Einstein Metrics with Prescribed Conformal Infinity on the Ball

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1. INTRODUCTION

In this paper we study a boundary problem for Einstein metrics. Let $M$ be the interior of a compact $(n+1)$-dimensional manifold-with-boundary $\overline{M}$, and $g$ a Riemannian metric on $M$. If $\hat{g}$ is a metric on $bM$, we say the conformal class $[\hat{g}]$ is the conformal infinity of $g$ if, for some defining function $\rho \in C^\infty(\overline{M})$ that is positive in $M$ and vanishes to first order on $bM$, $\rho^2g$ extends continuously to $\overline{M}$ and $\rho^2g|_{bM}$ is conformal to $\hat{g}$. It is clear that this an invariant notion, independent of the choice of defining function $\rho$.

In [FG] the following problem was posed: given a conformal class $[\hat{g}]$ on $bM$, find a metric $g$ on $M$ satisfying

(i) $\text{Ric}(g) = -ng$,

(ii) $g$ has $[\hat{g}]$ as conformal infinity.  

(1.1)

The constant $-n$ in (i) can be replaced by any other negative number, just by multiplying $g$ by a constant. No generality is gained, either, by replacing $\rho^2$ in the definition of conformal infinity by an arbitrary power of $\rho$, since an easy computation (cf. formula (2.2)) shows that if $g$ satisfies (1.1) then the power must be 2. The motivating example is hyperbolic space, in which $M$ is the unit ball $B^{n+1} \subset \mathbb{R}^{n+1}$ with defining function $\rho = \frac{1}{2}(1 - |x|^2)$, $h = \rho^{-2} \sum_i (dx^i)^2$ is the hyperbolic metric with constant curvature $-1$, and $\hat{h}$ is the usual metric on $S^n$.

In order to avoid possible topological complications, we will restrict attention here to the case $M = B^{n+1}$. For $n = 2$, any smooth conformal structure on $S^2$ is diffeomorphic to the usual one; extending the diffeomorphism of $S^2$ to a diffeomorphism of $\overline{B^2}$ and pulling back the hyperbolic metric provides a solution to the problem. Thus we always assume $n \geq 3$.

Our main result is the following theorem.

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THEOREM A. Let $M = B^{n+1}$ be the ball and $\hat{h}$ the standard metric on the sphere $S^n$. For any smooth Riemannian metric $\hat{g}$ on $S^n$ which is sufficiently close to $\hat{h}$ in $C_{2,\alpha}$ norm if $n \geq 4$, or $C_{3,\alpha}$ norm if $n = 3$, for some $0 < \alpha < 1$, there exists a smooth metric $g$ in $M$ satisfying (1.1).

We also study the boundary regularity of the solution $g$. If $\hat{g}$ is assumed close to $\hat{h}$ in $C_{k,\alpha}$ norm for $k$ sufficiently large, then the optimal regularity we obtain for $n \geq 4$ is $\rho^2 g \in C_{n-1,\gamma}(\hat{M})$ for $0 < \gamma < \gamma_n < 1$, and for $n = 3$, $\rho^2 g \in C_{4,\gamma}(\hat{M})$ for any $0 < \gamma < 1$. (See Theorem 4.1 for the precise statement.) The reason for the stronger hypothesis in dimension 3 is that our basic linear isomorphism is probably not sharp. In fact, it is not sharp in higher dimensions either, but in the higher-dimensional case this only weakens the boundary regularity results and not the basic existence theorem. See the remarks after Corollary 3.11 and in the proof of Theorem 4.1.

The formulation of conformal infinity that we use was introduced by R. Penrose in studying Ricci-flat Lorentz metrics (see [PR]). The case of negative Ricci curvature was considered by LeBrun [L], who showed using twistor methods that any real-analytic conformal structure in three dimensions is locally the conformal infinity of a self-dual Einstein metric defined on a four-dimensional collar neighborhood. Along these lines Pedersen [P] has found an explicit self-dual Einstein metric on $B^4$ whose conformal infinity is the conformal Berger sphere. Formal aspects of our problem were studied in [FG], where in particular high-order approximate solutions near $bM$ were constructed. Our work has also been motivated by that of Cheng and Yau [CY2], who constructed complete Kähler-Einstein metrics of negative Ricci curvature on pseudoconvex domains in complex manifolds. In that case the Einstein metric is asymptotic to the CR structure of the boundary at infinity.

As was pointed out in [FG], problem (1.1) can be alternately formulated as a characteristic boundary problem for a homogeneous Ricci-flat Lorentz metric in $\mathbb{R}^{n+2}$ generalizing the Minkowski metric. In the model case, hyperbolic space can be considered as arising from the Minkowski metric $\hat{h} = \sum_i (d\xi^i)^2 - dy^2$ on $\mathbb{R}^{n+2}$ with coordinates $(\xi, \eta) \in \mathbb{R}^{n+1} \times \mathbb{R}^1$, by restriction to the hyperboloid $S = \{\eta^2 = 1 + |\xi|^2, \eta > 0\}$. In fact, under the map $(\xi, \eta) \mapsto x = \xi/(\eta + 1)$ from $S$ to $B^{n+1}$, $\hat{h}|_S$ goes over to the hyperbolic metric $h$ described above. More generally, we seek a Lorentz metric defined on the interior $\mathcal{C}_+ = \{\eta > |\xi|\}$ of the cone $\mathcal{C} = \{\eta = |\xi| > 0\}$ in $\mathbb{R}^{n+2}$. Given a metric $\hat{g}$ on $S^n$, define a degenerate homogeneous metric $g_0$ on $\mathcal{C}$ by $g_0 = \eta^2 \pi^* \hat{g}$, where $\pi: \mathcal{C} \to S^n$ is $\pi(\xi, \eta) = \xi/\eta$. Under a conformal change of $\hat{g}$, $g_0$ simply rescales along the fibers of $\pi$, so is invariantly associated to $[\hat{g}]$ up to a homogeneous diffeomorphism of $\mathcal{C}$. For $s > 0$, let $\delta_s$ be the dilation $\delta_s(\xi, \eta) = (s\xi, s\eta)$. Given $\hat{g}$, one wants to find a Lorentz
metric $\tilde{g}$ on $\mathbb{C}_+$ which extends continuously to a nondegenerate metric on $\mathbb{C}_+ - \{0\} = \mathbb{C}_+ \cup \mathbb{C}$ and satisfies

\begin{align*}
(i) & \quad \delta^* \tilde{g} = s^2 \tilde{g}, \\
(ii) & \quad \tilde{g} |_{\mathbb{T}^n} = g_0, \\
(iii) & \quad \text{Ric}(\tilde{g}) = 0.
\end{align*}

(1.2)

In Section 5 we show how to construct solutions to (1.2) from solutions to (1.1). Thus we obtain the existence of solutions to (1.2) for $\tilde{g}$ sufficiently near the usual metric on $S^n$.

The first difficulty one encounters in studying (1.1) is the (by now familiar) problem that, due to their gauge invariance under the diffeomorphism group, the Einstein equations form a very degenerate system. There are two ways (at least) of dealing with this. The first goes back to the study of the Cauchy problem in general relativity: one fixes a gauge by writing the Ricci curvature operator in harmonic coordinates; this removes the degeneracy so that the resulting operator is elliptic or hyperbolic, depending on whether one in searching for a Riemannian or a Lorentzian metric. The second method is due to DeTurck [D1, D2]. In it one fixes a background metric $t$, and subtracts from the Ricci curvature operator $\text{Ric}(g)$ a second-order nonlinear operator $\Phi(g, t)$ depending on $t$ as well as $g$, cooked up so that $\text{Ric}(g) - \Phi(g, t)$ is elliptic or hyperbolic. In either method, one shows that the solution to the modified equation is actually Einstein by applying the Bianchi identity and invoking a uniqueness theorem for a linear system. It turns out, as we shall show at the end of the introduction, that the second method can be viewed as a generalization of the first, in which harmonic coordinates are replaced by a harmonic map.

We choose an appropriate auxiliary metric $t$ (which will depend on the boundary data $\tilde{g}$), and study the regularized Einstein equation

$$\text{Ric}(g) + ng - \Phi(g, t) = 0.$$  

(1.3)

This is a quasilinear elliptic system of equations for $g$, but is highly degenerate at $bM$ due to the singularity of $g$ there. Our basic method is first to find approximate solutions by formal calculations near $bM$, then to study the linearization of the problem about the hyperbolic metric and use the inverse function theorem in appropriate weighted degenerate Hölder spaces. It is possible to use the Fefferman–Graham formal solutions to construct approximate solutions to (1.3), but we have set up the problem a little differently and have included a complete derivation of our formal solutions in Section 2.

The linearization of the left-hand side of (1.3) about the hyperbolic metric $h$ is $\Delta_h - 2$ on the trace-free part of the argument and $\Delta_h + 2n$ on the
trace part, where $\Delta_h$ is the covariant Laplacian on symmetric 2-tensors. In a local coordinate system smooth up to $bM$, this is a system of differential operators which can be written as polynomials in $\rho \partial / \partial x^i$ with smooth coefficients. We call such operators uniformly degenerate. Operators of this type, which form a subring of the ring of totally characteristic operators of Mclrose [Mc], have been studied by Mazzeo [Ma] and by Mazzeo and Melrose [MM] in connection with other questions of analysis and geometry on a Riemannian manifold with a smooth conformal infinity.

If a scalar uniformly degenerate operator is applied to a radial function $u(\rho)$ on $B^{n+1}$, it reduces to an ordinary differential operator with a regular singular point at $\rho = 0$. Associated with any such operator are its characteristic exponents $s_1$ and $s_2$ (solutions of the indicial equation), and in the generic case any sufficiently regular solution of the operator must behave asymptotically like $\rho^{s_1}$ or $\rho^{s_2}$ near $\rho = 0$. For example, the characteristic exponents for $\Delta_h$ acting on scalar functions are $s_1 = 0$ and $s_2 = n$; the fact that the lowest exponent is zero means that the $\rho^0$ term in a formal power series solution to $\Delta_h$ can be prescribed arbitrarily, and reflects the well-known fact that the Dirichlet problem can be solved for $\Delta_h$ on the ball. The exponent $n$ represents a potential obstruction to the existence of an infinite-order formal power series solution. The linearization of (1.3) on symmetric 2-tensors has characteristic exponents $-2$ and $n - 2$ (at least on the tangential part of the metric), reflecting the fact that there exists a formal power series solution with arbitrary tangential $\rho^{-2}$ term, but the expansion is no longer formally determined at the $\rho^{n-2}$ term.

In Section 3 we consider the invertibility of uniformly degenerate operators on weighted Hölder spaces. From the discussion above, it is evident that their invertibility depends in an essential way on the asymptotic behavior of functions at the boundary. For $s \in \mathbb{R}$ we define spaces $A^s_{k,a}$ of functions that are $C^k_{k,a}$ in the interior and bounded by a multiple of $\rho^s$, with Hölder norms involving weights that degenerate at the boundary in a way that reflects the degeneracy of $\Delta_h$. To illustrate the sort of phenomena that occur, consider the case of $\Delta_h$ acting on scalar functions. A special case of our basic isomorphism theorem (Theorem 3.10) is that for $\kappa \in \mathbb{R}$,

$$\Delta_h + \kappa : A^s_{k+2,a} (B^{n+1}) \rightarrow A^s_{k,a} (B^{n+1})$$

is an isomorphism whenever $s_1 < s < s_2$, where $s_1, s_2$ are the characteristic exponents of $\Delta_h + \kappa$. (This is false if $s < s_1 < s_2$ or $s_1 < s_2 < s$.) Theorem 3.10 also yields a similar but slightly weaker result on symmetric 2-tensors.

The isomorphism theorem is proved by combining Schauder estimates for elliptic uniformly degenerate operators, which follow from usual Schauder estimates by rescaling, with an a priori $L^\infty$ estimate, which we prove by an application of the generalized maximum principle of Yau [Y].
using a special weight function. In deriving this a priori estimate we are in effect estimating the lowest eigenvalue of $\Delta_h$ on symmetric 2-tensors. At the end of Section 3, we use these methods to prove the curious result that $C_1$-small perturbations of the hyperbolic metric preserving the asymptotic behavior at infinity cannot decrease the lowest eigenvalue for the Laplacian on functions.

In Section 4 we set up the inverse function theorem and prove Theorem A.

Any diffeomorphism of $\bar{M}$ which is the identity on the boundary takes solutions of (1.1) to solutions; therefore any solution is unique at most up to such diffeomorphisms. Using this observation, it is possible to solve a seemingly stronger boundary problem. Suppose instead of just a boundary metric $\bar{g}$ we specify a positive definite section $\bar{g}$ of the bundle of symmetric 2-tensors on $TM|_{\partial M}$. This amounts to specifying both the boundary metric $\bar{g} - \bar{g}|_{\partial bM}$ and the $\bar{g}$-unit normal vector to $\partial M$. When $M$ is the ball, Theorem A yields a solution to the following problem for $\bar{g}$ sufficiently close to $\rho^2h|_{\partial M}$:

(i) $\text{Ric}(g) = -ng$;

(ii) $\rho^2g|_{\partial M}$ is conformal to $\bar{g}$.

In fact, let $g$ be a solution to (1.1) with $\bar{g} = \bar{g}|_{\partial bM}$; upon conformally changing $\bar{g}$ we can assume that $\rho^2g|_{\partial bM} = \bar{g}|_{\partial bM}$. Now choose any diffeomorphism $\varphi$ of $\bar{M}$ which is the identity on $\partial M$ and sends the $\bar{g}$-unit normal to $\partial M$ to the $\rho^2g$-unit normal; then $\varphi^*g$ solves (1.4). The solution to this stronger problem can be unique at most up to a diffeomorphism of $\bar{M}$ which agrees with the identity to first order along $\partial bM$.

It is clearly of interest to study the existence of solutions of (1.1) away from the hyperbolic metric, as well as the higher boundary regularity and uniqueness questions. We hope to return to these problems in the future.

We conclude the introduction by deriving the relationship between the two methods of breaking the gauge-invariance in the Einstein equations discussed earlier. In the harmonic-coordinate approach one fixes a coordinate chart and requires that the coordinate functions $x^i$ be harmonic with respect to the unknown metric; this is equivalent to requiring that $g$ satisfy $g^{ij}\Gamma^k_{ij} = 0$. Differentiating this identity with respect to $x^i$, lowering an index, symmetrizing and substituting into the Einstein equation yields an expression for the Ricci operator whose principal part is $-\frac{1}{2} g^{kl} \partial_k \partial_l g_{ij}$, which is nondegenerate as mentioned above.

This approach can be generalized: the coordinate expression for the Ricci tensor will have the same principal part if we require instead that $g^{ij}\Gamma^k_{ij} = b^k$ for some nonzero $b^k$, which can even depend on $g_{ij}$ but not on its derivatives. A particularly nice example of this arises if we fix a background metric $I_{ij}$.
and require that $g^{ij}T^k_{ij} = g^{ij}T^k_{ij}$, where $T^k_{ij}$ are the Christoffel symbols of $t$; this is equivalent to the condition that the identity $(M, g) \rightarrow (M, t)$ be a harmonic map. This condition is, of course, invariant if one transforms both $g$ and $t$ by a diffeomorphism, but fixing $t$ serves to break the gauge invariance. It is clearly a useful method for global problems in which global coordinates might not exist. The traditional harmonic coordinate normalization is the special case where $t$ is the Euclidean metric.

DeTurck's method consists of subtracting from $\text{Ric}(g)$ the operator $\Phi(g, t) = \delta^g g^{-1} \delta_g G_g t$, where $\delta_g$ and $G_g$ are defined in Section 2. Considered as a differential operator acting on $g$, $\text{Ric}(g) - \Phi(g, t)$ is a quasilinear operator whose principal part is also $-\frac{1}{2} g^{kl} \partial_k \partial_l g_{ij}$.

These two procedures are easily related: in fact, one calculates that $g^{ij}(T^k_{ij} - T^k_{ij}) = [t^{-1} \delta_g G_g t]^k$. Hence the identity is a harmonic map from $(M, g)$ to $(M, t)$ if and only if $\delta G_g t = 0$, and in this case $\Phi(g, t) = 0$. Thus DeTurck's technique essentially is the exploitation of the fact that $\text{Ric}(g) - \Phi(g, t)$ is an expression for the Ricci curvature in the gauge in which the identity is harmonic. We are grateful to Helmut Friedrich for suggesting to us that DeTurck's method might be related to harmonic maps.

2. Formal Calculations

We compute in this section the linearization of the regularized Einstein equation (1.3), and show how to construct formal power series solutions to the equation to high order along $bM$. First we fix some notational conventions.

Throughout this paper, we let $M$ denote the interior of a compact $(n+1)$-dimensional $C^\infty$ manifold $\bar{M}$ with boundary. We will assume that we are given a fixed defining function $\rho \in C^\infty(\bar{M})$ satisfying $\rho > 0$ in $M$, $\rho = 0$ and $d\rho \neq 0$ on $bM$. We will let $g$ denote an arbitrary $C^2$ Riemannian metric on $M$ such that $\rho^2 g$ extends continuously to a nondegenerate metric on $\bar{M}$: we call such a metric conformally compact (cf. [Ma]). For any conformally compact metric, we let

$$\tilde{g} = \rho^2 g \text{ on } \bar{M}; \quad \hat{g} = \tilde{g} \mid_{bM} \text{ on } bM.$$ 

Unless otherwise stated, all covariant derivatives will be taken with respect to $g$. We will denote the components of covariant derivatives of a tensor by indices preceded by a comma, so if $t$ is a 2-tensor, $\nabla^2 t$ is the 4-tensor with components $t_{i\ell,jk}$. For a scalar function $u$, we write $u_{ij} = \partial_j u$ and $u_{jk} = \partial_j \partial_k u = \Gamma^i_{jk} \partial_i u$. We observe the summation convention, and we use $g_{ij}$ and its inverse $g^{ij}$ to lower and raise indices, with one exception: $\tilde{g}^{ij}$ will
denote the inverse of $g_{ij}$, not the raised-index version. $\mathcal{S}^2$ will denote the bundle of symmetric 2-tensors on $M$, and $C_k(X, \mathcal{S}^2)$ the set of sections of $\mathcal{S}^2$ of class $C_k$ over $X$.

In our analysis of the behavior of the Ricci operator near $bM$, it is important to keep track of the form of certain nonlinear differential expressions. If $\bar{g}_1, \ldots, \bar{g}_N$ are metrics, all assumed to be of class $C_k$ on $M$, $\mathcal{S}^k(\bar{g}_1, \ldots, \bar{g}_N)$ will denote any tensor whose components in any coordinate system smooth up to $bM$ are polynomials, with coefficients in $C^\infty(M)$, in the components of the $\bar{g}_i, \bar{g}_i^{-1}$, and their partial derivatives, such that in each term the total number of derivatives of the $\bar{g}_i$ that appear is at most $k$. For example, $\mathcal{S}^0(g)$ is a polynomial in $g, g^{-1}$ alone; $\mathcal{S}^1(g, \bar{t})$ can contain terms that are linear in first derivatives of $g$ or $\bar{t}$; and $\mathcal{S}^2(g, \bar{t})$ can contain terms that are linear in second derivatives or quadratic in first derivatives. The same symbol $\mathcal{S}^k$ may denote a different such tensor each time it occurs.

**Lemma 2.1.** If $g$ is conformally compact, then

$$R_{jk} = \rho^{-2}(n g^{il} \rho_i \rho_l) \bar{g}_{jk} + \rho^{-1} \mathcal{S}^1(\bar{g}) + \mathcal{S}^2(\bar{g}).$$

**Proof.** We use the classical expression for the transformation of the Ricci tensor under a conformal change of metric: if $g = \rho^{-2} \bar{g}$, then

$$R_{jk} = -\rho^{-2}(n \bar{g}^{il} \rho_i \rho_l) \bar{g}_{jk} + \rho^{-1} \bar{R}_{jk} + \rho\bar{\rho}_{jk},$$

where $R_{jk}$, $\bar{R}_{jk}$ are the Ricci tensors of $g$ and $\bar{g}$, respectively, and the bar in $\bar{\rho}_{jk}$ indicates that the covariant derivatives are to be taken with respect to $\bar{g}$:

$$\bar{\rho}_{jk} = \partial_i \partial_j \rho - \bar{\Gamma}_{ij}^l \partial_l \rho.$$

Thus (2.1) follows. $lacksquare$

In particular, this means that if $\bar{g} \in C_2(M, \mathcal{S}^2)$, then $g = \rho^{-2} \bar{g}$ satisfies the Einstein equation $R_{jk} + n g_{jk} = 0$ near $bM$ modulo $O(\rho^{-1})$ if and only if

$$|d\rho|^2_{\bar{g}} = g^{il} \rho_i \rho_l = 1$$

on $bM$.

In fact, Mazzeo [Ma] has observed that if this holds, $g$ actually has asymptotically constant sectional curvature $-1$ near $bM$. Indeed, a calculation similar to the one above shows that

$$R_{ijkl} = -|d\rho|^2_{\bar{g}} (g_{ik} g_{jl} - g_{il} g_{jk}) + \rho^{-3} \mathcal{S}^1(\bar{g}) + \rho^{-2} \mathcal{S}^2(\bar{g}).$$

For this reason, we define an asymptotically hyperbolic metric to be a con-
formally compact metric $g$ on $M$ such that $\tilde{g} \in C_2(\overline{M}, \mathcal{P}^2)$ and $|d\rho|_\tilde{g}^2 = 1$ on $bM$. This is easily seen to be an invariant condition on $g$, independent of the choice of $\rho$.

Following DeTurck [D1], for any symmetric 2-tensor $t$, we define a symmetric 2-tensor $G_g t$ by

$$[G_g t]_i = t_i - \frac{1}{2} t_k g_{ik},$$

and a 1-form $\delta_g t$ (the divergence of $t$) by

$$[\delta_g t]_i = -t_i.$$

The formal adjoint of $\delta_g$ is then the operator from 1-forms to symmetric 2-tensors given by

$$[\delta_g^* \omega]_i = \frac{1}{2}(\omega_{i,j} + \omega_{j,i}).$$

We also define the covariant Laplacian $\Delta_g$ on tensors of any rank by

$$[\Delta_g t]_{i_1 \cdots i_q} = -t_{i_1 \cdots i_q,k}.$$

Let $\text{Ric}$ denote the nonlinear differential operator that takes metrics on $M$ to their Ricci tensors. In our notation, the contracted Bianchi identity can be written as

$$\delta_g G_g \text{Ric}(g) = 0,$$

and the linearization of $\text{Ric}$ (cf. [D1]) as

$$D(\text{Ric})_g r = \frac{1}{2} \Delta_g r - \delta_g^* \delta_g G_g r + \mathcal{R}(r),$$

where

$$[\mathcal{R}(r)]_{jk} = r''R_{jki} + \frac{1}{2}(R'_j r_{ik} + R'_k r_{ij}),$$

in which the curvature is that of $g$.

Now we define functions $\Phi$ and $Q$ which take a pair of metrics $(g, t)$ to symmetric 2-tensors by

$$\Phi(g, t) = \delta_g^* g t^{-1} \delta_g G_g t,$$

$$Q(g, t) = \text{Ric}(g) + n g - \Phi(g, t),$$

where $g t^{-1}$ is the endomorphism of $T^*M$ given by

$$[g t^{-1} \omega]_i = g_{ij} (t^{-1})^{jk} \omega_k.$$

We will attempt to solve the equation $Q(g, t) = 0$ for a suitable choice
of $t$. As the following lemma shows, as long as we know in advance that $g$ has strictly negative Ricci curvature and some minimal regularity at $bM$, this equation implies that $g$ is Einstein.

**Lemma 2.2.** Suppose $Q(g, t) = 0$, where $g, t$ are conformally compact and of class $C^3$ on $M$, $t \in C^2(M, \mathcal{S}^2)$, and in coordinates smooth up to the boundary, $\partial_k \tilde{g}_{ij}$ and $\rho \partial_k \partial_i \tilde{g}_{ij}$ are bounded. If $\text{Ric}(g)$ is strictly negative on $M$, then $\text{Ric}(g) = -ng$.

**Proof.** From (2.1) and our assumptions on $\tilde{g}$,

$$\text{Ric}(g)(V, V) = (-n|d\rho|^2_{\tilde{g}} + O(\rho))|V|^2_{\tilde{g}}$$

near $bM$, so there is a negative constant $K$ such that $\text{Ric}(g)(V, V) \leq K|V|^2_{\tilde{g}}$ for all $V \in TM$. Applying the Bianchi identity (2.3) to the equation $Q(g, t) = 0$, we find

$$0 = \delta_g G_g \Phi(g, t) = \delta_g G_g \delta^* \omega,$$

where $\omega$ is the 1-form

$$\omega = gt^{-1} \delta_g G_g t.$$

By the Ricci identity, $\delta_g G_g \delta^* \omega = 0$ can be written

$$0 = \frac{1}{4}(\omega_{i,j}^i + \omega_{j,i}^i - \omega_{j,j}^i) = \frac{1}{4}(\omega_{i,j}^i + R_{ij} \omega^j),$$

so the scalar function $|\omega|^2_{\tilde{g}} = g^{jk} \omega_j \omega_k$ satisfies

$$\Delta_{\tilde{g}} |\omega|^2_{\tilde{g}} = -2\omega_{i,k} \omega^i - 2\omega_{i,k} \omega^{i,k} = 2R_{ij} \omega^i \omega^j - 2\omega_{i,k} \omega^{i,k} \leq 2K|\omega|^2_{\tilde{g}}.$$

On the other hand, our assumptions on $g$ and $t$ imply that $|\omega|^2_{\tilde{g}}$ is bounded. It follows easily from the generalized maximum principle (Theorem 3.5) that $\omega = 0$, which in turn implies $\Phi(g, t) = \delta^* \omega = 0$, so $\text{Ric}(g) = -ng$. □

Observe that the proof of this lemma shows that $\delta_g G_g t = 0$. As we pointed out in the introduction, this is equivalent to the condition that the identity $(M, g) \rightarrow (M, t)$ be a harmonic map. Thus, under the hypotheses of Lemma 2.2, the equation $Q(g, t) = 0$ holds if and only if the identity $(M, g) \rightarrow (M, t)$ is harmonic and $\text{Ric}(g) = -ng$.

Let $D_t Q$ denote the Fréchet derivative of $Q$ with respect to its first
variable. For any pair \((g, t)\), \(D_1 Q_{(g, t)}\) is a linear partial differential operator from symmetric 2-tensors to symmetric 2-tensors, and for an arbitrary symmetric 2-tensor \(r\),

\[
D_1 Q_{(g, t)} r = D(\text{Ric})_g r + nr - D_1 \Phi_{(g, t)} r.
\]

**Lemma 2.3.** For metrics \((g, t)\) and a symmetric 2-tensor \(r\), we have

\[
D_1 \Phi_{(g, t)} r = -\delta^*_g \delta_g G_g r + \delta^*_g [\mathcal{Q}(r) - \mathcal{D}(r)] + \mathcal{B}(r),
\]  

(2.8)

where

\[
C^k_p = \frac{1}{2} (t^{-1})^{kl} (t_{jl,j} + t_{jl,l} - t_{jl,j});
\]

\[
D_k = g^{pq} C^k_{pq};
\]

\[
[\mathcal{B}(r)]_{jk} = \frac{1}{2} D^i (r_{k,i,j} + r_{j,i,k} - r_{k,j,i});
\]

\[
[\mathcal{Q}(r)]_i = g_{jk} C^k_{pq} r^{pq};
\]

\[
[\mathcal{D}(r)]_j = D^k r_{jk}.
\]

The covariant derivatives are with respect to \(g\).

**Proof.** Let \(r\) be any symmetric 2-tensor, \(g_s = g + sr\), and let a prime denote the \(s\)-derivative at \(s = 0\). Then

\[
D_1 \Phi_{(g, t)} r = \delta^*_g g t^{-1} \delta_g (G_{g, t})' + \delta^*_g g t^{-1} (\delta_g G_g) t' + \delta^*_g (g, t^{-1} \delta_g G_g) t' + (\delta^*_g g t^{-1} \delta_g G_g) t'
\]

\[
= I + II + III + IV. 
\]

(2.9)

Since all of these expressions are tensorial, we can simplify our computations by working in \(g\)-normal coordinates centered at an arbitrary point \(p \in M\), where the first derivatives and the Christoffel symbols of \(g\) vanish, and hence

\[
I^{kl}_{ij} = r^{kl} g_{ml} I^{ml}_{ij} + \frac{1}{2} g^{kl} (\partial_j r_{kl} + \partial_k r_{jl} - \partial_l r_{kj});
\]

\[
= \frac{1}{2} g^{kl} (r_{k,l,j} + r_{j,l,k} - r_{k,j,l}).
\]

For the first term of (2.9), we have

\[
[ G_{k,t} r ]_{ij} = (t_{ij} - \frac{1}{2} g^{kl} t_{kl} (g_{ij})')
\]

\[
= \frac{1}{2} (r^{kl} t_{kl} g_{ij} - g^{kl} t_{kl} r_{ij});
\]

and therefore

\[
[\delta_g (G_{g, t})']_j = -\frac{1}{2} (r^{kl} t_{kl} + r^{kl} t_{kl,j} - t^{kl,i} r_{ij} - t^{kl} r_{ij,l}).
\]  

(2.10)
Next, for any fixed symmetric 2-tensor $T$, at the origin in $g$-normal coordinates,
\[
[\delta_g T]_j' = -(g_s^{ik}[\partial_k T_{ij} - (\Gamma^q_{kl})_s T_{qj} - (\Gamma^q_{kj})_s T_{qi}])' \\
= \mu^{ik} T_{ij,k} + g^{ik} \Gamma^q_{kl} T_{qj} + g^{ik} \Gamma^q_{kj} T_{qi} \\
= \mu^{ik} T_{ij,k} + g^{ik} T_{qi} (\frac{1}{2} g^{qj} (r_{il,k} + r_{kl,i} - r_{kl,i})) \\
+ g^{ik} T_{qi} (\frac{1}{2} g^{qj} (r_{jl,k} + r_{kl,j} - r_{kl,i})) \\
= \mu^{ik} T_{ij,k} + T_{jl} r_{ik,k} - \frac{1}{2} T_{jl} r_{k,l} - \frac{1}{2} T_{jl} r_{l,k} + \frac{1}{2} T_{jl} r_{k,l}.
\]

Then substituting $T_{ij} = t_{ij} - \frac{1}{2} t_q g_{ij}$, we get
\[
[\delta_g G_s t]' = \mu^{ik} t_{ij,k} - \frac{1}{2} r_{jk} t_{i,k,l} + t_{jl} r_{ik,k} \\
- \frac{1}{2} t_{jl} r_{i,k,l} - \frac{1}{2} T_{jl} r_{k,l} + \frac{1}{2} T_{jl} r_{k,l}. \tag{2.11}
\]

Therefore, adding (2.10) and (2.11), we get
\[
[\delta_g (G_s t) + (\delta_g G_s t)']_j = \mu^{ik} t_{ij,k} - \frac{1}{2} r_{jk} t_{i,k,l} + t_{jl} r_{ik,k} - \frac{1}{2} t_{jl} r_{k,l} + \frac{1}{2} T_{jl} r_{k,l},
\]
and so
\[
I + II = -\delta^* \delta_g G_s t + \delta^* \mathcal{G}(r). \tag{2.12}
\]

Next, note that $D = -c^{-1} \delta_g G_s t$. Thus
\[
[g_s t^{-1} \delta_g G_s t]' = -\mathcal{D}(r),
\]
and so
\[
III = -\delta^* \mathcal{D}(r). \tag{2.13}
\]

Finally, the last term is (again at the origin in $g$-normal coordinates),
\[
IV = -[\delta^* gD]_{jk} = -\frac{1}{2}(\partial_k D_j + \partial_j D_k - 2(\Gamma^j_{jk})_s D_i)' \\
= \frac{1}{2} D_i g^{ij} (r_{kl,j} + r_{jl,k} - r_{kl,j}) \\
= [\mathcal{M}(r)]_{jk}. \tag{2.14}
\]

When we insert (2.12)–(2.14) into (2.9), we obtain (2.8).

The first main result of this section is the following explicit formula for the linearization of $Q$. 
PROPOSITION 2.4. For metrics $g$ and $t$ and a symmetric 2-tensor $r$, we have

$$D_1 Q_{(g,t)}^r = \frac{1}{2} A_g r + \mathcal{R}(r) + nr - \delta_g^* \mathcal{C}(r) + \delta_g^* \mathcal{D}(r) - \mathcal{B}(r),$$

(2.15)

where $\mathcal{R}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are defined in (2.5) and Lemma 2.3.

Proof. This follows from (2.4) and (2.8).

Thus the principal part of $D_1 Q_{(g,t)}$ is the same as that of one-half the covariant Laplacian $A_g$, which in local coordinates is

$$A_g = -g^{ij} \partial_i \partial_j + \text{lower order terms},$$

so $D_1 Q_{(g,t)}$ is elliptic in $M$. However, for conformally compact metrics, it is not uniformly elliptic up to $\partial M$, because $g^{ij} \to 0$ there. Thus our next task is to analyze the asymptotic behavior of $Q$ near $\partial M$.

PROPOSITION 2.5. Suppose $g$ and $t$ are conformally compact. Then

$$[\Phi(g, t)]_{jk} = \frac{1}{2} \rho^{-2} (B_k \rho_j + B_j \rho_k) + \rho^{-1} \mathcal{S}^1 (\tilde{g}, \tilde{t}) + \mathcal{S}^2 (\tilde{g}, \tilde{t}),$$

(2.16)

where

$$B = [(Tr_t \tilde{t}) \tilde{G}^{-1} - (n + 1)] d\rho.$$

(2.17)

Proof. To begin with,

$$[\delta_g G_{g,t}]_i = -g^{jk} (t_{i,jk}) + \frac{1}{2} (g^{jk} t_{j,k}),$$

Observe that

$$A^k_{jl} = \tilde{A}^k_{jl} - \rho^{-1} A^k_{jl},$$

where $\tilde{A}^k_{jl}$ is the Christoffel symbol of $\tilde{g}$, and

$$A^k_{jl} = \delta^k_i \rho_j + \delta^k_j \rho_i - \tilde{g}_{jl} \tilde{g}^{kq} \rho_q.$$ (2.18)

Thus $A^k_{jl} = \mathcal{S}^0 (\tilde{g})$, and $\tilde{A}^k_{jl} = \mathcal{S}^1 (\tilde{g})$, so

$$[\delta_g G_{g,t}]_i = -\rho^2 \tilde{g}^{jk} (\partial_k (\rho^{-2} t_{ij}) - (\tilde{A}^l_{jk} - \rho^{-1} A^l_{jk}) \rho^{-2} \tilde{t}_{lj})$$

$$- (\tilde{A}^l_{jk} - \rho^{-1} A^l_{jk}) \rho^{-2} \tilde{t}_{lj} + \frac{1}{2} \partial_l (\tilde{g}^{jk} \tilde{t}_{jk})$$

$$= -\rho^{-1} \tilde{g}^{jk} (-2 \rho_k \tilde{t}_{lj} + A^l_{ik} \tilde{t}_{lj} + A^l_{jk} \tilde{t}_{li}) + \mathcal{S}^1 (\tilde{g}, \tilde{t})$$

$$= -\rho^{-1} \tilde{g}^{jk} (\tilde{t}_{jk} \rho_i - (n + 1) \tilde{t}_{ij} \rho_k) + \mathcal{S}^1 (\tilde{g}, \tilde{t}),$$

where $\delta^k_i$ and $\mathcal{S}$ are defined in (2.5) and (2.6).
Then we have

$$\left[ gt^{-1} \delta_g G_g t \right]_k = -\rho^{-1} B_k + \epsilon^1(g, \bar{t}).$$

where $B$ is defined in (2.17). Finally, applying $\delta^*_{g}$ to this last expression yields (2.16).

Combining (2.1) and (2.16), we obtain the expression for $Q$

$$\left[ Q(g, t) \right]_{jk} = \rho^{-2} n(1 - \bar{\rho}^i \rho_j \rho_l) \bar{g}_{jk} - \frac{1}{2} (B_k \rho_j + B_j \rho_k))$$

$$+ \rho^{-1} \epsilon^1(g, \bar{t}) + \epsilon^2(g, \bar{t}),$$

(2.19)

where again $B$ is as in (2.17). In particular, this shows that

$$Q(g, t) = \rho^{-2} \epsilon^0(g, \bar{t}) + \rho^{-1} \epsilon^1(g, \bar{t}) + \epsilon^2(g, \bar{t}).$$

**Corollary 2.6.** Suppose $t$ is asymptotically hyperbolic and $g$ is conformally compact with $\bar{g} \in C_2(\bar{M}, \mathcal{S}^2)$. Then $Q(g, t) = O(\rho^{-1})$ near $bM$ if and only if the following hold on $bM$:

(a) $\text{Tr}_g \bar{t} = n + 1$, and

(b) $\bar{g}^{-1} d\rho = \bar{t}^{-1} d\rho$.

**Proof.** From (2.19), $Q(g, t) = O(\rho^{-1})$ if and only if

$$n(1 - \bar{g}^i \rho_j \rho_l) \bar{g}_{jk} - \frac{1}{2} (B_k \rho_j + B_j \rho_k) = 0$$

(2.20)

on $bM$. Using (2.17), (a) and (b) obviously imply $B = 0$ on $bM$, and (b) implies $|d\rho|_g = 1$ on $bM$, so (2.20) follows. Conversely, if (2.20) holds, since $\dim M \geq 3$, at any point of $bM$ we can contract (2.20) with a nonzero vector $V^k$ such that $V^k B_k = V^k \rho_k = 0$, and thus conclude $|d\rho|_g = 1$ on $bM$. This then implies that the matrix $(B_k \rho_j + B_j \rho_k)$ vanishes on $bM$, which can happen only if $B = 0$ since $d\rho \neq 0$ on $bM$. Since $\bar{g}$ is invertible on $bM$, this in turn implies that

$$(\text{Tr}_g \bar{t}) \bar{t}^{-1} d\rho = (n + 1) \bar{g}^{-1} d\rho,$$

and then contracting with $d\rho$ yields (a) and thus also (b).

If we fix $t$, Corollary 2.6 gives the conditions on $g$ which ensure that $Q(g, t) = O(\rho^{-1})$, i.e., that $g$ is a first approximation to a solution to $Q(g, t) = 0$. In order to construct higher-order approximate solutions we must study the higher asymptotics of $Q$. Using Proposition 2.4, this will reduce to computing an asymptotic expression for $A_g$ on tensors.
PROPOSITION 2.7. Suppose $g$ is an asymptotically hyperbolic metric on $M$, $\bar{q} \in C^2(\bar{M}, \mathcal{S}^2)$, and $f \in C^2(\mathbb{R}^+)$. Then
\begin{equation}
[A_g(f(\rho)\bar{q})]_{jk} = -\rho^2 f''(\rho) \bar{q}_{jk} + (n-5) \rho f'(\rho) \bar{q}_{jk} - f(\rho)(-2(n-1) \bar{q}_{jk} - (n+1)(\bar{q}_{jl} \bar{p}'_l \rho_k + \bar{q}_{kl} \bar{p}'_l \rho_j) + 2(\bar{g}^{jl} \bar{q}_{jl}) \rho_j \rho_k + 2(\bar{q}_{jl} \bar{p}'_l \bar{p}'_j) \bar{g}_{jk}) + \rho X_{jk}(f),
\end{equation}
where the ordinary differential operators $X_{jk}$ are second-order polynomials in $\rho d/\rho$ with bounded coefficients depending on $g$, $\bar{q}$, and we have set $\bar{p}' = \bar{g}^{ij} \rho_j = \rho^{-2} \rho'$.

Proof. Writing $g = \rho^{-2} \bar{g}$, we have
$$\Gamma_{jk}^{\bar{g}} = -\rho^{-1} A_{jk}^{\bar{g}} + \mathcal{E}^1(\bar{g}),$$
where $A_{jk}^{\bar{g}}$ is given by (2.18). Thus
\begin{align*}
\bar{q}_{jk,i} &= \rho^{-1}(2\bar{q}_{jk} \rho_i + \bar{q}_{kl} \rho_j + \bar{q}_{jl} \rho_k - g_{jl} \rho' \bar{q}_{ik} - g_{kj} \rho' \bar{q}_{li}) + O(1), \quad (2.22)
\end{align*}
and
\begin{align*}
\bar{q}_{jk,i} &= -\rho^{-2} \rho'(2\bar{q}_{jk} \rho_i + \bar{q}_{kl} \rho_j + \bar{q}_{jl} \rho_k - g_{jl} \rho' \bar{q}_{ik} - g_{kj} \rho' \bar{q}_{li}) \\
&\quad - g_{jl} \rho' \bar{q}_{ik} - g_{kj} \rho' \bar{q}_{li} \\
&\quad + \rho^{-1}(2\bar{q}_{jk,i} \rho_i + 2\bar{q}_{jk} \rho_i + \bar{q}_{kl} \rho_j + \bar{q}_{jl} \rho_k \\
&\quad + \bar{q}_{jl} \rho_k + \bar{q}_{jl} \rho_j - \rho' \bar{q}_{ik} - \rho' \bar{q}_{li} \\
&\quad - \rho' \bar{q}_{ik} - \rho' \bar{q}_{li} + \rho^{i'}(\bar{q}_{ik,j} + \bar{q}_{ij,k}) + O(\rho).
\end{align*}
Note that
\begin{align*}
\rho_i \rho' &= \rho^2 \rho_i \rho' = \rho^2 + O(\rho^3); \\
\rho' &= \rho^2 \bar{g}^{kl}(\bar{q}_k \bar{q}_l \rho - \Gamma_{kl}^i \rho_i) = \rho g^{kl} A_{kl,i} \rho + O(\rho^2) \\
&= (1-n) \rho + O(\rho^2); \\
p' \bar{q}_{jk,i} &= 2\rho \bar{q}_{jk} + O(\rho^2); \\
\bar{q}_{jl,i} &= \rho^{-1}(1-n) \bar{q}_{jl} \rho' + \bar{q}_{ij} \rho_j + O(\rho^2); \\
p' \bar{q}_{ij,k} &= \rho^{-1}(\bar{q}_{ij} \rho' \rho_k + \rho^2 \bar{q}_{jk} + \bar{q}_{ik} \rho' \rho_j - \bar{q}_{ik} \rho' \rho_j \bar{g}_{jk}) + O(\rho^2); \\
\end{align*}
Thus
\[
\tilde{q}_{jk,l} = -2(n-1) \tilde{q}_{jk} - (n+1)(\tilde{q}_{ji} \tilde{\rho}' \rho_k + \tilde{q}_{ki} \tilde{\rho}' \rho_j) \\
+ 2(\tilde{g}^{il} \tilde{q}_{il}) \rho_j \rho_k + 2(\tilde{q}_{li} \tilde{\rho}' \rho^l) \tilde{g}_{jk} + O(\rho).
\]
Substituting these relations into
\[
[D_g(f(\rho) \tilde{q})]_{jk} = -(f(\rho) \tilde{q}_{jk})_l \\
= -f''(\rho) \rho_j \rho^l \tilde{q}_{jk} - f'(\rho) \rho_l \tilde{q}_{jk} \\
- 2f'(\rho) \rho^l \tilde{q}_{jk,l} - f(\rho) \tilde{q}_{jk,l} \\
\]
yields (2.21).

**COROLLARY 2.8.** If \( g \) is an asymptotically hyperbolic metric on \( M \), \( \tilde{u} \in C_2(\tilde{M}), f \in C_2(\mathbb{R}^+), \) and \( \kappa \in \mathbb{R} \), then
\[
(\Delta_g + \kappa)(f(\rho) \tilde{u}) = (-\rho^2 f''(\rho) + (n-1) \rho f'(\rho) + \kappa f(\rho)) \tilde{u} + \rho X(f), \quad (2.23)
\]
where \( X \) is as in Proposition 2.7.

**Proof.** This can of course be computed directly by the same method as in the previous proposition, but it follows easily from (2.21). Just note that
\[
\Delta_g(f(\rho) \tilde{u} g) = \Delta_g(f(\rho) \tilde{u}) g,
\]
and so, substituting \( \rho^{-2} f(\rho) \) for \( f(\rho) \) and \( \tilde{q}_{jk} = \tilde{u} \tilde{g}_{jk} \) into (2.21), we get
\[
\Delta_g(f(\rho) \tilde{u}) \rho^{-2} \tilde{g}_{jk} \\
= -\rho^2(6\rho^{-4}f(\rho) - 4\rho^{-3}f'(\rho) + \rho^{-2}f''(\rho)) \tilde{u} \tilde{g}_{jk} \\
+ (n-5) \rho(-2\rho^{-3}f(\rho) + \rho^{-2}f'(\rho)) \tilde{u} \tilde{g}_{jk} \\
- \rho^{-2}f(\rho) - 2(n-1) \tilde{u} \tilde{g}_{jk} + 2\tilde{u} \tilde{g}_{jk} + \rho X(\rho^{-2}f) \tilde{g}_{jk} \\
= (-\rho^2f''(\rho) + (n-1) \rho f'(\rho)) \tilde{u} \rho^{-2} \tilde{g}_{jk} \\
+ \rho^{-1}X(f) \tilde{g}_{jk}.
\]

The ordinary differential operator
\[
I(f) = -\rho^2 f'' + (n-1) \rho f' + \kappa f
\]
which appears in this corollary is called the *indicial operator* for \( \Delta_g + \kappa \) acting on functions. It has a regular singular point at \( \rho = 0 \). The *characteristic exponents* of such an operator are the real numbers \( s \) for which \( I(\rho^s) = 0 \); in this case they are easily seen to be \( s_1, s_2 = \frac{1}{2}(n \pm \sqrt{n^2 + 4\kappa}) \). In
particular, if $\kappa = 0$, the characteristic exponents for $A$ are $s_1 = 0$ and $s_2 = n$. It follows from Corollary 2.8 that, if $s \neq s_1$ or $s_2$, there exists a solution $\tilde{u} \in C_2(\tilde{M})$ to the equation

$$(A_g + \kappa)(\rho^s \tilde{u}) = \rho^s \tilde{v} + O(\rho^{s+1})$$

for any $\tilde{v} \in C_2(\tilde{M})$: in fact, since $I(\rho^s) = (\kappa - s(n-s)) \rho^s$, we can just take $\tilde{u} = (\kappa - s(n-s))^{-1} \tilde{v}$.

For the covariant Laplacian acting on symmetric 2-tensors, the situation is somewhat more complicated. Suppose we want to solve the problem

$$(A_g + \kappa)(\rho^s \tilde{q}) = \rho^s \tilde{r} + O(\rho^{s+1}). \quad (2.25)$$

We can write $\mathcal{S}^2 = \mathcal{G} \oplus \mathcal{S}^2_0$, where $\mathcal{G}$ is the bundle of multiples of $g$ and $\mathcal{S}^2_0$ is the bundle of tensors that are trace-free (with respect to $g$, or equivalently $\tilde{g}$). It is easy to check that $A_g$ preserves sections of $\mathcal{G}$ and $\mathcal{S}^2_0$. If $\tilde{q}_{jk} = \tilde{u} g_{jk}$ is a section of $\mathcal{G}$ with $\tilde{u} \in C_2(\tilde{M})$, then (2.21) gives

$$(A_g + \kappa)(f(\rho) \tilde{u} g_{jk}) = I_0(f(\rho)) \tilde{u} g_{jk} + \rho X_{jk}(f). \quad (2.26)$$

where

$$I_0(f) = -\rho^2 f'' + (n - 5) \rho f' + (2n - 4 + \kappa) f,$$

with characteristic exponents

$$s_1, s_2 = \frac{1}{2}(n-4 \pm \sqrt{n^2 + 4\kappa}).$$

Thus we can solve the scalar part of the problem provided $s \neq s_1$ or $s_2$.

The bundle $\mathcal{S}^2_0 |_{bM}$ further decomposes, providing us with a diagonalization of the indicial operator given by (2.21). Define subbundles of $\mathcal{S}^2_0 |_{bM}$ as follows:

$$\mathcal{V}_1 = \{ \tilde{q}_{jk} : \tilde{r}' \tilde{q}_{jk} = 0 \text{ and } \tilde{g}^{jk} \tilde{q}_{jk} = 0 \};$$

$$\mathcal{V}_2 = \{ \tilde{q}_{jk} : \tilde{q}_{jk} = \lambda((n+1) \rho_j \rho_k - \tilde{g}_{jk}), \lambda \in \mathbb{R} \};$$

$$\mathcal{V}_3 = \{ \tilde{q}_{jk} : \tilde{q}_{jk} = v_j \rho_k + v_k \rho_j \text{ for } v \in T^* M |_{bM} \text{ with } \tilde{g}^{jk} v_k \rho_j = 0 \}.$$

It is easy to see that $\mathcal{S}^2_0 |_{bM} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$. For $1 \leq i \leq 3$, let $V_i$ denote the subspace of $C_2(\tilde{M}, \mathcal{S}^2_0)$ consisting of those tensors which lie in $\mathcal{V}_i$ at $bM$. For $\tilde{q} \in V_i$, $1 \leq i \leq 3$, a straightforward computation shows that (2.21) simplifies as follows.
Lemma 2.9. For $\tilde{q} \in V_1$ we have

$$(A_g + \kappa)(f(\rho) \tilde{q}) = I_1(f(\rho)) \tilde{q} + \rho X(f),$$

where X is as in Proposition 2.7, and $I_1$ and the corresponding characteristic exponents are

$$I_1(f) = -\rho^2 f'' + (n - 5) \rho f' + (2n - 2 + \kappa) f;$$

$$s_1, s_2 = \frac{1}{2}(n - 4 \pm \sqrt{n^2 + 4 \kappa + 8})$$

$$I_2(f) = -\rho^2 f'' + (n - 5) \rho f' + (4n - 2 + \kappa) f;$$

$$s_1, s_2 = \frac{1}{2}(n - 4 \pm \sqrt{n^2 + 8n + 4 \kappa + 8})$$

$$I_3(f) = -\rho^2 f'' + (n - 5) \rho f' + (3n - 1 + \kappa) f;$$

$$s_1, s_2 = \frac{1}{2}(n - 4 \pm \sqrt{n^2 + 4n + 4 \kappa + 12}).$$

Proposition 2.4 shows that the principal part of $D_1 Q_{(g, t)}$ is the same as that of $\frac{1}{2} A_g$, but the lower-order terms can be complicated in general. The next proposition shows that in the special case where $\tilde{g} = \tilde{t}$ on $bM$, the leading asymptotics of $D_1 Q_{(g, t)}$ reduce to operators of the form $A_g + \kappa$.

Proposition 2.10. Suppose $g, t$ are asymptotically hyperbolic metrics such that $\tilde{g} = \tilde{t}$ on $bM$. Write $r = \rho^s \tilde{q}$, where $\tilde{q} \in C_2(\tilde{M}, \mathcal{S}^2)$. If $r = ug + r_0$, where $r_0$ is trace-free, then

$$D_1 Q_{(g, t)} r = \frac{1}{2}((A_g + 2n)(ug) + (A_g - 2)(r_0)) + O(\rho^{s+1}).$$

(2.27)

Proof. Since $\tilde{g} = \tilde{t}$ on $bM$, if we write $t_{ij} = g_{ij} + v_{ij}$, then $v_{ij} = O(\rho^{-1})$, and it follows also that $t_{ij, k} = v_{ij, k} = O(\rho^{-2})$ and $t_{ij, kl} = O(\rho^{-3})$. Recalling the notation of Lemma 2.3, we obtain $C^{k}_{ij} = O(1)$, $C^{k}_{ij, l} = O(\rho^{-1})$, $D^k = O(\rho^2)$, and $D^k, l = O(\rho)$. Thus if $r = \rho^s \tilde{q}$ with $\tilde{q} \in C_2(\tilde{M}, \mathcal{S}^2)$, then $\delta^s_g[\mathcal{E}(r) - \mathcal{D}(r)]$ and $\mathcal{G}(r)$ are $O(\rho^{s+1})$. Since $g$ is asymptotically hyperbolic, we have

$$R_{ijkl} = -(g_{ik} g_{jl} - g_{il} g_{jk}) + O(\rho^{-3}),$$

so

$$\mathcal{G}(r) + nr = -r + Tr_g(r) g + O(\rho^{s+1}).$$

Formula (2.15) then gives

$$D_1 Q_{(g, t)} r = \frac{1}{2} A_g r - r + Tr_g(r) g + O(\rho^{s+1}),$$

so decomposing $r$ into its scalar and trace-free parts we obtain (2.27).
We now have all the information at hand to construct high-order formal solutions to $Q(g, t) = 0$. Our basic procedure is as follows. Given a metric $\hat{g}$ on $bM$, we choose as a first approximation some asymptotically hyperbolic metric $g = \rho^{-2} \hat{g}$ satisfying the boundary condition $\hat{g} |_{\partial M} = \hat{g}$. We also set $t = \rho$ and then leave $t$ fixed throughout the rest of the construction. By Corollary 2.6, $Q(g, t) = O(\rho^{-1})$ so this choice does indeed give a first approximate solution to $Q(g, t) = 0$. Now we use Lemma 2.9 and Proposition 2.10 to inductively modify $g$ to make $Q(g, t)$ vanish to higher order at $bM$.

As we will be using the inverse function theorem to obtain an exact solution it is important that our constructions depend smoothly on the data. So we define an extension operator $E$ from boundary metrics $\hat{g}$ to interior metrics such that $\rho^{-2} E(\hat{g})$ is always asymptotically hyperbolic. This operator is defined in terms of a fixed background asymptotically hyperbolic metric $h$, with $h = \rho^2 h \in C^\infty(M, \omega^2)$. In our eventual application, $h$ will be the hyperbolic metric itself on the ball. Let $v = v^j \partial_j$ denote the inward unit normal to $bM$ with respect to $h$; since $h$ is asymptotically hyperbolic we have $v^j = (h^{-1})^{ij} \rho_j$.

Choose a non-negative cutoff function $\phi \in C^\infty(M)$ which is 1 near $bM$ and is supported in the set on which the flow along $h$-geodesics normal to the boundary is a local diffeomorphism. Given any metric $\hat{g}$ on $bM$, we first extend $\hat{g}$ to a tensor $\tilde{\hat{g}}$ on $T\hat{M}|_{bM}$ by requiring that $v \cdot \hat{g} = dp$. Then extend $\tilde{\hat{g}}$ to a neighborhood of $bM$ in $\hat{M}$ by parallel translating along the $h$-geodesics normal to $bM$. Finally, define

$$E(\hat{g}) = \phi \tilde{\hat{g}} + (1 - \phi) h.$$  

Note that our construction ensures that $v^i \tilde{\hat{g}}^i_j = \rho_j$ on $bM$, or equivalently that $\tilde{\hat{g}}^i_j \rho_j = v^i$. Contracting with $\rho_j$, this shows that $\rho^{-2} E(\hat{g})$ is always asymptotically hyperbolic. Observe also that $E(h) = h$.

In order to keep track of the regularity of our approximate solution, we introduce some spaces of functions and tensors with asymptotic expansions. For $0 \leq k, m \in \mathbb{Z}, 0 \leq \alpha < 1$, let $A^{m, \alpha}_{k, \alpha}$ denote the space of functions that can be written as a sum

$$f = w_{k+m} + \rho w_{k+m-1} + \cdots + \rho^m w_k,$$

with each $w_j \in C_{j, \alpha}(\hat{M})$. The decomposition (2.28) is unique if we require for $j \neq k$ that $w_j = \phi v_j$, where $v_j$ is constant along the geodesics normal to $bM$ and $\phi$ is the cutoff function above. Then we take the norm on $A^{m, \alpha}_{k, \alpha}$ to be

$$\|f\|_{k, \alpha} = \sum_{j \geq k} \|w_j\|_{j, \alpha},$$

where $\|w_j\|_{j, \alpha}$ is the usual $C_{j, \alpha}$ norm on $\mathbb{M}$. It is a straightforward matter to check that these spaces are algebras under pointwise multiplication, that $A^{m, \alpha}_{k+1, \alpha} \subset A^{m+1, \alpha}_{k, \alpha}$, and that each $\partial_i$ is a bounded linear map from $A^{m, \alpha}_{k+1, \alpha}$ to $A^{m, \alpha}_{k, \alpha}$. If $j$ is a positive integer $\leq m$ and $f \in A^{m, \alpha}_{k, \alpha}$ is $O(\rho^j)$, then the
first \( j \) terms in (2.28) vanish, so \( A_{k,\alpha}^m \cap O(\rho^j) = \rho^j A_{k,\alpha}^{m-j} \) and the norms are equivalent. Thus \( \rho \partial_i \) maps \( A_{k+1,\alpha}^m \) to \( \rho A_{k,\alpha}^m \subset A_{k+1,\alpha}^m \) continuously. We also let \( A_{k,\alpha}^m(S^2) \) denote the space of symmetric 2-tensors whose coefficients in any coordinate system smooth up to \( bM \) are locally in \( A_{k,\alpha}^m \); in this case, we make the choices of \( w_j \) in (2.28) canonical by requiring that \( w_j = \varphi v_j \) for \( j \neq k \), where \( v_j \) is parallel along the geodesics normal to \( bM \). The reader can check that if \( \tilde{g} \in A_{k,\alpha}^m(S^2) \) is a nondegenerate metric on \( \tilde{M} \), then \( \tilde{g}^{-1} \) is in the same space, and \( \tilde{g} \mapsto \tilde{g}^{-1} \) is a smooth map of Banach spaces. Moreover, using our earlier notation \( \mathscr{G}^l \), for each \( k, m \geq 0 \) a (nonlinear) operator of the form \( \tilde{g} \mapsto \rho^j \mathscr{G}^l(\tilde{g}) \) maps \( A_{k+j,\alpha}^m(S^2) \) to \( A_{k,\alpha}^{m+j}(S^2) \) smoothly, and thus by (2.19) \( \mathcal{Q} \) is a smooth map from \( \rho^{-2} A_{k+1,\alpha}^m(S^2) \times \rho^{-2} A_{k+1,\alpha}^m(S^2) \) to \( \rho^{-2} A_{k,\alpha}^{2m+2}(S^2) \).

The second main result of this section follows.

**Theorem 2.11.** Suppose \( \tilde{g} \) is any metric on \( bM \) of class \( C_{k,\alpha}, \ k \geq 2, \ 0 \leq \alpha < 1 \), and set \( t = \rho^{-2} E(\tilde{g}) \). Then with \( m = \min(k-2, n-1) \) there exists an asymptotically hyperbolic metric \( g \in \rho^{-2} A_{k-m,\alpha}^m(S^2) \) on \( M \) such that \( \tilde{g} |_{bM} = \hat{g} \), and \( g \) satisfies

\[
\mathcal{Q}(g, t) = O(\rho^{-m-1}).
\]

The mappings \( \tilde{g} \mapsto g \) and \( \hat{g} \mapsto \mathcal{Q}(g, t) \), from a neighborhood of a fixed boundary metric in \( C_{k,\alpha}(bM, S^2(bM)) \) to \( \rho^{-2} A_{k-m,\alpha}^m(S^2) \) and \( \rho^{-m-1} A_{k-m-2,\alpha}^1(S^2) \), respectively, are smooth maps of Banach spaces.

**Proof.** We start with \( \tilde{g}_{-2} = E(\tilde{g}) \in C_{k,\alpha}(\tilde{M}, S^2) \), and set \( g_{-2} = t = \rho^{-2} \tilde{g}_{-2} \in \rho^{-2} A_{k,\alpha}^0(S^2) \). From (2.19), \( \mathcal{Q}(g_{-2}, t) \in \rho^{-2} A_{k-1,\alpha}^2(S^2) \), and it follows from Corollary 2.6 that \( \mathcal{Q}(g_{-2}, t) = O(\rho^{-1}) \), so in fact \( \mathcal{Q}(g_{-2}, t) \in \rho^{-1} A_{k-2,\alpha}^1(S^2) \) with smooth dependence on \( \tilde{g} \).

Suppose now that \( -1 \leq s \leq m-2 \leq n-3 \), and that we have constructed a metric \( g_{s-1} \in \rho^{-2} A_{k-s-1,\alpha}^{s+1}(S^2) \) such that \( \mathcal{Q}(g_{s-1}, t) = O(\rho^s) \). It follows that \( \mathcal{Q}(g_{s-1}, t) \in \rho^s A_{k-s-3,\alpha}^1(S^2) \). Consider the effect of adding a correction term \( r = \rho^s \tilde{q}, \tilde{q} \in \rho^s C_{k-s-2,\alpha}(\tilde{M}, S^2) \) to \( g_{s-1} \). We claim that \( \mathcal{Q}(g_{s-1} + \rho^s \tilde{q}, t) = \mathcal{Q}(g_{s-1}, t) + L(\rho^s \tilde{q}, t) + O(\rho^{s+1}) \), (2.29)

where \( L = D_1 \mathcal{Q}(g, t) \). To see why this is so, apply Taylor's formula to \( \mathcal{Q} \) about an asymptotically hyperbolic metric \( g \) to obtain

\[
\mathcal{Q}(g + r, t) = \mathcal{Q}(g, t) + D_1 \mathcal{Q}(g, t)r + \int_0^1 (1 - \lambda) D_1^2 \mathcal{Q}(g + \lambda r, t)(r, r) \, d\lambda.
\]

It follows from (2.19) that if \( \tilde{r} = \rho^2 r \in C_{2}(\tilde{M}, S^2) \), the second derivative \( D_1^2 \mathcal{Q}(g + \lambda r, t)(r, r) \) is \( \rho^{-2} \) times a homogeneous quadratic polynomial in \( \tilde{r} \),
$\rho \partial r$, and $\rho^2 \partial^2 r$, with coefficients that are continuous on $\bar{M}$, depending smoothly on $\lambda$. Therefore if we insert $r = \rho^s \bar{q}_s$,

$$Q(g + \rho^s \bar{q}_s, t) = Q(g, t) + D_1 Q_{(g, t)}(\rho^s \bar{q}_s) + O(\rho^{2s+2}).$$

Taking $g = g_{s-1}$ and noting that $2s + 2 \geq s + 1$ when $s \geq -1$, we get (2.29).

Now write $Q(g_{s-1}, t) = \rho^s \bar{w}_{k-s-2} + \rho^{s+1} \bar{w}_{k-s-3}$ as in (2.28). It follows from (2.29) that to solve $Q(g_{s-1} + \rho^s \bar{q}_s, t) = O(\rho^{s+1})$, we need to find $\bar{q}_s$ such that

$$L(\rho^s \bar{q}_s) = -\rho^s \bar{w}_{k-s-2} + O(\rho^{s+1}).$$

From Proposition 2.10, this can be done by decomposing $\bar{w}_{k-s-2}$ as

$$\bar{w}_{k-s-2} = \bar{w}^0 + \bar{w}^1 + \bar{w}^2 + \bar{w}^3,$$

where $\bar{w}^0 = \tilde{a}g_{s-1}$ is scalar and $\bar{w}^i \in V_i$ for $i = 1, 2, 3$, and setting $\bar{q}_s = \sum c_i \bar{w}^i$ near $\partial M$ for suitable constants $c_i$, provided that $s$ is not a characteristic exponent of $\Delta_k + 2n$ on sections of $\mathcal{G}$ or of $\Delta^k - 2$ on sections of $\mathcal{H}^2_0$. From (2.26) and Lemma 2.9, the spread between the characteristic exponents $s_1$ and $s_2$ is narrowest in $V_1$, for which

$$s_1 = -2, s_2 = n - 2.$$

Thus, since $-1 \leq s \leq n - 3$, there exists $\bar{q}_s \in C_{k-s-2,2}(\bar{M}, \mathcal{H}^2)$ such that $Q(g_{s-1} + \rho^s \bar{q}_s, t) = O(\rho^{s+1})$. We can choose $\bar{q}_s$ depending smoothly on $\tilde{g}$ so that $g_s = g_{s-1} + \rho^s \bar{q}_s \in \rho^{-2} A^i_{k-s-2,2}(\mathcal{H}^2)$ is a metric, and the induction is complete.

### 3. Estimates for the Laplacian on Hyperbolic Space

In this section we study $\Delta_h + \mathcal{X}$ acting on tensors, where $\Delta_h$ is the covariant Laplacian on hyperbolic space and $\mathcal{X}$ is a term of order zero. The main result is Theorem 3.10, which gives conditions under which $\Delta_h + \mathcal{X}$ is an isomorphism between suitable weighted Hölder spaces. The spaces we use are the usual dimensionless Hölder spaces used in interior Schauder estimates for elliptic equations, weighted to allow arbitrary rates of growth or vanishing at the boundary. They are invariantly defined on any smooth compact manifold with boundary; for our purposes, however, it will be sufficient to restrict attention to domains in $\mathbb{R}^{n+1}$.

Let $M$ be a bounded open set in $\mathbb{R}^{n+1}$ with $C^\infty$ boundary. For $0 \leq k \in \mathbb{Z}$ and $\Omega$ an open subset of $M$, denote by $C_k(\Omega) (= C_{k,0}(\Omega))$ the usual Banach space of $k$ times continuously differentiable functions on $\Omega$, and for $0 < \alpha < 1$ denote by $C_{k,\alpha}(\Omega)$ the subspace of functions whose $k$th
derivatives satisfy a uniform Hölder condition of order \( \alpha \), with their usual norms denoted \( \| \cdot \|_{k;\Omega} = \| \cdot \|_{k,0;\Omega} \) and \( \| \cdot \|_{k,\gamma;\Omega} \), respectively. We denote by \( C_k(\Omega) = C_{k,0}(\Omega) \) and \( C_{k,\alpha}(\Omega) \) the linear space of functions satisfying the corresponding estimates uniformly on compact subsets of \( \Omega \) (we do not define a norm on these spaces).

For \( x \in M \) let \( d_x \) denote the Euclidean distance from \( x \) to \( bM \). For \( s \in \mathbb{R} \) define

\[
\| u \|_{k,\alpha;\Omega}^{(s)} = \sum_{l=0}^{k} \sum_{|\gamma|=l} \| d^{-s+\frac{l}{k}} \partial^\gamma u \|_{L^\infty(\Omega)},
\]

where for any multi-index \( \gamma \), \( \partial^\gamma = \partial x^\gamma \); and for \( 0 < \alpha < 1 \) define

\[
\| u \|_{k,\alpha;\Omega}^{(s)} = \| u \|_{k,0;\Omega}^{(s)} + \sum_{|\gamma|=k} \sup_{x,y \in \Omega} \left[ \min(d^{-s+k+\alpha}, d^{-s+k+\alpha}) \left| \frac{\partial^\gamma u(x) - \partial^\gamma u(y)}{|x-y|^\alpha} \right| \right].
\]

The subspaces of \( C_k(\Omega) \) consisting of those functions for which the corresponding norm is finite are Banach spaces and are denoted \( A_{k,0}^s(\Omega) \), \( A_{k,\alpha}^s(\Omega) \), respectively. (\( M \) will be fixed throughout the discussion.)

Versions of the spaces \( A_{k,\alpha}^s(\Omega) \) are used in [DN] and [GT], and undoubtedly elsewhere too. However, in [GT] the definition is slightly different: in the definition of \( \| u \|_{k,\alpha;\Omega}^{(s)} \), where we have \( \min(d^{-s+k+\alpha}, d^{-s+k+\alpha}) \), Gilbarg and Trudinger use \( \min(d, d^{-s+k+\alpha}) \).

This definition has the unfortunate consequence that for \( s > k + \alpha \), the only function in \( A_{k,\alpha}^s(M) \) is 0, as can be seen by fixing \( x \in M \) and letting \( y \to bM \).

A useful property of the \( A_{k,\alpha}^s \) spaces is that the norm may be estimated in terms of the norms taken only over balls which are small near \( bM \). For \( x \in M \), let \( B_x \) be the open Euclidean ball with center \( x \) and radius \( \frac{1}{2}d_x \). It is clear that for \( x \in \Omega \),

\[
\| u \|_{k,\alpha;B_x \cap \Omega}^{(s)} \leq \| u \|_{k,\alpha;\Omega}^{(s)}.
\]

**Lemma 3.1.** For \( 0 \leq k \in \mathbb{Z}, s \in \mathbb{R}, \) and \( 0 \leq \alpha < 1 \), we have

\[
\| u \|_{k,\alpha;\Omega}^{(s)} \leq C \sup_{x \in \Omega} \| u \|_{k,\alpha;B_x \cap \Omega}^{(s)},
\]

where \( C \) depends only on \( k \).

**Proof.** Clearly for \( 0 \leq |\gamma| = l \leq k \),

\[
\| d^{-s+\frac{l}{k}} \partial^\gamma u \|_{L^\infty(\Omega)} = \sup_{x \in \Omega} \| d^{-s+\frac{l}{k}} \partial^\gamma u \|_{L^\infty(B_x \cap \Omega)} \leq \sup_{x \in \Omega} \| u \|_{k,0;B_x \cap \Omega}^{(s)}.
\]
Summing over $\gamma$ proves the result for $\alpha = 0$.

If $0 < \alpha < 1$, fix $\gamma$ with $|\gamma| = k$; we must estimate

$$\min(d_{x}^{-s+k+z}, d_{y}^{-s+k+z}) \frac{|\partial^{\gamma}u(x) - \partial^{\gamma}u(y)|}{|x - y|^z}$$

for $x, y \in \Omega$. If $|x - y| \leq \frac{1}{2} \max(d_{x}, d_{y})$, the estimate for either $B_{x}$ or $B_{y}$ applies directly. If, on the other hand, $|x - y| \geq \frac{1}{2} \max(d_{x}, d_{y})$, then the above expression is less than or equal to

$$2^{s} \min(d_{x}^{-s+k+z}, d_{y}^{-s+k+z}) \min(d_{x}^{-s}, d_{y}^{-s}) \mu_{\Omega}(x) - \mu_{\Omega}(y)$$

$$\leq 4 \| d^{-s+k} \partial^{\gamma}u \|_{L^{\infty}(\Omega)}$$

which we have already estimated above.

We will use a reformulated version of Lemma 3.1. Let $B_{0} \subset \mathbb{R}^{n+1}$ denote the open ball of radius $\frac{1}{2}$ centered at $0$, and for $x \in M$ define $\psi_{x} : B_{0} \to B_{x}$ by $\psi_{x}(z) = x + d_{x}z$. If $y \in B_{x}$, then $\frac{1}{2}d_{x} \leq d_{y} \leq \frac{3}{2}d_{x}$, so that

$$\| d^{-s+\ell} \partial^{\gamma}u \|_{L^{\infty}(B_{x} \cap \Omega)} \geq d_{x}^{-s+\ell} \| \partial^{\gamma}u \|_{L^{\infty}(B_{x} \cap \Omega)}.$$ 

Combining this with the observation that $(d^{\gamma}u) \circ \psi_{x} = d_{x}^{-s} (u \circ \psi_{x})$ for $|\gamma| = \ell$, one deduces that

$$\| u \|_{k, \alpha : B_{x} \cap \Omega} \approx d_{x}^{-s} \| u \circ \psi_{x} \|_{k, \alpha : \psi_{x}^{-1}(B_{x} \cap \Omega)}.$$ 

Consequently from Lemma 3.1 we obtain

**Proposition 3.2.** If $u \in C_{k}(\Omega)$, then $u \in \Lambda_{k, \alpha}^{s}(\Omega)$ if and only if

$$\sup_{x \in M} d_{x}^{-s} \| u \circ \psi_{x} \|_{k, \alpha : \psi_{x}^{-1}(B_{x} \cap \Omega)} < \infty,$$

and this supremum is comparable to $\| u \|_{k, \alpha, \Omega}^{(s)}$. 

The next proposition gives the basic properties of the spaces $\Lambda_{k, \alpha}^{s}(\Omega)$. Its proof involves standard techniques of estimation along with Lemma 3.1, and is thus left to the reader.

Fix a positive defining function $\rho$ for $bM$ as in Section 2. Let $0 \leq k \in \mathbb{Z}$, $s \in \mathbb{R}$, and $0 < \alpha < 1$. Unless otherwise indicated the norms of the linear maps below depend only on the function $\rho$, the diameter of $M$, and the parameters $k, k', \alpha, \alpha', s, s'$ that occur, but not on the domains $\Omega, \Omega'$. 

Proposition 3.3. (1) If $\Omega' \subset \Omega \subset M$, then $A^s_{k,\alpha}(\Omega) \subset A^s_{k,\alpha}(\Omega')$ continuously, with norm 1.

(2) $C_{k,\alpha}(\overline{\Omega}) \subset A^0_{k,\alpha}(\Omega)$ continuously.

(3) $A^s_{k,\alpha}(\Omega) \subset C_{k,\alpha}(\Omega)$ for all $s$.

(4) If $\Omega \in M$ and $0 < \alpha < 1$, then $A^s_{k,\alpha}(\Omega) \subset C_{k,\alpha}(\Omega)$ continuously, with norm depending on the distance from $\Omega$ to $bM$ as well as on $k, \alpha, s$, and the diameter of $M$.

(5) $A^s_{k,\alpha}(\Omega) \subset A^s_{k',\alpha'}(\Omega)$ continuously if $k' + \alpha' \leq k + \alpha$.

(6) $A^s_{k,\alpha}(\Omega) \subset A^s_{k,\alpha}(\Omega)$ continuously if $s > s'$.

(7) $A^{s+\alpha}_{k,\alpha}(\Omega) = C_{k,\alpha}(\Omega)$ continuously if $0 < \alpha < 1$.

(8) $\rho^s \in A^s_{k,\alpha}(\Omega)$ for all $k$ and $\alpha$.

(9) Pointwise multiplication maps $A^s_{k,\alpha}(\Omega) \times A^s_{k,\alpha}(\Omega) \to A^{s+s}_{k,\alpha}(\Omega)$, and

$$\|uv\|_{k,\alpha;\Omega} \leq C \|u\|_{k,\alpha;\Omega} \|v\|_{k,\alpha;\Omega}.$$

(10) $\rho^s \circ v$ maps $A^s_{k,\alpha}(\Omega)$ continuously.

(11) If $u \in A^s_{k,\alpha}(\Omega)$ and $\rho^{-s} u \geq 0$, then $u^{-1} \in A^{-s}_{k,\alpha}(\Omega)$; the map $u \mapsto u^{-1}$ is a smooth map of Banach spaces.

(12) $A^s_{k,\alpha} = A^0_{k,\alpha}(M)$ continuously for all $m$, where $A^m_{k,\alpha}$ is defined in (2.28).

(13) Suppose $M, M' \subset \mathbb{R}^{n+1}$ are bounded open sets with smooth boundaries, and $\Phi: M' \to M$ is a diffeomorphism of manifolds-with-boundary, with $\Phi$ and $\Phi^{-1}$ of class $C_{k,\alpha}$ if $k \geq 1$, or $C_1$ if $k = 0$. Let $\Omega \subset M$ be open and $\Omega' = \Phi^{-1}(\Omega)$. If $u \in A^s_{k,\alpha}(\Omega)$ then $u \circ \Phi \in A^s_{k,\alpha}(\Omega')$ and

$$\|u \circ \Phi\|_{k,\alpha;\Omega'} \leq C \|u\|_{k,\alpha;\Omega},$$

where $C$ depends on $k, \alpha, s$, the diameters of $M, M'$, and the $C_{k,\alpha}$ (or $C_1$) norms of $\Phi$ and $\Phi^{-1}$.

Several consequences of Proposition 3.3 are worth noting. First, by (8) and (9), multiplication by $\rho^s$ is an isomorphism from $A^0_{k,\alpha}(\Omega)$ to $A^s_{k,\alpha}(\Omega)$, and $\|u\|_{k,\alpha;\Omega} \approx \|\rho^{-s} u\|_{k,\alpha;\Omega}$. From (2) and (9) it follows that multiplication by functions in $C^\infty(\overline{\Omega})$ preserves $A^s_{k,\alpha}(\Omega)$.

We will also need weighted Hölder spaces of tensors on $M$. By $A^s_{k,\alpha}(\Omega, \mathcal{T}^p)$ we denote the space of covariant, rank $p$ tensors on $\Omega$ all of whose components in Euclidean coordinates are in $A^s_{k,\alpha}(\Omega)$, with the obvious norm. Proposition 3.3(13) shows that this space is independent of the choice of coordinates on $M$, and the components in any coordinate system smooth up to $M$ are in $A^s_{k,\alpha}(\Omega)$.

Finally, although we will not need it in this paper, we observe that by utilizing a partition of unity and coordinate charts in the usual way, $A^s_{k,\alpha}$
spaces can be defined on any $C^\infty$ compact manifold-with-boundary, and by Proposition 3.3(13) are independent of the choices of coordinates and thus invariantly defined.

Our main tools in studying the Laplacian on hyperbolic space will be Schauder estimates for elliptic uniformly degenerate operators and a generalized maximum principle. First we discuss the Schauder estimates. The usual proof of interior Schauder estimates for uniformly elliptic equations in terms of the $A^0_{k,\alpha}$ spaces applies just as well in the elliptic uniformly degenerate case; this is contained in the treatments in [DN], [GT]. However, since we need a slightly different formulation and generalization of the results, we include a proof here based on reduction to the usual elliptic estimates via rescaling and Proposition 3.2.

Let $M$ and $\rho$ be as above and consider a linear second-order $N \times N$ system $P$ of differential operators on $M$. We say $P$ is uniformly degenerate if for $u = (u', \ldots, u') \in C^1(M, \mathbb{R}^N)$,

$$(Pu)^i = \sum_{j=1}^{N} P^i_j(x, \rho \partial) u^j, \quad 1 \leq i \leq N,$$

where for $1 \leq i, j \leq N$, $P^i_j(x, \xi)$ is a real quadratic polynomial in $\xi$ with coefficients in $C^\infty(M)$. It follows from Proposition 3.3(10) and the remarks following it that $P$ maps $A^\alpha_{k+2,\alpha}(\mathbb{R}^N)$ continuously. (In fact, the same is true if the coefficients of $P^i_j$ are merely in $C^\alpha(M)$.)

We say $P$ is elliptic as a uniformly degenerate operator if the homogeneous quadratic principal part $p^i_j(x, \xi)$ satisfies

$$\det(p^i_j(x, \xi)) \geq K|\xi|^{2N} \quad \text{for all } x \in M, \xi \in \mathbb{R}^{n+1}, \quad (3.1)$$

for some constant $K > 0$.

**Proposition 3.4.** Let $0 \leq k \in \mathbb{Z}$, $s \in \mathbb{R}$, $0 < \alpha < 1$, and let $\Omega' \subset \Omega$ be open subsets of $M$ such that for all $x \in \Omega'$, $B_\varepsilon \subset \Omega$. Suppose that $P$ is an elliptic uniformly degenerate operator as above, and that $u \in C^2(\Omega'; \mathbb{R}^N) \cap A^\alpha_{k+2,\alpha}(\Omega'; \mathbb{R}^N)$ is such that $Pu \in A^\alpha_{k,\alpha}(\Omega; \mathbb{R}^N)$. Then $u \in A^\alpha_{k+2,\alpha}(\Omega'; \mathbb{R}^N)$ and

$$\|u\|_{k,\alpha;\Omega'} \leq C(\|Pu\|_{k,\alpha;\Omega} + \|u\|_{0,0;\Omega}), \quad (3.2)$$

where $C$ is independent of $u$, $\Omega$, and $\Omega'$.

**Proof.** First observe that on compact subsets of $M$, $P$ is an elliptic system in the usual sense, so certainly $u \in C^{k+2,\alpha}(\Omega; \mathbb{R}^N)$. Thus it is only the estimate (3.2) that concerns us.

We will reduce (3.2) to the following standard interior elliptic estimate: let $Q$ be a second-order $N \times N$ elliptic system on the ball
$B_0 = \{ |z| < \frac{1}{4} \} \subset \mathbb{R}^{n+1}$. Thus $(Qv)' = \sum_{j=1}^{N} Q_j'(z, \partial) v'$, where each $Q_j'(z, \partial)$ is a real second-order operator, and (3.1) holds for the principal part $q_j$. Let $B_0' = \{ |z| < \frac{1}{4} \}$, and suppose that $v \in C_{k}(B_0'; \mathbb{R}^N) \cap L^\infty(B_0'; \mathbb{R}^N)$ is such that $Qv \in C_{k,\alpha}(\overline{B}_0'; \mathbb{R}^N)$. Then, by standard Schauder theory, $v \in C_{k+2,\alpha}(\overline{B}_0'; \mathbb{R}^N)$ and

$$\|v\|_{k+2,\alpha; B_0'} \leq C(\|Qv\|_{k,\alpha; B_0} + \|v\|_{0,0; B_0}),$$

(3.3)

where $C$ depends only on $k, \alpha, N, n, K$, and the $C_{k,\alpha}(\overline{B}_0)$ norms of the coefficients of the operators $Q_j$.

To carry out the reduction, let $x \in \Omega'$, and let $B_x$ and $B_x'$ be the open balls about $x$ of radii $\frac{1}{4}d_x, \frac{1}{4}d_x$, respectively. Now $\psi_x$ maps $B_\alpha$ onto $B_x$; we transform the system $P$ via $\psi_x$ to a system $Q$ on $B_0$, defined by

$$Q(u \circ \psi_x) = (Pu) \circ \psi_x.$$

Since $(\psi_x)' \circ \partial = d_x \circ \partial$, one sees easily that

$$Q_j'(z, \partial) = P_j'(\psi_x(z), d_x^{-1} \rho \circ \psi_x(z) \partial).$$

But $\rho \approx d_x$ on $B_x$; it follows that $Q$ is an elliptic system on $B_0$, with ellipticity constant $K$ bounded from above and below independently of $x$. Similarly one sees that the $C_{k,\alpha}(\overline{B}_0)$ norms of the coefficients of $Q_j$ are bounded independently of $x$.

It is clear that Proposition 3.2 remains true using either family of balls $B_x$ or $B_x'$. Hence, applying Proposition 3.2 and (3.3), we obtain

$$\|u\|_{k+2,\alpha; \Omega'}^{(s)} \leq C \sup_{x \in \Omega'} d_x^{-s} \|u \circ \psi_x\|_{k+2,\alpha; \psi_x^{-1}(\Omega' \cap \Omega')} \leq C \sup_{x \in \Omega'} d_x^{-s} \|u \circ \psi_x\|_{k+2,\alpha; B_0} \leq C \sup_{x \in \Omega'} d_x^{-s}(\|Pu \circ \psi_x\|_{k,\alpha; B_0} + \|u \circ \psi_x\|_{0,0; B_0}) \leq C(\|Pu\|_{k,\alpha; \Omega}^{(s)} + \|u\|_{0,0; \Omega}^{(s)}),$$

which is (3.2).

Generalized maximum principles on complete Riemannian manifolds, originally formulated by Yau [Y], have been used extensively. (However, there appears to be an error in the proof of Theorem 2 of [Y], which is reproduced in Theorem 3 of [CY1].) For the class of manifolds that interest us, a simple proof is available, modeled on the corresponding argument in the Kähler case [CY2], so we include it here. We are grateful to S.-Y. Cheng for useful discussions concerning these maximum principles.
THEOREM 3.5. Let $M$ be the interior of a smooth compact manifold $M$ with boundary, $\rho$ a positive defining function for $bM$, $g$ a conformally compact metric on $M$ with $\rho \partial \overline{\partial}_{jk}$ bounded, and let $f \in C^2(M)$ be bounded above. Then there is a sequence $\{x_k\} \subset M$ such that

(i) $\lim_{k \to \infty} f(x_k) = \sup_M f$;

(ii) $\lim_{k \to \infty} |\nabla_g f(x_k)|_g = 0$;

(iii) $\liminf_{k \to \infty} A_g f(x_k) \geq 0$.

Proof. Let $L = \sup_M f$, and consider $F = L - f$. Then $F \geq 0$; we may assume that $F > 0$ on $M$, for otherwise by the usual maximum principle we can take $x_k = x$, where $F(x) = 0$. We need to find $x_k$ such that (i) $\lim F(x_k) = 0$, (ii) $\lim |\nabla_g F(x_k)|_g = 0$, and (iii) $\limsup_{k \to \infty} A_g F(x_k) \leq 0$. First choose $\{y_k\}$ such that $\lim F(y_k) = 0$; by passing to a subsequence we may assume that $y_k \to y \in bM$. Now introduce a coordinate system $(x^i)$ about $y$, and for fixed $k$ let $\varphi(x) = 1 - \delta^{-2} |x - y_k|^2$, where $2\delta$ is the Euclidean distance in these coordinates from $y_k$ to $bM$. If we set $D = \{\varphi > 0\}$, then it is clear that the partial derivatives of $\varphi$ satisfy

$$\sup_{D} |\partial_i \varphi| \leq \delta^{-1}, \quad \sup_{D} |\partial_i \partial_j \varphi| \leq 2\delta^{-2}. \tag{3.4}$$

Since $\rho \approx \delta$ on $D$, from the form of the metric $g$ we deduce that

$$\sup_{D} |\nabla_g \varphi|_g, \sup_{D} |A_g \varphi| \leq C, \tag{3.4}$$

where $C$ is independent of $k$.

Let $x_k \in D$ be a point where the function $F/\varphi$ attains its minimum in $D$. Then $(F/\varphi)(x_k) \leq (F/\varphi)(y_k) = F(y_k)$, so

$$F(x_k) \leq F(y_k). \tag{3.5}$$

Also, at $x_k$, $\nabla_g \log(F/\varphi) = 0$ and $A_g \log(F/\varphi) \leq 0$. Thus $(\nabla_g F/F)(x_k) = (\nabla_g \varphi/\varphi)(x_k)$, so

$$|\nabla_g F(x_k)|_g = \frac{F}{\varphi} (x_k) |\nabla_g \varphi(x_k)|_g \leq F(y_k) |\nabla_g \varphi(x_k)|_g. \tag{3.6}$$

Since $A_g (\log F) = A_g F/F + |\nabla_g F|^2/F^2$, we obtain $(A_g F/F)(x_k) \leq (A_g \varphi/\varphi)(x_k)$. Hence

$$A_g F(x_k) \leq \frac{F}{\varphi} (x_k) A_g \varphi(x_k) \leq F(y_k) |A_g \varphi(x_k)|. \tag{3.7}$$
Now letting \( k \to \infty \) in (3.5), (3.6), (3.7) and using (3.4) and \( F(y_k) \to 0 \) results in (i), (ii), (iii).

We now consider \( \mathcal{A}_g \) on tensors as a uniformly degenerate system. Unless otherwise stated, we assume in the rest of this section that \( g \) is a conformally compact metric with \( \tilde{g} \in C^\infty(\tilde{M}, \mathcal{G}^2) \). In a coordinate system which is smooth up to the boundary, consider the local coordinate expression for the covariant Laplacian:

\[
(\mathcal{A}_g u)_{i_1 \ldots i_p} = -u_{i_1 \ldots i_p, k}. 
\]

We may express the components of \( \mathcal{A}_g u \) in terms of differential operators applied to the components of \( u \): the second-order part is 

\[-g^{kl} \partial_k \partial_\ell u_{i_1 \ldots i_p} = -\rho^2 \tilde{g}^{kl} \partial_k \partial_\ell u_{i_1 \ldots i_p}. \]

The coefficient of any first derivative \( \partial_k u_{i_1 \ldots i_p} \) is a sum of terms of the form \( g^{kl} T_{r_1} \) (which we abbreviate \( g^{-1} T \)) for various choices of the indices; since \( \rho \Gamma \in C^\infty(\tilde{M}) \) it follows that these coefficients are all of the form \( \rho \cdot C^\infty(\tilde{M}) \). Similarly the coefficient of a zeroth order term \( u_{i_1 \ldots i_p} \) is a sum of terms of the form \( g^{-1} \partial_\Gamma g \) and \( g^{-1} \Gamma g \), so it is in \( C^\infty(\tilde{M}) \). It follows immediately that \( \mathcal{A}_g \) is a second-order uniformly degenerate elliptic system.

We wish to prove isomorphism theorems for \( \mathcal{A}_g + \kappa \), with \( \kappa \) constant. For future reference, we will consider the more general case of an operator of the form \( \mathcal{A}_g + \mathcal{K} \), where \( \mathcal{K} \) is a smooth self-adjoint endomorphism of the bundle \( \mathcal{F}^p \) of rank \( p \) tensors. The only missing ingredient is an a priori estimate for the \( \mathcal{A}_g \) norm of a tensor \( u \) in terms of \( (\mathcal{A}_g + \mathcal{K}) u \). The basic estimate we will need is given in the following definition.

**Definition 3.6.** Let \( s \in \mathbb{R}, 0 \leq p \in \mathbb{Z}, \) and \( \mathcal{K} \in C_0(M, \text{End}(\mathcal{F}^p)) \). We say that the basic estimate holds for \( \mathcal{A}_g + \mathcal{K} \) on \( \mathcal{A}^s(M, \mathcal{F}^p) \) if for all \( u \in C_2(\Omega, \mathcal{F}^p) \cap \mathcal{A}^s_{0,0}(\Omega, \mathcal{F}^p) \) such that \( (\mathcal{A}_g + \mathcal{K}) u \in \mathcal{A}^s_{0,0}(\Omega, \mathcal{F}^p) \) in either of the situations

(i) \( \Omega \subset M, u \in C_0(\Omega, \mathcal{F}^p), \) and \( u = 0 \) on \( \partial \Omega \), or

(ii) \( \Omega = M, \)

we have

\[
\| u \|_{0,0;0}^{(s)} \lesssim C \| (\mathcal{A}_g + \mathcal{K}) u \|_{0,0;0}^{(s)},
\]

where \( C \) is independent of \( u \) and \( \Omega \).

**Proposition 3.7.** Let \( M \) be a smoothly bounded domain in \( \mathbb{R}^{n+1}, s \in \mathbb{R}, 0 \leq p, k \in \mathbb{Z}, 0 < \alpha < 1; \) let \( \mathcal{K} \in C^\infty(\tilde{M}, \text{End}(\mathcal{F}^p)) \) be self-adjoint, and suppose the basic estimate holds for \( \mathcal{A}_g + \mathcal{K} \) on \( \mathcal{A}^s(M, \mathcal{F}^p) \). Then \( \mathcal{A}_g + \mathcal{K} : \mathcal{A}^s_{k+2,0}(M, \mathcal{F}^p) \to \mathcal{A}^s_{k,0}(M, \mathcal{F}^p) \) is an isomorphism.
Proof. Injectivity is an immediate consequence of case (ii) of (3.8). To prove surjectivity, let $f \in \Lambda_{k,2}^s(M, \mathcal{T}^p)$ be arbitrary. Let $\Omega \in M$ have smooth boundary; then $\Delta_g + \mathcal{K}$ acting on $p$-tensors with Dirichlet boundary conditions on $b\Omega$ is a self-adjoint operator bounded from below. If $u \in C_2(\Omega, \mathcal{T}^p)$ satisfies $(\Delta_g + \mathcal{K}) u = 0$, $u|_{b\Omega} = 0$, then since $C_0(\Omega, \mathcal{T}^p) \subset \Lambda_{0,0}^s(\Omega, \mathcal{T}^p)$ for all $s \in \mathbb{R}$, it follows from (3.8) that $u = 0$. Thus 0 is not a Dirichlet eigenvalue of $\Delta_g + \mathcal{K}$ acting on $p$-tensors on $\Omega$. Since $\Lambda_{k,2}^s(M, \mathcal{T}^p) \subset C_{k,2}^s(\Omega, \mathcal{T}^p)$, by the standard theory of elliptic boundary problems the Dirichlet problem $(\Delta_g + \mathcal{K}) u = f$, $u|_{b\Omega} = 0$ has a unique solution $u \in C_{k,2}^s(M, \mathcal{T}^p)$.

Let $\Omega_i$ ($i = 1, 2, \ldots$) be an exhaustion of $M$ by relatively compact subdomains with smooth boundaries, so that $\Omega_i \subset \Omega_{i+1} \subset M$ and $M = \bigcup_{i=1}^{\infty} \Omega_i$. For each $i$, let $u_i \in C_{k,2}^s(\Omega_i, \mathcal{T}^p)$ be the solution to $(\Delta_g + \mathcal{K}) u_i = f$ on $\Omega_i$, $u_i = 0$ on $b\Omega_i$. Then we have (3.8) for $u_i$ on $\Omega_i$. If we fix $j$, then for all $i$ sufficiently large and $x \in \Omega_j$ it will be the case that $B_x \subset \Omega_i$. Thus we may apply Proposition 3.4 with $P = \Delta_g + \mathcal{K}$, which in combination with (3.8) gives

$$\| u_i \|_{k,2,\Omega_j} \leq C \| f \|_{k,2,\Omega_j} \leq C \| f \|_{k,2,M}. \tag{3.9}$$

By Proposition 3.3(4), it follows that the $u_i$ are bounded in $C_{k,2}^s(\Omega_j, \mathcal{T}^p)$, so a subsequence converges in $C_{k,2}^s(\Omega_j, \mathcal{T}^p)$ by Arzelà-Ascoli. Now passing to a diagonal subsequence we obtain a limit function $u \in C_{k,2}^s(M, \mathcal{T}^p)$ and a subsequence of the $u_i$ which converge to $u$ in $C_{k,2}^s(\Omega_j, \mathcal{T}^p)$ for all $j$. Thus $(\Delta_g + \mathcal{K}) u = f$. Letting $i \to \infty$, then $j \to \infty$ in (3.9) it follows that $u \in \Lambda_{k,2}^s(M, \mathcal{T}^p)$ and that $\| u \|_{k,2,\Omega_j} \leq C \| f \|_{k,2,M}$, thus proving both surjectivity of $\Delta_g + \mathcal{K}$ and boundedness of $(\Delta_g + \mathcal{K})^{-1}$. \]

The next proposition reduces the problem of proving the basic estimate to proving the existence of a certain weight function. Let $\rho$ be a positive defining function for $bM$ as above.

PROPOSITION 3.8. Let $s \in \mathbb{R}$, let $\mathcal{K} \in C_0(M, \text{End}(\mathcal{T}^p))$ be any continuous endomorphism field, and define $K \in C_0(M)$ by

$$K(x) = \inf \{ (\mathcal{K}(x) u, u)_g : u \in \mathcal{T}^p, \ |u|_g = 1 \}. $$

Suppose there exists $\varphi \in C_2(M)$ such that $\rho^{-s} \varphi \in C_1(\tilde{M})$, $\rho^{-s} \varphi > 0$ in $\tilde{M}$, and

$$ (\Delta_g + K) \varphi \geq \delta \varphi \tag{3.10}$$

for some constant $\delta > 0$. Then the basic estimate holds for $\Delta_g + \mathcal{K}$ on
\( A^{1-p}(M, \mathcal{F}^p) \), and we can take the constant in (3.8) to be \( C = C'/\delta \), where \( C' \) is independent of \( \delta \) and \( \mathcal{K} \).

**Proof.** We first consider the case \( p = 0 \), which is somewhat simpler. In this case, \( \mathcal{K} = K \). Let \( \varphi \) be as in the hypothesis, and let \( u \in C_2(\Omega) \cap A^0_{0,0}(\Omega) \) be such that \((A_g + \mathcal{K})u \in A^0_{0,0}(\Omega) \). This means \( u/\varphi \in C_2(\Omega) \cap L^\infty(\Omega) \), and additionally assume that \( u/\varphi \) vanishes on \( \partial\Omega \) in case (i).

Direct computation shows that
\[
A_g \left( \frac{u}{\varphi} \right) = -\left( \frac{u}{\varphi} \right)_k = \frac{A_g u}{\varphi} - \left( \frac{A_g \varphi}{\varphi} \right) \frac{u}{\varphi} + V \left( \frac{u}{\varphi} \right),
\]
where \( V \) is the vector field \( V = (2/\varphi) \varphi^k \partial_k \). This can be written
\[
\left( A_g - V + K + \frac{A_g \varphi}{\varphi} \right) \frac{u}{\varphi} = \frac{(A_g + K) u}{\varphi}. \tag{3.11}
\]
By replacing \( u \) by \(-u\) if necessary we may assume that \( \sup_{\partial\Omega} |u/\varphi| = \sup_{\partial\Omega} (u/\varphi) > 0 \). In case (i), \( u/\varphi \) attains its maximum at some \( x \in \Omega \), and \( \nabla_g(u/\varphi)(x) = 0 \), \( A_g(u/\varphi)(x) \geq 0 \). Hence from (3.10) and (3.11) there follows
\[
\delta \frac{u}{\varphi}(x) \leq \frac{(A_g + K) u}{\varphi}(x),
\]
which immediately gives (3.8) since \( \varphi \approx \rho^* \). In case (ii) we apply Theorem 3.5 to \( f = u/\varphi \). Observe that for each \( k \), \( |\varphi^k/\varphi| = |g^k \partial_j \varphi/\varphi| \leq C \rho \), so \( |V(u/\varphi)| \leq C \rho \max_k |\partial_k (u/\varphi)| \leq C |\nabla_g(u/\varphi)| \). Hence upon evaluating (3.11) at \( x_k \) and letting \( k \to \infty \) one obtains
\[
\delta \sup_M \left| \frac{u}{\varphi} \right| \leq \sup_M \left| \frac{(A_g + K) u}{\varphi} \right|
\]
as desired.

For \( p > 0 \), we consider \( |u|_{g/\varphi} \), where
\[
|u|_{g/\varphi}^2 = u_{i_1 \cdots i_p} u_{i_1 \cdots i_p}. \tag{3.12}
\]
Since each \( u_{i_1 \cdots i_p} \in A^1_{0,0} \), it follows that \( |u|_{g/\varphi} \in L^\infty(\Omega) \). As in the case \( p = 0 \) we will apply the maximum principle to \( |u|_{g/\varphi} \). Now (omitting \( g \) from the notation for simplicity),
\[
A \left( \frac{|u|}{\varphi} \right) = - \left( \frac{(u_{i_1 \cdots i_p} u_{i_1 \cdots i_p})^{1/2}}{\varphi} \right)_k
\]
\[
= - \left( \frac{(u \nabla_k u)}{\varphi |u|} - \frac{|u| \varphi_k}{\varphi^2} \right)_k
\]
Hence

\[
(A - V + \delta) \frac{|u|}{\phi} \leq \left( A - V + K + \frac{\Delta \phi}{\phi} \right) \frac{|u|}{\phi} \leq \frac{(A + \mathcal{K}) u, u}{\phi |u|}.
\]  

(3.13)

In case (i), evaluate this at a point where \( \sup(u/\phi) \) is attained—note that even though \( |u|/\phi \) is not \( C_2 \) where \( u = 0 \), it is \( C_2 \) at its maximum. As in the case \( p = 0 \) we get

\[
\delta \sup_{\Omega} \frac{|u|}{\phi} \leq \sup_{\Omega} \frac{|(A + \mathcal{K}) u|}{\phi}.
\]

Since \( \sup_{\Omega}(|u|/\phi) \approx \|u\|_{0,0,\mathcal{K}}^{(s-p)} \) the result follows. In case (ii) we want to apply Theorem 3.5 to \( |u|/\phi \), but \( |u|/\phi \) is not \( C_2 \) where \( u = 0 \). However, it is easy to see by inspecting the proof of Theorem 3.5 that the sequence \( \{x_k\} \) may be chosen so that \( |u(x_k)| > 0 \) (unless, of course, \( u \equiv 0 \)), and thus Theorem 3.5 still applies. Then our estimate follows in the limit from (3.13) just as before.

It is worth observing that for \( s = 0 \) and \( \mathcal{K} \) uniformly positive definite, one may always take \( \phi = 1 \) in this proposition, and therefore the basic estimate always holds for \( A_g + \mathcal{K} \) on \( A^{-p}(M, \mathcal{F}^p) \). In particular, for any positive constant \( \kappa \) on any \( M \),

\[
A_g + \kappa : A^{-p}_{k+2,s}(M, \mathcal{F}^p) \to A^{-p}_{k,2}(M, \mathcal{F}^p)
\]

is always an isomorphism.

Now consider \( M = B^{n+1} \) with the hyperbolic metric \( h_{ij} = \rho^{-2} \delta_{ij} \), \( \rho(x) = \frac{1}{2}(1 - |x|^2) \). The next lemma proves that for all \( s \in \mathbb{R} \) there exists a weight function on \( B^{n+1} \) of the type required in Proposition 3.8 if \( \inf_{M} K > s(s-n) \).

**Proposition 3.9.** Let \( s \in \mathbb{R} \). There is a function \( \phi \in C^\infty(B^{n+1}) \) such that \( 0 < \rho^{-s} \phi \in C^\infty(B^{n+1}) \) and \( [A_h + s(s-n)] \phi \geq 0 \).
Proof. We compute
\[ \partial_j \rho = -x^j, \]
\[ \Gamma'_{kl} = \rho^{-1} (\delta_{jk} x^l + \delta_{jl} x^k - \delta_{kl} x^j), \]
\[ g^{kl} \rho_{kl} = \rho^2 \delta^{kl} (\partial_k \partial_l \rho - \Gamma'_{kl} \partial_i \rho) = -(n + 1) \rho^2 - (n - 1) \rho |x|^2 \]
\[ = (n - 3) \rho^2 - (n - 1) \rho. \]
Thus if \( \varphi = f \circ \rho \) is a radial function on the ball, with \( f \in C^2(0, \frac{1}{2}] \),
\[ (\Delta_h + s(s-n)) \varphi \]
\[ = -g^{kl} [(f'' \circ \rho) \rho_{kl} + (f' \circ \rho) \rho_{kl}] + s(s-n) f \circ \rho \]
\[ = (2\rho^3 - \rho^2) f'' \circ \rho + ((n-1) \rho - (n-3) \rho^2) f' \circ \rho + s(s-n) f \circ \rho. \]
Let \( F \) be the ordinary differential operator
\[ F(f) = (2t^3 - t^2) f'' + ((n-1) t - (n-3) t^2) f' + s(s-n) f. \]
Our result will be proved if we can show there exists \( f: (0, \frac{1}{2}] \to \mathbb{R} \) such that \( t^{-s} f \in C^\infty [0, \frac{1}{2}] \), \( t^{-s} f > 0 \), and \( F(f) \geq 0 \).
If \( 2s \geq n-1 \) or \( s \leq 0 \), we simply take \( f(t) = t^s \), which satisfies \( F(f) = s(2s-n+1) t^{s+1} \geq 0 \). On the other hand, if \( 0 < 2s < n-1 \), we begin by writing formally
\[ f(t) = t^s \left( \sum_{k \geq 0} a_k t^k \right), \]
and attempting to determine the \( a_k \) so that \( F(f) = 0 \). This equation has a regular singular point at \( t = 0 \). Substituting the power series for \( f \), we get
\[ F(f) = t^s \sum_{k \geq 1} \left( (k+s-1)(2k+2s-n-1) a_{k-1} \right. \]
\[ \left. - k(k+2s-n) a_k \right) t^k. \]
(3.14)
It follows that \( a_0 \) and \( a_{n-2s} \) are arbitrary, and the coefficients between 0 and \( n-2s \) are determined recursively by
\[ a_k = \frac{(k+s-1)(2k+2s-n-1)}{k(k+2s-n)} a_{k-1}, \quad 0 < k < n-2s. \]
(3.15)
Put \( a_0 = 1 \), define \( a_k \) for \( 1 \leq k < n-2s \) by (3.15), and take \( f \) to be a finite sum
\[ f(t) = t^s \sum_{k=0}^{N} a_k t^k. \]
with

\[ N = \left\lfloor \frac{n-1}{2} - s + 1 \right\rfloor < n - 2s. \]

Then for \( 1 \leq k \leq N, \)

\[ (2k + 2s - n - 1) \leq 0, \quad (k + s - 1) \geq 0, \quad (k + 2s - n) < 0, \]

so it follows from (3.15) that \( a_k \geq 0. \) Hence \( t^{-}f(t) > 0 \) for \( t \geq 0, \) and (3.14) gives

\[ F(f) = (N + s)(2N + 2s - n + 1) \, a_N t^{s + N + 1} \geq 0. \quad (3.16) \]

Combining the results of Propositions 3.7, 3.8, and 3.9, we immediately obtain the following theorem, which is the main result of this section.

**Theorem 3.10.** Let \( M = B^{n+1} \) denote the unit ball with the hyperbolic metric \( h. \) Let \( s \in \mathbb{R}, \) \( 0 \leq k, p \in \mathbb{Z}, \) \( 0 < \alpha < 1, \) and let \( \mathcal{H} \subset C^\infty(M, \text{End}(\mathcal{F}^p)) \) be a self-adjoint endomorphism field satisfying

\[ \inf_{x \in M} (\mathcal{H}(x) u, u)_x > s(s - n). \]

Then

\[ A_h + \mathcal{H} : A_{k+2,2}^{s-2}(B^{n+1}, \mathcal{F}^p) \to A_{k,2}^{s-2}(B^{n+1}, \mathcal{F}^p) \]

is an isomorphism.

We will use Theorem 3.10 in the case when \( p = 2 \) and \( \mathcal{H} \) is a constant \( \kappa > s(s - n). \) \( \mathcal{F}^2 \) splits as a direct sum \( \mathcal{F}^2 = \mathcal{S}^2 \oplus \mathcal{A}^2, \) where \( \mathcal{S}^2 \) and \( \mathcal{A}^2 \) are the bundles of symmetric and skew-symmetric 2-tensors, respectively. The operator \( A_h + \kappa \) preserves this splitting, so we deduce that in this case \( \mathcal{F}^2 \) may be replaced by \( \mathcal{S}^2 \) in Theorem 3.10. Similarly, \( \mathcal{S}^2 \) splits into \( \mathcal{B} \oplus \mathcal{S}^2_0, \) where \( \mathcal{B} \) is the bundle of multiples of the metric and \( \mathcal{S}^2_0 \) the bundle of trace-free symmetric 2-tensors, so we have the following corollary.

**Corollary 3.11.** With the notation as in Theorem 3.10, if \( \kappa > s(s - n), \)

\[ A_h + \kappa : A_{k+2,2}^{s-2}(B^{n+1}, \mathcal{S}^2_0) \to A_{k,2}^{s-2}(B^{n+1}, \mathcal{S}^2_0), \]

\[ A_h + \kappa : A_{k+2,2}^{s-2}(B^{n+1}, \mathcal{S}) \to A_{k,2}^{s-2}(B^{n+1}, \mathcal{S}) \]

are isomorphisms.
As we noted in §2, the indicial equation for $A_g + \kappa$ on functions is $s^2 - ns - \kappa = 0$, so the condition $\kappa > s(s-n)$ corresponds to the requirement that $s$ lie between the two characteristic exponents for $A_g + \kappa$ (at least when $\kappa \geq -n^2/4$ so that these characteristic exponents are real). This is the largest range of values of $s$ for which one can hope to invert $A_g + \kappa$ on functions: injectivity fails below the smaller exponent and surjectivity fails above the larger one. A similar remark holds for $A_g + \kappa$ on $\mathcal{D}^2$: if $u$ is a function then $(A_g + \kappa)(ug) = [(A_g + \kappa)u]g$, so since $g \approx \rho^{-2}$, on the space of multiples of $g$ the characteristic exponents of $A_g + \kappa$ are those on functions shifted by 2, and we have proved invertibility of $A_g + \kappa$ between these exponents. However, as we saw in Lemma 2.9, the narrowest range for the characteristic exponents for $A_g + \kappa$ on $\mathcal{D}^2$ is $|s - (n/2 - 2)| < \sqrt{n^2 + 4\kappa + 8}$. Thus one would hope for invertibility of $A_g + \kappa$: $A_{k+2,a}(M, \mathcal{D}^2) \to A_{k,a}(M, \mathcal{D}^2)$ for $s$ in this range, while we have only obtained the range $|s - (n/2 - 2)| < \sqrt{n^2 + 4\kappa + 8}$. Fortunately our weaker result suffices for our applications, but does cause a weakening of the final conclusion when $n=3$ and of the boundary regularity in general (see Theorem 4.1). Similarly, for $A_g + \kappa: A_{k+2,a}(M, \mathcal{F}^1) \to A_{k,a}(M, \mathcal{F}^1)$ one expects invertibility in the range $|s - (n/2 - 2)| < \sqrt{n^2 + 4\kappa + 4}$, while we have only obtained $|s - (n/2 - 2)| < \sqrt{n^2 + 4\kappa}$.

We conclude this section by deriving from Propositions 3.8 and 3.9 some conclusions regarding the spectrum of $A_g$. Actually Proposition 3.8 was itself inspired by work of Cheng and Yau [CY1] on estimates for the lowest eigenvalue of $A_g$ on functions on a complete Riemannian manifold. We first extend the Cheng–Yau results to tensors.

**Proposition 3.12.** Let $\Omega$ be the interior of a compact Riemannian manifold with boundary $(\bar{\Omega}, g)$ and $\delta > 0$. Suppose there exists $\varphi \in C^2(\Omega) \cap C_0(\bar{\Omega})$, $\varphi > 0$ in $\bar{\Omega}$, such that $A_g \varphi \geq \delta \varphi$. Then for all $0 < p \in \mathbb{Z}$, the first Dirichlet eigenvalue of $A_g$ on $\mathcal{D}^p$ satisfies $\lambda_1 \geq \delta$.

**Proof.** Let $0 \neq u \in C^2(\bar{\Omega}, \mathcal{D}^p)$ satisfy $A_g u = \lambda_1 u$, $u|_{\partial \Omega} = 0$. As in the proof of Proposition 3.8, at a point where $|u|/\varphi$ attains its maximum we have

$$( -\lambda_1 + \delta ) \frac{|u|}{\varphi} \leq \left( A_g - V - \lambda_1 + \frac{A_g \varphi}{\varphi} \right) \frac{|u|}{\varphi} \leq \frac{( (A_g - \lambda_1) u, u )_g}{\varphi |u|} = 0,$$

so $\lambda_1 \geq \delta$ as desired. $lacksquare$

It is customary to define the first eigenvalue of $A_g$ on $p$-tensors on a complete noncompact Riemannian manifold $M$ to be the infimum over smoothly bounded subdomains $\Omega \subset M$ of the lowest Dirichlet eigenvalue of
\( A_g \) on \( \mathcal{F}^p \). We then have the following immediate corollary of Proposition 3.12.

**Corollary 3.13.** Let \( M \) be a complete noncompact Riemannian manifold and suppose there exists a strictly positive function \( \varphi \in C_2(M) \) and \( 0 < \delta \in \mathbb{R} \) such that \( A_g \varphi \geq \delta \varphi \). Then for all \( 0 \leq p \in \mathbb{Z} \), the first eigenvalue of \( A_g \) on \( \mathcal{F}^p \) satisfies \( \lambda_1 \geq \delta \).

Our final proposition is a stability result for the lowest eigenvalue of asymptotically hyperbolic perturbations of the hyperbolic metric.

**Proposition 3.14.** Let \( M = B^{n+1} \), \( \rho = \frac{1}{2}(1 - |x|^2) \), and let \( h \) be the hyperbolic metric. There exists a constant \( \varepsilon > 0 \) such that for any asymptotically hyperbolic metric \( g \) with \( \| \tilde{g} - h \|_{1;M} < \varepsilon \) and any \( 0 \leq p \in \mathbb{Z} \), the first eigenvalue of \( A_g \) on \( p \)-tensors satisfies \( \lambda_1 \geq n^2/4 \).

**Proof.** Let \( \varphi = \rho^{n/2} \). The proof of Proposition 3.9 shows that

\[
\left( A_h - \frac{n^2}{4} \right) \varphi = \frac{n}{2} \rho^{n/2 + 1} \geq 0.
\]

We will show that \( \rho^{-n/2 - 1}(A_g - n^2/4) \varphi > 0 \) on \( \bar{M} \) for all asymptotically hyperbolic \( g \) for which \( \bar{g} \) is \( C_1 \)-close to \( h \), whereupon the result follows from Corollary 3.13.

Computing as in Proposition 2.7 and Corollary 2.8 gives

\[
(A_g + \kappa)(f(\rho)) = \left[ -\rho^2 f''(\rho) + (n-1) \rho f'(\rho) \right] |dp|_{\bar{g}}^2 + \rho^2 (A_g \rho) f'(\rho) + \kappa f(\rho).
\]

Hence, setting \( \kappa = -n^2/4 \) and \( f(\rho) = \rho^{n/2} \), we obtain

\[
\rho^{-n/2 - 1} \left( A_g - \frac{n^2}{4} \right) \varphi = \frac{n}{2} A_g \rho + \frac{n^2}{4} \left( |dp|_{\bar{g}}^2 - 1 \right).
\]

As we saw above, this expression equals \( n/2 \) when \( g = h \), and clearly varies continuously with \( C_1 \) variations of \( \bar{g} \) so long as \( |dp|_{\bar{g}}^2 = 1 \) on \( bM \). In fact, it is clear that explicit bounds on the size of the \( C_1 \) variation of \( \bar{g} \) can be given to ensure that \( \rho^{-n/2 - 1}(A_g - n^2/4) \varphi > 0 \).

It would be interesting to better understand the behavior of \( \lambda_1 \) for general asymptotically hyperbolic metrics. Observe that if an asymptotically hyperbolic \( g \) is too far away from \( h \) the conclusion of Proposition 3.14 can fail, since we can choose \( g \) arbitrarily on a compact subdomain and thereby obtain \( \lambda_1 \) as close to 0 as we desire.
4. PROOF OF THEOREM A

In this section we will use the linear isomorphism results of the preceding section together with the inverse function theorem to solve the regularized Einstein equation (1.3) and thus prove Theorem A of the introduction.

Let $B^{n+1}$ be the ball and $h$ the hyperbolic metric, with $\rho = \frac{1}{2}(1 - |x|^2)$ as in Section 1, and let $Q$ denote the nonlinear elliptic operator defined by (2.7). Write $L = D_{(h,h)}$, and consider formula (2.15) for $L$. Since $t = g = h$, the operators $R, G,$ and $D$ all vanish identically. Because $h$ has constant curvature $-1$, the operator $R$ defined by (2.5) reduces to

$$R(r) = (Tr_h r) h - (n+1) r.$$  

Thus, writing $r = uh + r_0$ where $r_0$ is trace-free, from Proposition 2.4 we obtain

$$Lr = \frac{1}{2}(\Delta h + 2n)(uh) + (\Delta_h - 2) r_0.$$  

Therefore the asymptotic formula (2.27) for $L$ is exact in this case.

Theorem A is an immediate consequence of the following result.

**THEOREM 4.1.** Let $M = B^{n+1}$ be the ball and $h$ the hyperbolic metric. Suppose $k \geq 2$ if $n \geq 4$, and $k = 3$ if $n = 3$, and let $0 < \alpha < 1$.

(a) There exists $\varepsilon > 0$ such that if $\tilde{g}$ is any smooth metric on $bM$ with $\| \tilde{g} - h \|_{k,\alpha} < \varepsilon$, there is a $C^\infty$ Einstein metric $g$ on $M$ such that $\rho^2 g$ is continuous on $\partial M$ and $\rho^2 g |_{\partial M} = \tilde{g}$.

(b) If $n = 3$ and $0 < \gamma < 1$, $g$ may be chosen so that $\rho^2 g \in C_{1,\gamma}(\bar{M}, \mathcal{S}^2)$. If $n \geq 4$, there is a number $\gamma_n$ with $0 < \gamma_n < 1$ so that if $0 < \gamma < \gamma_n$ and $k + \alpha \geq 2n - 2 + \gamma$, then $g$ may be chosen so that $\rho^2 g \in C_{n-1,\gamma}(\bar{M}, \mathcal{S}^2)$. In fact, one may take $\gamma_n = 1 - \frac{1}{2}(n - \sqrt{n^2 - 8})$.

**Remark.** When $n \geq 4$, our proof also gives intermediate boundary regularity of $\rho^2 g$ in terms of $k, \alpha$.

**Proof.** Set $m = \min(k - 2, n - 1)$. For any $C_{k,\alpha}$ boundary metric $\tilde{g}$, define $T(\tilde{g}) = \rho^{-2}E(\tilde{g})$, and let $S(\tilde{g})$ denote the metric given by Theorem 2.11 which satisfies $Q(S(\tilde{g}), T(\tilde{g})) = O(\rho^{m-1})$, where the background metric $h$ used in defining $S$ and $T$ is the hyperbolic metric. We want to choose $s \in \mathbb{R}$ as large as possible so that

$$L : A^{s-2}_{k-m,\alpha}(M, \mathcal{S}^2) \rightarrow A^{s-2}_{k-m-2,\alpha}(M, \mathcal{S}^2)$$  

is an isomorphism. By (4.1) and Corollary 3.11, this will be the case if $s(s-n) < -2$. We also want to have $Q(S(\tilde{g}), T(\tilde{g})) \in A^{s-2}_{k-m-2,\alpha}(M, \mathcal{S}^2)$, so in addition we require $s - 2 \leq m - 1$. Thus we choose $s$ as follows:
\[ 1 < s < 2 \quad \text{if } n = 3, \]
\[ s = m + 1 \quad \text{if } n \geq 4 \quad \text{and} \quad m < n - 1, \]
\[ n - 1 < s < n - 1 + \gamma_n \quad \text{if } n \geq 4 \quad \text{and} \quad m = n - 1, \]

where \( \gamma_n \) is as above. Then (4.2) is an isomorphism and \( s - 2 \leq m - 1 \). (It is at this point in the argument that our results are weakened by the application of the presumably nonsharp linear isomorphism theorem; see the remarks after Corollary 3.11. With a better isomorphism theorem we could allow \( k = 2 \) when \( n = 3 \), and for large \( k \) we could choose any \( s < n \).)

Define an open subset \( \mathcal{B} \subset C_{k, \alpha}(bM, \mathcal{S}^2(bM)) \times A_{k-m, \alpha}^{-2}(M, \mathcal{S}^2) \) by

\[
\mathcal{B} = \{ (\hat{g}, r) : \hat{g} \text{ is positive definite on } bM, \ S(\hat{g}) \text{ is defined}, \]
\[ \quad \text{and} \ S(\hat{g}) + r \text{ is positive definite on } M \}, \]

and a map \( \varphi : \mathcal{B} \to C_{k, \alpha}(bM, \mathcal{S}^2(bM)) \times A_{k-m-\gamma, \alpha}^{-2}(M, \mathcal{S}^2) \) by

\[
\varphi(\hat{g}, r) = (\hat{g}, Q(S(\hat{g}) + r, T(\hat{g}))). \]

As we intend to apply the inverse function theorem to \( \varphi \), we must check that it is a smooth map of Banach spaces. Since \( A_{k, \alpha}^i \subset A_{k, \alpha}^2(M) \) continuously by Proposition 3.3(12), it follows from Theorem 2.11 that \( S, \ T : C_{k, \alpha}(bM, \mathcal{S}^2(bM)) \to A_{k-m, \alpha}^{-2}(M, \mathcal{S}^2) \) are smooth and \( \hat{g} \mapsto Q(S(\hat{g}), T(\hat{g})) \) is smooth from \( C_{k, \alpha}(bM, \mathcal{S}^2(bM)) \) to \( A_{k-m-1, \alpha}^{-2}(M, \mathcal{S}^2) \), hence to \( A_{k-m-2, \alpha}^{-2}(M, \mathcal{S}^2) \) by Proposition 3.3(6). By Taylor's formula,

\[
Q(g + r, t) - Q(g, t) = \int_0^1 D_1 Q_{(g + tr, t)}(r) \, d\lambda. \]

From formula (2.15), if \( g, t \) and \( r \) are all in \( A_{k-m, \alpha}^{-2}(M, \mathcal{S}^2) \), then \( D_1 Q_{(g + tr, t)} \) is a (nonsmooth) uniformly degenerate linear operator whose coefficients are in \( A_{k-m-2, \alpha}^0(M) \), depending smoothly on \( g, t, r, \) and \( \lambda \), and thus \( D_1 Q_{(g + tr, t)} : A_{k-m, \alpha}^{-2}(M, \mathcal{S}^2) \to A_{k-m-2, \alpha}^{-2}(M, \mathcal{S}^2) \) continuously. It follows from Proposition 3.3 that \( \varphi \) is smooth.

As noted in Section 2, our recipe for extending boundary metrics to interior metrics gives \( E(\hat{h}) = \hat{h} \), so \( S(\hat{h}) = T(\hat{h}) = h \). Therefore \( \varphi(\hat{h}, 0) = (\hat{h}, 0) \), and the linearization of \( \varphi \) about \( (\hat{h}, 0) \),

\[
D_2 \varphi(\hat{h}, 0) : C_{k, \alpha}(bM, \mathcal{S}^2(bM)) \times A_{k-m, \alpha}^{-2}(M, \mathcal{S}^2) \rightarrow C_{k, \alpha}(bM, \mathcal{S}^2(bM)) \times A_{k-m-2, \alpha}^{-2}(M, \mathcal{S}^2),
\]
is given by
\[ D\mathcal{L}_{(\hat{h},0)}(\hat{\mathcal{q}}, r) = (\hat{\mathcal{q}}, D_1 Q_{(\hat{h}, h)}(DS_{\hat{h}} \mathcal{q} + r) + D_2 Q_{(\hat{h}, h)}(DT_{\hat{h}} \mathcal{q})) = (\hat{\mathcal{q}}, Lr + K\mathcal{q}), \]
where
\[ K\mathcal{q} = D_1 Q_{(\hat{h}, h)}(DS_{\hat{h}} \mathcal{q}) + D_2 Q_{(\hat{h}, h)}(DT_{\hat{h}} \mathcal{q}). \]

Since \( Q(S(\hat{g}), T(\hat{g})) \in A^{s-2}_{k-m-2,2}(M, \mathcal{S}^2) \) for every \( \hat{g} \), it follows by differentiation that \( K\mathcal{q} \in A^{s-2}_{k-m-2,2}(M, \mathcal{S}^2) \). Therefore the unique solution \((\hat{q}, r)\) to
\[ D\mathcal{L}_{(\hat{h},0)}(\hat{\mathcal{q}}, r) = (\hat{\mathcal{w}}, v) \]
is given by
\[ \hat{\mathcal{q}} = \hat{\mathcal{w}}, \quad r = L^{-1}(v - K\hat{\mathcal{w}}). \]

The mapping \((\hat{w}, v) \mapsto (\hat{q}, r)\) is bounded from \( C_{s,2}(bM, \mathcal{S}^*(bM)) \times A^{s-2}_{k-m-2,2}(M, \mathcal{S}^2) \) to \( C_{s,2}(bM, \mathcal{S}^2(bM)) \times A^{s-2}_{k-m,2}(M, \mathcal{S}^2) \), so by the inverse function theorem \( \mathcal{L} \) is locally invertible in some neighborhood of \((\hat{h}, 0)\). Thus if \( \hat{g} \) is sufficiently close to \( \hat{h} \), we can solve the equation
\[ \mathcal{L}(\hat{g}, r) = (\hat{h}, 0). \]

Clearly the solution must satisfy \( \hat{g}_1 = \hat{g} \) and \( Q(S(\hat{g}) + r, T(\hat{g})) = 0 \). Therefore \( g = S(\hat{g}) + r \) satisfies \( Q(g, T(\hat{g})) = 0 \).

Since \( g \) is smooth, it is clear that \( \rho^2 S(\hat{g}) \) and \( \rho^2 T(\hat{g}) \) are smooth in \( \hat{M} \). Thus \( g = S(\hat{g}) + r \in \rho^2 C^\infty(\hat{M}, \mathcal{S}^2) + \rho^{-2} A^{s-2}_{k-m-2,2}(M, \mathcal{S}^2) \); in particular \( g \in C_{s,2}(\hat{M}, \mathcal{S}^2) \). Since the equation \( Q(g, r) = 0 \) is a smooth nonlinear elliptic system for \( g \), it follows that \( g \in C^\infty(M, \mathcal{S}^2) \) by standard elliptic regularity [Mo].

As for boundary regularity, we always have \( s \geq 1 \) so that \( \hat{g} = \rho^2 g \in C^\infty(\hat{M}, \mathcal{S}^2) + A^1_{s,2}(M, \mathcal{S}^2) \). Thus \( \hat{g} \) always extends continuously to \( \hat{M} \); in fact, using Proposition 3.3 it follows that \( \partial_k \hat{g}_{ij} \) and \( \rho \partial_k \partial_i \hat{g}_{ij} \) are bounded, so \( \hat{g} \) is Lipschitz in \( \hat{M} \). If \( n = 3 \) and \( 0 < \gamma < 1 \), take \( s = 1 + \gamma \); then \( A^{s-2}_{k-m,2}(M, \mathcal{S}^2) \subset C_{s,2}(\hat{M}, \mathcal{S}^2) \) by Proposition 3.3(5) and 3.3(7), so \( \hat{g} \in C_{s,2}(\hat{M}, \mathcal{S}^2) \). And if \( n > 4 \), \( 0 < \gamma < \gamma_n \), and \( k + \alpha \geq 2n - 2 + \gamma \), take \( s = n - 1 + \gamma \); then \( m - n + 1 \) and this time Proposition 3.3(5) and 3.3(7) give \( A^{s-2}_{k-m,2}(M, \mathcal{S}^2) \subset C_{s,2}(\hat{M}, \mathcal{S}^2) \) as desired.

It remains only to show that \( g \) in Einstein. Shrinking the neighborhood of \( \hat{h} \) if necessary, \( \hat{g} \) may be made as close to \( \hat{h} \) in the \( A^0_{k-m,2}(M, \mathcal{S}^2) \) norm as desired. But Lemma 2.1 shows that
\[ \text{Ric}(g) = \rho^{-2}(\mathcal{S}^0(\hat{g}) + \rho \mathcal{S}^1(\hat{g}) + \rho^2 \mathcal{S}^2(\hat{g})), \]
so \( g \) will have strictly negative Ricci curvature. Thus Lemma 2.2 guarantees that \( g \) is Einstein.
5. Ricci-Flat Lorentz Metrics

In this section we show how to obtain a Lorentz metric \( \tilde{g} \) which solves (1.2) on the open cone \( S^*_+ = \{ \eta > |\xi| \} \) in \( \mathbb{R}^{n+2} \) from a sufficiently regular solution \( g \) to (1.1).

It is useful to blow up the origin by introducing coordinates \( x' = \xi' / \eta \) and \( t = \eta \), thus identifying \( S^*_+ \) with \( B^{n+1} \times \mathbb{R}_+ \). Then \( \tilde{g} \) will be homogeneous of degree 2 in \( t \) and will restrict to \( t^2 \tilde{g} \) on \( T(S^* \times \mathbb{R}_+) \). (We suppress \( \pi^* \) from here on.) Observe that in these coordinates the Minkowski metric \( h = \sum (dx')^2 - dt^2 \) becomes

\[
\tilde{h} = t^2 \sum_{i=1}^{n+1} (dx'^i)^2 - t \, dt \, dt' - \eta^2,
\]

where \( r = 1 - |x|^2 \).

The key observation is the following:

**Proposition 5.1.** Let \( g \) be a metric on \( B^{n+1} \). Then the Lorentz metric \( \tilde{g}_0 = s^2 g - ds^2 \) on \( B^{n+1} \times \mathbb{R}_+ \) satisfies \( \text{Ric}(\tilde{g}_0) = \text{Ric}(g) + ng \).

**Proof.** Set \( s = e^{-\gamma} \); then \( \tilde{g}_0 = e^{2\gamma} [g - dy^2] \) is a conformal multiple of a product metric. Under a conformal change \( \tilde{g}_0 = e^{2\gamma} g_1 \) in \( (n+2) \) dimensions, the Ricci tensor transforms by

\[
\text{Ric}(\tilde{g}_0) = \text{Ric}(g_1) - n(\nabla^2 y - dy^2) + (dy - n |dy|^2) g_1,
\]

where the operations on the right are with respect to \( g_1 \). For \( g_1 = g - dy^2 \) we have \( \text{Ric}(g_1) = \text{Ric}(g) \), \( \nabla^2 y = 0 \), \( \Delta y = 0 \), and \( |dy|^2 = -1 \). It follows immediately that \( \text{Ric}(\tilde{g}_0) = \text{Ric}(g) + ng \). \( \square \)

Of course \( \tilde{g}_0 = s^2 g - ds^2 \) is singular near \( S^n \times \mathbb{R}_+ \). But by pulling \( \tilde{g}_0 \) back by a homogeneous diffeomorphism of \( B^{n+1} \times \mathbb{R}_+ \) it is possible to obtain a metric which extends continuously to \( B^{n+1} \times \mathbb{R}_+ \). This diffeomorphism will be constructed using a special defining function for \( S^n \).

**Lemma 5.2.** Let \( g \) be an asymptotically hyperbolic metric on \( B^{n+1} \). Then there is a defining function \( \rho \) for \( S^n \) satisfying \( |d\rho|^2_{\tilde{g}} = 1 \) near \( S^n \), where \( \tilde{g} = \rho^2 g \).

**Proof.** Choose a fixed smooth defining function \( \rho_0 \) for \( S^n \), and set \( \tilde{g}_0 = \rho_0^2 g_0 \). Since \( g \) is asymptotically hyperbolic, \( |d\rho_0|^2_{\tilde{g}_0} = 1 \) on \( S^n \). If we set \( \rho = \rho_0 e^\omega \), then \( \tilde{g} = e^{2\omega} \tilde{g}_0 \) and \( d\rho = e^\omega (d\rho_0 + \rho_0 du) \), so

\[
|d\rho|^2_{\tilde{g}} = |d\rho_0 + \rho_0 du|^2_{\tilde{g}_0} = |d\rho_0|^2_{\tilde{g}_0} + 2\rho_0 (\nabla \rho_0)(u) + \rho_0^2 |du|^2_{\tilde{g}_0}.
\]
Thus the condition \( |d\rho|^2_{\tilde{g}} = 1 \) is equivalent to

\[
2(\text{grad}_{g_0}\rho_0)(u) + \rho_0 |du|_{g_0}^2 = \frac{1 - |d\rho_0|^2_{\tilde{g}}}{\rho_0}.
\]

Since the vector field \( \text{grad}_{g_0}\rho_0 \) is transverse to \( S^n \), this is a noncharacteristic first-order PDE for \( u \). It follows that there is a solution near \( S^n \); in fact \( u|_{S^n} \) can be arbitrarily prescribed.

The defining function of Lemma 5.2 determines for some \( \varepsilon > 0 \) an identification of \( S^n \times [0, \varepsilon) \) with a neighborhood of \( S^n \) in \( B^{n+1}_\varepsilon \); \((p, \lambda) \in S^n \times [0, \varepsilon) \) corresponds to the point obtained by following the integral curve of \( \text{grad}_g \rho \) emanating from \( p \) for \( \lambda \) units of time. Since \( |d\rho|^2_{\tilde{g}} = 1 \), the \( \lambda \)-coordinate is just \( \rho \), and \( \text{grad}_g \rho \) is orthogonal to the slices \( S^n \times \{\lambda\} \). Hence, identifying \( \lambda \) with \( \rho \), on \( S^n \times [0, \varepsilon) \) the metric \( \tilde{g} \) takes the form \( \tilde{g} = k_\rho + d\rho^2 \) for a 1-parameter family \( k_\rho \) of metrics on \( S^n \), and \( g = \rho^{-2}(k_\rho + d\rho^2) \). Since \( g \) has \( \{\tilde{g}\} \) as conformal infinity, it follows that \( k_\rho \) extends continuously to \( \rho = 0 \) and \( k_0 \in \{\tilde{g}\} \).

Consider now \( \tilde{g}_0 = s^2 \tilde{g} - ds^2 \). Under the change of variables \( s = t\rho \), \( \tilde{g}_0 \) becomes

\[
\tilde{g}_0 = t^2k_\rho - 2\rho t dt d\rho - \rho^2 dt^2,
\]

which is homogeneous of degree 2 in \( t \). In these coordinates \( \tilde{g}_0 \) at least extends continuously to \( B^{n+1}_\varepsilon \times \mathbb{R}_+ \), but degenerates on \( S^n \times \mathbb{R}_+ \). However, if we set \( \rho^2 = r \), then \( \tilde{g}_0 = t^2k\sqrt{r} - t dt dr - r dt^2 \), and the degeneracy at \( \rho = 0 \) has been removed. Thus choose a diffeomorphism \( \varphi : B^{n+1}_r \rightarrow B^{n+1}_1 \) which, under the identification of a neighborhood of the boundary with \( S^n \times [0, \varepsilon) \) of the previous paragraph, takes the form \( \varphi(p, r) = (p, \rho) \) with \( \rho^2 = r \). Then define a diffeomorphism \( \Phi : B^{n+1}_1 \times \mathbb{R}_+ \rightarrow B^{n+1}_1 \times \mathbb{R}_+ \) by

\[
\Phi(x, t) = (\varphi(x), s), \quad \text{where } s = t\rho(\varphi(x)).
\]

It follows that \( \Phi^*\tilde{g}_0 \) extends continuously to a homogeneous non-degenerate Ricci-flat metric on \( B^{n+1}_1 \times \mathbb{R}_+ \) whose restriction to \( T(S^n \times \mathbb{R}_+) \) is \( t^2k_0 \). As \( k_0 \) is conformal to \( \tilde{g} \), we can finally pull back \( \Phi^*\tilde{g}_0 \) by a homogeneous diffeomorphism of \( B^{n+1}_1 \times \mathbb{R}_+ \) which smoothly rescales the \( \mathbb{R}_+ \) fibers to obtain \( \tilde{g} \).

**Remark.** At first glance it appears that the change of variables \( \rho = \sqrt{r} \) in the above argument destroys the differentiability of \( \tilde{g} \) at \( bB^{n+1}_1 \times \mathbb{R}_+ \). However, this is not the case. By explicitly computing the Ricci tensor of a metric of the form \( g = \rho^{-2}(k_\rho + d\rho^2) \), one can show that if \( \text{Ric}(g) = -ng \), then the Taylor expansion of \( k_\rho \) about \( \rho = 0 \) can involve only even powers of \( \rho \) through the \( \rho^n \) terms. Thus if \( k_\rho \) is sufficiently regular, \( \tilde{g} \) will be differentiable to any order less than \( n/2 \).
REFERENCES


