Rings with Periodic Symmetric or Skew Elements

I. N. HERSTEIN

Department of Mathematics, University of Chicago, Chicago, Illinois 60637 and Weizmann Institute of Science, Rehovot, Israel

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Recently there have been extensions to the case of rings with involution of the classical theorem of Jacobson [6, 1] which asserts that if every element \( x \) in a ring \( R \) is periodic, in the sense that \( x^{n(x)} = x \) for some \( n(x) > 1 \), then \( R \) must be commutative.

Now, as is easy to see, if \( F \) is a subfield of the algebraic closure of a finite field of characteristic not 2, then the ring \( R \) of all \( 2 \times 2 \) matrices over \( F \) relative to the symplectic involution defined via \( (a, b)^* = (b, -a) \) satisfies the relation \( s^{n(s)} = s \), \( n(s) > 1 \), for every \( s \in R \) such that \( s^* = s \). With \( F \) and \( R \) as above but using the involution given by the transpose we see that \( k^{n(k)} = k \), \( n(k) > 1 \), for every \( k \in R \) such that \( k^* = -k \).

Thus, imposing the Jacobson condition of periodicity on all the symmetric elements of a ring with involution, or on all the skew elements of such a ring, is clearly not sufficient to force the commutativity of that ring. However, there is a great deal that one can say such rings. For division rings with involution Herstein and Montgomery [4] showed that when the symmetric elements, or the skew elements, satisfied Jacobson’s condition the division ring was indeed commutative. Montgomery [9, 10] then showed that the structure of a semiprime ring with periodic symmetric elements is that of a subdirect sum of fields (algebraic over finite fields) and of \( 2 \times 2 \) matrix rings over such fields. The case of periodic skew elements, however, was left open.

In this paper we show that Montgomery’s results (except for some statements of finiteness of fields) also hold for the structure of rings whose skew elements are periodic. In doing this we give, at the same time and via the same proof, a new proof of Montgomery’s theorems. While her proof made use of some results in Jordan algebras due to Osborn, we shall have no need to use or cite these results.

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An interesting consequence of the results obtained for the skew elements relates to a question raised by Jacobson [7] for restricted Lie algebras. He asked: let $L$ be a restricted Lie algebra in which applying the restriction map a sufficient number of times on an element brings you back to the element (i.e., $a^{p^{n(a)}} = a$ for every $a \in L$); must $L$ then be abelian? In case $L$ is the set of all the skew elements in a ring with involution the answer to Jacobson’s question turns out to be yes. More precisely, let $K = \{x \in R \mid x^* = -x\}$ for $R$ a ring with involution $*$; suppose that $k^{n(k)} = k$, $n(k) > 1$, for every $k \in K$, then $ab = ba$ for all $a, b \in K$.

In all that follows $R$ will be an associative ring with involution $*$, and $S = \{x \in R \mid x^* = x\}$, $K = \{x \in R \mid x^* = -x\}$ will denote the sets of symmetric and skew elements of $R$ respectively.

We begin with

**Lemma 1.** Let $F$ be a field of characteristic $p \neq 0$ which is algebraic over the prime field. Suppose that in $F_n$, the ring of $n \times n$ matrices over $F$, there is an involution $*$ such that $xx* \neq 0$ if $x \neq 0 \in F_n$. Then $n = 1$ or $2$. Finally, if $n = 2$ then the characteristic of $F$ cannot be 2.

**Proof.** The lemma is pretty well known linear algebra, but we shall do it explicitly here.

The $*$ on $F_n$ induces an automorphisms $ -$ on $F$. Let $F_0 = \{\alpha \in F \mid \alpha = \bar{\alpha}\}$.

Since $xx* \neq 0$ if $x \neq 0$ it is easy to see that we can pick matrix units $e_{ij}$ in such a way that $e_{ii}^* = e_{ii}$ for $i = 1, 2, \ldots, n$. Since $e_{ij} = e_{ij}e_{ij}e_{ij}$, applying $*$ and using $e_{ii}^* = e_{ii}$, $e_{ij}^* = e_{ij}$ we obtain that $e_{ii}^* = \alpha_{ii}e_{ii}$ for $i \neq j$, where $\alpha_{ij} \in F$. It is easy to verify that $\bar{\alpha}_{ij} = \alpha_{ij}$ and that $\alpha_{ij}^* = \alpha_{ij}$.

Suppose that $n \geq 3$. Since we are working in a finite field, if $\beta = \alpha_{12}$ and $\gamma = \alpha_{3}$ then in the field, $F_1$, generated by $\beta$ and $\gamma$ over the prime field we can find $a_1, a_2, a_3$, not all 0, such that $a_1^2 + \beta a_2^2 + \gamma a_3^2 = 0$. Since $F_1 \subset F_0$, $\bar{a}_i = a_i$. Consider the matrix

$$0 \neq x = \begin{pmatrix} a_1 & a_2 & a_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \text{ in } F_n,$$

then

$$xx^* = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ \beta a_2 & 0 & \cdots & 0 \\ \gamma a_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \text{ and } xx^* = (a_1^2 + \beta a_2^2 + \gamma a_3^2) e_{11} = 0.$$
With this contradiction—\( xx^* = 0 \) with \( x \neq 0 \)—we conclude that \( n = 1 \) or \( 2 \).

In fact, we can even say that if the characteristic of \( F \) is 2 then \( n = 2 \) is not possible. For then \((x_0, x_1)(x_0, x_2) = (x_1^2 + ax_2, x_0)\) \( e_{11} = 0 \) for appropriate choice of \( x_1, x_2 \in F_0 \), since every element in a finite field of characteristic 2 is a square.

We continue with

**Lemma 2.** Let \( R \) be a finite ring with involution * such that \( xx^* \neq 0 \) if \( x \neq 0 \) in \( R \). Then \( R \) is the direct sum of finite fields and rings of all \( 2 \times 2 \) matrices over finite fields, (of characteristic not 2 in the latter case).

**Proof.** Since \( xx^* \neq 0 \) for \( x \neq 0 \), \( R \) cannot have any nonzero symmetric elements which are nilpotent. Since the radical \( N \) of \( R \) must be nilpotent—\( R \), after all, is finite—and if \( a \in N \) then \( aa^* \) must be a nilpotent symmetric element, we get that \( N = 0 \), that is, that \( R \) is semisimple.

Thus \( R \) is the direct sum of finite fields and total matrix rings over finite fields. If \( R \) is commutative then no matrix ring can be present in this direct sum, and the theorem is true in that situation. Suppose then that \( R \) is not commutative. We intend to show that these total matrix rings must all be rings of \( 2 \times 2 \) matrices.

Now \( R = R_1 \oplus \cdots \oplus R_k \) where the \( R_i \) are simple rings; in fact they are the minimal ideals of \( R \). From \( xx^* \neq 0 \) for \( x \neq 0 \) we have that \( R_iR_i^* \neq 0 \). The minimality of \( R_i \) then assures us that \( R_i^* = R_i \). In short, the simple components of \( R \) are invariant with respect to *.

If some \( R_i = F_n \), the ring of \( n \times n \) matrices over a finite field \( F \), with \( n > 1 \), then \( F_n \) inherits from \( R \) an involution * such that \( xx^* \neq 0 \) if \( x \neq 0 \). By Lemma 1 we then know that \( n = 2 \) and that \( F \) is of characteristic not 2.

The theorems which we shall prove are of most interest when stated for the set of symmetric elements \( S \) or the set of skew elements \( K \). However, the same proofs yield the results when we impose the conditions on certain, select subsets of \( S \) and \( K \), rather than on all of \( S \) or on all of \( K \). So we introduce these subsets now.

**Definition.** The set of traces \( T \) in \( R \) is defined by \( T = \{ x + x^* \mid x \in R \} \).

Clearly \( T \) is a subset of \( S \).

**Definition.** The set of skew traces \( K_0 \) in \( R \) is defined by \( K_0 = \{ x - x^* \mid x \in R \} \).

Clearly \( K_0 \) is a subset of \( K \).

In order to minimize the writing out of a long hypothesis several times we introduce formal names for them, and refer to them as such.
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Condition I: \( s^{n(s)} = s, \ n(s) > 1, \) for all \( s \in S. \)

Condition I': \( t^{n(t)} = t, \ n(t) > 1, \) for all \( t \in T. \)

Condition II: \( k^{n(k)} = k, \ n(k) > 1, \) for all \( k \in K. \)

Condition II': \( k^{n(k)} = k, \ n(k) > 1, \) for all \( k \in K_0. \)

We proceed to

**Lemma 3.** Let \( R \) be a semiprime ring with involution in which all the symmetric idempotents are central. Suppose that \( R \) satisfies Condition I' or that \( R \) satisfies Condition II'. Then \( R \) is a subdirect sum of commutative rings and orders in \( 2 \times 2 \) matrices.

**Proof.** Let \( P \) be a prime ideal of \( R. \) If \( P^* \neq P \) then in \( R/P, \) the non-zero ideal \( P^* = (P + P^*/P) \) is commutative. For, since every element \( x \in P^* \) is an image of an element \( x \pm x^* \) where \( x^* \in P \) so either in Condition I' or II', \( x^{n(x)} = x \) hence, by Jacobson's theorem, \( P^* \) is commutative. Since \( R \) is a prime ring with a nonzero commutative ideal \( P^* \), \( R \) itself is commutative.

On the other hand, if \( P^* = P \) then in \( R/P, \) if \( 0 \neq \tilde{x} \in \tilde{T} \) or \( \tilde{x} \in \tilde{K}_0, \) respectively it is an image of an element \( x \) in \( T \) or \( K_0 \) respectively. But \( x^{n(x)} = x, \) so that \( x^{n(x)} - 1 \) is a symmetric idempotent; by hypothesis, \( e = x^{n(x)} - 1 \) is central. Thus in \( R, \) \( x^{n(x)} - 1 \) is a central idempotent. But \( R \) is prime, hence \( e = 1, \) since \( e \neq 0. \) Thus every nonzero element in \( T, \) or \( K_0, \) respectively, is invertible. By a result of Herstein and Montgomery [5], \( R \) is commutative or the \( 2 \times 2 \) matrices over a field. Since \( R \) is a subdirect sum of the \( R/P \)'s, our lemma is proved.

Theorem 1, which we are about to prove, is the crucial step in the proof of the theorems that are to ensue. In its proof we make use of the following theorem proved recently by Montgomery [11]: let \( R \) be a ring and suppose that \( a \in R \) is such that \( a^n \in Z, \) where both \( a \) and \( n \) are invertible in \( R. \) If \( C_R(a) \) satisfies a polynomial identity, then \( R \) itself must satisfy a polynomial identity.

We now prove

**Theorem 1.** Let \( R \) be a primitive ring with involution \( * \) satisfying Condition I' or satisfying Condition II'. Then \( R \) is a field or the ring of all \( 2 \times 2 \) matrices over a field. Furthermore, if Condition II' holds, then any two elements of \( K_0 \) commute.

**Proof.** In case Condition I' holds, there are no nonzero nilpotent traces; in case Condition II' holds, there are no nonzero nilpotent skew traces. In either case it follows from Theorem 4 and its Corollary in [2] that either \( R \) is an order in the ring of \( 2 \times 2 \) matrices over a field, or else \( xx^* = 0 \) implies
that \( x = 0 \) in \( R \). In the first case \( R \), as a subring of \( 2 \times 2 \) matrices over a field, must satisfy a polynomial identity, those of the \( 2 \times 2 \) matrices. By Kaplansky's theorem, \( R \) must then be a simple algebra of dimension 4 over its center. If \( R \) is a division ring, then by the results of [4], \( R \) must be a field. Hence we get, in the first case, that \( R \) must indeed be the ring of \( 2 \times 2 \) matrices over a field, one of the desired conclusions. We leave to the very end of the proof the verification that in the ring of \( 2 \times 2 \) matrices over a field any two skew traces must commute if we assume that Condition \( II' \) holds for the involution in that ring.

Thus we have progressed slightly, namely, in order to prove the theorem we may assume that \( xx^* = 0 \) forces \( x = 0 \) in \( R \).

We also assert that we may assume that \( K_0 \neq 0 \) and that \( T \neq 0 \). If \( K_0 = 0 \) then \( x^* = x \) for every \( x \in R \), hence, since \( * \) is an involution, \( R \) must be commutative; as a commutative primitive ring \( R \) must be a field. On the other hand, if \( T = 0 \), then \( x^* = -x \) for every \( x \in R \); but then \( x^2 = (x^*)^2 = (x^2)^* = -x^2 \), which leads to \( 2x^2 = 0 \). If the characteristic of \( R \) is 2, then \( T = K_0 \) and we are back to the previous situation. If the characteristic is not 2 then we get that \( x^2 = 0 \) for every \( x \in R \); this implies that \( R^0 = 0 \), contrary to the primitivity of \( R \). Hence we may assume that both \( K_0 \neq 0 \) and \( T \neq 0 \).

An immediate consequence of this is that \( R \) must be of characteristic \( p \), \( p \neq 0 \). For either in Condition \( I' \) or \( II' \), there is an element \( a \neq 0 \) (\( a \) in \( T \) or \( K_0 \)) with \( a^n = a \), \( (2a)^m = 2a \), \( n > 1 \), \( m > 1 \). From this we get that \( (2^n - 2)a = 0 \) where \( q = (n - 1)(m - 1) + 1 \), and so \( R \) must be of prime characteristic \( p \), \( p \neq 0 \).

If \( a \neq 0 \) is in \( T \) and \( a^n = a \) then \( e = a^{n-1} \neq 0 \) is a symmetric idempotent; if \( b \neq 0 \) is in \( K_0 \) and \( b^n = b \) then \( e = b^{m-1} \neq 0 \) is a symmetric idempotent. Thus, either in Condition \( I' \) or \( II' \), \( R \) must have a nonzero symmetric idempotent. If \( I \) is the only nonzero symmetric idempotent, then, in Condition \( I' \), every nonzero element of \( T \) must be invertible, and in Condition \( II' \), every nonzero element of \( K_0 \) must be invertible. Making use of the results of [5], we conclude that \( R \) must be a division ring or the ring of all \( 2 \times 2 \) matrices over a field. If \( R \) is a division ring, then by the main theorems of [4], \( R \) must be a field. In other words, we would be done. So we may suppose that there are nonzero symmetric idempotents \( e \) in \( R \) with \( e \neq 1 \).

Let \( e \neq 0 \), \( 1 \) be a symmetric idempotent. If \( e \) commutes with all symmetric elements then \( e \) centralizes \( S \), the subring generated by \( S \). Now, by Theorem 1.4 and 1.6 of [3] and one of the results of [8], either \( S \subseteq Z \), the center of \( R \) or \( S \) contains a nonzero ideal \( U \) of \( R \). If \( S \subseteq Z \) then since \( e \in S \subseteq Z \), \( e \) must be in \( Z \); in a prime ring (and so in a primitive ring) there are no nontrivial central idempotents, contradicting \( e^2 = e \neq 0 \), \( 1 \) and \( e \in Z \). On the other hand, if \( S \) contains the ideal \( U \neq 0 \) of \( R \), then \( e \) centralizes \( U \); however, in a prime ring,
the centralizer of a nonzero ideal must be in the center. So again we end up with the contradiction that $e \in Z$. In short, there must be a symmetric element $b \in S$ such that $c = eb - be \neq 0$.

Since $c \in K_0$, when we are in Condition II' then $c^n = c$ for some $n > 1$. On the other hand, if we are in Condition I' but not II', the characteristic of $R$ must be different from 2. In that case, since $c^2 \in S$, $2c^2$ is in $T$, hence we can find an integer $k > 1$ such that both $2k \equiv 2 \mod p$ and $(2c^2)^k = 2c^2$. The net outcome of this is that $2(2c^2 - c^2) = 0$, and so $c^{2k} - c^2 = 0$. Thus $(c^{2k-1} - c)^2 = 0$, which is to say, $(c^{2k-1} - c)(c^{2k-1} - c)^* = 0$. Since $xa^* \neq 0$ if $x \neq 0$, we get that $c^{2k-1} = c$. Hence in both Condition I' and Condition II' we have that $c^n = c$ for some $n > 1$.

Note, however, that because $c = eb - be$,

$$ec + ce = e(2b - be) + (eb - be)e = eb - be = c.$$  

This immediately gives that $c^2 = ec^2$. From these relations we deduce that the ring $A_0 = \{q_1(e) + q_2(e)e \mid q_1, q_2 \text{ polynomials over } P\}$, where $P$ is the prime field having $p$ elements, is a finite subring of $R$ (it is finite since $c^n = c$). Since $e^* = -c$ and $e^* = e$, $A_0$ must be invariant relative to $*$. Moreover, since $xx^* \neq 0$ for $x \neq 0$ in $R$, this also holds true in $A_0$. Finally, $A_0$ is not commutative, for if it were, since $e, c \in A_0$ and $ec + ce = c$, we would have $2ec = c$, and so $e(2ec) - ec$, that is, $ec - 0$, and so $c - 2ec - 0$. We have all the hypotheses of Lemma 2 satisfied for $A_0$. Thus $A_0$ is the direct sum of fields and of $2 \times 2$ matrix rings over fields, with each direct summand invariant under $*$. Since $A_0$ is not commutative, one of these direct summands of $A_0$, call it $B$, must be a $2 \times 2$ matrix ring over a finite field $F$ (of characteristic not 2).

We look at the nature of $*$ on $B = F_2$. As we saw in Lemma 1, we may assume that $e_{11}^* = e_{11}, e_{22}^* = e_{22}$ and $e_{12}^* = ae_{21}, e_{21}^* = \alpha^{-1}e_{12}$ where $\alpha^* = \alpha \in F$. Since $F$ is finite, $\alpha^n = 1$ with $n > 1$. We claim that we may assume that $\alpha^{2n} = 1$. For write $m = 2nt$ where $t$ is odd; then $\alpha = \alpha_{\beta}$ where $\alpha_{\beta}$, $\beta$ are powers of $\alpha$, $\alpha_{\beta} = 1$ and $\beta^t = 1$. Since $\beta$ is of odd order, $\beta = \delta$, $\delta \in F_2$, (in fact, $\delta$ is a power of $\beta$, so of $\alpha$). Let $f_{10} = \delta e_{10}, f_{11} = \delta^{-1}e_{21}, f_{12} = e_{12}, f_{20} = e_{20};$ these form a set of matrix units for $F_2$, and as is verified easily $f_{12}^* = \alpha_1 f_{21}, f_{21}^* = \alpha_1^{-1}f_{12}$ where now we know that $\alpha_1^{2n} = 1$. Thus we may assume that $\alpha^{2n} = 1$.

Since $B$ is a direct summand of $A_0$, and $A_0$ is semisimple, $B = fA_0$ where $f^2 = f = f^* \neq 0$. Let $R_0 = fRf$; then $R_0$ is primitive, invariant with respect to $*$ and since its $K_0$ lies in $K_0 \cap R_0$ and its $T$ lies in $T \cap R_0$, $R_0$ inherits the hypothesis of $R$. Our aim is to show that $R_0$ must be the ring of all $2 \times 2$ matrices over a field.

Consider $C_{R_0}(\alpha) = \{x \in R_0 \mid x\alpha = \alpha x\};$ we know that $C_{R_0}(\alpha)$ contains the
2 \times 2 \text{ matrix ring over } Z(\alpha), \text{ the field obtained by joining } \alpha \text{ to } Z, \text{ the center of } R_0. \text{ Condition } I' \text{ or } II' \text{ forces } Z \text{ to be algebraic over the prime field, so } K = Z(\alpha) \text{ is algebraic over the prime field. Now, } C_{R_0}(\alpha) \subset K_2, \text{ the } 2 \times 2 \text{ matrices over } K, \text{ and } K_2 \text{ and } C_{R_0}(\alpha) \text{ have the same unit element } f, \text{ and } C_{R_0}(\alpha) \text{ is an algebra over } K. \text{ By a classic theorem in matrix theory, } C_{R_0}(\alpha) = K_2 \otimes_K G, \text{ where } G \text{ is the centralizer in } C_{R_0}(\alpha) \text{ of } K_2. \text{ Since } K_2^* = K_2, \text{ its centralizer must also be invariant with respect to } *, \text{ that is, } G^* = G. \text{ Since in } G, x^*x = 0 \text{ forces } x = 0, G \text{ is semiprime. Also } \alpha^{2n} \in Z \text{ and the characteristic of } R_0 \text{ is not } 2.

If in } G, e^* = e = e^2 \text{ implies that } es = se \text{ for all } s = s^* \text{ in } G, \text{ then if } k \in K \cap G, \ v = k e - ke \in (ek - ke)c. \text{ Let } d(x) = xe - ex; \text{ this would translate the above into } d(s) = 0, d^2(k) = 0 \text{ in } G. \text{ Hence } d^2(x) = 0 \text{ for all } x \in G. \text{ Since } e^2 = e, d^2(x) = d(x) \text{ for all } x. \text{ The net result of all this is that } d(x) = 0 \text{ for all } x \in G, \text{ whence } e \in Z(G). \text{ By Lemma 3 we would have that } G \text{ satisfies a polynomial identity, hence } C_{R_0}(\alpha) = K_2 \otimes_K G \text{ must satisfy a polynomial identity. But the conditions of Montgomery's theorem cited earlier now apply to } R_0, \text{ yielding that } R_0 \text{ satisfies a polynomial identity. Because } R_0 \text{ is primitive, it then must be finite dimensional over its center. In short, } R_0 \text{ is simple artinian.}

On the other hand, if there exists an } e^* = e = e^2 \text{ implies that } es = se \text{ for all } s = s^* \text{ in } G, \text{ then if } k \in K \cap G, \ v = k e - ke \in (ek - ke)c. \text{ Let } d(x) = xe - ex; \text{ this would translate the above into } d(s) = 0, d^2(k) = 0 \text{ in } G. \text{ Hence } d^2(x) = 0 \text{ for all } x \in G. \text{ Since } e^2 = e, d^2(x) = d(x) \text{ for all } x. \text{ The net result of all this is that } d(x) = 0 \text{ for all } x \in G, \text{ whence } e \in Z(G). \text{ By Lemma 3 we would have that } G \text{ satisfies a polynomial identity, hence } C_{R_0}(\alpha) = K_2 \otimes_K G \text{ must satisfy a polynomial identity. But the conditions of Montgomery's theorem cited earlier now apply to } R_0, \text{ yielding that } R_0 \text{ satisfies a polynomial identity. Because } R_0 \text{ is primitive, it then must be finite dimensional over its center. In short, } R_0 \text{ is simple artinian.}

If } h^2 = h \in R_0 \text{ is such that } hR_0 \text{ is a minimal right ideal of } R_0, \text{ then } hR_0h \text{ is a division ring. But } fh = hf = h, \text{ hence } hR_0h = hfRfh = hRh, \text{ and so } hRh \text{ is a division ring. But then } hR \text{ is a minimal right ideal of } R. \text{ Thus } R \text{ is a primitive ring with involution } * \text{ having a minimal right ideal.}

However, primitive rings with involution having minimal right ideals have a fairly sharply described structure (see [6], Section 12, Chapter IV). In fact, if the ring is not the } 2 \times 2 \text{ matrix ring over a field, then by the general structure theorem cited, } R \text{ would contain a subring invariant with respect to } * \text{ which is isomorphic to } D_3, \text{ the } 3 \times 3 \text{ matrices over a division ring } D. \text{ Moreover, } D \text{ has an involution induced by } *. \text{ Thus } D \text{ satisfies Condition } I' \text{ or } II' \text{ according as } R \text{ does. By the results of [4] for the division ring case, } D \text{ must be a field.}

Thus if } R \text{ is not isomorphic to the } 2 \times 2 \text{ matrices over a field, then for
some field $K$ of characteristic $p \neq 0$ which is algebraic over the prime field, 
$K_3$ would have an involution $*$ such that $xx^* \neq 0$ if $x \neq 0$. Lemma 1 assures us that this cannot happen. Hence $R$ must be the $2 \times 2$ matrices over a field, or a field itself. If the characteristic of $R$ is 2, the proof shows that $R$ must be a field.

What remains is to show that if Condition II' holds, that is, if $(x - x^*)^{n(x)} = x - x^*$ for all $x \in F_2$, then any two skew traces must commute.

We claim that we may assume that $*$ is of transpose type. For suppose the involution is symplectic. If the characteristic is 2, then $x + x^*$ is a scalar, so any two such elements commute, that is, any two elements of $K_3$ commute. If the characteristic is not 2, the element $(i i)$ is skew and nilpotent, hence is certainly not periodic.

Therefore for $a, b, c, d \in F$, $(\frac{a}{c} \frac{b}{d})^* = (\frac{d}{a} \cdot \frac{1}{c} \cdot b)$ where $\cdot$ indicates the automorphism on $F$ induced by $*$ and where $\overline{\alpha} = \alpha \in F$.

If $\overline{a} = a$ for every $a \in F$ then the skew traces in $F_2$ under $*$ are all of the form $(\frac{0}{a} \frac{1}{b})$, and any two such elements commute.

To finish we merely must rule out the possibility that $\overline{a} \neq a$ for some $a \in F$. If so, let $b = a - \overline{a}$. The field $L = P(a, b)$ generated by $a$ and $b$ over the prime field $P$ is finite since $a$ and $b$ are algebraic over $P$; moreover, since $\overline{\alpha} = \alpha$ and $b = -b$, $L$ must be invariant under $-$. Let $L_0 = \{x \in L \mid \overline{x} = x\}$. If the characteristic of $P$ is not 2, then $[L : L_0] = 2$, and since we are in a finite field, every element in $L_0$ is a norm of an element in $L$, that is, every element in $L_0$ is of the form $u\overline{a}$ with $u \in L$. Since $-\alpha^{-1} \in L_0$, $-\alpha^{-1} = \beta \overline{\beta}$ for some $\beta \in L \subset F$. On the other hand, if the characteristic of $P$ is 2, every element in $L$ is a square and $L = L_0$, so that $-\alpha^{-1} = \beta \overline{\beta}$ for some $\beta \in L$. Thus in both cases, $-\alpha^{-1} = \beta \overline{\beta}$ with $\beta \in L$. The matrix $x = (\frac{0}{a} \frac{1}{b})$ is then a skew trace. In characteristic not 2 this is automatic since $x$ is skew. In characteristic 2 we merely note that $x = z + z^*$ where $z = (\frac{0}{a} \frac{1}{b})$ and so is a skew trace. But as is easily verified, $x^3 = 0$. This is a contradiction. With this the theorem is completely proved.

With Theorem 1 out of the way we are free to go on to get fairly general structure theorems for rings with periodic symmetric or skew elements.

The next three theorems are due to Montgomery [9, 10].

**Theorem 2.** Let $R$ be a semisimple ring with involution $*$ such that for every $x \in R$, $(x + x^*)^{n(x)} = x + x^*$, $n(x) > 1$. Then $R$ is isomorphic to a subdirect sum of fields and $2 \times 2$ matrices over fields. In particular, $R$ satisfies the standard identity in 4 variables.

**Proof.** Let $P$ be a primitive ideal of $R$. If $P = P^*$ then $R/P$ inherits the property of $R$. Since $R/P$ is primitive, by applying Theorem 1, we have that $R/P$ is either a field or the $2 \times 2$ matrices over a field.
If $P \neq P^*$ then in the ring $\bar{R} = R/P$, every element in the ideal $A = (P + P^*)/P$ is an image of an element of the form $x + x^*$, with $x \in P$, hence if $a \in A$ then $a^{n(a)} = a$ for some $n(a) > 1$. By Jacobson's theorem, $A$ must be commutative. Thus the primitive ring $\bar{R}$ has a nonzero commutative ideal $A$. This forces $\bar{R}$ to be commutative; as a commutative, primitive ring, $R$ must be a field.

Since $R$ is semisimple, $R$ is a subdirect sum of the $R/P$ where $P$ ranges over the primitive ideals of $R$. By the above argument each $R/P$ is a field on a $2 \times 2$ matrix ring over a field. This proves the main part of the theorem.

Since fields and $2 \times 2$ matrices over fields satisfy the standard identity in 4 variables, the rest of the theorem is clear.

By strengthening the hypothesis on $R$ we can get a slightly stronger theorem

**Theorem 3.** Let $R$ be a semiprime ring with condition $*$ such that every $a \in S$ satisfies $a^{n(a)} = a$ with $n(a) > 1$. Then $R$ is isomorphic to a subdirect sum of fields and $2 \times 2$ matrices over fields. Thus $R$ satisfies the standard identity in 4 variables. If $R$ is of characteristic 2 it must be commutative.

**Proof.** We first assert that $R$ must be semisimple. If $a \in J \cap S$ then $a^n = a$, $n > 1$, so $e = a^{n-1} \in J$ is an idempotent. Since $J$ has no nonzero idempotents, $e = 0$. Thus $a = a^{n-1}a = 0$. In short $J \cap S = 0$.

Now $J^* = J$, so if $a \in J$ then $a + a^* \in J \cap S = 0$. Hence $a^* = -a$. $a^2 \in J \cap S = 0$, whence $a^2 = 0$ for every $a \in J$. If $x \in R$ then $ax \in J$, hence $(ax)^* = -ax$, that is, $x^*a = ax$; thus $0 = x^*a^2 = axa$. The ideal $RaR$ is therefore nilpotent. Since $R$ is semiprime, we get that $a = 0$ and so $J = 0$.

Since we now know that $R$ is semisimple, by Theorem 2 it is a subdirect sum of fields and $2 \times 2$ matrices over fields. We still have to show that if the characteristic of $R$ is 2 then the $2 \times 2$ matrices cannot arise. This reduces to showing that in the ring of $2 \times 2$ matrices over a finite field $F$ of characteristic 2 the condition $s^{h(a)} = s$ is not possible for the symmetric elements.

The proof given at the end of Theorem 1, shows that we can rule out the symplectic involution and involutions of transpose type in which the elements of $F$ are not fixed, for then we produced nilpotent symmetric. Thus we may assume that $(a \ b)_{\alpha \beta}^* = (a \ b)_{\alpha \beta}^{-1}$. Since $\alpha \in F$, and $F$ is a finite field of characteristic 2, $\alpha = \beta^2$ for some $\beta \in F$. The matrix $(a \ b)_{\alpha \beta}$ is then symmetric and nilpotent, hence is certainly not periodic. With this the proof is complete.

We keep on in this vein.

**Theorem 4.** Let $R$ be a ring with involution $*$ in which every $a \in S$ satisfies $a^{n(a)} = a$ with $n(a) > 1$. Then $J$, the radical of $R$, satisfies $J^3 = 0$. $J(xy - yx) = 0$ for all $x, y \in R$; furthermore $R/J$ is a subdirect sum of fields and $2 \times 2$ matrices over fields. Thus $R$ satisfies the polynomial identity $S(x_1, x_2, x_3, x_4)$ where $S(x_1, ..., x_4)$ is the standard identity in 4 variables.
Proof. As in the proof of Theorem 3, \( J \cap S = 0 \), hence \( a^2 = 0 \) for \( a \in J \), and \( a^* = -a \). If \( 2x = 0 \) for \( x \in J \) then \( x = -x = -(x^*) = x^* \), putting \( x \in J \cap S = 0 \). Since \( J \) is 2-torsion free and every element in \( J \) has square 0, \( J^2 = 0 \) follows.

Since \( a \in J \) implies \( a^* = -a \), if \( x \in R \) then since \( ax \in J \), \((ax)^* = -ax \). Thus gives \( x^*a = ax \). Hence \((xy)^*a = axy \) for \( x, y \in R \); but \((xy)^*a = y^*x^*a = y^*ax = ayx \). Thus \( a(x^* - y^*) = 0 \), and so \( J(xy - yx) = 0 \).

Finally, \( R - R/J \) is semisimple, has the * of \( R \) induced in it, and all \( x + x^* \), being images of \( x + x^* \), are periodic. Applying Theorem 2 gives us that \( R/J \) is a subdirect sum of fields and \( 2 \times 2 \) matrices, and \( R/J \) satisfies \( S(x_1, \ldots, x_4) \). But then in \( R \), \( S(a_1, \ldots, a_4) \in J \) hence has square 0. In fact it is immediate from the proof, since \( J(xy - yx) = 0 \), that \( R \) satisfies \( S(a_1, \ldots, a_4) \).

We now turn to the analogous theorems for the skew elements.

Theorem 5. Let \( R \) be a semisimple ring with involution * in which 
\[(x - x^*)^{n(x)} = x - x^* \quad \forall x \in R, \ n(x) > 1. \]
Then \( R \) is a subdirect sum of fields and \( 2 \times 2 \) matrices over fields. Moreover, if \( a^* = -a, b^* = -b \) in \( R \) then \( ab = ba \).

Proof. The proof is similar to that of Theorem 2. If \( P \) is a primitive ideal of \( R \) and if \( P^* = P \), then \( R/P \) inherits the * and hypothesis of \( R \). By Theorem 1, \( R \) is a field or the \( 2 \times 2 \) matrices over a field; moreover any two skew elements of \( R/P \) commute. So, if \( a, b \) in \( R \) are skew, \( ab - ba \in P \).

If \( P^* \neq P \) then every element in \( A = (P + P^*)/P \) is the image of an element of the form \( x - x^* \) with \( x \in P \). Thus \( A = 0 \) is a commutative ideal in the primitive ring \( R/P \). Hence \( R/P \) must be commutative, so a field. In consequence, \( xy - yx \in P \) for all \( x, y \in R \).

Thus for any primitive ideal, \( P \), of \( R \) the ring \( R/P \) is either a field or the \( 2 \times 2 \) matrices over a field. Since \( R \) is semisimple, \( R \) is a subdirect sum of these \( R/P \) and that part of the theorem is proved. Also, if \( a, b \in R \) are both skew, we saw that \( ab - ba \) must be in all the primitive ideals \( P \). Since \( \bigcap P = 0 \) because \( R \) is semisimple, we have that \( ab - ba = 0 \), that is \( a \) and \( b \) commute.

We finish the paper by showing that in any ring with periodic skew traces any two skew elements commute. This verifies Jacobson's question [7] which was cited earlier, about restricted Lie algebras in the special context where the Lie algebra is the set of skew elements in a ring with involution.

Theorem 6. Let \( R \) be a ring involution * such that 
\[(x - x^*)^{n(x)} = x - x^* \quad \forall x \in R, \ n(x) > 1. \]
If \( a^* = -a, b^* = -b \) in \( R \), then \( ab = ba \).

Proof. Since \( J^* = J \) where \( J \) is the radical of \( R \), \( R/J \) inherits the hypothesis of \( R \). Thus any two skew elements in \( R/J \) commute. If \( a^* = -a, b^* = -b \)
this says that \( ab - ba \in J \). But \( ab - ba = ab - (ab)^* \) so \( (ab - ba)^n = ab - ba \) for some \( n > 1 \). Thus \( e = (ab - ba)^{n-1} \) is an idempotent lying in \( J \). Because \( J \) is the radical of \( R \), this forces \( e = 0 \). Thus

\[
ab - ba = (ab - ba)^{n-1}(ab - ba) = e(ab - ba) = 0.
\]

REFERENCES

11. SUSAN MONTGOMERY, Centralizers satisfying polynomial identities, to appear.