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# Numerical Mountain Pass Periodic Solutions of a Nonlinear Spring Equation

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**Abstract**—In this paper, we show how the introduction of a nonlinear term in the classic spring model can produce dramatic results. We compute a large amplitude solution which is drastically different from the known linear, small amplitude solution. A dual variational formulation is given, recasting the problem as one in which saddle points correspond to solutions of the differential equation. Our computations are based on the numerical mountain pass algorithm developed by Choi and McKenna which was inspired by the theorems of Ambrosetti, Rabinowitz and Ekeland. © 1998 Elsevier Science Ltd. All rights reserved.

**Keywords**—Dual variational formulation, Nonlinear spring, Mountain pass algorithm.

## 1. INTRODUCTION

Consider a mass attached to a vertical spring with a rubber band providing additional support. While the spring imparts force in both the upwards and downwards directions, the rubber band imparts force only in the upwards direction. Let $u(t)$ denote the displacement of mass from equilibrium at time $t$, where $u = 0$ denotes the position of the mass before elongation. Thus, we may model the restoring force with a nonlinear Hooke's Law given by $au^+ - bu^-$ where

$$u^+ := \begin{cases} u, & \text{if } u > 0, \\ 0, & \text{if } u \le 0, \end{cases} \tag{1}$$

$$u^- := \begin{cases} 0, & \text{if } u > 0, \\ -u, & \text{if } u \le 0, \end{cases} \tag{2}$$

and $a > b$. If we suppose a small additional force $f(t) = \epsilon \sin(\mu t)$ is added and use Newton's Law, then $u(t)$ must satisfy

$$u'' + au^+ - bu^- = 10 + \epsilon \sin(\mu t). \tag{3}$$

We look for $2\pi$ periodic solutions.

Here, we take $a = 17$, $b = 13$, and $\mu = 4$ as in [1]. Equation (3) has a near-equilibrium positive solution $u(t)$, easily obtained by inspection, given by

$$u(t) = \frac{10}{17} + \epsilon \sin(4t). \tag{4}$$

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This solution is small in amplitude and is often referred to as the linear solution. There is mathematical proof that large amplitude solutions exist, however the proofs are not constructive. Here, we compute some large amplitude solutions so that we may have quantitative information about the behavior of the physical system.

Thus far, the numerical methods which have been used to compute solutions to boundary value problems include continuation methods, shooting schemes, and a method which can be termed "hit or miss."

While continuation is a powerful tool, it requires a great deal of computational power and precise refinement. The method we employ is robust and global so we choose it instead.

Shooting consists of numerically solving the associated initial value problem for various initial values until the solution computed exhibits the appropriate boundary condition. Although the way in which the initial values used at each step depend on the previous initial values tried, and often convergence is achieved, a reasonable initial guess for the initial conditions is required. Since we are interested in periodic solutions, determining a good initial guess is quite risky.

The "hit or miss" method of finding periodic solutions essentially consists of solving the associated initial value problem, obtaining a function $u(t)$, then letting $t$ grow until the solution settles into a periodic function of the appropriate period. Again, a reasonable initial guess is required, and of course a damping term would be required to extinguish blow-up.

Here, we employ a global method for obtaining large-amplitude periodic solutions to (3) which requires no initial guess or damping term. We use the mountain pass algorithm based on the work of Choi and McKenna [2]. The mountain pass algorithm is a constructive implementation of the mountain pass theorems of Ambrosetti, Rabinowitz and Ekeland which are primarily used to prove the existence of critical points of functionals defined on Banach Space. The algorithm was suggested by a heuristic view of the mountain pass theorem, so it is here that we begin the description.

## 2. MATHEMATICAL BACKGROUND FOR THE MOUNTAIN PASS ALGORITHM

The mountain pass theorem is used to prove the existence of critical points of functionals, $I \in C^1(E, \mathbf{R})$ which satisfy the *Palais-Smale* (PS) condition, a compactness condition which occurs repeatedly in critical point theory. We say that $I$ satisfies the *Palais-Smale* condition if any sequence $\{u_m\} \subset E$, for which $I(u_m)$ is bounded and $I'(u_m) \to 0$ as $m \to \infty$ possesses a convergent subsequence.

MOUNTAIN PASS THEOREM. *Let $E$ be a real Banach space, $I \in C^1(E, \mathbf{R})$ satisfying the (PS) condition, and $B_\rho$ be a ball with radius $\rho$ in $E$. Suppose*

I. *for some $w \in E$, there are constants $\rho, \alpha > 0$ such that*

$$I|_{w+\partial B_\rho} \geq I(w) + \alpha; \tag{5}$$

II. *there is an $e \in E \backslash \overline{B}_\rho$ such that*

$$I(e) \leq I(w). \tag{6}$$

*Then $I$ possesses a critical value $c \geq I(w) + \alpha$. Moreover, $c$ can be characterized as*

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} |(u),$$

*where*

$$\Gamma = \{g \in C([0,1], E) \mid g(0) = w, \; g(1) = e \}.$$

*Basically, if $I$ is a functional defined on $E$, a Banach Space with a known critical point $w$, which is a local minimum, and another point $e$ such that $I(e) < I(w)$, the mountain pass theorem states (with some additional technicalities) if one considers all the continuous paths joining $w$ and $e$ and looks at the infimum of the maxima of the functional $I$ along those paths, one obtains a critical value.*

To illustrate the use of this theorem [3], consider the semilinear elliptic equation

$$\Delta u + u^3 = 0, \qquad \text{in } \Omega \tag{7}$$

with Dirichlet boundary conditions $u = 0$ on $\partial\Omega$, where $\Omega$ is a nice bounded region in $\mathbf{R}^2$.

Solutions of this equation correspond to critical points of the functional

$$I(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \frac{u^4}{4}\right) dx$$

defined on the Hilbert space $H_0^1(\Omega)$.

It is clear that $u = 0$ is a solution of the equation. In fact, this solution corresponds to a local minimum of the functional $I$. One can also see that if we choose another positive function $v$ in the Banach space, then $I(cv) \to -\infty$ as $c \to +\infty$. Thus, we have the ingredients of the mountain pass theorem, and it can be concluded that an additional solution $u_1$ exists, with the property that $I(u_1) > 0$.

### 2.1. The Mountain Pass Algorithm: A Brief Description

The basic idea of the associated numerical algorithm is the following. On a finite-dimensional approximating subspace, take a piecewise linear path joining the local minimum and a point $e$ whose image is lower. Calculate the maximum of $I$ along the path. Then deform the path by pushing the point at which the maximum is located in the direction of steepest descent. Repeat this step, stopping only when a critical point is reached.

In [2], Choi and McKenna showed that when carefully implemented, this process gives rise to an extremely robust algorithm which is globally convergent and which will always converge to a solution with the required mountain pass property. A detailed description of the algorithm can be found in [2,4]. Since its implementation in 1991 by Choi and McKenna, the mountain pass algorithm has been used in many different nonlinear problems including: elliptic equations, wave equations, a suspension bridge equation, and as a tool for finding traveling wave solutions and water waves [2,4–7]. The algorithm is now one of the major tools in the numerical analysis of partial differential equations.

## 3. BASIC THEORY FOR THE NONLINEAR SPRING EQUATION

We look for $2\pi$-periodic solutions of the nonlinear spring equation as given in (3). Define a subspace of the Hilbert space $L^2$ as follows:

$$S := \left\{ u \mid u \text{ is } 2\pi \text{ periodic on } \mathbf{R} \text{ and } \int_0^{2\pi} u < \infty \right\}.$$

Let the linear operator $A$ acting on $S$ be given by $Au = -u''$. The eigenvalue problem $Au = \lambda u$ is known to have eigenfunctions $\phi_n = \sin nt,\ \cos nt,\ (n = 1, 2, 3, \dots)$. Corresponding eigenvalues are $\lambda_n = n^2$.

There has been considerable theoretical work on equations of the type

$$y''(t) + f(y(t)) = s(1 + \epsilon h(t)).$$

We provide a summary of the theorems that pertain to (3) For more details see [8,9].

THEOREM 3.1. *If $a > b > 0$ and $1/\sqrt{a} + 1/\sqrt{b} \neq 2/N$ for $N = 1, 2, 3, \ldots$, then the piecewise linear homogeneous equation*

$$y'' + ay^+ - by^- = 0$$

*has no nonconstant $2\pi$-periodic solutions.*

This can be seen by investigating the phase-plane for the differential equation: all periodic solutions will have period $\pi/\sqrt{a} + \pi/\sqrt{b}$.

THEOREM 3.2. *Let $a > b > 0$ and $2\pi/\sqrt{a} < 2\pi/\mu < \pi/\sqrt{a} + \pi/\sqrt{b}$, then the piecewise linear equation*

$$y'' + ay^+ - by^- = 10 + \epsilon \sin \mu t$$

*has three $2\pi$-periodic solutions.*

One of the solutions guaranteed by this theorem is positive and small in amplitude and thus, can be found by solving

$$y'' + ay = 10 + \epsilon \sin \mu t.$$

The other two solutions are large in amplitude, hence they change sign.

Heuristically, these theorems address a cause and effect relationship between $a$ and $b$ "straddling" an eigenvalue of the operator $A$ and the existence of solutions to the associated differential equation.

## 4. A DUAL VARIATIONAL FORMULATION

To justify application of the mountain pass theorem to the nonlinear spring equation we study here, we need to reformulate the problem somewhat. We have three basic tasks to perform: we must verify that Conditions I and II are satisfied for suitable points in our function space, and we must verify that the (PS) condition is satisfied.

The major impediment to a **direct** variational formulation for the problem (3) is that periodic solutions will not be of minimum or mountain pass type. However, we show that this problem can be reformulated as a dual variational problem as in [4,5]. In doing so, we manipulate the positive, near equilibrium solution into a strict local minimum of the corresponding functional. Using this formulation, we find other periodic solutions.

Let $\beta$ be a given constant satisfying

$$\beta > a \quad \text{and} \quad \beta \neq \lambda_n. \tag{8}$$

Define

$$\begin{aligned} Lu &= u'', \\ B_\beta(u) &= (\beta - a)u^+ - (\beta - b)u^-, \\ f(t) &= 10 + \epsilon \sin 4t. \end{aligned}$$

Equation (3) can be recast as

$$(L + \beta)u - B_\beta(u) \;=\; f.$$

Direct analysis shows that $(L + \beta)^{-1}$ exists and is compact from $\overline{S}$ to $\overline{S}$, so for any sufficiently smooth $v \in \overline{S}$, we can find a unique $u \in \overline{S}$ such that

$$(L + \beta)u \;=\; v. \tag{9}$$

After making this transformation, our equation becomes

$$v - B_\beta(u) = f.$$

Since $\beta > a$, the operator $B_\beta : \mathbf{R} \to \mathbf{R}$ is strictly monotone increasing and is hence, invertible. We can see that

$$B_\beta^{-1}(v - f) = u \tag{10}$$

and combining (9) and (10) yields

$$-(L + \beta)^{-1}v + B_\beta^{-1}(v - f) = 0. \tag{11}$$

As can be seen from Lemma 4.2 below, solutions of this equation are critical points of the functional $I : \overline{S} \to \mathbf{R}$ defined by:

$$I(v) = \int_0^{2\pi} \left[ -\frac{1}{2}(L + \beta)^{-1}v \cdot v + J_\beta^*(v - f) \right] dt, \tag{12}$$

where

$$J_\beta^*(u) := \begin{cases} \dfrac{1}{2(\beta - a)}u^2, & \text{if } u > 0, \\ \dfrac{1}{2(\beta - b)}u^2, & \text{if } u \le 0. \end{cases}$$

We note that $(J_\beta^*)' = B_\beta^{-1}$.

Lemma 4.1. *Let condition (8) be satisfied. If $v \in \overline{S}$ satisfies (11) and $u$ is defined by (9), then $u \in H^2(0, 2\pi)$ and $u$ is a solution to equation (3) subject to the periodic boundary conditions.*

We refer to [5] for a proof of a similar lemma.

Since our nonlinearity $(au^+ - bu^-)$ behaves linearly at infinity, a standard calculation proves the following lemma.

Lemma 4.2. *$I$ is continuous and Fréchet differentiable with*

$$\begin{aligned} I'(v)\psi &= \int_0^{2\pi} -(L + \beta)^{-1}v \cdot \psi + B_\beta^{-1}(v - f)\psi \\ &= \int_0^{2\pi} -(L + \beta)^{-1}v \cdot \psi + \frac{1}{\beta - a}(v - f)^+\psi - \frac{1}{\beta - b}(v - f)^-\psi. \end{aligned} \tag{13}$$

*For any $v, \psi \in H$.*

Under the transformation (9), the exact solution $u$ given by (4) becomes

$$v_{\min} = \frac{10}{17}\beta + (\beta - 16)\epsilon \sin 4t. \tag{14}$$

Remark. By condition (8) near $v_{\min}$, all terms in $I$ are positive, so $J_\beta^*(v - f)$ behaves "like" $1/(2(\beta - a))v^2$. Therefore, if $\beta$ is chosen close to $a$, this will be the dominant term in the functional, and we have arranged for $v_{\min}$ to be a strict local minimum. In addition, the solution was verified numerically to be a minimum.

We will now turn our attention to showing that $I$ satisfies Palais-Smale and that Condition II of the mountain pass theorem is satisfied. A similar version of the following can be found in [5].

Lemma 4.3. *Let $16 < a < 25$ and $9 < b < 16$. In $H^2(0, 2\pi) \cap \overline{S}$, the Palais-Smale condition for the functional $I$ is satisfied.*

Proof. Given any sequence $\{v_n\}$ in $\overline{S}$ such that $I(v_n)$ is bounded and $I'(v_n) \to 0$, we claim that $\|v_n\|_{L^2(0,2\pi)}$ is bounded. Assume the contrary: $\|v_n\| \to \infty$ as $n \to \infty$. Since $I'(v_n) \to 0$, we have

$$-(L + \beta)^{-1}v_n + B_\beta^{-1}(v_n - f) \to 0. \tag{15}$$

Let $w_n \equiv (v_n/\|v_n\|)$, then

$$-(L+\beta)^{-1}w_n + B_\beta^{-1}\left(w_n - \frac{f}{\|v_n\|}\right) \to 0.$$

Since $\|w_n\| = 1$, there exists a subsequence, which we still call $\{w_n\}$, such that $w_n \to w$ weakly in $L^2$ for some $w$ in $L^2(\Omega)$. Due to the compactness of the operator $(L+\beta)^{-1}$ which maps weakly convergent sequences to convergent sequences, we have

$$B_\beta^{-1}\left(w_n - \frac{f}{\|v_n\|}\right) \to (L+\beta)^{-1}w.$$

Since the growth of $B_\beta$ near positive and negative infinity is linear, a standard calculation shows that

$$w_n - \frac{f}{\|v_n\|} \to B_\beta\left((L+\beta)^{-1}w\right).$$

That is

$$w_n \to B_\beta\left((L+\beta)^{-1}w\right). \tag{16}$$

Since the weak limit is unique, $w = B_\beta((L+\beta)^{-1}w)$, which is the same as

$$-(L+\beta)^{-1}w + B_\beta(-w) = 0. \tag{17}$$

Now, define $u = (L+\beta)^{-1}w$. By Lemma 4.1, we have that $u \in H^2 \cap \overline{S}$ and $u$ satisfies $Lu + au^+ - bu^- = 0$. By Theorem 3.1, this implies that $u = 0$, and therefore, $w = 0$. Equation (16) then implies $w_n \to 0$ in the $L^2$ norm. This contradicts the fact that $\|w_n\| = 1$. Hence, we have justified our claim and $\|v_n\|$ has an *a priori* bound.

With the established *a priori* bound, there exists a $v \in L^2$ such that $v_n \to v$ weakly in $L^2$. Using condition (15) and the compactness of the operator $(L+\beta)^{-1}$, we have $B_\beta^{-1}(v_n - f) \to (L+\beta)^{-1}v_n$. Again, due to the linear growth rate of $B_\beta$ at infinity

$$v_n \to B_\beta\left((L+\beta)^{-1}v\right) + f, \qquad \text{in } L^2 \text{ norm}.$$

Thus, $v_n$ converges in the $L^2$ norm and the proof is complete. ∎

## 5. RESULTS

Using the mountain pass algorithm, we located various large amplitude solutions. The first point of the path was always placed at the minimum, $v_{\min}$. The other endpoint, the "lower point" $e$, was placed at the point $e$ indicated in the caption.

After a solution was obtained using the algorithm, we plugged in its initial data to a fourth-order Runge-Kutta scheme to verify its accuracy. Below, we plot the mountain pass solution as a solid curve, and the Runge-Kutta solution as a dashed curve. The graphs are almost identical, indicating the accuracy of the mountain pass algorithm.

We note that the second solution is essentially a translate of the first, but to which translate the algorithm converges depends on the choice of $e$. This feature of the algorithm is really a feature of the associated functional. Certain symmetries are preserved by the algorithm due to the symmetric nature of the functional, and so, one can almost specify the type of solution desired. This feature was first noted and proven in [2] and can be proven for this functional in the same fashion, so we omit the proof.

Here, we let $a$ and $b$ straddle 9, the third eigenvalue of $A$ and proceed in a similar fashion.

We also considered the two-dimensional bridge problem, i.e., we take $b = 0$ so that there is an upwards force, but only slack downwards. We converged to a large amplitude solution, but the Runge-Kutta solution is not as convincing here as it was before for values of $t$ close to $2\pi$. This

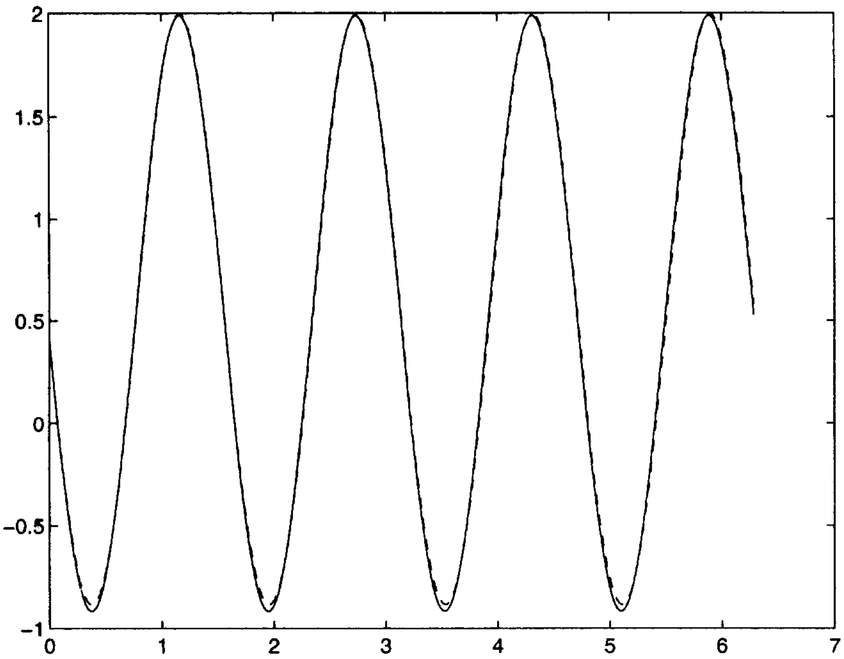


Figure 1. The solution converged to using $e = v_{\min} - 50\sin(4t)$ is shown in solid; the computed Runge-Kutta solution is shown dashed. We took $a = 17$, $b = 13$, $\beta = 20.8$.

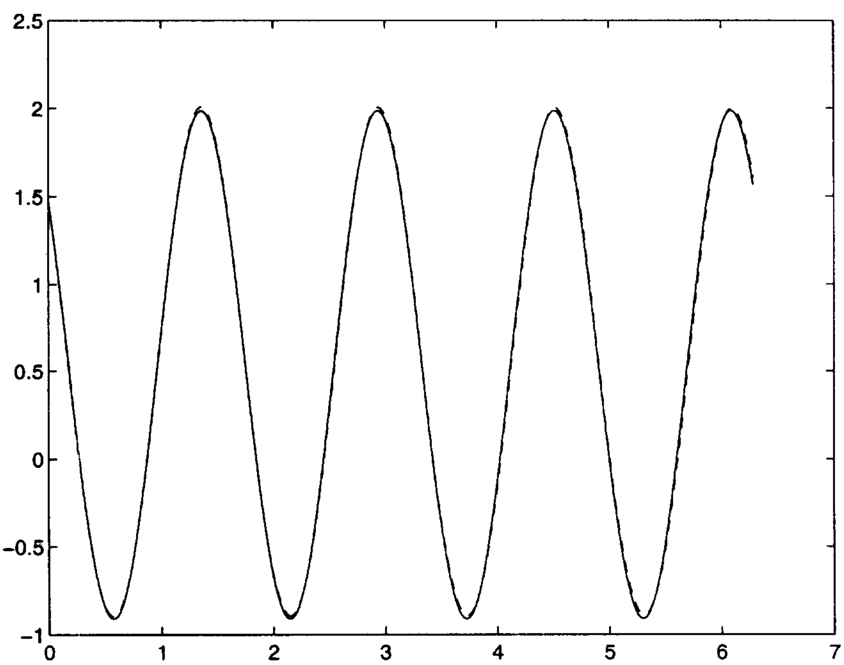


Figure 2. The solution converged to using $e = v_{\min} - 30\sin(4t) + 30\cos(4t)$ is shown in solid; the computed Runge-Kutta solution is shown dashed. We took $a = 17$, $b = 13$, $\beta = 20.8$.

is expected since the derivative of the nonlinearity $f(t,u) = au^{+} - bu^{-}$ has a greater discontuity than in previous cases. Here, the jump is from 0–17, instead of just over an eigenvalue. This discontinuity causes inaccuracy in the Runge-Kutta solver. Error estimates in Runge-Kutta apply only when $f \in C^5$. Since this check was not sufficiently persuasive, we halved the mesh size, reran the mountain pass algorithm and converged to the same solution, a check we feel provides reasonable confidence.

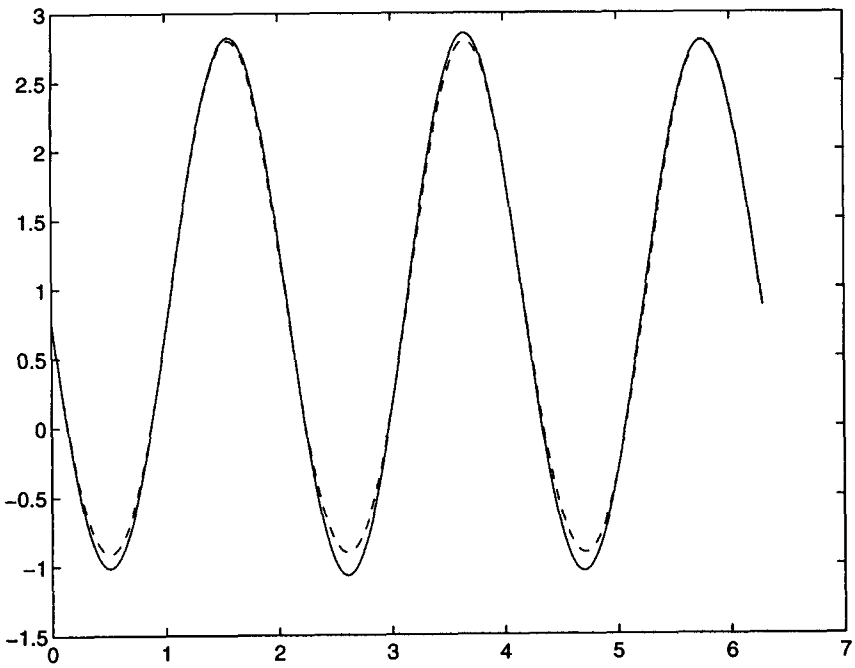


Figure 3. The solution converged to using $e = v_{\min} - 35\sin(3t)$ is shown in solid; the computed Runge-Kutta solution is shown dashed. We took $a = 10$, $b = 5$, $\beta = 14.8$.

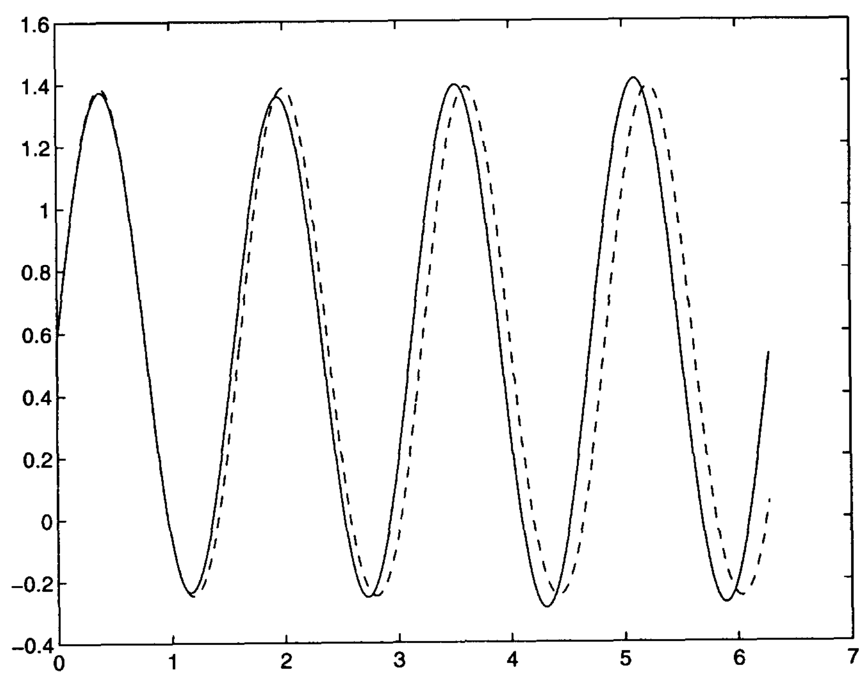


Figure 4. The solution converged to using $e = v_{\min} - 35\sin(4t)$ is shown in solid; the computed Runge-Kutta solution is shown dashed. Here we took $a = 17$, $b = 0$, $\beta = 20.8$.

## 6. CONCLUSION

To indicate how different the large amplitude solution is from the small linear solution, we show both graphs here. Despite a common forcing term, the solutions are radically different. This is discussed by Blanchard, Devaney and Hall in [10, p. 445–446] as an initial value problem and is quite appropriate and illuminating for an undergraduate differential equations class.

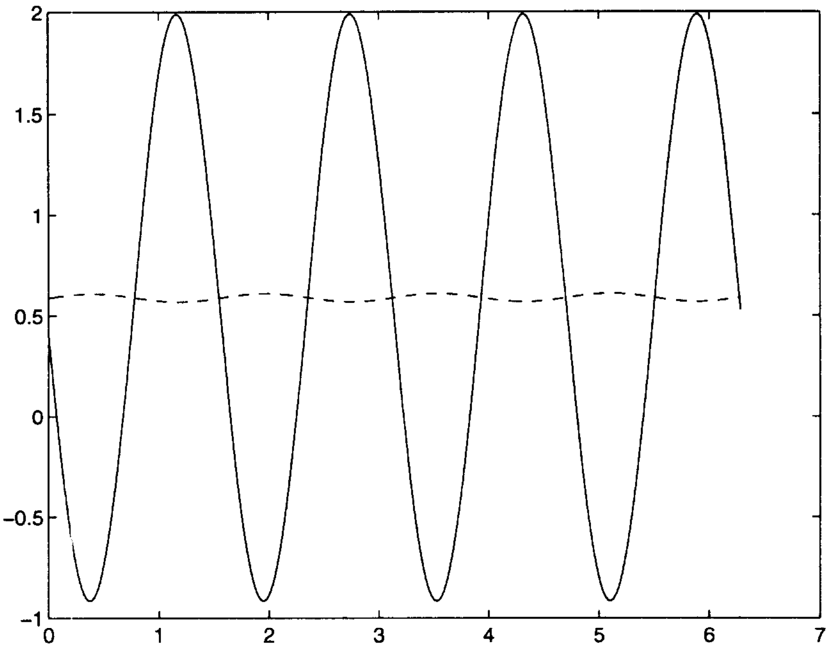


Figure 5. The solution converged to using $e = v_{\min} - 50\sin(4t)$ is shown in solid; the known, small amplitude solution is shown dashed.

The theorems cited in the background section indicate that there are three distinct $2\pi$ periodic solutions. We show the small, linear solution, and a large amplitude solution. There is however, one unaccounted for—a solution which is neither a local minimum, nor of mountain pass type. For a further discussion of the unstable solution, see [11].

We reiterate that the algorithm we used required no judicious initial guess or forcing term. This leads us to believe that this approach would be appropriate when the model is not just one mass attached to a nonlinear spring, but perhaps a few masses—which leads to a system of equations. Since thus far few methods exist to compute periodic solutions for systems, we view the work presented here as a first step towards such work.

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