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On Large Deviations of Kolmogorov–Smirnov–Renyi Type Statistics

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One result of Smirnov's important paper [*Uspehi Mat. Nauk.* **10**, 179–206, (in Russian)] yields exponential bounds for the large deviations of his one-sided Smirnov statistic and the two-sided Kolmogorov statistic. In the present paper exponential bounds are given for the large deviations of a wide class of Kolmogorov–Smirnov–Renyi type statistics. As a by-product, exponential bounds for the large deviations of the corresponding limit distributions are obtained.

1. INTRODUCTION

Let $X_1, X_2, \ldots$ be independent identically distributed random variables with a common continuous distribution function F. Without loss of generality we may assume that $F(t) = t$ $(0 \leqslant t \leqslant 1)$ holds. Let F_n be the empirical distribution function and put $G_n(t) = n^{1/2}(F_n(t) - t)$ $(0 \leqslant t \leqslant 1)$. Let α, β, u, v be real constants satisfying (a) $0 \leqslant \alpha < \beta \leqslant 1$, (b) $u \geqslant 0$, $u + v \geqslant 0$, and (c) $u + vt > 0$ for all $t \in [\alpha, \beta]$. For a Brownian bridge $\{B(t), 0 \leqslant t \leqslant 1\}$ we define for arbitrary $a, b \in R$

$$P(\alpha, \beta, a, b) = P(|B(t)| \leqslant a + bt \text{ for all } t \in [\alpha, \beta]), \tag{1.1}$$

$$P^+(\alpha, \beta, a, b) = P(B(t) \leqslant a + bt \text{ for all } t \in [\alpha, \beta]). \tag{1.2}$$

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Put $\varphi(t) = (u + vt)(t(1 - t))^{-1/2}$ and for arbitrary $x > 0$

$$\begin{aligned} A(x; \alpha, \beta, u, v) &= \frac{1}{x} \exp\left(-\frac{1}{2} x^2(\varphi(\alpha))^2\right), && \text{if } \frac{v}{u} > \frac{1-2\alpha}{\alpha} \\ &= \exp(-2x^2 u(u+v)), && \text{if } \frac{1-2\beta}{\beta} \leqslant \frac{v}{u} \leqslant \frac{1-2\alpha}{\alpha} \\ &= \frac{1}{x} \exp\left(-\frac{1}{2} x^2(\varphi(\beta))^2\right), && \text{if } \frac{v}{u} < \frac{1-2\beta}{\beta} \end{aligned}$$

$$(0 < \alpha < \beta < 1), \quad (1.3)$$

$$\begin{aligned} A(x; 0, \beta, u, v) &= \exp(-2x^2 u(u+v)), && \text{if } \frac{v}{u} \geqslant \frac{1-2\beta}{\beta} \\ &= \frac{1}{x} \exp\left(-\frac{1}{2} x^2(\varphi(\beta))^2\right), && \text{if } \frac{v}{u} < \frac{1-2\beta}{\beta} \end{aligned}$$

$$(0 < \beta < 1), \quad (1.4)$$

$$\begin{aligned} A(x; \alpha, 1, u, v) &= \frac{1}{x} \exp\left(-\frac{1}{2} x^2(\varphi(\alpha))^2\right), && \text{if } \frac{v}{u} > \frac{1-2\alpha}{\alpha} \\ &= \exp(-2x^2 u(u+v)), && \text{if } \frac{v}{u} \leqslant \frac{1-2\alpha}{\alpha} \end{aligned}$$

$$(0 < \alpha < 1), \quad (1.5)$$

$$A(x; 0, 1, u, v) = \exp(-2x^2 u(u+v)). \quad (1.6)$$

We are going to deal with the Kolmogorov–Smirnov–Renyi type statistics

$$T_n(\alpha, \beta, u, v) = \sup_{\alpha \leqslant t \leqslant \beta} \frac{|G_n(t)|}{u + vt}, \quad (1.7)$$

$$T_n^+(\alpha, \beta, u, v) = \sup_{\alpha \leqslant t \leqslant \beta} \frac{G_n(t)}{u + vt}. \quad (1.8)$$

Smirnov [8, Theorem 3] proved for $x_n \geqslant x_0 > 0$ and $x_n = o(n^{1/6})$

$$\frac{P(T_n^+(0, 1, 1, 0) > x_n)}{1 - P^+(0, 1, x_n, 0)} = 1 + o(1). \quad (1.9)$$

Gnedenko, Koroluk, and Skorokhod [5, p. 154] reported results of Smirnov, Karplevskaia, and Chan Li–Tsian sharpening (1.9). Noting that $A(x; 0, 1, 1, 0) = 1 - P^+(0, 1, x, 0)$ and

$$P(T_n^+(0, 1, 1, 0) > x) \leqslant P(T_n(0, 1, 1, 0) > x) \leqslant 2P(T_n^+(0, 1, 1, 0) > x)$$

holds, we derive from (1.9) the existence of constants d_1, d_2, d_3, $d_4 > 0$ such that for sufficiently large n

$$d_1 \leqslant \frac{P(T_n^+(0, 1, 1, 0) > x_n)}{A(x_n ; 0, 1, 1, 0)} \leqslant d_2, \tag{1.10}$$

$$d_3 \leqslant \frac{P(T_n(0, 1, 1, 0) > x_n)}{A(x_n ; 0, 1, 1, 0)} \leqslant d_4. \tag{1.11}$$

The following theorem supplies exponential inequalities for the limit distributions of (1.7) and (1.8).

THEOREM 1. *If $x_n \to \infty$, then there exist constants d_1, d_2, d_3, $d_4 > 0$ such that for sufficiently large n*

$$d_1 \leqslant \frac{1 - P^+(\alpha, \beta, ux_n, vx_n)}{A(x_n ; \alpha, \beta, u, v)} \leqslant d_2, \tag{1.12}$$

$$d_3 \leqslant \frac{1 - P(\alpha, \beta, ux_n, vx_n)}{A(x_n ; \alpha, \beta, u, v)} \leqslant d_4. \tag{1.13}$$

With aid of Theorem 1 we are able to prove the following generalization of (1.10) and (1.11), which gives exponential bounds for the large deviations of the statistics (1.7) and (1.8).

THEOREM 2. *If $x_n \to \infty$ and $x_n = o(n^{1/6})$, then there exist constants d_1, d_2, d_3, $d_4 > 0$ such that for sufficiently large n*

$$d_1 \leqslant \frac{P(T_n^+(\alpha, \beta, u, v) > x_n)}{A(x_n ; \alpha, \beta, u, v)} \leqslant d_2, \tag{1.14}$$

$$d_3 \leqslant \frac{P(T_n(\alpha, \beta, u, v) > x_n)}{A(x_n ; \alpha, \beta, u, v)} \leqslant d_4. \tag{1.15}$$

Theorem 2, which is of some interest by itself, can be used to prove the general form of the law of the iterated logarithm for (1.7). This is done in [7].

Notations. Let Z and R be the set of integers and real numbers, respectively. Also, let $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp(-\frac{1}{2}t^2)\, dt$, $b(k; n, p) = \binom{n}{k} p^k(1-p)^{n-k}$, and $B(x; n, p) = \sum_{k=0}^{[x]} b(k; n, p)$.

2. Proofs of the Theorems

In order to prove Theorem 1 put $\psi(t) = (u(2t-1)+vt)(t(1-t))^{-1/2}$, $s_0 = [\beta(1-\alpha)(\alpha(1-\beta))^{-1}]^{1/2}\varphi(\alpha)x_n$, $t_0 = [\beta(1-\alpha)(\alpha(1-\beta))^{-1}]^{1/2}\psi(\alpha)x_n$, $s_1 = \varphi(\beta)x_n$, $t_1 = \psi(\beta)x_n$, $s_2 = \varphi(\alpha)x_n$, $t_2 = \psi(\alpha)x_n$,

$$A(t) = [\alpha(1-\beta)(\beta-\alpha)^{-1}]^{1/2}(s_0 - t), \quad B(t) = [\alpha(1-\beta)(\beta-\alpha)^{-1}]^{1/2}(t-t_0),$$

and

$$I_1 = 1 - (2\pi)^{-1/2}\int_{-\infty}^{s_1} \exp(-\tfrac{1}{2}t^2)\,\Phi(A(t))\,dt,$$

$$I_2 = (2\pi)^{-1/2}\int_{-\infty}^{t_1} \exp(-\tfrac{1}{2}t^2)\,\Phi(B(t))\,dt,$$

$$I_3 = \exp(-2x_n u(u+v))I_2 .$$

We shall prove (1.12) and (1.13) only for the case $0 < \alpha < \beta < 1$. After a corresponding modification of formula (2.1) below, the other cases are treated analogously. Eicker [2, Appendix B] proved for $0 < \alpha < \beta < 1$

$$1 - P^{+}(\alpha, \beta, u, v) = I_1 + I_3 . \tag{2.1}$$

In order to use this formula we list some properties of the terms defined above, which can be easily verified:

In $[0, 1]$ $\varphi(t)$ has exactly one minimum in $\bar{t} = u/(2u+v)$ and $\varphi(\bar{t}) = 2(u(u+v))^{1/2}$,

$$(\varphi(t))^2 - (\psi(t))^2 = 4u(u+v) \qquad \text{for all} \quad t \in [\alpha, \beta], \tag{2.2}$$

$$\exp(-\tfrac{1}{2}(t^2 + (A(t))^2)) = \exp(-\tfrac{1}{2}s_2^2)\exp\left(-\frac{\beta(1-\alpha)}{2(\beta-\alpha)}(t-s_2)^2\right), \tag{2.3}$$

$$\exp(-\tfrac{1}{2}(t^2 + (B(t))^2)) = \exp(-\tfrac{1}{2}t_2^2)\exp\left(-\frac{\beta(1-\alpha)}{2(\beta-\alpha)}(t-t_2)^2\right), \tag{2.4}$$

$$\begin{aligned} A(s_1) &= [(\beta-\alpha)(\alpha\beta)^{-1}]^{1/2}\, ux_n , \qquad B(t_1) \doteq [(\beta-\alpha)(\alpha\beta)^{-1}]^{1/2}\, ux_n , \\ A(s_2) &= [(\beta(1-\alpha))^{1/2} - (\alpha(1-\beta))^{1/2}]\,(\beta-\alpha)^{-1/2}\,\varphi(\alpha)x_n , \end{aligned} \tag{2.5}$$

$$B(t_2) = -[(\beta(1-\alpha))^{1/2} - (\alpha(1-\beta))^{1/2}]\,(\beta-\alpha)^{-1/2}\,\psi(\alpha)x_n , \tag{2.6}$$

$$B(-t_2) = -[(\beta(1-\alpha))^{1/2} + (\alpha(1-\beta))^{1/2}]\,(\beta-\alpha)^{-1/2}\,\psi(\alpha)x_n , \tag{2.7}$$

$$0 < s_2 < s_1 < s_0 \,, \text{ if } \frac{v}{u} > \frac{1-2\alpha}{\alpha}, \quad 0 < s_1 < s_2 < s_0 \,, \text{ if } \frac{v}{u} < \frac{1-2\beta}{\beta}, \tag{2.8}$$

$$0 < t_2 < t_0 < t_1 \,, \text{ if } \frac{v}{u} > \frac{1-2\alpha}{\alpha}, \quad t_0 < t_1 \,, t_2 < 0, \text{ if } \frac{v}{u} < \frac{1-2\beta}{\beta}. \tag{2.9}$$

Now we are going to give upper and lower bounds for the terms I_1 and I_2 separately. For this, the cases $v/u > (1-2\alpha)/\alpha$, $(1-2\beta)/\beta \leqslant v/u \leqslant (1-2\alpha)/\alpha$ and $v/u < (1-2\beta)/\beta$ must be considered separately. The calculations in these different cases are almost analogous. Therefore in this paper we only deal with the case $v/u > (1-2\alpha)/\alpha$.

1. *Upper bounds.* From (2.3), (2.5), (2.8) and the well-known inequality

$$\frac{1}{x}(1-x^{-2}) \leqslant (2\pi)^{1/2} \exp(\tfrac{1}{2}x^2)(1-\Phi(x)) \leqslant \frac{1}{x} \qquad (x > 0) \tag{2.10}$$

we obtain

$$\begin{aligned} I_1 &\leqslant 1-\Phi(s_2) + (2\pi)^{-1/2}(A(s_2))^{-1}\exp(-(1/2)s_2^{\,2}) \\ &\quad \times (2\pi)^{-1/2}\int_{-\infty}^{s_2} \exp(-(\beta(1-\alpha)/2(\beta-\alpha))(t-s_2)^2)\,dt \\ &\leqslant c_1 x_n^{-1}\exp(-(1/2)\,x_n^{\,2}(\varphi(\alpha))^2) \qquad (c_1 > 0, \text{ const}). \end{aligned} \tag{2.11}$$

From (2.4), (2.6), (2.9) and (2.10) we get

$$\begin{aligned} I_2 &\leqslant (2\pi)^{-1/2}\int_{-\infty}^{t_2} \exp(-(1/2)t^2)(1-\Phi(-B(t)))\,dt \\ &\quad + (2\pi)^{-1/2}\int_{t_2}^{t_1} \exp(-(1/2)t^2)\,\Phi(B(t))\,dt \leqslant 1-\Phi(t_2) \\ &\quad + (2\pi)^{-1/2}(|\,B(t_2)|)^{-1}\exp(-(1/2)t_2^{\,2}) \\ &\quad \times (2\pi)^{-1/2}\int_{-\infty}^{t_2} \exp(-(\beta(1-\alpha)/2(\beta-\alpha))(t-t_2)^2)\,dt \\ &\leqslant c_2 x_n^{-1}\exp(-(1/2)\,x_n^{\,2}(\psi(\alpha))^2) \qquad (c_2 > 0, \text{ const}) \end{aligned}$$

and thus from (2.2)

$$I_3 \leqslant c_2 x_n^{-1}\exp(-(1/2)\,x_n^{\,2}(\varphi(\alpha))^2). \tag{2.12}$$

Combining (2.11) and (2.12) we finally obtain from (1.3) and (2.1)

$$1 - P^{+}(\alpha, \beta, ux_n\,, vx_n) \leqslant d_2 A(x_n; \alpha, \beta, u\ v) \qquad (d_2 > 0, \text{ const}). \tag{2.13}$$

2. *Lower bounds.* Using again (2.10) we get for sufficiently large n

$$I_1 \geqslant 1 - \Phi(s_1) \geqslant c_3 x_n^{-1} \exp(-(1/2)\, x_n^2(\varphi(\beta))^2) \qquad (c_3 > 0, \text{ const}). \quad (2.14)$$

From (2.4), (2.6), (2.7), (2.9), and (2.10) we obtain for sufficiently large n

$$\begin{aligned} I_2 &\geqslant (2\pi)^{-1/2} \int_{-t_2}^{t_2} \exp((-(1/2)t_2^2)\,(1 - \Phi(-B(t)))\, dt \\ &\geqslant (2\pi)^{-1/2}\,(|\,B(-t_2)|)^{-1}\,(1 - (B(t_2))^{-2}) \exp(-(1/2)t_2^2) \\ &\quad \times (2\pi)^{-1/2} \int_{-t_2}^{t_1} \exp(-(\beta(1-\alpha)/2(\beta-\alpha))\,(t - t_2)^2)\, dt \\ &\geqslant c_4 x_n^{-1} \exp(-(1/2)\, x_n^2(\psi(\alpha))^2) \qquad (c_4 > 0, \text{ const}) \end{aligned}$$

and thus from (2.2)

$$I_3 \geqslant c_4 x_n^{-1} \exp(-\tfrac{1}{2}(\varphi(\alpha))^2). \qquad (2.15)$$

From (1.3), (2.1), (2.14) and (2.15) we get

$$1 - P^{+}(\alpha, \beta, ux_n, vx_n) \geqslant d_1 A(x_n; \alpha, \beta, u, v) \qquad (d_1 > 0, \text{ const}). \quad (2.16)$$

Combination of (2.13) and (2.16) yields (1.12). From (1.12) and the inequality

$$1 - P^{+}(\alpha, \beta, ux, vx) \leqslant 1 - P(\alpha, \beta, ux, vx) \leqslant 2(1 - P^{+}(\alpha, \beta, ux, vx))$$

we get (1.13). Thus the proof of Theorem 1 is established.

In order to prove Theorem 2 we put $\kappa' = n\alpha + n^{1/2}x_n(u + \alpha v)$, $\kappa = [\kappa']$, $\lambda' = n\beta + n^{1/2}x_n(u + \beta v)$, $\lambda = [\lambda']$,

$$\begin{aligned} \lambda^* &= n, && \text{if } \lambda' > n \\ &= \lambda, && \text{if } \lambda' < n, \ \lambda' \notin Z, \\ &= \lambda - 1, && \text{if } \lambda' \leqslant n, \ \lambda' \in Z \end{aligned}$$

$$H(t) = (2\pi t(1-t)^3)^{-1/2} x_n(u+v) \exp(-\tfrac{1}{2}x_n^2(\varphi(t))^2), \qquad (2.17)$$

$$\begin{aligned} K(t) &= (2\pi)^{-1}[(1-\beta)\, t^{-1}(1-t)^{-2}(\beta - t)^{-1}]^{1/2} \\ &\quad \times \exp(-\tfrac{1}{2}x_n^2((\varphi(t))^2 + (u+v)^2(\beta - t)(1-t)^{-1}(1-\beta)^{-1})), \end{aligned} \qquad (2.18)$$

$$L(t) = \Phi(x_n(u+v)[(\beta - t)(1-t)^{-1}(1-\beta)^{-1}]^{1/2}), \qquad (2.19)$$

$$\begin{aligned} p(s) &= (\lambda' - s)(n - s + n^{1/2}x_n(u+v))^{-1}, \ r(s) = (s - n^{1/2}x_n u) \\ &\quad \times (n + n^{1/2}x_n v)^{-1}, \ R(s) = b(s; n, r(s))\{n^{1/2}x_n(u+v) \\ &\quad \times [n - s + n^{1/2}x_n(u+v)]^{-1} B(\lambda^* - s - 1; n - s - 1, p(s)) + (n + n^{1/2}x_n v) \\ &\quad \times (1-\beta)[n - s + n^{1/2}x_n(u+v)]^{-1} b(\lambda^* - s; n - s - 1, p(s))\}. \end{aligned}$$

From (1.17d) of [3] we get

$$P(T_n^+(\alpha, \beta, u, v) \leqslant x_n) = B(\lambda^*; n, \beta) - \sum_{s=\kappa+1}^{\lambda^*} R(s). \tag{2.20}$$

As in the proof of Theorem 1 we only deal with the case $0 < \alpha < \beta < 1$. The other cases are treated analogously. Using Stirling's formula, Taylor expansion of the logarithm and the Berry–Esseen theorem, the binomial terms in (2.20) can be replaced by the corresponding normal terms. Since these estimations are rather long and elementary we omit them in the present paper. The interested reader should refer to Anhang [6, A, B, C], where these estimations are given in detail for the cases $u, v \geqslant 0$ and $v = -u$. It is not hard to see that the case $-u < v < 0$ can be treated analogously. We get the following estimates, where the Landau symbol $O(\cdot)$ stands exclusively for positive sequences:

$$\begin{aligned}
&B(\lambda^*; n, \beta) - P(T_n^+(\alpha, \beta, u, v) \leqslant x_n) \\
&\quad \geqslant (1 - O(x_n^2 n^{-1/3})) \sum_{t=(\kappa+1)/n}^{\beta_0} (1/n)\, H(t) L(t) \\
&\qquad + (1 - O(x_n n^{-1/6})) \sum_{t=(\kappa+1)/n}^{\beta_0} (1/n)\, K(t) \\
&\quad \geqslant \int_\alpha^\beta (H(t) L(t) + K(t))\, dt - A'(x_n\,; \alpha, \beta, u, v)\, O(x_n n^{-1/3} + n^{-1/6}) \\
&\qquad - O\left(x_n^2 n^{-1/3} \int_\alpha^\beta H(t) L(t)\, dt\right) - O\left(x_n n^{-1/6} \int_\alpha^\beta K(t)\, dt\right)
\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}
&B(\lambda^*; n, \beta) - P(T_n^+(\alpha, \beta, u, v) \leqslant x_n) \\
&\quad \leqslant (1 + O(x_n^2 n^{-1/3})) \sum_{t=(\kappa+1)/n}^{\beta_0} (1/n)\, H(t) L(t) \\
&\qquad + (1 + O(x_n^3 n^{-1/2})) \sum_{t=(\kappa+1)/n}^{\beta_0} (1/n)\, K(t) \\
&\qquad + \sum_{s=n\beta_0+1}^{\lambda^*} R(s) \leqslant \int_\alpha^\beta (H(t) L(t) + K(t))\, dt \\
&\qquad + A'(x_n\,; \alpha, \beta, u, v)\, O(n^{-1/6}) + O\left(x_n n^{-1/3} \int_\alpha^\beta H(t) L(t)\, dt\right) \\
&\qquad + O\left(x_n^3 n^{-1/2} \int_\alpha^\beta K(t)\, dt\right)
\end{aligned} \tag{2.22}$$

with

$$\beta_0 = (1/n)[n(\beta - n^{-1/3})] \qquad A'(x; \alpha, \beta, u, v) = \sup_{\alpha \leqslant t \leqslant \beta} \exp(-(1/2)\, x^2(\varphi(t))^2).$$

Using a well-known result due to Cramer [1] concerning the large deviations in the central limit case (cf. [4, p. 552]) we obtain

$$\frac{1 - B(\lambda^*; n, \beta)}{1 - \Phi(\varphi(\beta))} = 1 \pm O(x_n{}^3 n^{-1/2}). \tag{2.23}$$

Korollar 1.1 of [6] yields

$$P^+(\alpha, \beta, ux_n, vx_n) = \Phi(\varphi(\beta)) - \int_\alpha^\beta (H(t)L(t) + K(t))\, dt. \tag{2.24}$$

From (2.20)–(2.24) and by comparison of (1.3) and $A'(x; \alpha, \beta, u, v)$ we get

$$\frac{P(T_n{}^+(\alpha, \beta, u, v) > x_n)}{1 - P^+(\alpha, \beta, ux_n, vx_n)} = 1 \pm O(x_n n^{-1/6})$$

and thus (1.14), with the aid of Theorem 1.

In order to prove (1.15) we use [3, Corollary 1.2], which yields

$$\begin{aligned} &P(T_n{}^+(\alpha, \beta, u, v) > x_n) \\ &\quad \leqslant P(T_n(\alpha, \beta, u, v) > x_n) \\ &\quad \leqslant P(T_n{}^+(\alpha, \beta, u, v) > x_n) + P(T_n{}^+(1 - \beta, 1 - \alpha, u + v, -v) > x_n). \end{aligned} \tag{2.25}$$

From (1.14), (2.25) and $A(x; 1 - \beta, 1 - \alpha, u + v, -v) = A(x; \alpha, \beta, u, v)$ we obtain (1.15). Thus the proof of Theorem 2 is established.

3. Examples

1. On account of assumption (c), in the cases $u = 0$, $v = 1$ and $u = 1$, $v = -1$ we have $\alpha > 0$ and $\beta < 1$, respectively. We get

$$A(x; \alpha, \beta, 0, 1) = \frac{1}{x} \exp\left(-\frac{\alpha}{2(1 - \alpha)}\, x^2\right),$$

$$A(x; \alpha, \beta, 1, -1) = \frac{1}{x} \exp\left(-\frac{1 - \beta}{2\beta}\, x^2\right)$$

and thus from Theorem 2 for the Renyi statistics

$$R_n(\alpha, \beta) = T_n(\alpha, \beta, 0, 1) \quad \text{and} \quad R_n'(\alpha, \beta) = T_n(\alpha, \beta, 1, -1)$$

$$d_1 \leqslant P(R_n(\alpha, \beta) > x_n)\, x_n \exp\left(\frac{\alpha}{2(1-\alpha)}\, x_n^2\right) \leqslant d_2,$$

$$d_1 \leqslant P(R_n'(\alpha, \beta) > x_n)\, x_n \exp\left(\frac{1-\beta}{2\beta}\, x_n^2\right) \leqslant d_2 \; (d_1, d_2 > 0, \text{const})$$

for sufficiently large n ($x_n \to \infty$, $x_n = o(n^{1/6})$).

2. For the Kolmogorov–Manija statistic $K_n(\alpha, \beta) = T_n(\alpha, \beta, 1, 0)$ we get for sufficiently large n ($x_n \to \infty$, $x_n = o(n^{1/6})$)

$$d_1 \leqslant P(K_n(\alpha, \beta) > x_n)\, x_n (A(x_n; \alpha, \beta, 1, 0))^{-1} \leqslant d_2 \qquad (d_1, d_2 > 0, \text{const})$$

with

$$\begin{aligned} A(x; \alpha, \beta, 1, 0) &= \frac{1}{x} \exp\left(-\frac{1}{2\alpha(1-\alpha)}\, x^2\right), \quad \text{if } \alpha > \frac{1}{2}. \\ &= \exp(-2x^2), \quad \text{if } \alpha \leqslant \tfrac{1}{2} \leqslant \beta \\ &= \frac{1}{x} \exp\left(-\frac{1}{2\beta(1-\beta)}\, x^2\right), \quad \text{if } \beta < \frac{1}{2}. \end{aligned}$$

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