and Rauch [3] and in the semilinear case by O. Guès [11], [12]. In that case, using methods of nonlinear geometric optics, one can study the interaction between high frequency oscillations propagating along the boundary and the boundary layer [13]. In the quasilinear case, a special case of a totally characteristic boundary was treated by Grenier [9], [10]. A very challenging problem with a characteristic boundary is the problem of the convergence of the solution of the Navier-Stokes equations with homogeneous Dirichlet condition to the solution of the Euler equations with natural slip boundary condition. For analytic data, such a result has been obtained by Asano [1] and Caflisch and M. Sanmartino [4], [5].

2. STATEMENTS OF THE RESULTS

1.1. Assumptions

For every $b \in \mathbb{R}^N$, $u \in \mathbb{R}^N$, we call $\mathcal{H}(b, u)$ the following first order operator in $\mathbb{R} \times \mathbb{R}_x^n$ (we note $x = (x_1, ..., x_n)$, $\partial_j = \partial/\partial t$, $\partial_j = \partial/\partial x_j$):

$$\mathcal{H}(b, u) = \partial_t + \sum_{j=1}^n A_j(b, u) \partial_j + B(b, u),$$  \hspace{1cm} (1)

where $A_j(b, u)$, $j = 0, ..., n$, $B(b, u)$ are $N \times N$ real matrices depending smoothly on variables $b \in \mathbb{R}^N$, $u \in \mathbb{R}^N$. In the following we will note $p = (b, u)$ the dependent variables in $\mathbb{R}^N \times \mathbb{R}^N$. The first assumption is hyperbolicity of $\mathcal{H}(b, u)$:
Assumption 1.1. There exists an $N \times N$ symmetric matrix $S(p)$, depending smoothly on $p \in \mathbb{R}^{N+N}$ such that, for each $p \in \mathbb{R}^{N+N}$, $S(p)$ is positive definite and $SA_j(p)$ is symmetric, $j = 1, ..., N$.

The second assumption concerns the matrix $A_d(p)$. It states that the negative invariant subspace of $A_d(p)$, $N(p) = \sum_{\lambda \leq 0} \ker(A_d(p) - \lambda I)$, has a constant dimension $d$, and that the hypersurfaces $\{x_n = \text{cst}\}$ are not characteristic for $\mathcal{W}(p)$:

Assumption 1.2. There is a $C^\infty$ basis $(r_1(p), ..., r_N(p))$ of $\mathbb{R}^N$, such that $r_j(p)$, $j = 1, ..., N$, are eigenvectors of $A_d(p)$ corresponding to respective eigenvalues $\lambda_j(p)$ which satisfy $\lambda_j(p) \neq 0, \forall p \in \mathbb{R}^{N+N}, \forall j \in \{0, ..., N\}$.

This assumption implies that the eigenvalues depend smoothly on $p$ and that the number of negative eigenvalues is constant: $\sharp\{j\mid \lambda_j(p) < 0\} = d$ for some $d \in \{0, 1, ..., N\}$ independent of $p$. This, in turn, implies that $N(p)$ depends smoothly on $p$, having constant dimension $d$. For example, Assumption 1.2 is satisfied when the system is strictly hyperbolic (that is $\lambda_1 < \cdots < \lambda_N$) with nonvanishing eigenvalues; it is also satisfied in the more general case of nonvanishing eigenvalues with constant multiplicities. However, Assumption 1.2 allows the crossing of smooth eigenvalues.

1.2. Setting of the Problem and Main Results

We fix a function $b(t, x) \in H^\infty(\mathbb{R}^{1+n}; \mathbb{R}^N)$ for all of the following. For $T > 0$, let $\Omega_T = \{(t, x) \in \mathbb{R}^{1+n} \mid -1 < t < T, x_n > 0\}$ and $\Gamma_T = \{(t, x) \in \mathbb{R}^{1+n} \mid -1 < t < T, x_n = 0\}$.

Let $f(t, x) \in H^\infty(\mathbb{R}^{1+n}; \mathbb{R}^N)$ be a given function satisfying $f(t, x) = 0$ if $t \leq 0$, which will play the role of a source term. For $\varepsilon > 0$ and $T > 0$, we are interested in the following mixed problem:

$$P(\varepsilon, T) \left\{ \begin{array}{ll}
-\varepsilon A_d u' + \mathcal{W}(b, u') u' = f \\
|u'|_{\Omega_T} = 0, & u'|_{\partial \Omega_T} = 0.
\end{array} \right. \quad (2)$$

Classical results on nonlinear parabolic problems [17, 20] assert that for every $\varepsilon > 0$ there is some $T_\varepsilon > 0$ and a unique $u' \in H^\infty(\Omega_T)$ solution of $P(\varepsilon, T)$. Our first result is that the life span of this regular solution does not shrink to zero as $\varepsilon$ goes to 0:

**Theorem 1.3.** There exists $T_0 > 0$ and $\varepsilon_0 > 0$ such that, for every $\varepsilon \in [0, \varepsilon_0]$, the problem $P(\varepsilon, T_0)$ has a unique solution $u' \in H^\infty(\Omega_{T_0})$.  


We want now to describe the behavior of $u'$ as $\varepsilon$ goes to 0. To this end, we consider the following system of ordinary differential equations where the unknown is $V \in C^\infty([0, +\infty[; \mathbb{R}^N)$, while $b \in \mathbb{R}^N$, $v \in \mathbb{R}^N$ are parameters. This two points boundary value problem (as we shall see in details in Section 2) arises when looking formally for a solution of $P(\varepsilon, T)$ of the form $U(t, x, \varepsilon) = V(b(t, x), z) + \varepsilon(t, x)$ where the smooth boundary profile $V(b, z)$ vanishes at $z = +\infty$:

$$\begin{align*}
V' &= A_s(b, V + v)V', \\
V(0) &= -v, \\
\lim_{z \to \infty} V &= 0, \\
V &\in C^\infty([0, +\infty[; \mathbb{R}^N).
\end{align*}$$

(3)

The same system was introduced in the works of Gisclon and Serre for the study of a system of conservation laws in one dimension. In that case, because of the conservative structure, an integration leads to a first order system instead of (3). However, despite the fact that such an integration cannot be done in system (3), a similar analysis to that of Gisclon and Serre can be carried out.

Since the function $b(t, x)$ belongs to $H^s(\mathbb{R}^1; \mathbb{R}^N)$ it is bounded, and we fix $\rho > 0$ such that $b(t, x)$ takes its values in $B_\rho$, the open ball of radius $\rho$ and center 0 in $\mathbb{R}^N$. Following Gisclon and Serre, we introduce for every $b \in B_\rho$ the set

$$\varnothing(b) = \{ v \in \mathbb{R}^N \text{ such that (3) has one solution } \}.$$

Proposition 1.4. There is a neighborhood $\mathcal{V}$ of 0 in $\mathbb{R}^N$ such that, for every $b$ in $B_\rho$, $\varnothing(b)$ is a smooth manifold in $\mathcal{V}$, depending smoothly on $b$ in $B_\rho$, passing through 0 in $\mathcal{V}$ with tangent space at this point $T_0 \varnothing(b) = N(b, 0)$. Moreover, the function $V(b, v; z)$ extends to a smooth map $V : B_\rho \times \mathcal{V} \times [0, +\infty[ \to \mathbb{R}^N$ and for any multi-index $\alpha \in \mathbb{N}^N$, $\beta \in \mathbb{N}^N$, $\gamma \in \mathbb{N}$, $k = |\alpha| + |\beta| + |\gamma|$, there are positive constants $c_k$, $\tau_k$ such that

$$|\partial^\alpha_x \partial^\beta_v \partial^\gamma_z V(b, v; z)| \leq c_k \exp(-\tau_k z),$$

in $B_\rho \times \mathcal{V} \times [0, +\infty[$.

Proof. We write the system in a first order form and consider the following dynamical system with parameters $b \in \mathbb{R}^N$, $v \in \mathbb{R}^N$, $u = (u', u'') \in \mathbb{R}^N \times \mathbb{R}^N$

$$U'' = A(b, v; U), \quad U(0) = u,$$  

(4)
where

\[ U = \begin{pmatrix} V \\ W \end{pmatrix} \in \mathbb{R}^{2N}, \quad A(b, v; U) = \begin{bmatrix} 0 & I_N \\ 0 & A_d(b, V + v) \end{bmatrix}. \]  

(5)

Choose \( b \in B_p \) and \( v \in \mathbb{R}^d \). The matrix

\[ A(b, v; 0) = \begin{pmatrix} 0 & I \\ 0 & A_d(b, v) \end{pmatrix} \]

has null space

\( \mathcal{N} = \{ U_{N+1} = \cdots = U_{2N} = 0 \} = \{ W = 0 \} \).

If \( r_j(b, v) \) are the eigenvalues of \( A_d(b, v) \), \( r_1(b, v), \ldots, r_d(b, v) \) being a smooth basis of \( \mathbb{R}^d \), it follows that \( A(b, v; 0) \) has eigenvalues 0 (multiplicity \( N \)) and \( \lambda_1, \ldots, \lambda_d(b, v) \) with corresponding eigenvectors \( R_j(b, v) = (r_j, \lambda_j r_j) \). The invariant manifolds theorems [15, 16, 23] tell that system (4) has, near 0, a center manifold of dimension \( N \), tangent to \( \mathcal{N} \) at 0. Since every point of \( \mathcal{N} \) is a critical point of (4), \( \mathcal{N} \) is contained in every center manifold near 0 [16] which implies that \( \mathcal{N} \) is the unique center manifold locally in 0. Let us call \( W_{h,v}^s \) the stable manifold of 0 for (4). Since there is no solution \( U \) of (4) with data on \( \mathcal{N} \) such that \( U(+\infty) = 0 \), we know that, in a neighborhood of 0 in \( \mathbb{R}^{2N} \),

\( W_{h,v}^s = \{ v \in \mathbb{R}^{2N} | (4) \text{ has a solution with } \lim_{+\infty} U = 0 \} \).

Now the tangent space at 0 to \( W_{h,v}^s \) is

\[ T_0 W_{h,v}^s = \sum_{1 \leq j \leq d} \mathbb{R} \cdot R_j(b, v), \]

and for any \( u' \in N(b, v) \), there is one and only one \( u'' \in \mathbb{R}^N \) such that \((u', u'') \in T_0 W_{h,v}^s\). This implies that, in a neighborhood of 0 in \( \mathbb{R}^{2N} \), \( p: (u', u'') \mapsto u'' \) induces a one to one projection of \( W_{h,v}^s \) on a smooth manifold \( \Sigma_{h,v} \in \mathbb{R}^N \) whose tangent space at 0 is \( T_0 \Sigma_{h,v} = N(b, v) \). Summing up, \( \Sigma_{h,v} \) is a smooth manifold of dimension \( d \) in a neighborhood \( C_{h,v} \) of 0 in \( \mathbb{R}^N \), such that

\[ \Sigma_{h,v} = \{ u'' \in C_{h,v} \mid \text{the system } V' = A_d(b, V + v) V', \ V(0) = u', \ V(+\infty) = 0, \]

has a unique smooth solution}.  

By smooth dependence on the parameters, letting \( v \) vary in a bounded set \( \mathscr{V} \), we can choose \( C_{h,v} = C \) independent of \((b, v) \in B_p \times \mathscr{V} \). At this stage, we repeat the argument in ([7]). Taking \( \mathscr{V} \) small enough, the set
\[ \Sigma_b^b : \{(v, w) \in v' \times v'' | v, w \in \Sigma(b, v') \} \] is a smooth submanifold of \( \mathbb{R}^n \times \mathbb{R}^n \), depending smoothly on \( b \in B_p \), and whose tangent space at \( 0 \in \mathbb{R}^{2N} \) is \( T_{\Sigma(b, 0)} = T_{\Sigma(b, 0)}^\mathbb{R}^n = N(b, 0) \times \mathbb{R}^n \). It follows that \( \Sigma_b^b \) is transverse to the diagonal \( d = \{(v, w) \in \mathbb{R}^{2N}, v + w = 0 \} \), and the transverse intersection \( \mathcal{J}_b := \Sigma_b^b \cap d \) is a smooth manifold, in a neighborhood of \( 0 \in \mathbb{R}^{2N} \), whose tangent space at \( 0 \) is \( T_{\mathcal{J}_b(0)} = \{(v, -v), v \in N(b, 0)\} \). Now, taking again \( v' \) small enough, the set
\[ \mathcal{J}(b) = \{v \in v' | -v \in \Sigma_b^b \} \]
writes \( \mathcal{J}(b) = p(\mathcal{J}_b) \) and is then a smooth manifold passing through \( 0 \in \mathbb{R}^n \) with tangent space at \( 0 \) is \( T_{\mathcal{J}(b, 0)} = p(T_{\mathcal{J}_b(0)}) = N(b, 0) \). Smoothness of \( V(b, v; z) \) and exponential decay are classical results for dynamical systems [15, 16] and the extension of \( V \) to \( B_p \times v' \times [0, +\infty[ \) is straightforward.

**Proposition 1.5.** There is a neighborhood \( \omega \) of \( 0 \) in \( \mathbb{R}^n \) such that, for every \( b \in B_p \) and \( u \in \mathcal{J}(b) \cap \omega \), the tangent space \( T_{\mathcal{J}(b)} \) is a maximal negative subspace for the quadratic form: \( \zeta \in \mathbb{R}^n \rightarrow \langle SA_b(u), \zeta, \zeta \rangle \).

Recall that a subspace \( N \) of \( \mathbb{R}^n \) is maximal negative for a quadratic form \( q(\cdot) \) if \( N \) is a subspace of maximal dimension such that \( q(\zeta) \leq 0 \) for every \( \zeta \in N \) ([19]). This maximal dimension is also the number of nonpositive eigenvalues of the (symmetric) matrix of \( q \), in any basis of \( \mathbb{R}^n \) ([2]). It is called maximal strictly negative when the restriction of \( q \) to the space \( N \cap (\ker q)^\perp \) is negative definite. In the case when \( q \) is a definite quadratic form, this last condition reduces to: \( q|_N \) is (negative) definite.

**Proof.** We prove in fact a stronger result: one can find \( \omega \) such that \( T_{\mathcal{J}(b)} \) is a maximal strictly negative subspace for \( SA_b(u,u) \), uniformly with respect to \( (b, u) \in B_p \times (\mathcal{J}(b) \cap \omega) \). Choose any regular equation of \( \mathcal{J}(b) \) in \( B_p \times \omega \) for some small enough neighborhood \( \omega \) of \( 0 \), that is a smooth map \( \Phi \in C^\infty(B_p \times \omega, \mathbb{R}^n) \) with rank \( \Phi' = N - d \) in \( B_p \times \omega \) and such that \( \mathcal{J}(b) = \{v \in \Phi(b, v) = 0\} \). The property we have to prove means that there is \( c_0 > 0 \) such that for every \( b \in B_p \), \( u \in \omega \) with \( \Phi(b, u) = 0 \):
\[ \langle A_b, u, \zeta, \zeta \rangle \leq -c_0 \|\zeta\|^2 \]
for every \( \zeta \in \mathbb{R}^n \) such that \( \Phi'(b, u) \cdot \zeta = 0 \). When \( u = 0 \), since \( \ker \Phi'(b, 0) = N(b, 0) \), it is a lemma by Bardos et al. [2]. Further properties concerning this strict negativeness can be found in the paper by Rauch [23]. Then, by continuity and compactness, the property remains true in \( B_p \times \omega \) for small enough \( \omega \).

It is then a classical result that the quasilinear hyperbolic problem with the nonlinear boundary conditions \( u'(t, x) \in \mathcal{J}(b(t, x)), (t, x) \in \Gamma_T \), is well posed [21, 22, 24].
Theorem 1.6. There exists $T_1 > 0$ and a unique $u^0 \in H^\omega(\Omega_{T_1}; \omega)$ solution of the following mixed hyperbolic problem:

\begin{align}
\mathcal{A}(b, u^0)u^0 &= f & \text{in } \Omega_{T_1} \\
u^0 &\in \mathcal{C}(b) & \text{on } \Gamma_{T_1}, \\
u^0_{|\partial_\omega} &= 0. \tag{6}
\end{align}

The proof of Theorem 1.6 uses a simple iterative scheme and a priori estimates in Sobolev spaces for the linearised mixed problem [24], which is a classical maximal dissipative hyperbolic problem in the sense of Lax and Phillips [19]. When the linearized boundary conditions are maximal strictly dissipative (as is, in fact, the case for (6)) the more general uniform Lopatinsky condition is satisfied, and a general proof in this context can be found in [21] or [22]. An extension of such result to the case of maximal dissipative conditions on a characteristic boundary can be found in [14].

We state now the main results of the paper. Recall that for every $(b, v) \in B_\rho \times \omega$ with $v \in \mathcal{C}(b)$, $V(b, v; z)$ is the unique solution of (3) given by Proposition 1.4. Let us call $y = (x_1, \ldots, x_{n-1})$ and

$$\tilde{V}(t, y; z) = V(b(t, y, 0), u^0(t, y, 0); z).$$

Theorem 1.7. There exists $T_2 > 0$ with $0 < T_2 \leq \min(T_0, T_1)$ such that, for any $\epsilon \in ]0, \epsilon_0]$, $u^0(t, x) = \tilde{V}(t, y; x_n/\epsilon) + \mathcal{C}(\epsilon)$ in $L^\omega \cap L^2(\Omega_{T_2}).$ \tag{7}

Corollary 1.8. $u^0 \to u^0$ in $L^2(\Omega_{T_2})$ as $\epsilon \to 0$.

Similar results have been obtained by Gisclon and Serre in one dimension. We observe that, like in the linear and semilinear cases, a boundary layer of characteristic size $\epsilon$ is created as $\epsilon$ goes to zero, when the boundary is noncharacteristic for $\mathcal{A}$. An important difference is that in the quasilinear case the boundary layer is nonlinear, when in the semilinear one, the boundary layer (in the noncharacteristic case) is linear. However, when the boundary is characteristic, nonlinear boundary layers are present in the semilinear case ([12]) and in the quasilinear one [4, 5, 10].

The next result gives more information about $T_2$.

Theorem 1.9. There exists a constant $\kappa > 0$ depending only on the matrices $A_j, B, E_{ij}, 1 \leq i, j \leq N$ and $b(t, x)$ such that for every $f \in H^\omega(\mathbb{R}^{1+n})$ with $f = 0$ for $t < 0$, if $T \leq T_1$ satisfies the condition

$$\sup_{-T \leq \tau \leq T} \sum_{|\beta| + |\gamma| + 1 < 1} \int_{|x_{n+1}| < 1} \left| z^x \partial_x^\beta \partial_z^\gamma \tilde{V}(t, y, z) \right| dz < \kappa,$$ \tag{9}

then one can take $T_0 = T_2 = T$ in Theorem 1.3 and Theorem 1.7.
In the next section we show, with an example, that condition (9) is actually relevant and cannot be avoided in general.

We prove Theorems 1.3, 1.7, and 1.9 simultaneously. In fact, Theorem 1.3 is an obvious consequence of Theorem 1.7 but we do not know how to prove the Theorem 1.3 without using Theorem 1.7, that is, without performing a boundary layer analysis.

The proof is in two steps, in the same spirit as [11, 12]. In the first step, for some $T^* > 0$, we construct an infinite sequence of smooth profiles $U^j(t, x, z)$, $(t, x) \in \Omega_{T^*}$, $z \in [0, +\infty)$ such that for any $M \in \mathbb{N}$, the finite sum

$$a'(t, x) = \sum_{j=1}^{M} \varepsilon^j U^j(t, x, x_n/\varepsilon)$$

(10)

satisfies $P(\varepsilon, T^*)$ up to an $O(\varepsilon^M)$ error term in $L^\infty(\Omega_{T^*})$ (and in $e^{-m^2}H^m(\Omega_{T^*})$, $\forall m \in \mathbb{N}$). Profiles $U^j$ are of the form

$$U^j(t, x; z) = \bar{U}^j(t, x) + \tilde{U}^j(t, y, z),$$

(11)

where

$$\lim_{z \to +\infty} \tilde{U}^j(t, y; z) = 0,$$

(12)

and they solve a collection of profile equations of BKW type (Section 2). The first profile in this expansion, $U^0$, is with the notations already introduced

$$U^0_0(t, x; z) = U_0(t, x) + \bar{U}(t, y; z).$$

(13)

As usual in BKW expansions, the profiles $U^j$ for $j \geq 1$ are given by linear equations. In the second step, taking $M$ big enough and using energy methods where condition (9) plays an important place, we prove that the exact solution $u'$ exists on $\Omega_{T^*}$ and, at the same time that $u' - a' = O(a')$ in $\Omega_{T^*}$, for some $l$ smaller than $M$.

Remark 1.10. In fact we prove a stronger result than Theorem 1.7, namely that $u'(t, x)$ admits, in $\Omega_{T^*}$, the complete asymptotic expansion

$$u'(t, x) \sim \sum_{j=0}^{M} \varepsilon^j U^j(t, x, x_n/\varepsilon)$$

(14)

in the sense that, for any $M \in \mathbb{N}$ we have

$$u'(t, x) = \sum_{j=0}^{M} \varepsilon^j U^j(t, x, x_n/\varepsilon) + \varepsilon^{M+1} R(t, x),$$

(15)
where \( \| R \|_{L^\infty(\Omega_0)} \leq \text{cst}, \| \partial_x R \|_{L^2(\Omega_0)} \leq C \| \psi \| e^{-(|x|/2)}, \forall x \in \mathbb{R}^{n+1}, \forall \text{e} \in [0, \epsilon_0] \). However, for the sake of simplicity, we just give the first order expansion in Theorem 1.7, which only involves the first profile

\[
U^0(t, x; z) = u^0(t, x) + \tilde{v}(t, y; z).
\]  

Remark 1.11. We state the results in the case where \( u \) vanishes in the past: \( u|_{t=0} = 0 \). However, all the results extend to the case where \( u \) is a given solution of \( P(\alpha, 0) \) admitting an asymptotic expansion (14) where the profiles \( U^j \) satisfies equations \( F^j = 0 \) (of Section 2), \( j = -1, 0, \ldots \), and if condition (9) is satisfied.

Remark 1.12. All the results have an extension to the case where \( f \) has a limited regularity in the past, that is \( f \in H^m(\Omega_T) \) for \( m = m_0 \) where \( m_0 \) is some big enough integer depending only on \( n \). The conclusion of Theorem 1.3 is then \( u^\alpha \in H^m(\Omega_T) \) instead of \( u^\alpha \in H^m(\Omega_T) \), while Theorems 1.7 and 1.9 remain valid without change. In that case one construct only a finite number (big enough) of profiles (with limited regularity), the energy estimates of Section 3 being unchanged. This remark applies also to the situation in Remark 1.11.

1.3. Example of an Unstable Boundary Layer

In this section we give an example showing that the condition (9) cannot be avoided in general. We take \( n = 1 \) and choose a function \( f(t, x) = g(t) h(x) \) where \( h \in C^\infty(\mathbb{R}) \) with \( h(0) \neq 0 \) and \( g \) is in some Sobolev space \( H^m(\mathbb{R}) \) with \( m \) big enough and such that \( g|_{r<0} = 0, g|_{r>0} \in C^m([0, +\infty]), g(0) = \cdots = g^{(m-1)}(0) = 0, \) and \( g^{(m)}(0) \neq 0 \).

Consider the following \( 2 \times 2 \) nonlinear system in one space dimension, where \( \lambda \) is a real parameter:

\[
\begin{align*}
-\varepsilon \partial_x^2 v + \partial_t v + \partial_x v - v \partial_x w &= f & \text{in } x > 0 \\
-\varepsilon \partial_x^2 w + \partial_t w - \partial_x w &= 0 \\
v|_{x=0} &= w|_{x=0} = 0 \\
v|_{r<0} &= 0, \quad w|_{r<0} &= \lambda (1 - \varepsilon^{-x^2}).
\end{align*}
\]  

We note \( u := (v, w) \). In order to make explicit computations, we have fixed a solution in the past which is independent of \( t \), but depends on \( \varepsilon > 0 \), and admits an exponential boundary layer. The matrix \( A_1 \) is

\[
A_1(v, w) = \begin{pmatrix} 1 & -v \\ 0 & -1 \end{pmatrix}
\]
which has two eigenvalues \( \pm 1 \). Assumptions (1.1, 1.2) are satisfied with

\[
S(v, w) = \begin{pmatrix} 4 & -2v \\ -2v & 1 \end{pmatrix}.
\]

As it is explained in Remarks 1.10 and 1.11, Theorem 1.7 extends to problem (17) if \( m \) is chosen large enough and if condition (9) is satisfied, that is: if \(|\lambda|\) is small enough. There is some \( \lambda_0 > 0 \) such that for \(|\lambda| < \lambda_0\), the set \( \mathcal{C} \) reduces to

\[
\mathcal{C} = \{ (v, w) \in \mathbb{R}^2 : v = 0 \}.
\]

The limit zero-viscosity hyperbolic problem corresponding to (6) writes, with \( \Omega_T = [-1, T] \times \mathbb{R}^+ \)

\[
\begin{align*}
\partial_t v^0 + \partial_1 v^0 - v \partial_1 w &= f, \quad \text{in } \Omega_T \\
\partial_t w^0 - \partial_1 w^0 &= 0, \quad \text{in } \Omega_T \\
v^0|_{x = 0} = 0, \quad v^0|_{\Omega_0} = 0, \quad w^0|_{\Omega_0} = \lambda,
\end{align*}
\]

which can be explicitly solved for any \( T > 0 \), that is in \( \mathbb{R}_x \times \mathbb{R}^+ \). Theorems 1.7 and 1.9 tell that, for some small enough \( \lambda_0 > 0 \): for any \(|\lambda| < \lambda_0\) there exists some \( T > 0 \) such that

\[
u^0 := (v^0, w^0) = (v^0, w^0) + \lambda(0, e^{-\frac{x}{\varepsilon}}) + O(\varepsilon) \quad \text{in } L^\infty(\Omega_T).
\]

where \((v^0, w^0)\) is the solution of the limit hyperbolic problem (19) on \( \Omega_T \). In this case, the boundary layer profile is

\[
\tilde{F}(t; z) = \lambda(0, e^{-z})
\]

and it is polarized on the \( w \)-components as it is expected since the vector field \( \partial_t - \partial_1 \) is outgoing.

**Theorem 1.13.** For any \( \lambda \) with \(|\lambda| < \lambda_0\) there is some \( T > 0 \) such that

\[
\|u^0 - u^0\|_{L^2(\Omega_T)} \to 0 \quad \text{as } \varepsilon \to 0,
\]

where \( u^0 \) is the solution of the limit hyperbolic problem (18) in \( \Omega_T \).

On the other hand, for large values of \( \lambda \), the behavior of \( u^0 \) is very different, the description (19) being no more true: for every \( T > 0 \), \( \|v^0\|_{L^2(\Omega_T)} \) blows up faster than \( \exp(\varepsilon/\varepsilon) \) for some \( c = c(T) > 0 \), as \( \varepsilon \to 0 \).

**Theorem 1.14.** There exists \( \lambda_1 > 0 \) such that for any \( \lambda > \lambda_1 \) and \( T > 0 \), there are constants \( C > 0 \) and \( \sigma > 0 \) such that

\[
\|v^0(t, \cdot)\|_{L^2(\mathbb{R}_x^+)} \geq Ce^{\sigma/\varepsilon}, \quad \forall t \in [0, T], \quad \forall \varepsilon \in [0, \varepsilon_0],
\]

(20)
This result proves that the simple existence of the profiles $\tilde{U}'$ is not sufficient to ensure the convergence, or even the uniform boundedness, of the solution $u'$ as $\varepsilon \to 0$; some additional condition is needed. The condition (9) actually appears to be such a (sufficient) condition.

**Remark 1.15.** As it is quoted [25, pp. 293–294], an inspection of the linearised equation of the two points boundary value problem (3), suggests that such an unstable boundary layer should exist, causing the blow up of the exact solution itself. Actually, in the system (17) when $\lambda \geq \lambda_1$, the boundary layer profile is *linearly unstable* in the sense of Serre [25, pp. 293–294] and provides an example of the blow up mechanism expected.

**Remark 1.16.** The hyperbolic part of system (17) is not conservative. We do not know if such an example of instability exists for a system of conservation laws.

**Proof.** Let us call $L_\varepsilon$ the second order operator on $[0, +\infty[$ defined by

$$L_\varepsilon \phi = \frac{d^2}{dz^2} \phi + \frac{d\phi}{dz} + \lambda \varepsilon^{-z} \phi.$$

**Lemma 1.17.** There is $\lambda_1 > 0$ such that for any $\lambda > \lambda_1$, $L_\varepsilon$ has a positive eigenvalue with smooth positive exponentially decreasing eigenfunction $\Psi(z)$, with $\Psi(0) = 0$.

**Proof of the Lemma.** We call $H_\varepsilon$ the Schrödinger operator on $[0, +\infty[$ defined by

$$H_\varepsilon \phi = \phi'' + \lambda \varepsilon^{-z} \phi$$

with domain $\mathcal{D} = H^2([0, +\infty[) \cap H^1([0, +\infty[)$. Pick an arbitrary function $\chi \in C^\infty([0, +\infty[) \cap \mathcal{D}$, $\chi \neq 0$ and take $\lambda_1$ such that

$$(H_\varepsilon \chi, \chi)_{L^2} = -\|\chi''\|_{L^2}^2 + \lambda_1 \|e^{-\varepsilon^2 z} \chi\|_{L^2}^2 = \frac{1}{4} \|\chi\|_{L^2}^2.$$

Then, for any $\lambda > \lambda_1$, the real

$$\lambda = \sup \{ (H_\varepsilon \phi, \phi)_{L^2} / \|\phi\|_{L^2}^2 : \phi \in \mathcal{D} \} > \frac{1}{4}$$

is a positive eigenvalue of $H_\varepsilon$ associated with a smooth positive eigenfunction $\phi_1$ with exponential decay [6]

$$|d/dz|^p \phi_1 = o(e^{-\rho z}), \quad z \to +\infty, \quad \forall \rho < \lambda.$$

Now the function $\Psi(z) = \exp(-z/2) \phi_1(z)$ satisfies

$$L_\varepsilon \Psi = e^{-z^2/4} (H_\varepsilon - \frac{1}{4}) \phi_1 = (\lambda - \frac{1}{4}) \Psi = \mu \Psi.$$
and is an eigenfunction of $L_\lambda$, with positive eigenvalue $\mu = \alpha - 1/4$, and this proves the Lemma.

We turn to the proof of the proposition. The explicit calculation shows that $w'(t, x) = \lambda (1 - \exp(-x/\varepsilon))$ for any $t, x$, and that $v'(t, x)$ is given by the linear equation

$$
\begin{cases}
-\varepsilon \partial_t^2 v' + \partial_t v' + \partial_1 v' - \frac{\lambda}{\varepsilon} e^{-x/\varepsilon} v' = f \\
v'|_{x=0} = 0, \quad v'|_{t=0} = 0.
\end{cases}
$$

(21)

Fix $\lambda > \lambda_1$ and $\psi$ like in the lemma. Consider

$$I(t) = \int_0^{+\infty} v'(t, x) \psi(x/\varepsilon) \, dx.$$

Then, using Eq. (21), one finds

$$
\frac{dI}{dt}(t) = \int f(t, x) \psi(x/\varepsilon) \, dx + \int \left( \varepsilon \partial_t^2 v' - \partial_1 v' + \frac{\lambda}{\varepsilon} e^{-x/\varepsilon} v' \right) \psi(x/\varepsilon) \, dx
$$

$$= g(t) \left[ \int h(x) \psi(x/\varepsilon) \, dx + \frac{1}{\varepsilon} \int v'(t, x)(L_\lambda \psi)(x/\varepsilon) \, dx \right]
$$

$$= g(t) \beta(\varepsilon) + \frac{\mu}{\varepsilon} I(t),$$

where

$$\beta(\varepsilon) = \int_0^{+\infty} h(x) \psi(x/\varepsilon) \, dx.$$

We obtain

$$I(t) = \beta(\varepsilon) e^{\mu t/\varepsilon} \int_0^t e^{-\mu s/\varepsilon} g(x) \, dx.$$

Now, as $\varepsilon \to 0$, $\beta(\varepsilon) \sim \varepsilon \beta_0$ where

$$\beta_0 = h(0) \int_0^{+\infty} \psi(z) \, dz \neq 0,$$
and
\[ \int_0^t e^{-\mu s} g(s) \, ds \sim c_1 e^{m+1}, \]
where
\[ c_1 = \frac{g^{(m)}(0^+)}{m!} \cdot \int_0^{+\infty} e^{-\nu y^m} \, dy \neq 0. \]
This implies
\[ R(t) \sim \beta_0 c_1 e^{m+2} e^{\nu/\varepsilon} \]
and since \( \|\Psi(x/\varepsilon)\|_{L^2[0,1]} = \varepsilon c_2, \) \( c_2 = \|\Psi\|_{L^2}, \) we deduce from the Cauchy Schwartz inequality that
\[ \|v'(t, \cdot)\|_{L^2} \geq \beta_0 c_1 c_2^{-1} e^{m+1} e^{\nu/\varepsilon} (1 + o(1)). \]
Then for any \( \sigma < \mu, \) the inequality (20) follows.

2. BOUNDARY LAYER PROFILES

In this section, we construct an infinite sequence of profiles \( U_j(t, x; z), j \in \mathbb{N}, \) such that partial sums (10) are approximate solutions of \( P(\varepsilon, T). \)

2.1. Spaces of Profiles

In order to describe the boundary layers, we call \( \mathbb{B}(T) \) the set of smooth functions \( V(t, y; z) : T \times [0, +\infty) \to \mathbb{R}^N, \) with uniform (in \( t, y \)) exponential decay as \( z \) goes to \( +\infty: \)
\[ |\partial_\gamma^\alpha \partial_{\gamma'}^\beta \partial_z^\gamma V(t, y; z)| \leq c_k e^{-\mu z}, \quad \alpha + |\beta| + \gamma = k, \quad k \in \mathbb{N}, \quad z \geq 0. \]
Note that if \( V \in \mathbb{B}(T), \) a change of variables shows that
\[ \|\partial_\gamma^\alpha (V(t, y; x_n/\varepsilon))\|_{L^2(\mathbb{R}^N)} \leq c_n (\sqrt{\varepsilon})^{-|\alpha|}, \quad \varepsilon > 0. \]
(22)
We call \( \mathbb{P}(T) \) the space of functions \( U(t, x; z) \) which write
\[ U(t, x; z) = \bar{U}(t, x) + \tilde{U}(t, y; z). \]
where \( U \in H^\infty(\Omega_T; \mathbb{R}^N) \) and \( \tilde{U} \in \mathcal{B}(T) \). We say that a family of smooth functions \( u^\varepsilon(t, x) \), \( \varepsilon > 0 \), satisfies

\[
 u^\varepsilon \sim \sum_0^\infty \varepsilon^j U^j(t, x; x_n/\varepsilon) \tag{23}
\]

for a given sequence \( U^j \in \mathcal{P}(T) \), if for any positive integer \( M \) the difference

\[
 r_M^\varepsilon = u^\varepsilon - \sum_0^M \varepsilon^j U^j(t, x; x_n/\varepsilon)
\]

satisfies

\[
 \| (\varepsilon \partial_t)^\ast r_M^\varepsilon \|_{L^\infty} + \| (\sqrt{\varepsilon} \partial_x)^\ast r_M^\varepsilon \|_{L^\infty} \leq C \varepsilon^{M+1}, \quad \varepsilon \in \mathbb{N}, \quad \varepsilon > 0. \tag{24}
\]

The reason why \( \sqrt{\varepsilon} \partial_x \) in the \( L^2 \) norm is the estimate (22). The space \( \mathcal{P}(T) \) is not an algebra, since it is not stable for multiplication. However, the property for a family \( u^\varepsilon(t, x) \) of admitting an asymptotic expansion (23) with profiles in \( \mathcal{P}(T) \), is preserved by composition with nonlinear (smooth) functions as stated in the next proposition. For a function \( f \in H^\infty(\Omega_T) \), the restriction \( f|_{\Omega_T} \) is a well defined function on \( \Omega_T \). In order to simplify the redaction, we will still denote by \( f|_{\Omega_T} \) the function extended to \( \Omega_T \) independently of \( x^\varepsilon_0(t, x) \in \Omega_T \rightarrow f(t, y, 0) \).

**Proposition 2.1.** Let \( u^\varepsilon, \varepsilon > 0 \), be a family of functions in \( H^\infty(\Omega_T) \) satisfying (23), with \( U^j \in \mathcal{P}(\Omega_T) \). Then, for any function \( f(t, x, u) \in C^\infty(\Omega_T \times \mathbb{R}^N; \mathbb{R}^N) \) with \( f(t, x, 0) = 0 \), the family \( f(t, x, u^\varepsilon) \) is in \( H^\infty(\Omega_T) \) and there is a sequence \( V^j \in \mathcal{P}(T) \), \( j \in \mathbb{N} \), such that

\[
 f(u^\varepsilon) \sim \sum_0^\infty \varepsilon^j V^j(t, x; x_n/\varepsilon).
\]

Moreover, \( V^0 = f(U^0) \) and \( \tilde{V}^0 = f(U^0|_{\Omega_T}) - f(U^0|_{\Omega_T}) \).

**Proof.** Let \( a, b \) be two functions \( a \in H^\infty(\Omega_T) \), \( b \in \mathcal{B}(T) \) and \( \beta = b(t, y; x_n/\varepsilon) \). First order Taylor expansion for \( f \) write \( f(a + \varepsilon) = f(a) + \varepsilon \cdot g(a, \varepsilon) \), \( g \in C^\infty \). Taylor expansion in \( x_n \) for \( a(t, x) \) writes \( a = a^0 + x_n a^1 \), \( a^0 = a(t, y, 0), a^1 \in H^\infty(\Omega_T) \). Then \( f(a) = f(a^0) + x_n a^1 g(a^0, x_n a^1) \), \( f(a + \varepsilon) = f(a^0 + b) + x_n a^1 g(a^0 + b, x_n a^1) \) and the difference writes \( f(a + \varepsilon) - f(a) = f(a^0 + b) - f(a^0) + x_n h(a^0, b, x_n a^1) \) where \( h \in C^\infty \). Now the point is that since \( b \in \mathcal{B}(T) \) the function \( b^\varepsilon = zb(t, y; z) \) is still in \( \mathcal{B}(T) \). Writing \( x_n \beta^\varepsilon = z^\varepsilon b^\varepsilon(t; y; x_n/\varepsilon) \), we obtain

\[
 f(a + \beta^\varepsilon) - f(a) = f(a^0 + b) - f(a^0) + \varepsilon R_\varepsilon, \tag{25}
\]
where \( R_\varepsilon = [b^\xi a^1 b(a^0, b, x_0 a^1)](t, x, x_0 / \varepsilon) \) satisfies
\[
\| (\varepsilon \partial)^\alpha R_\varepsilon \|_{L^\infty} + \| (\sqrt{\varepsilon} \partial)^\alpha R_\varepsilon \|_{L^2} \leq C_\varepsilon, \quad \alpha \in \mathbb{N}, \quad \varepsilon > 0, \tag{26}
\]
because of the exponential decay of \( b^\xi \). Turning now to the proposition, \( u' = U^0(t, x, x_0 / \varepsilon) + \varepsilon Q_0 \) where \( Q_0 \) satisfies the same estimate \( (26) \). Using \( (25) \) with \( a = U^0 \) and \( b = \bar{U}^0 \), we obtain
\[
f(u') = \bar{V}^0(t, y; x_0 / \varepsilon) + \varepsilon R_\varepsilon,
\]
where \( V^0 = f(a^0), \bar{V}^0 = f(a^0 + b) - f(b) \in \mathcal{D}(T) \) and \( R_\varepsilon = R_\varepsilon + Q_0 \) satisfies the same estimate \( (26) \) as \( R_\varepsilon \). This proves the estimate \( (23) \) for \( f(u') \) when \( M = 0 \). Using higher order Taylor expansion give the general asymptotic expansion and estimate.

### 2.2. Profiles Equations

Plugging the asymptotic expansion \( u' \sim \sum_{\alpha} \varepsilon^\alpha U^\alpha(t, x, x_0 / \varepsilon) \) into \( P(u', T) \) we find, thanks to Proposition 2.1, that
\[
-\varepsilon Au' + \mathcal{H}(b, u') u' - f \sim \sum_{\alpha} \varepsilon^\alpha F^\alpha(t, x, x_0 / \varepsilon),
\]
where \( F^j \in \mathcal{D}(T), j = -1, 0, \ldots \). The function \( F^{-1} \) is given by
\[
F^{-1} = 0,
\]
\[
\bar{F}^{-1} = -\varepsilon^2 \bar{U}^0 + A_d(b, U^0) U^0 + \bar{U}^0 \partial_x \bar{U}^0.
\]
To write down \( F^0 \) we note \( A^0 = A_d(b, U^0) \partial_x U^0 \) and call \( A(t, y; z) \) the \( N \times N \) matrix defined by
\[
A v = \sum_{j=1}^N v_j \partial U_j(b, U^0_{|T}) \partial_x U^0, \quad v \in \mathbb{R}^N.
\]
The function \( F^0 \) is given by
\[
F^0 = \mathcal{H}(b, U^0) U^0 - f
\]
\[
F^0 = -\varepsilon^2 \bar{U}^0 + A^0 \partial_x \bar{U}^1 + A \bar{U}^1 + A U^0_{|T} + Q^0,
\]
where \( Q^0 = \bar{Q}_0 \in \mathcal{D}(T) \) depends only on \( U^0 \). Let us call \( L^0 \) the linearisation of \( \mathcal{H} \) on \( U^0 \), that is
\[
L^0(v) = H(b, U^0) v + \sum_{j=1}^N v_j \partial B(b, U^0) \cdot U^0.
\]
Then, the function $F^j$, $j \geq 1$, is given by the formula

$$F^j = L^j(U^j) - Q^j$$

$$\bar{F}^j = -\partial_z U^{j+1} + A^j \partial_z \bar{U}^{j+1} + A_{U^j}^{j+1} - \bar{Q}^j$$

where the $Q^j \in \mathbb{P}(T)$, $j \in \mathbb{N}$, and $Q^j$ depends only on $\{ U^k, 0 \leq k \leq j \}$. We state now the existence of profiles solving the equations $F^j = 0$, $j \geq -1$.

**Proposition 2.2.** Let $T_1$ be fixed like in Theorem 1.6. There exists an infinite sequence of profiles $U^j \in \mathbb{P}(T_1)$, $j \in \mathbb{N}$, such that $U^j|_{\Omega^{c} = 0}$ and $F^k = 0$, for every $k \geq -1$.

**Proof.** Let us call $(S^j)$ the system

$$(S^j) \begin{cases} \bar{F}^j = 0, \quad F^j = 0, \quad U^j|_{\Omega^{c} = 0} = 0, \quad U^j|_{t < 0} = 0 \end{cases}$$

We show that the sequence $(S^j)$, $j \in \mathbb{N}$, can be solved by induction. The profile $U^0 = u^0 + \bar{V}$, where $u^0$ is given by Theorem 1.6 and $\bar{V}$ by (7), solves $(S^0)$. Now, suppose that $U^k$, $k = 0, ..., j - 1$ solve $(S^0), ..., (S^{j-1})$. Let us introduce the set $\mathcal{Q}_j$ of the $\ell \in \mathbb{R}^N$ such that the two point linear problem

$$-\partial_z^2 V + A^j \partial_z V + AV = \bar{Q}^j - A\ell$$

$$\bar{V}|_{z=0} = -\ell, \quad \bar{V}(+\infty) = 0,$$

has a solution. This set $\mathcal{Q}_j$ is a linear submanifold of $\mathbb{R}^N$ whose direction is the linear subspace $\mathcal{T}$ of the $\ell \in \mathbb{R}^N$ such that the homogeneous problem

$$-\partial_z^2 V + A^j \partial_z V + AV = -A\ell$$

$$\bar{V}|_{z=0} = -\ell, \quad \bar{V}(+\infty) = 0,$$

has a solution. Observe now that, for any $b \in B_x$ and $v \in \mathcal{Q}(b) \cap \partial\omega$, if $V_b$ is the solution of the dynamical system (3) and if $\mathcal{X} = \ell \partial V_1 + \cdots + \ell_N \partial V_N$ is a tangent vector to $\mathcal{Q}(b)$ at the point $V_b(0)$, applying $\mathcal{X}$ to system (3) we obtain the following equation, where $W = \mathcal{X}V$ and $A_{(b, v)}W = \sum_{j=1}^{N} w_j \partial A_{u_j \partial u_j}(b, V + v) \partial V$:

$$-\partial_z^2 W + A_{(b, v)}D_z D_z W + A_{(b, v)}W = -A_{(b, v)}\ell$$

$$W|_{z=0} = -\ell, \quad W(+\infty) = 0.$$
following linear hyperbolic problem with maximal dissipative boundary conditions
\[ L^j(U^j) = Q^j \]
\[ U^j_{|\Gamma} \in \mathcal{E}_j, \quad U^j_{|\Omega_0} = 0. \]

Then \( \tilde{U}^j \) is the unique solution of the linear ordinary differential equation
\[ -\partial_z^2 \tilde{U}^j + A \partial_z \tilde{U}^j + A \tilde{U}^j = \tilde{Q}^{j-1} - A U^j_{|\Gamma}, \]
\[ \tilde{U}^j_{|z=0} = -U^j_{|\Gamma}, \quad \tilde{U}^j \in \mathcal{B}(T), \]
which has a unique solution in \( \mathcal{B}(T) \) since the initial data \( U^j_{|\Gamma} \) lies in \( \mathcal{E}_j \).

3. ENERGY ESTIMATES

In this section, using energy estimates, we will prove Theorems 1.3 to 1.9, and in fact the more precise result described in 1.10. As the estimates involve a lot of elementary steps, we will begin by the study of a linear system, closely related to \( P(e, T) \).

3.1. A Linear System

Let \( y = (x_2, \ldots, x_d) \). Let \( T > 0 \). In order to split the difficulties, we begin with a linear system
\[ \partial_s u^s + \sum_{i=1}^n A^i_s(t, x) \partial_s u^s - A u^s = f^s \]  
(27)
\[ u^s = 0 \quad \text{on} \quad \partial \Omega \]  
(28)
\[ u^s(t) = 0 \quad \text{for} \quad t \leq 0, \]  
(29)

where \( A^i_s \) are matrices, and \( f^s \) is a bounded family of \( H^s([-1, T] \times \Omega) \) functions, with \( f^s(t, x) = 0 \) for \( t \leq 0 \) (in order to ensure \( u^s(t, x) = 0 \) for \( t \leq 0 \)), and \( \Omega = \mathbb{R}^{n-1} \times \mathbb{R}_+ \). Let us assume that (27) is symmetrizable: there exists a positive definite matrix \( A^0_s(t, x) \), with
\[ A^0_s(t, x) > \delta I > 0 \]
such that
\[ A^0_s(t, x) A_s^i(t, x) \]
are symmetric matrices for \( i = 1, \ldots, n \).
We will consider the following norms \([3, 10, 12]\): let \(x \geq 0\), \(\beta = (\beta_1, ..., \beta_{n-1})\) and \(\gamma \geq 0\) and
\[
\|w\|_{x, \beta, \gamma}^2 = e^{2x + 2|\beta| + 2\gamma} \int \phi^{2\gamma}(x_n) \partial_x^{\beta_n} \partial_{y_1}^{\beta_1} \cdots \partial_{y_{n-1}}^{\beta_{n-1}} \partial_{\gamma}^w A \phi^{2\gamma} \partial_x^{\beta_n} \partial_{y_1}^{\beta_1} \cdots \partial_{y_{n-1}}^{\beta_{n-1}} \partial_{\gamma}^w w, 
\]
where
\[
\phi(x_n) = \frac{x_n}{x_n + 1}
\]
and
\[
\|w\|_{x, \beta, \gamma}^2 = \sum_{\alpha + |\beta| + \gamma \leq x} C_1^{\alpha - |\beta| - \gamma} \|w\|_{x, \beta, \gamma}^2,
\]
where the constant \(C_1\) will be fixed later. We will shorten \(\partial_{y_1}^{\beta_1} \cdots \partial_{y_{n-1}}^{\beta_{n-1}}\) in \(\partial^\beta\).

The weight \(\phi^x\) is the classical weight to handle conormal derivatives, while the weight \(\phi^{x+|\beta|+\gamma}\) takes into account the singular dependence on \(\varepsilon\).

**Theorem 3.1.** Let us assume that there exists a constant \(C_2\) such that for every \(x, \beta, \gamma\), with \(x + |\beta| + \gamma \leq s\),
\[
|\phi^x(x_n) \partial_x^{\beta_n} \partial_{y_1}^{\beta_1} \cdots \partial_{y_{n-1}}^{\beta_{n-1}} A| \leq C_2
\]
for every \(-1 \leq t \leq T\), \(x \in \Omega\), \(0 < \varepsilon \leq 1\), and \(0 \leq i \leq n\). Let us assume that there exists \(\delta > 0\) such that
\[
|\det A_x| \geq \delta > 0
\]
for every \(-1 \leq t \leq T\) and \(x \in \Omega\), and that \(A_0^x A^x\) can be split in two symmetric matrices
\[
A_0^x A^x = A_{x, \text{mat}}^x + A_{x, b}^x,
\]
where
\[
|\partial_x^{\beta_n} \partial_{y_1}^{\beta_1} \cdots \partial_{y_{n-1}}^{\beta_{n-1}} A_{x, \text{mat}}^x | \leq C_2
\]
for \(x + |\beta| + \gamma \leq s\) and \(A_{x, b}^x\) satisfies (33) and (if \(x \geq 1\))
\[
|\phi^{x-1}(x_n) \partial_x^{\beta_n} \partial_{y_1}^{\beta_1} \cdots \partial_{y_{n-1}}^{\beta_{n-1}} A_{x, b}^x | \leq C_2.
\]
Let
\[ C_0 = \sup_{-1 < t < T, \alpha + |\beta| + \gamma < 1} \sum_{0}^{\infty} \int_{0}^{x} \left( \frac{\partial^\alpha \partial^\beta \partial^\gamma \overline{A}_n(x_1, ..., x_n) \right) dx_n. \] (38)

There exists a constant $\delta > 0$ independent of $s$ and $C_2$ such that if $C_0 < \delta$, and if $C_1$ is large enough, system (27, 28, 29) is well posed on $[-1, T]$, and there exists $C_1$ independent on $\varepsilon$, depending only on $C_0$, $C_1$, and $C_2$ such that for $\varepsilon \in [0, 1]$, \[ \partial_\tau \| u' \|_s \leq C_1 \| u \|_s + \| f \|_s. \] (39)

**Proof.** Let us estimate $\partial_\tau \| u' \|_s^2$. Equation (27) can be rewritten
\[ \sum_{i=0}^{n} \overline{A}_i(t, x) \partial_\tau u' - e\overline{A}_0 \partial_\tau u = \bar{f}^s, \] (40)
where $\overline{A}_0 = \overline{A}_n, \overline{A}_i = \overline{A}_n \overline{A}_r$ for $1 \leq i \leq n$ and $\bar{f}^s = \overline{A}_0 f^s$. Let $\alpha, \beta, \gamma \geq 0$ with $\alpha + |\beta| + \gamma < s$. We have
\[ \overline{A}_n \partial_\tau^{\alpha+1} \partial^\beta \partial^\gamma \overline{A}_n \partial_\tau u' + \overline{A}_n \partial_\tau^{\alpha+1} \partial^\beta \partial^\gamma \overline{A}_n \partial_\tau u' + \sum_{i=0}^{n-1} \overline{A}_i \partial_\tau^{\alpha+1} \partial^\beta \partial^\gamma \overline{A}_i \partial_\tau u' + \sum_{i=0}^{n-1} \left[ \partial_\tau^{\alpha+1} \partial^\beta \partial^\gamma \overline{A}_n \partial_\tau u + \partial_\tau^{\alpha+1} \partial^\beta \partial^\gamma \overline{A}_n \partial_\tau u \right] u \]
\[ + \sum_{i=0}^{n-1} \left[ \partial_\tau^{\alpha+1} \partial^\beta \partial^\gamma \overline{A}_n \partial_\tau u - e\partial_\tau^{\alpha+1} \partial^\beta \partial^\gamma \overline{A}_0 \partial_\tau u \right] = \partial_\tau^{\alpha+1} \partial^\beta \partial^\gamma \bar{f}^s. \] (41)

So we have to estimate eight integrals $I_1, ..., I_8$. Let us begin with $I_1$.

\[ I_1 = \int \phi^{2s-2|\beta| + 2s} \partial_\tau^{\alpha+1} \partial^\beta \partial^\gamma \overline{A}_n \partial_\tau u \overline{A}_n \partial_\tau^{\alpha+1} \partial^\beta \partial^\gamma \overline{A}_n \partial_\tau u = I_{1,1} + I_{1,2}, \] (42)
where
\[ I_{1,1} = -\frac{1}{2} \int \phi^{s+|\beta| + 2s} \partial_\tau^{s+|\beta| + 2s} \partial_\tau \overline{A}_n \partial_\tau u \overline{A}_n \partial_\tau^{s+|\beta| + 2s} \partial_\tau u \] and

\[ I_{1,2} = -\phi^{s+|\beta| + 2s} \partial_\tau^{s+|\beta| + 2s} \partial_\tau u \overline{A}_n \partial_\tau^{s+|\beta| + 2s} \partial_\tau u \]
since $\phi = 0$ and $\partial^\beta \partial^\gamma \mu = 0$ for every $\beta, \gamma \geq 0$ on $\partial \Omega$, and since $A_n^{\ast}$ is symmetric. We immediately get

$$|I_{t,1}| \leq C \|u\|^2_\ast$$

since $|\partial_u A_n^{\ast}|_{L^\infty} \leq C$.

To bound $I_{t,2}$ (if $\alpha \geq 1$), we have to take care that $A_n^{\ast}$ does not vanish on the boundary and that $\phi^{\ast -1} \partial_u^{\beta} \partial^\gamma \mu$ is not uniformly bounded. Therefore we use the equation to express $\partial_u^{\beta} \partial^\gamma \mu$: this will decrease the number of normal derivatives to $\phi^{\ast -1}$, except for the viscosity term, where we will have $\alpha + 1$ normal derivatives, but also an extra $e$. We have

$$A^*_n \partial_n u = -\partial_n u - \sum_{i=1}^{n-1} A^*_i \partial_i u + e \partial_n u + f^\ast,$$

so $(A^*_n)^{-1}$ is uniformly bounded by (34),

$$\partial_n u = -(A^*_n)^{-1} \partial_n u - \sum_{i=1}^{n-1} (A^*_n)^{-1} A^*_i \partial_i u + e (A^*_n)^{-1} \partial_n u + (A^*_n)^{-1} f^\ast. \quad (43)$$

Therefore,

$$\partial_u^{\beta} \partial^\gamma \mu = -\partial_u^{\beta - 1} \partial^\gamma \mu [(A^*_n)^{-1} \partial_n u] - \sum_{i=1}^{n-1} \partial_u^{\beta - 1} \partial^\gamma \mu [(A^*_n)^{-1} A^*_i \partial_i u]$$

$$+ e \partial_u^{\beta - 1} \partial^\gamma \mu [(A^*_n)^{-1} \partial_n u] + \partial_u^{\beta - 1} \partial^\gamma \mu [(A^*_n)^{-1} f^\ast].$$

Now,

$$\int e^{x + |\beta| + r \phi u^* \partial_u^{\beta + |\beta|} u A_n \partial_n u} \partial_u^{\beta} \partial^\gamma \mu [(A^*_n)^{-1} \partial_n u]$$

contains $2\alpha - 1$ normal derivatives which can be completely absorbed by $\phi^{2\alpha - 1}$. For this we expand $\partial_u^{\beta - 1} \partial^\gamma \mu$: it is a sum of terms of the form

$$\int e^{x + |\beta| + r \phi u^* \partial_u^{\beta + |\beta|} u A_n \partial_n u} \times [(\phi \partial_u^{\beta - 1} \partial^\gamma (A^*_n)^{-1})(e^{x + |\beta| + r \phi u^* \partial_u^{\beta + |\beta|} u} \partial_u^{\beta} \partial^\gamma \mu [(A^*_n)^{-1} \partial_n u])]$$

for $0 \leq \alpha' \leq \alpha - 1$, $0 \leq \beta' \leq \beta$ and $0 \leq \gamma' \leq \gamma$, terms which are bounded by

$$C \|u\|^2_{\alpha + |\beta| + \gamma'}.$$
The second right hand term of (44) is similar. It remains to study

$$
\varepsilon \int \phi' (e^{x+|\rho|} + i\phi n \partial^\beta \partial^\gamma u) e^{x+|\rho|} + \gamma \phi^{n-1} A_{n,\text{int}}^{-1} \partial^\beta \partial^\gamma [(A_n)^{-1} \partial u].
$$

Here there are 2n + 1 normal derivatives, with a weight $\phi^{2n-1}$, which enables us to use the small control provided by the viscosity. It is a sum of terms of the form

$$
e \int \phi' e^{x+|\rho|} + \gamma \phi^{n-1} A_{n,\text{int}}^{-1} \partial^\beta \partial^\gamma u \partial^\alpha \partial^\alpha (A_n)^{-1} (e^{x+|\rho|} + \gamma \phi^{n-1} \partial^\beta \partial^\gamma \partial^\alpha \partial^\alpha u),$$

(1 \leq i \leq n - 1), which are bounded by

$$C \sqrt{\varepsilon \|u\|}, \|\sqrt{\varepsilon} e^{x+|\rho|} + \gamma \phi^{n-1} \partial^\beta \partial^\gamma \partial^\alpha \partial^\alpha u \|_L^2$$

and

$$e \int \phi' e^{2n+2|\rho|} + \gamma \phi^{n-1} A_{n,\text{int}}^{-1} \partial^\beta \partial^\gamma u \partial^\alpha \partial^\alpha (A_n)^{-1} (e^{x+|\rho|} + \gamma \phi^{n-1} \partial^\beta \partial^\gamma \partial^\alpha \partial^\alpha u).$$

(45)

Integrating by parts in $\partial_n$, this term gives birth to

$$e \int \phi' (\phi^{x+|\rho|} + \gamma \phi^{n-1} A_{n,\text{int}}^{-1} \partial^\beta \partial^\gamma u$$

$$\times \partial^\alpha \partial^\alpha (A_n)^{-1} (e^{x+|\rho|} + \gamma \phi^{n-1} \partial^\beta \partial^\gamma \partial^\alpha \partial^\alpha u),$$

$$e \int \phi' \phi^{x+|\rho|} + \gamma \phi^{n-1} A_{n,\text{int}}^{-1} \partial^\beta \partial^\gamma u$$

$$\times \partial^\alpha \partial^\alpha (A_n)^{-1} (e^{x+|\rho|} + \gamma \phi^{n-1} \partial^\beta \partial^\gamma \partial^\alpha \partial^\alpha u),$$

$$e \int \phi' \phi^{x+|\rho|} + \gamma \phi^{n-1} A_{n,\text{int}}^{-1} \partial^\beta \partial^\gamma u$$

$$\times \partial^\alpha \partial^\alpha (A_n)^{-1} (e^{x+|\rho|} + \gamma \phi^{n-1} \partial^\beta \partial^\gamma \partial^\alpha \partial^\alpha u),$$

$$e \int \phi' \phi^{x+|\rho|} + \gamma \phi^{n-1} A_{n,\text{int}}^{-1} \partial^\beta \partial^\gamma u$$

$$\times \partial^\alpha \partial^\alpha (A_n)^{-1} (e^{x+|\rho|} + \gamma \phi^{n-1} \partial^\beta \partial^\gamma \partial^\alpha \partial^\alpha u),$$

$$e \int \phi' \phi^{x+|\rho|} + \gamma \phi^{n-1} A_{n,\text{int}}^{-1} \partial^\beta \partial^\gamma u$$

$$\times \partial^\alpha \partial^\alpha (A_n)^{-1} (e^{x+|\rho|} + \gamma \phi^{n-1} \partial^\beta \partial^\gamma \partial^\alpha \partial^\alpha u).$$

(45)
which can be bounded by terms of the form

$$C \| \sqrt{e^x + x + f(x)} \|_{L^2} + \| \sqrt{e^x + x + f(x)} \|_{L^2}$$

and

$$C \| \sqrt{e^x + x + f(x)} \|_{L^2} + \| \sqrt{e^x + x + f(x)} \|_{L^2},$$

for

$$x \leq x - 1, \beta \leq \beta'$$

and

$$\gamma', \gamma, \gamma, \gamma, \gamma, \gamma,$$

which ends the bound of $I_1$:

$$|I_1| \leq C \| u \|_{L^2}^2 + \eta \| L_{x + \beta} + \gamma \| \| L_{x + \beta} + \gamma \|,$$

(46)

where

$$L_m = \sum_{\alpha + \beta \leq m} \sum_{i=1}^{n} (e^x + x + f(x)) \| \partial_x^{\alpha} \partial_t^{\beta} \partial_i^{\gamma} u \|_{L^2}^2,$$

(47)

and where $\eta$ will be chosen later.

Let us bound $I_2$.

$$I_2 = \int e^{x + 2\beta} \partial_x^{\alpha} \partial_t^{\beta} \partial_i^{\gamma} u \partial_x^{\alpha} \partial_t^{\beta} \partial_i^{\gamma} u.$$

(48)

We have

$$|I_2| \leq \frac{1}{2} \int |e^{x + 2\beta} \partial_x^{\alpha} \partial_t^{\beta} \partial_i^{\gamma} u|^2 \partial_{x}. (49)$$

In the first right hand side term, $\partial_x A_{x, b}$ is of order $e^{-1}$, and in the second term we have $2x$ normal derivatives and only $\phi^{2x-1}$ to control them. We will use the control by the viscosity in order to get a very bad pointwise control on $\phi^{2x-1} \partial_x \partial_t^{\beta} \partial_i^{\gamma} u$, control which, however, appears to be sufficient provided $A_{x, b}$ decreases fast enough (here will appear the condition on $C_0$).

We have
\[
\phi^x \partial_x^{\gamma} \partial_y u(t, x_1, ..., x_n)
\]
\[
= \int_0^{x_n} \phi'(\tilde{x}_n) \partial_n^{x+1} \partial_n^{y} u(t, x_1, ..., x_{n-1}, \tilde{x}_n) \, d\tilde{x}_n
\]
\[
+ \int_0^{x_n} \phi'(\tilde{x}_n) \phi^{x-1}(\tilde{x}_n) \partial_n^{x-1} \partial_n^{y} u(t, x_1, ..., x_{n-1}, \tilde{x}_n) \, d\tilde{x}_n.
\]
Therefore,
\[
|\phi^x \partial_x^{\gamma} \partial_y u(t, x_1, ..., x_n)|
\]
\[
\leq C \sqrt{\frac{X_n}{n}} \left( \sqrt{\phi^x \partial_x^{x+1} \partial_y u} + \sqrt{\phi^{x-1} \partial_x^{x-1} \partial_y u} \right)^{1/2}.
\]
Thus
\[
\int |e^{x+|\gamma|} + \phi^x \partial_x^{\gamma} \partial_y u|^2 |\partial_n A_n|_h
\]
\[
\leq C \sup_{-T \leq t \leq T} \int_0^{+\infty} \frac{X_n}{\phi} |\partial_n A_n|_h \, dx_n
\]
\[
\times (\sqrt{\phi^{x+|\gamma|} \partial_x^{x+1} \partial_y u} + \sqrt{\phi^{x-1} \partial_x^{x-1} \partial_y u})^2
\]
and
\[
\int_{x_n < 1} |e^{x+|\gamma|} + \phi^x \partial_x^{\gamma} \partial_y u| |e^{x+|\gamma|} + \phi^{x-1} \partial_x^{x-1} \partial_y u| |A_n|_h
\]
\[
= \int_{x_n < 1} |e^{x+|\gamma|} + \phi^x \partial_x^{\gamma} \partial_y u|^2 \left| A_n \right| \frac{1}{\phi}
\]
\[
\leq C \sup_{-T \leq t \leq T} \int_0^{+\infty} \frac{1}{\phi} |A_n|_h \, dx_n
\]
\[
\times (\sqrt{\phi^{x+|\gamma|} \partial_x^{x+1} \partial_y u} + \sqrt{\phi^{x-1} \partial_x^{x-1} \partial_y u})^2,
\]
since
\[
\frac{X_n}{\phi(x_n)} \leq 2
\]
for \( x_n \leq 1 \) (the integral for \( x_n \geq 1 \) is straightforward), which leads to
\[
|I_2| \leq C \left( \| e^{x_n + |\beta|} | \phi \phi^* \nabla^2 \partial^\gamma u \|_{L^2} \right)^2 \times \left( \sup_{T \leq t \leq T_y} \frac{x_n}{\varepsilon} |\nabla_n \mathcal{A}_{n,b}| \ dx_n \right) \times \left( \sup_{T \leq t \leq T_y} \int_{-\infty}^{+\infty} \frac{1}{\varepsilon} |\mathcal{A}_{n,b}| \ dx_n \right).
\]
(50)

Let us turn to \( I_3 \). Let \( 1 \leq i \leq n - 1 \).
\[
I_3 = \int e^{x_n + 2|\beta| + 2\phi^2 \nabla^2 \partial^\gamma u \mathcal{A}_n \partial^\gamma \nabla^2 \partial^\gamma u}.
\]
(51)

This term is straightforward since the weight \( \phi^2 \) is sufficient to control the \( 2x \) normal derivatives. Integrating by parts, we get
\[
|I_3| \leq \frac{1}{2} \int e^{x_n + |\beta| + \phi^2 \nabla^2 \partial^\gamma u} |\mathcal{A}_n \partial^\gamma \nabla^2 \partial^\gamma u| \leq C \| u \|_2^2.
\]
(52)

Notice that for \( i = 0 \) we have
\[
I_i = \int e^{x_n + 2|\beta| + 2\phi^2 \nabla^2 \partial^\gamma u \mathcal{A}_n \partial^\gamma \nabla^2 \partial^\gamma u}.
\]
(53)

which is precisely the term we want to bound.

Let us turn to \( I_4 \),
\[
I_4 = \int e^{x_n + 2|\beta| + 2\phi^2 \nabla^2 \partial^\gamma u \mathcal{A}_n \partial^\gamma \nabla^2 \partial^\gamma u} (A_{n,\text{int}} \nabla^2 \partial^\gamma u).
\]
(54)

sum of terms of the form
\[
J = \int e^{x_n + 2|\beta| + 2\phi^2 \nabla^2 \partial^\gamma u \mathcal{A}_n \partial^\gamma \nabla^2 \partial^\gamma u} (A_{n,\text{int}} \nabla^2 \partial^\gamma u) \partial^\beta \partial^\gamma \partial^\gamma \partial^{\gamma'} u.
\]
(54)

with \( \alpha + \beta + \gamma' \geq 1 \). As for \( I_{1,2} \), we use (43) in order to decrease the total amount of normal derivatives by one. We have to bound three kinds of terms. The first type is
\[
\int e^{x_n + |\beta| + \phi^2 \nabla^2 \partial^\gamma u} (A_{n,\text{int}} \nabla^2 \partial^\gamma u) \partial^\beta \partial^\gamma \partial^\gamma \partial^{\gamma'} u
\times (e^{x_n + |\beta| + \phi^2 \nabla^2 \partial^\gamma u} (A_{n,\text{int}} \nabla^2 \partial^\gamma u) \partial^\beta \partial^\gamma \partial^\gamma \partial^{\gamma'} u).
\]
which is bounded by $C \|u\|_{a + |\beta| + \gamma}^2$. The second type is similar and the third type is

$$
\int e^{x + |\beta| + \gamma} \phi^x \partial_x \partial_y \partial_z \partial_t^\gamma u \left( \partial_x \partial_y \partial_z^\gamma \bar{A}_{n,m} \right) \times \left( e^{x + |\beta| + \gamma} \phi^{x - \frac{\gamma}{2}} \partial_x^{x - \frac{\gamma}{2}} \partial_y^{y - \frac{\gamma}{2}} \partial_z^{z - \frac{\gamma}{2}} \partial_t^\gamma \left[ (A_n^x)^{-1} A_t \right] \right),
$$

which can be bounded as (45).

Let us turn to $I_3$.

$$I_3 = \int e^{x + 2|\beta| + 2\beta \phi^{x} \partial_x \partial_y \partial_z \partial_t^\gamma u \left( \partial_x \partial_y \partial_z^\gamma \bar{A}_{n,h} \right)} u \left( \partial_x \partial_y \partial_z^\gamma \bar{A}_{n,h} \right) (55)
$$

sum of terms of the form

$$J = \int e^{x + 2|\beta| + 2\beta \phi^{x} \partial_x \partial_y \partial_z \partial_t^\gamma u \left( \partial_x \partial_y \partial_z^\gamma \bar{A}_{n,h} \right)} u \left( \partial_x \partial_y \partial_z^\gamma \bar{A}_{n,h} \right) \partial_x^{x - \frac{\gamma}{2}} \partial_y^{y - \frac{\gamma}{2}} \partial_z^{z - \frac{\gamma}{2}} \partial_t^\gamma u.
$$

with $0 \leq x' \leq x$, $0 \leq \beta' \leq \beta$, and $0 \leq \gamma' \leq \gamma$, and with $x' + |\beta'| + \gamma' \geq 1$.

$$J = \int e^{x + |\beta| + \gamma} \phi^x \partial_x \partial_y \partial_z \partial_t^\gamma u \left( e^{x + |\beta| + \gamma} \phi^x \partial_x \partial_y \partial_z \partial_t^\gamma \bar{A}_{n,h} \right) \times e^{x - x' + 1} \partial_x \partial_y \partial_z^\gamma u (56)
$$

The control of $\partial_x \partial_y \partial_z^\gamma \bar{A}_{n,h}$ by $\phi^{x - 1}$ is not sufficient. However, if $x' + |\beta'| + \gamma' \geq 2$ we can use the extra weight $e^{x + |\beta'| + \gamma' - 1}$,

$$|e^{x' - 1 + |\beta'| + \gamma'} \psi^{x - 1} \partial_x^{x - 1} \partial_y^{\gamma'} \partial_z^\gamma \bar{A}_{n,h}| = e^{x + |\beta| + \gamma - 2} e^{x - 1 \partial_x^{x - 1} \partial_y^{\gamma'} \partial_z^\gamma \bar{A}_{n,h}}$$

is bounded uniformly in $\varepsilon$, hence

$$|J| \leq C \|u\|_{a + |\beta| + \gamma}^2.
$$

It remains to bound $|J|$ for $x' + |\beta'| + \gamma' = 1$. Let us bound the integral for $x_n \leq 1$ (the integral for $x_n \geq 1$ being straightforward). We will again use the control provided by the viscosity term

$$|J| \leq C \left\{ \sum_{1 \leq x' + |\beta'| + \gamma' \leq 2} \|e^{x + |\beta| + \gamma} \phi^x \partial_x^{x - 1} \partial_y^{\gamma'} \partial_z^\gamma u\|_{L^2}^2 \right\} \times \sum_{x' + |\beta'| + \gamma' \leq 1} \sup_{T_0 \leq T, \gamma' \leq 2} \int_T^{T_0} \frac{1}{\varepsilon} \| \partial_x^{x - 1} \partial_y^{\gamma'} \partial_z^\gamma \bar{A}_{n,h} \|_d x_n, (57)
$$

and similarly for $|I_3|$.
Now let, for $0 \leq i \leq n-1$,

$$I_6 = \int e^{2x + 2|\beta| + 2i|\beta|} u[\partial^x \partial^\beta \partial_{y_i} \hat{A}_{y_i}] u$$  \hfill (58)$$

sum of terms of the form

$$J = \int e^{2x + 2|\beta| + 2|\beta|} u[\partial^x \partial^\beta \partial_{y_i} \hat{A}_{y_i}] \partial^x \partial^\beta \partial \partial_{y_i} \gamma' \partial \partial_{y_i} \gamma' \partial u$$  \hfill (59)$$

where $x' + \beta + \gamma' \geq 1$. We have

$$|J| \leq \|e^{x + |\beta| + \phi} \partial^\phi \partial \partial_{y_i} \partial_{y_i} \partial_{y_i} \gamma \|_{L^2} \|e^{x + |\beta| + \phi} \partial^\phi \partial \partial \partial_{y_i} \gamma \|_{L^2}$$

but

$$|\partial^\phi \partial \partial_{y_i} \partial \partial_{y_i} \gamma \|_{L^2} \leq C_2$$

which enables us to bound $I_6$ by $C \|u\|_{L^2}^2$.  

Now let

$$I_7 = e \int e^{2x + 2|\beta| + 2|\beta|} u[\partial^x \partial^\beta \partial_{y_i} \hat{A}_{y_i}] u$$  \hfill (60)$$

Let us begin with

$$I_{7, n} = e \int e^{2x + 2|\beta| + 2|\beta|} u[\partial^x \partial^\beta \partial_{y_i} \hat{A}_{y_i}] u$$  \hfill (61)$$

Integrating by parts, as the boundary integral vanishes since $\partial^\beta \partial_{y_i} u = 0$ and $\phi = 0$ on $\partial \Omega$, $I_{7, n}$ is the sum of

$$e \int e^{2x + 2|\beta| + 2|\beta|} u[\partial^x \partial^\beta \partial_{y_i} \hat{A}_{y_i}] u$$

and of quantities of the form ($0 \leq x' \leq x$, $0 \leq \beta' \leq \beta$, $0 \leq \gamma' \leq \gamma$ and $x' + |\beta'| + |\gamma'| \geq 1$)

$$e \int e^{2x + 2|\beta| + 2|\beta|} u[\partial^x \partial^\beta \partial_{y_i} \hat{A}_{y_i}] u$$
and

\[ e \int e^{2x + 2|\beta| + 2y} e^{2x + 2\beta} \partial_x^\alpha \partial_y^\beta \partial_x^\gamma \partial_y^\delta \partial_i^j u \partial_x^{\alpha + 1} \partial_y^{\beta} \partial_x^{\gamma + 1} \partial_y^{\delta} \partial_i^j A_i \partial_x^s - x + 1 \partial_y^t - y \partial_i^j u, \]

which are bounded by \( e^{L_x + |\beta| + \gamma} L_x + |\beta| + \gamma - 1 \).

The terms of the form

\[ e \int e^{2x + 2|\beta| + 2y} e^{2x + 2\beta} \partial_x^\alpha \partial_y^\beta \partial_x^\gamma \partial_y^\delta \partial_i^j u \partial_x^{\alpha + 1} \partial_y^{\beta} \partial_x^{\gamma + 1} \partial_y^{\delta} \partial_i^j [e A_i \partial_x^s u] \]

can be treated similarly, and we get

\[ I_7 + \frac{e}{2} \sum_{i=1}^n |e^{2x + 2|\beta| + 2y} e^{2x + 2\beta} \partial_x^\alpha \partial_y^\beta \partial_x^\gamma \partial_y^\delta \partial_i^j u \partial_x^{\alpha + 1} \partial_y^{\beta} \partial_x^{\gamma + 1} \partial_y^{\delta} \partial_i^j u \leq C e^{L_x + |\beta| + \gamma - 1}. \tag{62} \]

It remains to bound \( I_8 \).

\[ I_8 = \int e^{2x + 2|\beta| + 2y} e^{2x + 2\beta} \partial_x^\alpha \partial_y^\beta \partial_x^\gamma \partial_y^\delta \partial_i^j u \partial_x^{\alpha + 1} \partial_y^{\beta} \partial_x^{\gamma + 1} \partial_y^{\delta} \partial_i^j u, \tag{63} \]

and

\[ |I_8| \leq ||u|| \cdot ||f||. \tag{64} \]

Let us sum all these estimates

\[ \partial_i \| u \|_{n, \beta, \gamma} = e^{2x + 2|\beta| + 2y} \int e^{2x + 2|\beta| + 2y} e^{2x + 2\beta} \partial_x^\alpha \partial_y^\beta \partial_x^\gamma \partial_y^\delta \partial_i^j w \partial_x^{\alpha + 1} \partial_y^{\beta} \partial_x^{\gamma + 1} \partial_y^{\delta} \partial_i^j w \]

\[ + 2e^{2x + 2|\beta| + 2y} \int e^{2x + 2|\beta| + 2y} e^{2x + 2\beta} \partial_x^\alpha \partial_y^\beta \partial_x^\gamma \partial_y^\delta \partial_i^j w \partial_x^{\alpha + 1} \partial_y^{\beta} \partial_x^{\gamma + 1} \partial_y^{\delta} \partial_i^j w \]

\[ \leq C \| u \|_{n, \beta, \gamma} + 2e^{2x + 2|\beta| + 2y} \]

\[ \times \int e^{2x + 2|\beta| + 2y} e^{2x + 2\beta} \partial_x^\alpha \partial_y^\beta \partial_x^\gamma \partial_y^\delta \partial_i^j w \partial_x^{\alpha + 1} \partial_y^{\beta} \partial_x^{\gamma + 1} \partial_y^{\delta} \partial_i^j w. \]

Therefore, using all the previous estimates,

\[ \partial_i \| u \|_{n, \beta, \gamma} = \frac{e^{2x + 2|\beta| + 2y}}{2} \sum_{i=1}^n \int e^{2x + 2|\beta| + 2y} e^{2x + 2\beta} \partial_x^\alpha \partial_y^\beta \partial_x^\gamma \partial_y^\delta \partial_i^j u \partial_x^{\alpha + 1} \partial_y^{\beta} \partial_x^{\gamma + 1} \partial_y^{\delta} \partial_i^j u \]

\[ \leq C \| u \|_{n, \beta, \gamma} + CC_\varepsilon \]

\[ \times \sum_{1 \leq s + |\beta| + \gamma \leq 2} \| e^{x + |\beta| + \gamma} e^{s - s + 1} e^{2 - s} \partial^\beta \partial x - s - \gamma \partial_i^j u \|_{L^2}^2 \]

\[ + C \eta L_x + |\beta| + \gamma - 2 \| u \|_n \cdot \| f \|_r. \tag{65} \]
where
\[ C_0 = \sup_{-T \leq t \leq T} \sum_{\gamma + |\beta| + \eta \leq s} \int_0^\infty \frac{\alpha^n}{\nu} |\partial_x^n \partial_t^{\gamma} \partial_\gamma A_{n,k}|. \] (66)

Let \( C_0 \) be small enough (\( C_0 N \leq \varepsilon^4 \), where \( N \) is the number of solutions of \( \gamma + |\beta| + s \leq 2 \). Notice that this condition is independent on \( s \).

Summing (65) for \( \gamma + |\beta| + s \leq s \) for \( \varepsilon \) small enough and for \( C_1 \) large enough then gives
\[ \partial_t \|u\|^2 \leq C_0 \|u\|^2 + 2 \|u\|, \] (67)
which ends the proof of Theorem 3.1.

### 3.2. End of the Proof

Let \( \alpha^s \) be a family of approximate solutions of the form (0). Let \( s \geq d \) be an integer. We know that for \( T \) small enough (depending on \( \varepsilon \)), problem \( P(\varepsilon, T) \) has a smooth solution \( u^\varepsilon \). Let \( v^\varepsilon = u^\varepsilon - \alpha^s \). Let \( T_\varepsilon \) be the maximum of all the possible times \( T \) such that
\[ \sup_{t \leq T} \|v^\varepsilon(t, \cdot)\|_{H^s} \leq \varepsilon N, \] (68)
where \( N \) is an integer which will be fixed later. We want to prove that there exists \( T_\varepsilon \) such that \( T_\varepsilon \geq T_0 \) for \( \varepsilon \) small enough.

On \( v^\varepsilon \), we have
\[ \partial_t v^\varepsilon + \sum_{i=0}^n A_j(a + v) \partial_j v^\varepsilon - \varepsilon A v^\varepsilon \]
\[ = -\partial_t a - \sum_{j=1}^n A_j(a + v) \partial_j a + \varepsilon A a - [B(b, a) - B(b, a)] \]
\[ = e^{M^2} \partial_\varepsilon + \sum_{j=1}^n [A_j(a) - A_j(a + v)] \partial_j a + C(b, a, v) \] (69)
with
\[ C(b, a, v) = [B(b, a + v) - B(b, a)]. \] (70)
by construction of \( \alpha^s \), where \( \partial_\varepsilon \) is uniformly bounded in \( L^\infty(\Omega_T^\varepsilon) \) and in \( e^{-m^2} H^m(\Omega_T^\varepsilon) \). Multiplying by \( A_j(a + v) \), we get
\[ \sum_{j=0}^n A_j(a + v) \partial_j v^\varepsilon - \varepsilon A_j(a + v) A v^\varepsilon = e^{M^2} \partial_\varepsilon \]
\[ + \sum_{j=1}^n A_j(a + v)(A_j(a) - A_j(a + v)) \partial_j a + C(b, a, v), \] (71)
where \( \mathcal{R} = A_0(a + v)^{\mathcal{R}} \) and \( \tilde{C}(b, a, v) = A_0(a + v) \ C(b, a, v) \). Let

\[
a_{an}(t, x) = \sum_{j=0}^{M} e^{jU}(t, x)
\]

(72)

and

\[
a_{b}(t, x) = \sum_{j=0}^{M} e^{jU}(t, x_1, ..., x_{n-1}, \frac{x_n}{\epsilon})
\]

(73)

Notice that

\[
|\partial_\nu^\alpha \partial_{\xi}^\beta \partial_\eta^\gamma a_{an}| \leq C,
\]

\[
|\phi \partial_\nu^{\alpha-1} \partial_{\xi}^\beta \partial_\eta^\gamma a_{b}| \leq C,
\]

and

\[
|\phi^{\alpha-1} \partial_\nu^\alpha \partial_{\xi}^\beta \partial_\eta^\gamma a_{b}| \leq C
\]

for \( \alpha + |\beta| + \gamma \leq s \).

Equation (71) leads to

\[
\tilde{A}_n(a_{an}) \ \hat{\partial}_n v + [\tilde{A}_n(a) - \tilde{A}_n(a_{an})] \ \hat{\partial}_n v + \sum_{j=0}^{n-1} \tilde{A}_j(a) \ \hat{\partial}_j v
\]

\[
+ A_0(a + v)[A_n(a + v) - A_n(a)] \ \hat{\partial}_n v
\]

\[
+ \sum_{j=0}^{n-1} A_j(a + v)[A_j(a + v) - A_j(a)] \ \hat{\partial}_j v - \epsilon \tilde{A}_0 A v
\]

\[
= e^{M\mathcal{R}} + A_0(a + v)(A_n(a) - A_n(a + v)) \ \hat{\partial}_n a_{an}
\]

\[
+ A_0(a + v)(A_n(a) - A_n(a + v)) \ \hat{\partial}_n a_{b}
\]

\[
+ \sum_{j=0}^{n-1} A_j(a + v)(A_j(a) - A_j(a + v)) \ \hat{\partial}_j a + \tilde{C}(b, a, v).
\]

(74)

Let us bound \( \tilde{\mathcal{R}}_j \|a\|_2 \). As \( \tilde{A}_j(a) \) for \( 1 \leq j \leq n - 1 \) and \( \tilde{A}_n(a) = \tilde{A}_n(a_{an}) + [A_n(a) - A_n(a_{an})] \) satisfy assumptions (33), (36), and (37), the corresponding integrals can be bounded as in the previous section. The Laplace term can be treated as previously and \( \mathcal{R}_j \) is straightforward. Let

\[
\tilde{A}_j(t, x, v) = A_0(t, x, a(t, x) + v)(A_j(t, x, a(t, x) + v) - A_j(t, x, a(t, x))
\]

(75)
for $1 \leq j \leq n$. Notice that

$$
\bar{A}_n(t, x, v) = A_0(t, x, a(t, x) + v) \int_0^1 \partial_v A_n(t, x, a(t, x) + \tau v) \, d\tau,
$$

hence

$$
|\bar{A}_n(t, x, v)| \leq |v| C(|v|_{L^\infty}).
$$

More generally,

$$
|\phi^* \partial_n \partial_v \partial_v \partial_v \bar{A}_n| \leq |v| C(|v|_{L^\infty}),
$$

and

$$
|\phi^* \partial_n \partial_v \partial_v \partial_v \bar{A}_n| \leq C(|v|_{L^\infty}).
$$

Moreover

$$
\partial_v [\bar{A}_n(t, x, v(t, x))] = \int_0^1 \partial_v [\partial_n \partial_v A_n(t, x, a + \tau v(t, x)) v(t, x)] \, d\tau
$$

therefore

$$
|\partial_v [\bar{A}_n(t, x, v(t, x))]|_{L^\infty} \leq \varepsilon^{-1} \|v\|_{H^\sigma} C(|v|_{L^\infty}, |v|_{L^\infty})
$$

for $\sigma > d/2 + 1$, and

$$
|\bar{A}_n(t, x, v)| \leq \varepsilon^{-1} x_n \|v\|_{H^\sigma} C(|v|_{L^\infty}, |v|_{L^\infty}), \tag{76}
$$

since $\bar{A}_n = 0$ if $x_n = 0$.

Let us turn to $\bar{A}_n(v) \partial_n v$.

$$
I_1 = \int e^{2v + 2|\beta| + 2} \phi^* \partial_n \partial_v \partial_v \partial_v \bar{A}_n(v) \partial_n \partial_v \partial_v \partial_v \bar{A}_n(v) \, d\tau v
$$

$$
= -\frac{1}{2} \int e^{2v + 2|\beta| + 2} \phi^* \partial_n \partial_v \partial_v \partial_v \bar{A}_n(v) \partial_n \partial_v \partial_v \partial_v \bar{A}_n(v).
$$

Therefore by (76)

$$
|I_1| \leq C\varepsilon^{-1} \|v\|_{H^\sigma} C(|v|_{L^\infty}, |v|_{L^\infty}). \tag{77}
$$

This bound is very weak, however it is sufficient since $\varepsilon^{-1}$ can be absorbed in $\|v\|_{H^\sigma} \|v\|_{H^\sigma}$ which is cubic in $v$.

Let

$$
I_2 = \int e^{2v + 2|\beta| + 2} \phi^* \partial_n \partial_v \partial_v \partial_v \bar{A}_n(t, x, v) \partial_n v, \tag{78}
$$
sum of terms of the form
\[
J = \int e^{2\alpha + 2|\beta| + 2\rho \phi_n \partial^\rho \partial^\rho \partial_i \xi_v \partial^\rho \partial_i [\tilde{A}_n(t, x, v)]} \partial_n^{x + 1} \partial^\rho \partial_i^{-\gamma} v.
\]

But \(e^{x + \beta + \gamma \phi_n \partial^\rho \partial^\rho \partial_i [\tilde{A}_n(t, x, v)]}\) is a sum of terms of the form
\[
[e^{x + \beta} + \gamma \phi_n \partial^\rho \partial_i^{-\gamma} v] \times e^{x + \beta + \gamma \phi_n \partial^\rho \partial_i^{-\gamma} v} \ldots e^{x + \beta + \gamma \phi_n \partial^\rho \partial_i^{-\gamma} v},
\]
where \(x \in x_1 + \ldots + x_m = x'\) and similarly for \(\beta'\) and \(\gamma'\), and where \(\delta^*_n = m\).

As \(J\) contains \(2x + 1\) normal derivatives whereas \(\phi\) appears only at the power \(2x\), we need to express \(\partial_n^{x + 1} \partial^\rho \partial_i^{-\gamma} v\) using (74) as for (44), in order to decrease the total number of normal derivatives to \(x\) (except for the viscosity terms).

We have for instance to bound terms of the form
\[
J_1 = \int e^{x + \beta} \phi_n \partial^\rho \partial_i ^{-\gamma} v [\tilde{A}_n(t, x, v)] \times \partial_n^{x + 1} \partial^\rho \partial_i^{-\gamma} v [(A_n(t, x, a + v))^{-1} \partial_i v].
\]

Now the weight \(\phi_n\) is sufficient to control the \(2x\) normal derivatives. More precisely,
\[
e^{x + \beta - \sigma - \gamma} \phi_n \partial^\rho \partial_i ^{-\gamma} v [\tilde{A}_n(t, x, v)] \times \partial_n^{x + 1} \partial^\rho \partial_i^{-\gamma} v [(A_n(t, x, a + v))^{-1} \partial_i v]
\]
is a sum of terms of the form
\[
[e^{x + \beta} \phi_n \partial^\rho \partial_i ^{-\gamma} v] \times e^{x + \beta + \gamma \phi_n \partial^\rho \partial_i^{-\gamma} v} \ldots e^{x + \beta + \gamma \phi_n \partial^\rho \partial_i^{-\gamma} v},
\]
where \(\tilde{x} \in \tilde{x}_1 + \ldots + \tilde{x}_m = x - x'\) and similarly for \(\tilde{\beta}'\) and \(\tilde{\gamma}'\), and where \(\tilde{\delta}^*_n = \tilde{m}\).

At most two sums of indices \(x + \beta + \gamma, x_1 + \beta_1 + \gamma_1, \ldots, \tilde{x} + \tilde{\beta}_1 + \tilde{\gamma}_1, \ldots\) are greater than \(s/2\). These two indices are bounded by \(\|v\|_2^s\) and the others by \(\|v\|_2^s\),

\[
\sup_{|\sigma| + \gamma < s/2} |\partial_n^{x + 1} \partial^\rho \partial_i^{-\gamma} v|_{L^\infty}
\]

which leads to
\[
|J_1| \leq C \|v\|_2^s C \sup_{|\sigma| + \gamma < s/2} |\partial_n^{x + 1} \partial^\rho \partial_i^{-\gamma} v|_{L^\infty}.
\]
The other terms can be treated in a similar way, which leads to a similar bound for $|I_2|$

$$|I_2| \leq C \left\| v \right\|_2^2 C \left( \sup_{\varepsilon+|\beta|+1 \leq s \leq \infty} |\partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} v|_{L^\infty} \right) + \eta u L_{\infty} + |\beta| + \gamma$$

$$+ \frac{5}{\eta} C \left( \sup_{\varepsilon+|\beta|+1 \leq s \leq \infty} |\partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} v|_{L^\infty} \right) L_{\infty} + |\beta| + \gamma + 1.$$ 

The integrals $I_1$ and $I_2$ coming from $\tilde{A}_j(t)\partial_j$ for $1 \leq j \leq n - 1$ can be treated in a similar way.

Let us turn to

$$I_5 = \int (\varepsilon^{2n+2} + 2|\beta| + 2)\phi^{2\varepsilon^{2n+2}} \partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} \tilde{A}_j(t, x, v) \partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} a_{\nu},$$

where $\partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} [\tilde{A}_j(t, x, v)]$ is a sum of terms of the form

$$\left[ \partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} \tilde{A}_j \right] \partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} v \ldots \partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} v.$$

We have only to take care of the integral between 0 and 1. Here again, we have $2x + 1$ normal derivatives, whereas $\phi$ appears only at the power $2x$.

The additional derivative will be controlled either using the extra weight $\varepsilon^{2n+2} + 2|\beta| + 2$ when possible or the viscosity term.

$I_5$ is a sum of terms of the form

$$I_5 = \int (\varepsilon^{2n+2} + 2|\beta| + 2)\phi^{2\varepsilon^{2n+2}} \partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} \tilde{A}_j(t, x, v) \partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} a_{\nu},$$

where $\partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} [\tilde{A}_j(t, x, v)]$ is a sum of terms of the form

$$\left[ \partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} \tilde{A}_j \right] \partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} v \ldots \partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} v.$$

If $\partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} \tilde{A}_j(t, x, v)$ is a sum of terms of the form

$$\left[ \varepsilon^{2n+2} + 2|\beta| + 2 \phi^{2\varepsilon^{2n+2}} \partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} \tilde{A}_j \right] \partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} v \ldots$$

$$\times (\varepsilon^{2n+2} + 2|\beta| + 2)\phi^{2\varepsilon^{2n+2}} \partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} a_{\nu}.$$

If $\partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} \tilde{A}_j(t, x, v)$ is a sum of terms of the form

$$\left[ \varepsilon^{2n+2} + 2|\beta| + 2 \phi^{2\varepsilon^{2n+2}} \partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} \tilde{A}_j \right] \partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} v \ldots$$

$$\times (\varepsilon^{2n+2} + 2|\beta| + 2)\phi^{2\varepsilon^{2n+2}} \partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} a_{\nu}.$$

If $\partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} \tilde{A}_j(t, x, v)$ is a sum of terms of the form

$$\left[ \varepsilon^{2n+2} + 2|\beta| + 2 \phi^{2\varepsilon^{2n+2}} \partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} \tilde{A}_j \right] \partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} v \ldots$$

$$\times (\varepsilon^{2n+2} + 2|\beta| + 2)\phi^{2\varepsilon^{2n+2}} \partial^{\gamma}_{\alpha} \partial^{\rho}_{\beta} \partial^{\omega}_{\gamma} a_{\nu}.$$
since \( \alpha' + |\beta - \beta'| + \gamma - \gamma' \geq 1 \), and if \( \delta'' = 0 \), we use
\[
\phi^{\alpha'} e^{(\beta'|1' + |\gamma'|1')} |\partial_\alpha^n \partial^\beta \partial^\gamma \tilde{A}_n| \leq |v| C(v|L^q)
\]
to get
\[
|I'_3| \leq C \left( \sup_{\alpha' + |\beta'| + \gamma' \leq \zeta_2} \| \partial_\alpha^n \partial^\beta \partial^\gamma v \|_{L^q} \right) \sup_{\alpha'} \left( \prod_{1'=1}^{\alpha'} |v| \right) C(v|L^q).
\]
(80)

If \( \alpha' = \alpha \), \( \beta' = \beta \) and \( \gamma' = \gamma \), we have to bound
\[
I_3 = \int e^{2\alpha + 2|\beta| + 2\gamma} \partial_\alpha^n \partial^\beta \partial^\gamma \tilde{A}_n(t, x, v) \partial_\alpha \varepsilon.
\]
We use the control provided by the viscosity
\[
e^{\alpha + |\beta| + \gamma} \phi \partial_\alpha^n \partial^\beta \partial^\gamma \tilde{A}_n(t, x, v)
\]
\[
e^{\alpha + |\beta| + \gamma} \int_0^{\Gamma} \phi \partial_\alpha^{n+1} \partial^\beta \partial^\gamma \tilde{A}_n(t, x, v) \, dx_n
\]
\[
+ \alpha e^{\alpha + |\beta| + \gamma} \int_0^{\Gamma} \phi \partial_\alpha^{n+1} \partial^\beta \partial^\gamma \tilde{A}_n(t, x, v) \, dx_n.
\]
Let us consider only the first right hand side term (the second being similar). It is a sum of terms of the form
\[
\int_0^{\Gamma} \frac{1}{\varepsilon} \left[ e^{\alpha + |\beta| + \gamma} \phi \partial_\alpha^n \partial^\beta \partial^\gamma \tilde{A}_n(t, x, v) \right] e^{\alpha + |\beta| + \gamma} \partial_\alpha^{n+1} \partial^\beta \partial^\gamma \tilde{A}_n(t, x, v) \cdots,
\]
where \( \alpha' + \alpha + \cdots + \alpha_m = \alpha + 1 \). Notice that \( \partial_\alpha^{n+1} \partial^\beta \partial^\gamma \tilde{A}_n \) appears once and
that only one sum \( \alpha + |\beta| + |\gamma| + |\gamma| \geq s/2 \), therefore
\[
|\phi \partial_\alpha^{n+1} \partial^\beta \partial^\gamma \tilde{A}_n(t, x, v)|
\]
\[
\leq \frac{C}{\varepsilon} \left( \sup_{\alpha' + |\beta' + \gamma' | \leq \zeta_2} |\partial_\alpha^n \partial^\beta \partial^\gamma v|_{L^q} \right)
\]
\[
\times \left[ \sqrt{\varepsilon} \phi \partial_\alpha^{n+1} \partial^\beta \partial^\gamma v \right]^{2}_{L^2} + C_L \left( |\beta| + |\gamma| \right)^{1/2}.
\]
Hence
\[
|I_3| \leq C \left( \sup_{\alpha' + |\beta' + \gamma' | \leq \zeta_2} |\partial_\alpha^n \partial^\beta \partial^\gamma v|_{L^q} \right) \sup_{\alpha} \left( \prod_{1'=1}^{\alpha} |v| \right) \left( \frac{C_L}{\varepsilon} \int_0^{\Gamma} \left| \partial_\alpha \varepsilon \right| \, dx_n \right)
\]
\[
\times \left[ \sqrt{\varepsilon} \phi \partial_\alpha^{n+1} \partial^\beta \partial^\gamma v \right]^{2}_{L^2} + \left[ \sqrt{\varepsilon} \phi \partial_\alpha^{n+1} \partial^\beta \partial^\gamma v \right]^{2}_{L^2} + C_L \left( |\beta| + |\gamma| \right)^{1/2}.
\]
Let us now study

\[ I_6 = \int e^{2n + \frac{1}{2} + \frac{\gamma}{2}} \partial^2 \partial^{\gamma} v \partial^2 \partial^{\gamma} [A_n(v) \partial_n a_{\mu}] \]  

(81)

As

\[ |\partial^2 \partial^{\gamma} a_{\mu}|_{L^2} \leq C \]  

(82)

for \( \alpha + |\beta| + \gamma \leq s + 1 \), we immediately get

\[ |I_6| \leq C \left( \sup_{\alpha + |\beta| + \gamma \leq s + 1} |\partial^2 \partial^{\gamma} v|_{L^2} \right) \|v\|_{\alpha, \beta, \gamma}^2 \]  

(83)

It remains

\[ I_7 = \int e^{2n + \frac{1}{2} + \frac{\gamma}{2}} \partial^2 \partial^{\gamma} v \partial^2 \partial^{\gamma} [A_n(v) \partial_n a] \]  

(84)

for \( 1 \leq j \leq n - 1 \), which can be handled as \( I_5 \) and \( I_6 \), and

\[ I_8 = \int e^{2n + \frac{1}{2} + \frac{\gamma}{2}} \partial^2 \partial^{\gamma} v \partial^2 \partial^{\gamma} C(b, a, v) \]  

(85)

is straightforward.

So, for \( \alpha + |\beta| + \gamma \leq s \),

\[ \partial \|v\|_{\alpha, \beta, \gamma}^2 + \sum_{j=1}^{\infty} \int e^{2n + \frac{1}{2} + \frac{\gamma}{2}} \partial^2 \partial^{\gamma} v \partial^2 \partial^{\gamma} [A_n(v) \partial_n a] \]  

\[ \leq \|v\|_{\alpha, \beta, \gamma}^2 \sup_{\alpha + |\beta| + \gamma \leq s + 2} |\partial^2 \partial^{\gamma} v|_{L^2} \]  

\[ + e^M \|v\|_{\alpha + |\beta| + \gamma + \delta} \sup_{\alpha + |\beta| + \gamma + \delta \leq s + 2} \|A_n(v)\|_{H^\infty} + C \|v\|_{\alpha + |\beta| + \gamma + \delta} \]  

\[ + C_0 e \sup_{\alpha + |\beta| + \gamma \leq s + 2} |\partial^2 \partial^{\gamma} v|_{L^2} \]  

(86)

where

\[ C_0 = \sup_{-T \leq \tau \leq T, \alpha + |\beta| + \gamma \leq s + 2} \sum_{1 \leq k = 1}^{\infty} \int_0^1 \frac{1}{\mu} |\partial^2 \partial^{\gamma} a_{\mu}| \, dx. \]
It remains to control

$$\sup_{x'+|\beta|+\gamma' \leq s/2} |\partial_x^\alpha \partial_t \partial_x^\beta \partial_t^\gamma v|_{L^s}.$$  

For that notice that we can use the equation to bound $e At$ with the help of $\|v\|_s$. This provides a control on $\|e^{x-2s/2} \partial_x^\alpha \partial_t \partial_x^\beta \partial_t^\gamma v\|_{L^2}$ for $\alpha \geq 2$. By recurrence, one can control

$$\|e^{x-2s/2} \partial_x^\alpha \partial_t \partial_x^\beta \partial_t^\gamma v\|_{L^2}$$

for $\alpha \leq \alpha'$, and thus $\|v\|_{H^s}$ by $e^{-2s/2} \|v\|_s$. By standard Sobolev injections, for $s$ large enough (depending on $d$),

$$\sup_{x'+|\beta|+\gamma' \leq s/2} |\partial_x^\alpha \partial_t \partial_x^\beta \partial_t^\gamma v|_{L^s} \leq Ce^{-s/2} \|v\|_s, \quad (87)$$

Now let

$$z = e^{-s/2} v. \quad (88)$$

We have

$$\partial_t \|z\|_s^2 + \frac{\varepsilon}{2} \sum_{\gamma=1}^N e^{s + 2|\beta| + \gamma} \|\partial_x^\alpha \partial_t \partial_x^\beta \partial_t^\gamma z\|_s \leq \|e^{s/2} \partial_x^\alpha \partial_t \partial_x^\beta \partial_t^\gamma z\|_s \leq C\|z\|_s$$

$$+ e^{s/2} \|z\|_s \bigg( \|\partial_x^\alpha \partial_t \partial_x^\beta \partial_t^\gamma z\|_s + \sum_{i=1}^N e^{s + 2|\beta| + \gamma} \|\partial_x^\alpha \partial_t \partial_x^\beta \partial_t^\gamma z\|_s \bigg)$$

$$\times \sum_{1 \leq |\beta|+\gamma' \leq 2} \|e^{s + 2|\beta| + \gamma} \partial_x^\alpha \partial_t \partial_x^\beta \partial_t^\gamma z\|_{L^2}, \quad (89)$$

and

$$\|z(0)\|_s \leq Ce^{-s/2}. \quad (90)$$

By definition of $T_\varepsilon$, we have $z(t) \leq C$ for $0 \leq t \leq T_\varepsilon$, provided that $N' > s + 2$. On $[0, T_\varepsilon]$, we therefore have, for $C_1$ large enough, summing (89) with $\eta$ small enough

$$\partial_t \|z\|_s \leq C \|z\|_s + C, \quad (91)$$

provided $M > s + 2$, which implies that there exists $T_0 > 0$ such that $T_\varepsilon \geq T_0$ for $\varepsilon$ small enough.
This ends the proof of the assertion claimed in Remark 1.10. Notice that the smallness of $C_0$ is linked to the smallness of

$$\sup_{-T \leq t \leq T, \ y} \sum_{|\alpha| + |\beta| \leq 1} \int_0^1 z^a |\partial_z^a \partial_t^b \tilde{U}^i(t, y, z)| \, dx_\alpha$$

$$= \sup_{-T \leq t \leq T, \ y} \sum_{|\alpha| + |\beta| \leq 1} \int_0^1 z^a |\partial_z^a \partial_t^b \tilde{U}^0(t, y, z)| \, dx_\alpha + C_0(\epsilon),$$

where $C_0(\epsilon)$ goes to zero with $\epsilon$. Therefore the condition of smallness on $C_0$ reduces to a condition of smallness on the first boundary profile $\tilde{U}^0$. Theorems 1.3 to 1.9 are then straightforward.

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