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Decomposition of a Monic Matrix Polynomial into a Product of Linear Factors

by I. KRUPNIK[10]

It is proved that a monic matrix polynomial with all its elementary divisors of degree not more than 2 can be decomposed into a product of linear factors. (It is well known that such decomposition is possible in the case when all the elementary divisors are linear.) An example which shows that the number 2 mentioned above cannot be replaced by 3 is also given.

0. *Notation*

The matrix-valued function

$$L(\lambda)=\sum_{k=0}^{n-1}\lambda^k A_k+\lambda^n E \qquad \left(A_k, E\in \mathbf{C}^{m\times m}\right) \tag{1}$$

is said to be a monic matrix polynomial of degree n, of size $m\times m$ (where $\mathbf{C}^{m\times m}$ denotes the set of all complex matrices of size $m\times m$, and E denotes the unit matrix). A number $\lambda_0\in\mathbf{C}$ is called an eigenvalue of $L(\lambda)$ if the equation

$$L(\lambda_0)g_0=0$$

has a nonzero solution g_0. Such a vector g_0 is called an eigenvector of $L(\lambda)$ corresponding to λ_0. Vectors $g_1,\dots,g_{s-1}$ are said to be associated with an eigenvector g_0 if

$$\sum_{k=0}^{j}\frac{1}{k!}L^{(k)}(\lambda_0)g_{j-k}=0 \qquad (j=1,\dots,s-1).$$

The number s is called the length of the chain $g_0,\dots,g_{s-1}$. The maximal length of a chain composed of an eigenvector g_0 and vectors associated with it is denoted by

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$s(g_0)$. A canonical system of eigenvectors and associated vectors of $L(\lambda)$ corresponding to the number λ_0 is defined to be a system

$$g_0^{(k)}, g_1^{(k)}, \dots, g_{s_k-1}^{(k)} \qquad (k = 1, \dots, d)$$

where the vectors $\{g_0^{(k)}\}_1^d$ form a basis in Ker $L(\lambda_0)$, $s(g)^{(k)} = s_k$, $g_1^{(k)}, \dots, g_{s_k-1}^{(k)}$ is a chain of vectors associated with $g_0^{(k)}$,

$$s_1 = \max_{0 \neq g \in \operatorname{Ker} L(\lambda_0)} s(g), \qquad s_k = \max_{0 \neq g \in M_k} s(g) \quad (k > 1),$$

and M_k is some direct complement in Ker $L(\lambda_0)$ in the subspace spanned by $g_0, \dots, g_0^{(k-1)}$. The number

$$s(\lambda_0, L(\lambda)) = \sum_{k=1}^{d} s_k$$

is called the multiplicity of the eigenvalues λ_0 of $L(\lambda)$.

We note that $s(\lambda_0, L(\lambda))$ coincides with the multiplicity of λ_0 as a zero of Det $L(\lambda_0)$.

1. *Motivation*

We deal here with the possibility of decomposition of $L(\lambda)$ into a product of linear factors as follows:

$$L(\lambda) = \prod_{k=1}^{n} (\lambda E - Z_k) \qquad (Z_k \in \mathbf{C}^{m \times m}). \tag{2}$$

It is well known that this decomposition in the matrix case is not always possible.

The following result was achieved in [2–4].

THEOREM 1.1. *Suppose that $L(\lambda)$ is a monic matrix polynomial. If the lengths of all its Jordan chains equal* 1, *then $L(\lambda)$ can be decomposed into a product of linear factors.*

The following statement was proved in [5].

THEOREM 1.2. *Suppose that $L(\lambda)$ is a monic matrix polynomial. If all the multiplicities of zeros of the characteristic polynomial Det $L(\lambda)$ are not greater than* 2, *then $L(\lambda)$ can be decomposed into a product of linear factors.*

Comparing these two results, a natural question arises. Is the decomposition (2) possible in the case when the lengths of all Jordan chains of $L(\lambda)$ are not greater than 2? A positive answer to this question is the main result of this work (Theorem 2.2). We give also an example (Example 1) showing that the restrictive number 2 is precise in Theorem 2.2.

We construct this example in $\mathbf{C}^{3\times 3}$. An analogous decomposition theorem (Theorem 2.1) for monic matrix polynomials of size 2×2, but with the restrictive number 3, completes the description, since the example

$$L_1(\lambda) = \begin{pmatrix} \lambda^2 & -1 \\ 0 & \lambda^2 \end{pmatrix}$$

shows that the restrictive number 3 is precise in Theorem 2.1.

2. *Method and Results*

The main result of this article is contained in the next two theorems:

THEOREM 2.1. *Suppose that $L(\lambda)$ is a 2×2 monic matrix polynomial. If the lengths of all its Jordan chains are not greater than 3, then $L(\lambda)$ can be decomposed into a product of linear factors.*

THEOREM 2.2. *Suppose that $L(\lambda)$ is a monic matrix polynomial. If the lengths of all its Jordan chains are not greater than 2, then $L(\lambda)$ can be decomposed into a product of linear factors.*

The following result of H. Langer [1] is used in the proofs of these theorems:

LEMMA 1. *Let $L(\lambda)$ be a monic matrix polynomial. $L(\lambda)$ admits a factorization*

$$L(\lambda) = N(\lambda)(\lambda E - Z) \tag{3}$$

if and only if there exists a basis in $\mathbf{C}^m$ formed by some Jordan chains of $L(\lambda)$ (not necessarily of the maximal lengths).

Since the degree of $N(\lambda)$ is $n - 1$, we can use mathematical induction. Thus, the main point of the proof is the choice of a basis satisfying the conditions of Lemma 2.1. In order to prove Theorem 2.2 we use the following

LEMMA 2. *Let $L(\lambda)$ be a monic matrix polynomial of degree n, of size $m \times m$. Let Σ be its canonical system of Jordan chains. Then there exists a monic matrix polynomial $M(\lambda)$ of degree n, of size $m \times m$, such that Σ is its canonical system of Jordan chains, and different chains from Σ correspond to different eigenvalues of $M(\lambda)$.*

If $L(\lambda)$ satisfies the conditions of Theorem 2.2, then $M(\lambda)$ satisfies the conditions of Theorem 1.2. Theorem 1.2 implies that $M(\lambda)$ admits the factorization (3). By Lemma 1, there exists a basis in $\mathbf{C}^m$ formed by Jordan chains of $M(\lambda)$. This means that there exists a basis in $\mathbf{C}^m$ formed by Jordan chains of $L(\lambda)$.

To conclude we give the abovementioned example.

EXAMPLE 1. Let

$$L_2(\lambda) = \begin{pmatrix} \lambda^2 - \lambda & 1 & -1 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^2 - 2\lambda + 1 \end{pmatrix}.$$

It is readily checked that a canonical system of this matrix polynomial can be chosen as follows:

the first chain:

$$\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix};$$

the second chain:

$$\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

The maximal length of Jordan chains is 3 here. It is evident that there is no basis in $\mathbf{C}^3$ satisfying Lemma 1. Thus, $L_2(\lambda)$ cannot be decomposed into a product of linear factors.

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On the Classical Theory of Elementary Spinors

by FRANCO PIAZZESE[11]

1

Spinors are a mathematical tool long employed in theoretical physics to describe the wave function of a quantum system with spin: Pauli described the wave function of a spinning electron with an elementary (2-component) spinor in his nonrelativistic theory; Dirac employed to a 4-component spinor in his very successful relativistic theory.

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