Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics Letters 20 (2007) 1070–1074
Applied Mathematics Letters
www.elsevier.com/locate/aml

Periodic solutions of logistic equations with time delay

Benedetta Lisena

Dipartimento di Matematica, Universitá degli studi di Bari, via Orabona 4, 70125 Bari, Italy

Received 6 February 2006; received in revised form 19 October 2006; accepted 10 November 2006

Abstract

A new criterion is established for the existence of positive periodic solutions to the following delay logistic equation:

$$u'(t) = u(t)[r(t) - a(t)u(t) + b(t)u(t-\tau)]$$

where $r(t)$, $a(t)$, $b(t)$ are periodic continuous functions, $a(t) > 0$, $b(t) \geq 0$ and $r(t)$ has positive average.

Keywords: Delay equations; Logistic equation; Periodic solution

1. Introduction

Differential equations with time delays have been studied by many authors as models of population growth (see [4, 5]).

In particular, for delayed logistic equations with periodic coefficients, the existence of a periodic positive solution has been investigated (see, for example, [1–3,7]). In [7], by using the method of coincidence degree, Li and Kuang have obtained sufficient conditions for the existence of positive periodic solutions to the Lotka–Volterra equation

$$u'(t) = u(t)\left[k(t) - a(t)u(t) + \sum_{i=1}^{n} b_i(t)u(t-\tau_i) - \sum_{j=1}^{m} c_j(t)u(t-\sigma_j)\right] \tag{1.1}$$

with periodic coefficients. Eq. (1.1) can model the dynamics of population size of a species in a time-fluctuating environment.

The simpler periodic equation,

$$u'(t) = u(t)\left[k(t) - a(t)u(t) + b(t)u(t-\tau)\right], \tag{1.2}$$

was previously considered by Freedman and Wu [3], who gave sufficient conditions for the existence of a positive periodic solution. In addition, they employed the Lyapunov–Razumikhin technique to obtain an attracting region.

Apparently, the above existence results, relating to Eqs. (1.1) and (1.2), are not related. Therefore, the author has been motivated to seek a relationship between them. Indeed, if we take into account Eq. (1.2), Theorem 2.4 indicates the connection between them, by means of a more general criterion.

E-mail address: lisena@dm.uniba.it.

0893-9659/$ - see front matter
doi:10.1016/j.aml.2006.11.008

2. Existence of periodic solutions

Let $a(t), b(t), r(t)$ be T-periodic continuous functions. The objective of this work is to derive sufficient conditions for the existence of a positive periodic solution to the delay logistic equation

$$u'(t) = u(t)[r(t) - a(t)u(t) + b(t)u(t-\tau)], \tag{2.1}$$

where $a(t) > 0$, $b(t) \geq 0$ and

$$m[r] = \frac{1}{T}\int_0^T r(s)\mathrm{d}s > 0.$$

The above model (2.1) of single-species growth has been utilized by many authors since it takes into account both the seasonality of environment and the effects of time delay.

It is well known that, for any continuous $\phi(t)$, there exists a unique solution of (2.1) which satisfies the initial condition

$$u(t) = \phi(t), \quad t \in [-\tau, 0].$$

Moreover, if $\phi(t) \geq 0$ and $\phi(0) > 0$, then $u(t) > 0$ for $t > 0$.

We will call such solutions positive. As regards the existence of a periodic solution in Eq. (2.1), the investigations of Li and Kuang [7] lead to the following

Theorem 2.1. *Suppose that*

$$a(t) > b(t), \quad t \in [0, T]. \tag{2.2}$$

Then Eq. (2.1) *has a positive, T-periodic solution.*

Proof. When $r(t) > 0$ our statement can be directly derived from the results in [7] for the more general equation (1.1). It is not hard to check that the proof given by Li and Kuang may be easily adapted under the weaker assumption $m[r] > 0$ (see also [2]). □

Previously, Freedman and Wu [3] proved an existence result for Eq. (1.2) introducing a different assumption.

Theorem 2.2. *Assume that $k(t) > 0$, $a(t) > 0$, $b(t) \geq 0$ and suppose that the delayed functional equation*

$$k(t) - a(t)\psi(t) + b(t)\psi(t-\tau) = 0$$

has a positive, T-periodic, continuously differentiable solution. Then (1.2) *has a positive, T-periodic solution.*

Two remarks are in order. First, we must observe that Theorem 2.1 in [3] included a global asymptotic stability result. In the second place, the cited theorem required that all coefficients were continuously differentiable, but the authors do not use such a hypothesis in the part of the proof concerning existence.

Lemma 2.1. *Let $a(t), b(t), r(t), k(t)$ be continuous, T-periodic functions such that*

$$m[r] > 0, \qquad a(t), k(t) > 0, \qquad b(t) \geq 0, \qquad k(t) > r(t).$$

Suppose that the differential equation

$$u'(t) = u(t)(k(t) - a(t)u(t) + b(t)u(t-\tau)) \tag{2.3}$$

has a positive T-periodic solution $\mathring{v}(t)$ and let $\mathring{U}(t)$ denote the positive periodic solution of the logistic equation

$$U'(t) = U(t)(r(t) - a(t)U(t)).$$

If $u(t)$ is a solution of Eq. (2.1) *for $t \geq 0$, such that*

$$\mathring{U}(s) \leq u(s) \leq \mathring{v}(s), \quad s \in [-\tau, 0],$$

then

$$\mathring{U}(t) \le u(t) \le \mathring{v}(t), \quad t \in [0, T].$$

Proof. First note that

$$\mathring{U}(t) < \mathring{v}(t), \quad t \in \mathbf{R}. \tag{2.4}$$

In fact the following inequality:

$$m[a(t)\,\mathring{U}(t)] = m[r(t)] < m[k(t)] < m[k(t)] + m[b(t)\,\mathring{v}(t)] = m[a(t)\,\mathring{v}(t)]$$

implies the existence of a $t_0 > 0$ such that

$$\mathring{U}(t_0) < \mathring{v}(t_0).$$

Since for $u(t) > 0$

$$r(t) - a(t)u(t) < k(t) - a(t)u(t) + b(t)u(t-\tau),$$

we obtain

$$\mathring{U}(t) < \mathring{v}(t), \quad t > t_0.$$

The periodicity of both $\mathring{U}(t)$ and $\mathring{v}(t)$ ensures (2.4).

Now let $u(t)$ be a solution of (2.1) such that

$$\mathring{U}(s) \le u(s) \le \mathring{v}(s), \quad s \in [-\tau, 0].$$

Therefore

$$u'(t) \ge u(t)(r(t) - a(t)u(t)), \qquad \mathring{U}(0) \le u(0).$$

By a known comparison theorem

$$\mathring{U}(t) \le u(t), \quad t \ge 0.$$

Furthermore, for $t \in [0, \tau]$

$$u'(t) \le u(t)(r(t) - a(t)u(t) + b(t)\,\mathring{v}(t-\tau)) \le u(t)(k(t) - a(t)u(t) + b(t)\,\mathring{v}(t-\tau))$$

and

$$u(0) \le \mathring{v}(0).$$

As before,

$$u(t) \le \mathring{v}(t), \quad t \in [0, \tau].$$

The extension to all intervals $[(n-1)\tau, n\tau]$ follows by induction. In particular

$$\mathring{U}(t) \le u(t) \le \mathring{v}(t), \quad t \in [0, T]$$

as required. □

The following result is our major tool used in guaranteeing the existence of a T-periodic solution.

Theorem 2.3. *Assume that $\mathring{v}(t)$ and $\mathring{U}(t)$ are just the periodic functions considered in Lemma 2.1. Then Eq. (2.1) has a positive T-periodic solution $\mathring{u}(t)$ verifying*

$$\mathring{U}(t) \le \mathring{u}(t) \le \mathring{v}(t), \quad t \in [0, T].$$

Proof. Take the linear space

$$X = \{\phi(t) \in C(\mathbf{R}, \mathbf{R}) : \phi \ T\text{-periodic and positive}\}$$

with the norm

$$\|\phi\| = \max_{t\in[0,T]} \phi(t) \, .$$

Define the continuous map

$$F : X \longrightarrow X$$

by $F(\phi)(t) = u(t)$, where $u(t)$ is the unique positive, T-periodic solution of the logistic equation

$$u' = u(r(t) + b(t)\phi(t-\tau) - a(t)u(t)), \quad t \geq 0.$$

Let $M = \max_{t\in[0,T]} \mathring{v}(t)$ and

$$A = M \cdot \max_{t\in[0,T]} \{|r(t)| + b(t)M + a(t)M\}.$$

Our aim is to prove that F has a fixed point. Define

$$S = \{\phi \in X : \mathring{U}(t) \leq \phi(t) \leq \mathring{v}(t)\}.$$

S is a convex closed subset of the normed linear space X.

Now take $\phi \in S$; then by Lemma 2.1, $u(t) = F(\phi)(t)$ verifies

$$\mathring{U}(t) \leq u(t) \leq \mathring{v}(t),$$

that is $u(t) \in S$. Furthermore, by the Ascoli–Arzelá theorem $F(S)$ is a relatively compact subset of S. In fact

$$\begin{aligned} |u'(t)| &\leq u(t)(|r(t)| + b(t)\phi(t-\tau) + a(t)u(t)) \\ &\leq \mathring{v}(t)(|r(t)| + b(t)\,\mathring{v}(t-\tau) + a(t)u(t)) \\ &\leq M(|r(t)| + b(t)M + a(t)M) \leq A. \end{aligned}$$

Then every family of functions in $F(S)$ is uniformly bounded and equicontinuous on $[0, T]$. By applying Schauder's second theorem (see [6]), there exists $\mathring{u} \in S$ such that

$$F(\mathring{u}) = \mathring{u},$$

that is, there is a positive periodic solution of (2.1) with

$$\mathring{U}(t) \leq \mathring{u}(t) \leq \mathring{v}(t), \quad t \in [0, T]. \quad \square$$

We are in a position to prove the main result of this paper.

Theorem 2.4. *Assume that* $a(t) > 0$, $b(t) \geq 0$, $m[r] > 0$ *and suppose that there exists a positive,* T*-periodic, continuously differentiable function* $\phi(t)$ *such that*

$$a(t)\phi(t) - b(t)\phi(t-\tau) > 0, \quad t \in [0, T]. \tag{2.5}$$

Then delay equation (2.1) *admits a positive,* T*-periodic solution.*

Proof. Take

$$\lambda > \max\left(\frac{r(t)}{a(t)\phi(t) - b(t)\phi(t-\tau)}\right) \tag{2.6}$$

and set

$$k(t) = a(t)(\lambda\phi(t)) - b(t)(\lambda\phi(t-\tau))$$

so that the functional equation

$$k(t) - a(t)\psi(t) + b(t)\psi(t-\tau) = 0$$

has $\lambda\phi(t)$ as a positive, T-periodic, continuously differentiable solution.

By Theorem 2.2, we can consider $\overset{\circ}{v}(t)$, a positive and T-periodic solution of the differential equation

$$u'(t) = u(t)(k(t) - a(t)u(t) + b(t)u(t-\tau)).$$

Since, by (2.6)

$$k(t) = \lambda(a(t)\phi(t) - b(t)\phi(t-\tau)) > r(t),$$

an application of the previous Theorem 2.3 concludes the proof. □

Remark. Note that our hypothesis (2.5) is weaker and more feasible than the one in Theorem 2.2 given by Freedman and Wu. Indeed the previous authors require the existence of an appropriate $\phi(t)$ for which

$$a(t)\phi(t) - b(t)\phi(t-\tau) = k(t) > 0, \quad t \in [0, T].$$

Concerning the result given by Li and Kuang within an equation of type (2.1), it is obvious that the inequality

$$a(t) > b(t), \quad t \in [0, T]$$

implies (2.5), choosing $\phi(t) = 1$.

Therefore our Theorem 2.4 improves Theorem 2.1, too.

References

[1] F.D. Chen, J. Shi, Periodicity in a logistic type system with several delays, Comput. Math. Appl. 48 (2004) 35–44.
[2] Y. Chen, Periodic solutions of a delayed periodic logistic equation, Appl. Math. Lett. 16 (2003) 1047–1051.
[3] H.I. Freedman, J. Wu, Periodic solutions of single-species models with periodic delay, SIAM J. Math. Anal. 23 (1992) 689–701.
[4] K. Gopalsamy, Stability and Oscillations in Delay Differential Equations of Population Dynamics, Kluwer Academic, Boston, 1992.
[5] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Academic Press, Boston, 1993.
[6] G.S. Ladde, V. Lakshmikantham, B.G. Zhang, Oscillation Theory of Differential Equations with Deviating Arguments, Marcel Dekker, Boston, 1987.
[7] Y. Li, Y. Kuang, Periodic solutions of periodic delay Lotka–Volterra equations and systems, J. Math. Anal. Appl. 255 (2001) 260–280.