A sequential coloring algorithm for finite sets

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Received 29 May 1997; revised 14 September 1998; accepted 21 September 1998

Abstract

Let X be a finite set and P a hereditary property associated with the subsets of X. A partition of X into n subsets each with property P is said to be a P-n-coloring of X. The minimum n such that a P-n-coloring of X exists is defined to be the P-chromatic number of X. In this paper we give a sequential coloring algorithm for P-coloring X. From the algorithm we then get a few upper bounds for the P-chromatic number. In particular, we generalize the Welsh–Powell upper bound for ordinary chromatic number to the case of P-chromatic number of any finite set X. © 1999 Elsevier Science B.V. All rights reserved

Keywords: Finite set; Hereditary property; Sequential coloring; Conditional chromatic number; Algorithm

1. Introduction

A simple but useful way of coloring the vertices of a graph G is as follows. First order the vertices, say $x_1 < \cdots < x_p$, and then color them one by one: color $x_1$ with 1, then color $x_2$ with 1 if $x_1$ and $x_2$ are not adjacent and 2 otherwise, and so on; color each vertex with the smallest color it can have at that stage (see [18] for techniques based on this method). In this paper, we will show how the similar method works for conditional colorings of a finite set.

Let X be a finite set whose elements are called vertices. Let P be a property associated with the subsets of X. If $Y \subseteq X$ possesses P, we call Y a P-set; otherwise a $\overline{P}$-set. In the following we always suppose that P is hereditary in the sense that whenever Y is a P-set then each subset of Y is also a P-set. We also suppose without mentioning that each singleton $\{x\}$ for $x \in X$ is a P-set. A P-n-coloring of X is an assignment of n colors 1, 2, \ldots, n to the vertices of X such that for each color i the
subset of all vertices colored with \( i \) is a \( P \)-set. Equivalently, a \( P \)-\( n \)-coloring of \( X \) is a partition of \( X \) into \( n \) \( P \)-sets. This concept of \( P \)-coloring of a finite set can date back to [7] when Cockayne, Miller and Prins proved the interpolation theorem for complete \( P \)-\( n \)-colorings. In [20], the \( P \)-chromatic number of \( X \), denoted by \( \chi_P(X) \), was defined to be the minimum \( n \) such that a \( P \)-\( n \)-coloring of \( X \) exists. Since each \( \{x\} \) is a \( P \)-set, \( \chi_P(X) \) is well-defined. If \( Y \) is a \( \overline{P} \)-set but \( Y \setminus \{x\} \) is a \( P \)-set for each \( x \in Y \), then we say that \( Y \) is a \textit{minimal} \( \overline{P} \)-set. For each \( x \in X \), we define the \( P \)-degree of \( x \) in \( X \), denoted by \( d_P(x, X) \), to be the largest number of members in a family \( \mathcal{A}_x \) of minimal \( \overline{P} \)-sets of \( X \) with the property that \( A \cap A' = \{x\} \) for any distinct \( A, A' \in \mathcal{A}_x \). Let \( \delta_P(X) \) and \( \Delta_P(X) \) be, respectively, the \textit{minimum} \( P \)-degree and the \textit{maximum} \( P \)-degree of the vertices of \( X \). Note that when restricted to a subset \( Y \) of \( X \), \( P \) is also a hereditary property associated with the subsets of \( Y \). Thus, \( \chi_P(Y) \), \( d_P(x, Y) \), \( \delta_P(Y) \) and \( \Delta_P(Y) \) are well-defined, where \( x \in Y \). If \( \chi_P(X \setminus \{x\}) < \chi_P(X) \) for each \( x \in X \), then \( X \) is said to be \( P \)-critical. We use \( p \) to denote the cardinality of \( X \).

The main purpose of this paper is to give an algorithm for sequentially \( P \)-coloring the finite set \( X \). By using this algorithm we derive several upper bounds for \( \chi_P(X) \). In particular, we generalize the well-known Welsh-Powell upper bound [3,17] for ordinary chromatic number to the case of \( P \)-chromatic number of any finite set. From this we can get upper bounds for some known graphical invariants such as the arboricity, vertex arboricity, and so on. Also, the upper bounds obtained for \( P \)-chromatic number imply a theorem of Tomescu [16] and an upper bound [2, Corollary 1, p. 117] for the chromatic number of a hypergraph.

2. The algorithm and its consequences

\textbf{Algorithm SC(O)}

1. Suppose \( O : x_1 < \cdots < x_p \) is a given order of the vertices of \( X \). Color \( x_1 \) with color 1.

2. Generally suppose \( x_1, \ldots, x_{i-1} \) have been colored, let \( X_i = \{x_1, \ldots, x_{i-1}, x_i\} \). If \( X_i \) is a \( P \)-set, then color \( x_i \) with any color that has been used already; otherwise \( X_i \) contains minimal \( \overline{P} \)-sets and we define

\( \mathcal{A}_i = \{A \subseteq X_i : A \text{ is a minimal } \overline{P} \text{-set containing } x_i \text{ and } A \setminus \{x_i\} \text{ is monochromatically colored}\}, \)

\( \mathcal{B}_i = \{B \subseteq X_i : B \text{ is a minimal } \overline{P} \text{-set containing } x_i \text{ and } B \setminus \{x_i\} \text{ is not monochromatic}\}, \)

and

\( \mathcal{C}_i = \{C \subseteq X_i : C \text{ is a minimal } P \text{-set not containing } x_i\}. \)

3. If there exists a color used by a vertex in \( \bigcup_{C \in \mathcal{C}_i} C \) but not used by any vertex in \( \bigcup_{A \in \mathcal{A}_i} A \setminus \{x_i\} \), then assign this color to \( x_i \).

4. If all colors used in \( \bigcup_{C \in \mathcal{C}_i} C \) are also used in \( \bigcup_{A \in \mathcal{A}_i} A \setminus \{x_i\} \), then we distinguish two cases:

4a. If there is a color used in \( \bigcup_{B \in \mathcal{B}_i} B \) but not in \( \bigcup_{A \in \mathcal{A}_i} A \), then assign this color to \( x_i \).
(4b) otherwise, assign \( x_i \) a new color.

(5) If \( i = p \), stop; otherwise set \( i := i + 1 \) and go to (2).

Inductively, one can show that the subset of the vertices in \( X_i \) with any one color contains no minimal \( \bar{P} \)-sets. Therefore, the algorithm \( \text{SC}(O) \) gives rise to a \( P \)-coloring of \( X \) for any given order \( O \). The algorithm is justified also by the fact that it gives an optimal \( P \)-coloring (that is, a \( P \)-coloring using \( \chi_p(X) \) colors) for some order \( O \) of \( X \). In fact, let \( n = \chi_p(X) \) and \( \{A_1, \ldots, A_n\} \) be a \( P \)-\( n \)-coloring of \( X \). Let us order the vertices of \( X \) in such a way that \( x \prec y \) whenever \( x \in A_i \), \( y \in A_j \) with \( i < j \). For such an order \( O_M : x_1 < \ldots < x_p \) one can prove by induction on \( j \) that if \( x_i \in X_i \), then \( \text{SC}(O_M) \) uses at most \( i \) colors for \( \{x_1, \ldots, x_i\} \). Thus, \( \text{SC}(O_M) \) exploits at most \( n \) colors for coloring all the vertices in \( X \). By the minimality of \( n \), \( \text{SC}(O_M) \) gives a \( P \)-\( n \)-coloring of \( X \).

Let \( O \) be the given order as in the algorithm. In applying \( \text{SC}(O) \), we denote \( d_{S_i}(x) = 0 \) for all \( x \in X \) and, if \( x_1, \ldots, x_{i-1} \) have been colored, define \( d_{S_i}(x) \) to be the number of distinct colors that have been assigned to vertices in \( \cup \{A \backslash \{x\} \} \), where the union is taken over all minimal \( \bar{P} \)-sets \( A \subseteq \{x_1, \ldots, x_{i-1}, x\} \) such that \( x \in A \) and \( A \backslash \{x\} \) is monochromatic. Define

\[
\begin{align*}
\text{DS}(O) &= \max_{1 \leq i \leq p} \min\{i, d_{S_i}(x) + 1\} = \max_{1 \leq i \leq p} d_{S_i}(x) + 1, \\
\text{B}(O) &= \max_{1 \leq i \leq p} \min\{i, d_{P}(x, X) + 1\} = \max_{1 \leq i \leq p} d_{P}(x, X) + 1, \\
\text{WP}(O) &= \max_{1 \leq i \leq p} \min\{i, d_{P}(x, X) + 1\}.
\end{align*}
\]

Note that if \( A, A' \subseteq \{x_1, \ldots, x_{i-1}, x\} \) are minimal \( \bar{P} \)-sets containing \( x \) such that \( A \backslash \{x\}, A' \backslash \{x\} \) are monochromatic and \( A, A' \) contain a common vertex other than \( x \), then the colors used for \( A \backslash \{x\}, A' \backslash \{x\} \) are the same. Thus, \( d_{S_i}(x) \leq d_{P}(x, X) \). This, together with \( d_{P}(x, X) \leq d_{P}(x, X) \), implies that \( \text{DS}(O) \leq \text{B}(O) \leq \text{WP}(O) \). From the algorithm \( \text{SC}(O) \) we get the following upper bounds for \( P \)-chromatic number.

**Theorem 1.** Let \( P \) be a hereditary property associated with the subsets of a finite set \( X \). Then for any order \( O \) of the vertices of \( X \), we have

\[ \chi_p(X) \leq \text{DS}(O). \]

In particular, we have

\[ \chi_p(X) \leq \text{B}(O), \]

and

\[ \chi_p(X) \leq \text{WP}(O). \]

**Proof.** It suffices to prove that \( \text{SC}(O) \) uses at most \( \text{DS}(O) \) colors for coloring all the vertices in \( X \). Let \( c_i \) be the number of colors used in coloring \( X_i = \{x_1, \ldots, x_i\} \)
by SC(O). We prove \( c_i \leq DS(O) \), \( 1 \leq i \leq p \), by induction on \( i \). Clearly, \( c_i \leq 1 \). If (3) or (4a) happens, then \( c_i = c_{i-1} \leq DS(O) \) by the induction hypothesis. If (4b) occurs, then \( c_i \leq |s_i'| \), where \( s_i' \subseteq s_i \) is such that for distinct \( A, A' \in s_i' \) the colors used for \( A \setminus \{x_i\}, A' \setminus \{x_i\} \) are different and \( s_i' \) is maximal with respect to this property. Thus, \( c_i \leq c_{i-1} + 1 \leq |s_i'| + 1 \leq DS(O) \). This completes the proof. \( \square \)

If \( x_i \prec x_j \) in the order \( O \) and \( d_P(x_i, X) < d_P(x_j, X) \), then we can exchange the positions of \( x_i \) and \( x_j \) and thus get a new order \( O' \). It can be verified that \( WP(O') \leq WP(O) \). So \( WP(O) \) is minimized when \( X \) is ordered as \( x_1 \prec \cdots \prec x_p \) such that \( d_P(x_1, X) \geq \cdots \geq d_P(x_p, X) \). From Theorem 1, we have

**Corollary 1.** Suppose \( X = \{x_1, \ldots, x_p\} \) with \( d_P(x_1, X) \geq \cdots \geq d_P(x_p, X) \). Then

\[
\chi_P(X) \leq \max_{1 \leq i \leq p} \min\{i, d_P(x_i, X) + 1\}.
\]

Note that if \( X \) is the vertex set of a graph \( G \) and \( P \) is the property of being a vertex independent set of \( G \), then \( \chi_P(X) \) is the ordinary chromatic number \( \chi(G) \) of \( G \) and, for each \( x \in X \), the \( P \)-degree \( d_P(x, X) \) is just the degree \( d(x) \) of \( x \) in \( G \). So Corollary 1 is a generalization of the well-known Welsh–Powell bound [3, 17]: If the vertices \( x_1, x_2, \ldots, x_p \) of a graph \( G \) are indexed in such a way that \( d(x_1) \geq \cdots \geq d(x_p) \), then

\[
\chi(G) \leq \max_{1 \leq i \leq p} \min\{i, d(x_i) + 1\}.
\]

An \( n \)-**coloring** of a hypergraph \( H \) [2] is a partition of the set of vertices of \( H \) into \( n \) subsets each contains no edges with cardinality greater than one. The **chromatic number** \( \chi(H) \) of \( H \) [2] is the minimum \( n \) such that an \( n \)-coloring of \( H \) exists. The \( \beta \)-**degree** \( d^\beta_H(x) \) of \( x \) in \( H \) [2] is the maximum number of edges different from \( \{x\} \) whose pairwise intersections are \( \{x\} \). Theorem 1 implies also the following theorem of Tomsescu which is stronger than Corollary 1.

**Corollary 2.** (Tomsescu [16]). Let \( \{A_1, \ldots, A_n\} \) be an \( n \)-coloring of a hypergraph \( H \), and \( d_i = \max_{x \in A_i} d^\beta_H(x), 1 \leq i \leq n \). Then we have

\[
\chi(H) \leq \max_{1 \leq i \leq n} \min\{i, d_i + 1\}.
\]

**Proof.** Set \( X \) to be the vertex set of \( H \) and call \( Y \subset X \) a **P-set** if it contains no edges of \( H \) with cardinality greater than one. Then \( \chi(H) = \chi_P(X) \). Define an order of the vertices in \( X \) such that \( x \prec y \) whenever \( x \in A_i, y \in A_j, i < j \). For such an order \( O : x_1 \prec \cdots \prec x_p \), it can be proved that if \( x_i \in A_i \) then \( d_{S_i}(x_i) + 1 \leq \min\{i, d_i + 1\} \). Thus, \( DS(O) \leq \max_{1 \leq i \leq n} \min\{i, d_i + 1\} \) and the result follows from \( \chi_P(X) \leq DS(O) \). \( \square \)
The order $O$ in $SC(O)$ is arbitrarily set beforehand. To make the number of colors used reasonably small, an effective way is to use the so-called smallest-last technique for determining an order $O_{SL}$, as in the case of ordinary vertex colorings of graphs \[14\]. More precisely, we construct an order $O_{SL}$ in the following way:

1. Let $x_p$ be a vertex of $X$ with the minimum $P$-degree in $X_p = X$;
2. for $i = p - 1, \ldots, 1$, let $x_i$ be a vertex with the minimum $P$-degree in $X_i = X \setminus \{x_p, \ldots, x_{i+1}\}$ when $x_p, \ldots, x_{i+1}$ have been chosen.

Denote

$$A(X) = \max_{Y \subseteq X} \min_{x \in Y} \delta_P(x, Y) + 1 = \max_{Y \subseteq X} \delta_P(Y) + 1.$$ 

Since $\delta_P(x_i, X_i) = \min_{x \in X_i} \delta_P(x, X_i)$, we have $B(O_{SL}) \leq A(X)$. On the other hand, let $Y_0$ be a subset of $X$ which attains the maximum in the definition of $A(X)$, and let $x_i$ be the last vertex of $Y_0$ in the order $O_{SL}$. Then

$$d_P(x_i, X_i) + 1 \geq d_P(x_i, Y_0) + 1 \geq \min_{x \in Y_0} d_P(x, Y_0) + 1 = A(X).$$ 

Thus, $B(O_{SL}) \geq A(X)$ and hence $B(O_{SL}) = A(X)$. The use of this smallest-last technique leads to the upper bounds (i) and (ii) in the following corollary, where (ii) is a generalization of the known result $\chi(G) \leq 1 + \Delta(G)$ for the chromatic number $\chi(G)$ of a graph $G$ to the case of $P$-chromatic number. The equivalent hypergraph forms of these two upper bounds can be found in [2, pp. 116–117].

**Corollary 3.** Suppose $X$ is a finite set and $P$ is a hereditary property associated with the subsets of $X$. Then the following (i)–(iii) hold.

(i) $\chi_P(X) \leq 1 + \max_{Y \subseteq X} \delta_P(Y)$.
(ii) $\chi_P(X) \leq 1 + \Delta_P(X)$.
(iii) If $X$ is $P$-critical, then $\chi_P(X) \leq 1 + \delta_P(X)$.

**Proof.** We have proved that $B(O_{SL}) = A(X)$. So (i) follows from $\chi_P(X) \leq B(O_{SL})$ immediately. For each $x \in Y$, we have $d_P(x, X) \leq d_P(x, X_i)$. Hence $\delta_P(Y) \leq \Delta_P(X)$ and (ii) follows from (i). For (iii), if there exists a vertex $x \in X$ with $d_P(x, X) < n - 1$, where $n = \chi_P(X)$, then since $X$ is $P$-critical there exists a $P$-$(n - 1)$-coloring $\{A_1, \ldots, A_{n-1}\}$ of $X \setminus \{x\}$. Let $O$ be the order of $X$ such that the vertices in $A_i$ precede those in $A_j$ whenever $i < j$ and all vertices in $X \setminus \{x\}$ precede $x$. Clearly, $DS(O) \leq n - 1$ and hence $\chi_P(X) \leq n - 1$ by Theorem 1. This contradiction shows that $d_P(x, X) \geq \chi_P(X) - 1$ for each $x \in X$ and hence proves (iii). \[\square\]

3. Concluding remarks

If $P$ is a hereditary graphical property and $X$ is the vertex set of a graph $G$, then $\chi_P(X)$ is the $P$-chromatic number [9] of $G$; if $P$ is an edge hereditary graphical property [11] and $X$ is the edge set of $G$, then $\chi_P(X)$ is just the $P$-chromatic index.
A study on $P$-chromatic number of a graph was conducted in [4]. As shown in the literature (see e.g. [1,9,19,20]), a large number of known graphical invariants such as the ordinary chromatic number, edge chromatic number, thickness, arboricity, vertex arboricity [6], linear arboricity [10], vertex linear arboricity [13], unicyclicity [10], biparticity [10], $n$-th chromatic number [5], cochromatic number [12], chromatic partition number [15], subchromatic number [1], partite chromatic number [8] can be expressed as $\chi_P(X)$, where $X$ is the vertex set or edge set of the graph and $P$ is a specific graphical property. The algorithm in previous section can be applied to all these invariants and a Welsh-Powell-type upper bound can be obtained from Corollary 1 for each of them. As an example, we consider the vertex arboricity $a(G)$ which is precisely the $P$-chromatic number $\chi_P(X)$, where $X$ is the vertex set of $G$ and $P$ is the property such that $Y \subseteq X$ is a $P$-set if and only if the subgraph $G[Y]$ induced by $Y$ is a forest. Note that the minimal $P$-sets are those $Y \subseteq V(G)$ such that $G[Y]$ is a chordless cycle. For each $x \in X$, let $c(x)$ be the maximum number of chordless cycles containing $x$ such that any two of them have no common vertex other than $x$. Then $c(x) = d_P(x,X)$ and hence Corollary 1 gives $a(G) \leq \max_{1 \leq i \leq p} \min\{i, c(x_i) + 1\}$, where $x_1, \ldots, x_p$ are the vertices of $G$ and $c(x_1) \geq \ldots \geq c(x_p)$. This new upper bound for vertex arboricity is sharp in some cases since, for example, if $G$ is the Petersen graph then $c(x) = 1$ for all vertices $x$ and both sides of the inequality above are equal to 2.

The reader is referred to a subsequent paper [21] for an analysis on the structure of a $P$-critical set and for the relationship between $\chi_P(X)$ and the domination number of an associated graph.

Acknowledgements

The author would like to thank the referees for their valuable comments and suggestions. He also appreciates Professor Yu-Yuan Qin for his inspiring support in the preparation of a preliminary version of this paper.

References