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Integral operators on the halfspace in generalized Lebesgue spaces $L^{p(\cdot)}$, part I

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Abstract

In this paper we generalize a version of the classical Calderón–Zygmund theorem on principle value integrals in generalized Lebesgue spaces $L^{p(\cdot)}$ proved in [J. Reine Angew. Math. 563 (2003) 197–220], to kernels, which do not satisfy standard estimates on $\mathbb{R}^{d+1}$. This result will be used in part II of this paper to prove the classical theorem on halfspace estimates of Agmon, Douglis, and Nirenberg [Comm. Pure Appl. Math. 12 (1959) 623–727] for generalized Lebesgue spaces $L^{p(\cdot)}$.

Keywords: Calderón–Zygmund theorem; Singular integral operator; Generalized Lebesgue spaces $L^{p(\cdot)}$

1. Introduction

Motivated by the study of electrorheological fluids the authors have been interested in transferring techniques known for generalized Newtonian fluids to the case of electrorheological fluids (see, e.g., Málek et al. [16], Frehse et al. [13], Růžička [19], and Diening [9] on a survey on existence and regularity results for generalized Newtonian flu-

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ids and electrorheological fluids). More precisely, the motion of generalized Newtonian fluids is governed by (cf. [17] for a detailed discussion of generalized Newtonian fluids)

$$\partial_t \mathbf{v} - \operatorname{div} \mathbf{S} + [\nabla \mathbf{v}]\mathbf{v} + \nabla \pi = \mathbf{f}, \qquad \operatorname{div} \mathbf{v} = 0, \tag{1}$$

where the extra stress tensor $\mathbf{S}$ is given by $\mathbf{S} = \mu(1 + |\mathbf{D}|^2)^{(p-2)/2}\mathbf{D}$ and $p \in (1,\infty)$ is a given material constant. Thus the natural energy space for the system of generalized Newtonian fluids is $W^{1,p}(\Omega)$. The motion of electrorheological fluids is governed by a system similar to (1), however the extra stress tensor is now given by (cf. [19])

$$\mathbf{S} = \alpha_{21}\big(\big(1 + |\mathbf{D}|^2\big)^{(p-1)/2} - 1\big)\mathbf{E} \otimes \mathbf{E} + \big(\alpha_{31} + \alpha_{33}|\mathbf{E}|^2\big)\big(1 + |\mathbf{D}|^2\big)^{(p-2)/2}\mathbf{D} \\ + \alpha_{51}\big(1 + |\mathbf{D}|^2\big)^{(p-2)/2}(\mathbf{DE} \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{DE}),$$

where α_{ij} are constants and $p = p(|\mathbf{E}|^2)$ is a given material function satisfying

$$1 < p^- \leqslant p\big(|\mathbf{E}|^2\big) \leqslant p^+ < \infty.$$

Therefore the natural energy space for the system of electrorheological fluids is the generalized Sobolev space $W^{1,p(\cdot)}(\Omega)$. One of the main issues in the study of the above systems is to prove the existence of solutions, where the values of p and p^-, respectively, are as small as possible. The applied techniques essentially use optimal estimates for solutions of linear elliptic equations and systems, e.g., the Laplace equation, the Stokes system and the divergence equation. These estimates are classical results in the usual Lebesgue spaces. However, in generalized Lebesgue spaces only little is known. The divergence equation has already been treated in Růžička and Diening [5]. In that paper the classical theorem by Calderón and Zygmund [3] on principal value integrals and the continuity of classical Calderón–Zygmund operators has also been extended to generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^d)$. With the help of these results one can easily show interior regularity for elliptic equations and systems in generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$. In order to treat the regularity near the boundary in these spaces, one needs corresponding results for the halfspace. It is the purpose of parts I and II of this paper to establish these results. In the present part we generalize a version of the classical Calderón–Zygmund theorem on principle value integrals in generalized Lebesgue spaces $L^{p(\cdot)}$ proved in [5], to kernels, which do not satisfy standard estimates on $\mathbb{R}^{d+1}$. Based on this result we prove the analogue of Lemma 3.2 in [1]. This result will be used in part II of the paper [6] to establish the analogue of the halfspace estimates by Agmon et al. [1].

2. A Calderón–Zygmund type result on $\mathbb{R}^{d+1}$

Let us introduce some notation. Points in $\mathbb{R}^{d+1}$ will be denoted by $P := (x,t)$, $Q := (y,s)$ and $R := (z,u)$, with $x, y, z \in \mathbb{R}^d$. We set $|x| := (\sum_{i=1}^d x_i^2)^{1/2}$ and $|P| := (|x|^2 + t^2)^{1/2}$. For all $P \in \mathbb{R}^{d+1}$ holds $\frac{1}{2}(|x| + |t|) \leqslant |P| \leqslant |x| + |t|$. By $\mathbb{R}^{d+1}_{\geqslant} := \{P \in \mathbb{R}^{d+1} \mid t \geqslant 0\}$ and $\mathbb{R}^{d+1}_{\leqslant} := \{P \in \mathbb{R}^{d+1} \mid t \leqslant 0\}$ we denote halfspaces and by $\mathbb{R}^{d+1}_{>}$ (respectively $\mathbb{R}^{d+1}_{<}$) the corresponding counterparts with strict inequalities. For a function $f : \mathbb{R}^{d+1} \to \mathbb{R}$ we denote the partial derivatives with respect to the ith variable, $i = 1, \ldots, d$, by $\partial_i f$, while the partial derivative with respect to the $(d+1)$-variable is denoted by $\partial_t f$. The gradient ∇f stands for $\nabla f := (\partial_1 f, \dots, \partial_d f, \partial_t f)$.

We will now introduce the spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$. Let Ω be a measurable subset of $\mathbb{R}^{d+1}$. For a measurable function $p : \mathbb{R}^{d+1} \to [1, \infty)$ (called the exponent) we define $L^{p(\cdot)}(\Omega)$ to consist of measurable functions $f : \Omega \to \mathbb{R}$ such that the modular $\rho_p(f) := \int_\Omega |f(Q)|^{p(Q)}\, dQ$ is finite. If $p^+ := \sup p < \infty$ (called a bounded exponent), then the expression $\|f\|_{p(\cdot)} := \inf\{\lambda > 0 \mid \rho_p(\lambda^{-1} f) < 1\}$ defines a norm on $L^{p(\cdot)}(\Omega)$. This makes $L^{p(\cdot)}(\Omega)$ a Banach space. If $p^- := \inf p > 1$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex and reflexive, and the dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $1/p(\cdot) + 1/p'(\cdot) = 1$. Further, let $W^{1,p(\cdot)}(\Omega)$ denote the space of measurable functions $f : \Omega \to \mathbb{R}$ such that f and the distributional derivative ∇f are in $L^{p(\cdot)}(\Omega)$. The norm $\|f\|_{1,p(\cdot)} := \|f\|_{p(\cdot)} + \|\nabla f\|_{p(\cdot)}$ makes $W^{1,p(\cdot)}(\Omega)$ a Banach space. By $W_0^{1,p(\cdot)}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. We refer to Hudzik [14], Kováčik and Rákosník [15], Samko [20], Edmunds et al. [10], Růžička [19], Edmunds and Rákosník [11], Fan et al. [12], Diening [7–9] for a detailed discussion of the spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$.

By B we denote an arbitrary ball in $\mathbb{R}^{d+1}$. We write $B(P)$ for a ball centered at P and B_r for a ball with radius r. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{d+1})$ we set

$$M_B f := \fint_B \left| f(Q) \right| dQ,$$

where $\fint_B$ is the mean value integral over B. By Mf we denote the Hardy–Littlewood maximal function of f, i.e.,

$$Mf(P) := \sup_{B(P)} M_{B(P)} f,$$

where the supremum is taken over all balls centered at P. By $\mathcal{P}(\mathbb{R}^{d+1})$ we denote the set of bounded exponents p such that M is bounded on $L^{p(\cdot)}(\mathbb{R}^{d+1})$. In particular, if $p \in \mathcal{P}(\mathbb{R}^{d+1})$ then $C_0^\infty(\mathbb{R}^{d+1})$ is dense in $W^{k,p(\cdot)}(\mathbb{R}^{d+1})$, $k \in \mathbb{N}_0$ (cf. [7]).

It has been shown by Diening [7] that if p satisfies $1 < p^- \leqslant p^+ < \infty$, the uniform, local continuity condition

$$\left| p(P) - p(Q) \right| \leqslant A_1 \left| \ln |P - Q| \right|^{-1}, \quad P, Q \in \mathbb{R}^{d+1}, \tag{2}$$

where A_1 is a given constant, and p in addition is constant outside some large ball $B_{R_0}(0)$, then $p \in \mathcal{P}(\mathbb{R}^{d+1})$. Later is was shown by Nekvinda [18] that the condition that p is constant outside some large ball $B_{R_0}(0)$ can be weakened to the integral condition: there exists a constant $\gamma > 0$ and $p_\infty \in [p^-, p^+]$ such that $\int_{\mathbb{R}^{d+1}} \gamma^{1/|p(P) - p_\infty|}\, dP < \infty$. In particular, if p satisfies the decay condition

$$\left| p(P) - p_\infty \right| \leqslant \frac{A_2}{\ln(e + |P|)}, \quad P \in \mathbb{R}^{d+1}, \tag{3}$$

where $p_\infty \in [p^-, p^+]$ and $A_2 > 0$ are given constants, one easily checks that the integral condition above is fulfilled (cf. [4] for a different proof of the same result). Thus we have $p \in \mathcal{P}(\mathbb{R}^{d+1})$ if the conditions (2) and (3) are satisfied for all $P, Q \in R^{d+1}$.

We also need the following maximal type operators: Let $0 < \alpha < \infty$ and $f \in L^{\alpha}_{\mathrm{loc}}(\mathbb{R}^{d+1})$. Then for all balls B we define

$$M_{\alpha,B} f := \left(\fint_B \left| f(Q) \right|^{\alpha} dQ \right)^{1/\alpha}, \qquad M_\alpha f(P) := \sup_{B(P)} M_{\alpha,B(P)} f,$$

$$M^{\#}_{\alpha,B} f := \left(\fint_B \left| f(Q) - f_B \right|^{\alpha} dQ \right)^{1/\alpha}, \qquad M^{\#}_\alpha f(x) := \sup_{B(P)} M^{\#}_{\alpha,B(P)} f,$$

where $f_B := \fint_B f(Q)\, dQ$. The operator $M^{\#}_1$ is called sharp operator. Note that $M_1 f = Mf$ and that for all $\alpha_1 \leqslant \alpha_2$ there holds $M_{\alpha_1} f \leqslant M_{\alpha_2} f$ and $M^{\#}_{\alpha_1} f \leqslant M^{\#}_{\alpha_2} f$ due to Jensen's inequality. In Diening and Růžička [5] it is shown that for all $f \in L^{p(\cdot)}(\mathbb{R}^{d+1})$

$$c\|f\|_{p(\cdot)} \leqslant \left\| M^{\#}_1 f \right\|_{p(\cdot)} \leqslant C\|f\|_{p(\cdot)},$$

whenever $p, p' \in \mathcal{P}(\mathbb{R}^{d+1})$ and $1 < p^- \leqslant p^+ < \infty$. This equivalence is crucial for proving the continuity of Calderón–Zygmund operators in generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^{d+1})$.

The aim of this section is to generalize Corollary 4.12 in [5], which is the version of the classical Calderón Zygmund theorem on principal value integrals in $L^{p(\cdot)}(\mathbb{R}^{d+1})$, to kernels, which do not satisfy standard estimates on $\mathbb{R}^{d+1}$. For that we need to generalize the notion of a standard kernel as follows:

Definition 2.1. Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a set. A kernel K on Ω is a locally integrable real-valued function defined on $\Omega \backslash \{0\}$. We say that K satisfies standard estimates of degree $-m$ on $\Omega_0 \subseteq \Omega$ if there exist $\delta > 0$ and $A_4 > 0$, such that for all $P, Q \in \Omega_0 \backslash \{0\}$ with $|P - Q| < \frac{1}{2}|Q|$ and all $R \in \Omega_0 \backslash \{0\}$ holds

$$\left| K(R) \right| \leqslant A_4 |R|^{-m}, \tag{4a}$$

$$\left| K(P) - K(Q) \right| \leqslant A_4 |P - Q|^{\delta} |Q|^{-m-\delta}. \tag{4b}$$

Note that (4a) and (4b) imply that K is δ-Hölder continuous on $\Omega_0 \backslash \{0\}$ and bounded on every sphere $\Omega_0 \cap \{P \mid |P| = r_0\}$, $0 < r_0 < \infty$. The sets Ω and Ω_0 in the above definition will usually be one of the sets $\mathbb{R}^{d+1}_{>}$, $\mathbb{R}^{d+1}_{<}$, $\mathbb{R}^{d+1}_{\geqslant}$ or $\mathbb{R}^{d+1}$.

We say that a operator T is associated to a kernel K on $\mathbb{R}^{d+1}$ if

$$Tf(P) = \int_{\mathbb{R}^{d+1}} k(P - Q) f(Q)\, dQ$$

holds for a.e. P outside the support of $f \in C^{\infty}_0(\mathbb{R}^{d+1})$. T is said to be a singular integral operator if T is associated to a kernel on $\mathbb{R}^{d+1}$, which satisfies standard estimates of degree $-(d + 1)$ on $\mathbb{R}^{d+1}$. If in addition T extends to a bounded, linear operator on $L^2(\mathbb{R}^{d+1})$, then we call T a Calderón–Zygmund operator.

Since we are interested in kernels, like

$$K(P) = \mathrm{sgn}(t)|P|^{-d-1}, \tag{5}$$

which satisfy standard estimates on $\mathbb{R}^{d+1}_>$ and $\mathbb{R}^{d+1}_<$, but not on $\mathbb{R}^{d+1}$, we need a modification of a definition of Alvarez and Pérez [2], which reads as follows:

Definition 2.2. For a kernel K on $\mathbb{R}^{d+1}$ we define for all $r > 0$ and all $Q \in \mathbb{R}^{d+1} \setminus \{0\}$

$$F_r K(Q) := \fint_{B_r(0)} \fint_{B_r(0)} \left| K(P - Q) - K(R - Q) \right| dR\, dP.$$

For $\alpha \geqslant 1$ we say that the kernel K satisfies condition (D_α) if and only if there are constants $A_5, N > 0$ such that

$$\sup_{r>0} \int_{|Q|>Nr} \left| f(Q + P_0) \right| F_r K(Q)\, dQ \leqslant A_5 M_\alpha f(P_0) \tag{D_α}$$

holds for all $f \in C_0^\infty(\mathbb{R}^{d+1})$ and $P_0 \in \mathbb{R}^{d+1}$.

Note that for $\alpha = 1$ this is exactly condition (D) of Alvarez and Pérez [2].

Lemma 2.3. *Let K be a homogeneous kernel of degree $-(d+1)$ on $\mathbb{R}^{d+1}$, which satisfies standard estimates on $\mathbb{R}^{d+1}_>$ and on $\mathbb{R}^{d+1}_<$ of degree $-(d+1)$. Then K satisfies condition (D_α) for all $\alpha > 1$.*

Proof. From the definition of $F_r K$ and the homogeneity of K we easily compute that for all $r > 0$ holds

$$F_r K(rP) = r^{-(d+1)} F_1 K(P),$$

and thus we have for all $f \in C_0^\infty(\mathbb{R}^{d+1})$

$$\int_{|Q|>Nr} \left| f(Q + P_0) \right| F_r K(Q)\, dQ = \int_{|Q|>N} \left| f_r(Q + P_0) \right| F_1 K(Q)\, dQ, \tag{6}$$

where $f_r(Q + P_0) := f(rQ + P_0) \in C_0^\infty(\mathbb{R}^{d+1})$. In order to verify condition (D_α), it suffices to show that for all $g \in C_0^\infty(\mathbb{R}^{d+1})$ holds

$$\int_{|Q|>N} \left| g(Q + P_0) \right| F_1 K(Q)\, dQ \leqslant C M_\alpha g(P_0). \tag{7}$$

Indeed, we choose $g = f_r$ in (7), use (6), take the supremum over $r > 0$ and use that $M_\alpha f_r(P_0) = M_\alpha f(P_0)$. For $|P|, |R| \leqslant 1$ and $Q = (y, s)$ we see that $R - Q, P - Q \in \mathbb{R}^{d+1}_<$ if $s > 1$ and that $R - Q, P - Q \in \mathbb{R}^{d+1}_>$ if $s < -1$. Thus we can rewrite the left-hand side of (7) as

$$\int_{\substack{|Q|>N \\ s>1}} \left| g(Q + P_0) \right| F_1 K(Q)\, dQ + \int_{\substack{|Q|>N \\ s<-1}} \left| g(Q + P_0) \right| F_1 K(Q)\, dQ$$
$$+ \int_{\substack{|Q|>N \\ |s|\leqslant 1}} \left| g(Q + P_0) \right| F_1 K(Q)\, dQ =: I_1 + I_2 + I_3.$$

For the terms I_1 and I_2 we use (4b) since for $N > 5$ and $|P|, |R| \leqslant 1$ we have $|P - Q| > 2|(P - Q) - (R - Q)|$. Moreover, we have $|P - Q| > |Q|/2$ and thus we can estimate

$$
\begin{aligned}
I_1 + I_2 &\leqslant 2A_4 \int\limits_{|Q|>N} \bigl|g(Q + P_0)\bigr| \fint\limits_{B_1(0)} \fint\limits_{B_1(0)} \frac{|P - R|^\delta}{|P - Q|^{d+1+\delta}}\, dR\, dP\, dQ \\
&\leqslant 2^{d+2+2\delta} A_4 \int\limits_{|Q|>N} \frac{|g(Q + P_0)|}{|Q|^{d+1+\delta}}\, dQ \\
&\leqslant c(A_4, \delta) \sum_{j=0}^{\infty} \int\limits_{2^j N<|Q|\leqslant 2^{j+1}N} \frac{|g(Q + P_0)|}{(2^j N)^{d+1+\delta}}\, dQ \\
&\leqslant c(A_4, \delta, d) \sum_{j=0}^{\infty} \frac{1}{(2^j N)^\delta} \fint\limits_{B_{2^{j+1}N}(0)} \bigl|g(Q + P_0)\bigr|\, dQ \\
&\leqslant c(A_4, \delta, d, N) Mg(P_0) \leqslant c(A_4, d, \delta, N) M_\alpha g(P_0).
\end{aligned}
\tag{8}
$$

For the term I_3 we use (4a) and $|P - Q|, |R - Q| > |Q|/2$ to derive

$$
\begin{aligned}
I_3 &\leqslant A_4 2^{d+2} \int\limits_{\substack{|Q|>N \\ |s|\leqslant 1}} \frac{|g(Q + P_0)|}{|Q|^{d+1}}\, dQ \\
&\leqslant c(A_4, d) \sum_{j=0}^{\infty} \frac{1}{(2^j N)^{d+1}} \int\limits_{B_{2^{j+1}N}(0)} \bigl|g(Q + P_0)\bigr| \chi_{N_j}(Q)\, dQ \\
&\leqslant c(A_4, d) \sum_{j=0}^{\infty} \frac{1}{(2^j N)^{d+1}} \Biggl(\int\limits_{B_{2^{j+1}N}(0)} \bigl|g(Q + P_0)\bigr|^\alpha\, dQ \Biggr)^{1/\alpha} \operatorname{vol}(N_j)^{1-1/\alpha} \\
&\leqslant c(A_4, d) \sum_{j=0}^{\infty} \frac{1}{(2^j N)^{d+1}} M_\alpha g(P_0) \operatorname{vol}(N_j)^{1-1/\alpha} \operatorname{vol}\bigl(B_{2^{j+1}N}(0)\bigr)^{1/\alpha} \\
&= c(A_4, d) M_\alpha g(P_0) \sum_{j=0}^{\infty} \frac{1}{(2^j N)^{1-1/\alpha}} \leqslant c(A_4, d, N, \alpha) M_\alpha g(P_0),
\end{aligned}
\tag{9}
$$

where we used $\alpha > 1$ and where χ_{N_j} is the characteristic function of the set $N_j := \{Q = (y, s) \mid 2^j N < |Q| \leqslant 2^{j+1}N,\ |s| \leqslant 1\}$. Estimates (8) and (9) imply (7) and thus the lemma is proved. □

For a kernel K on $\mathbb{R}^{d+1}$ we define the truncated kernels K_ε for $\varepsilon > 0$ through

$$
K_\varepsilon(P) := \begin{cases} K(P) & \text{for } |P| > \varepsilon, \\ 0 & \text{for } |P| \leqslant \varepsilon. \end{cases}
$$

Furthermore, we define for $\varepsilon > 0$

$$T_\varepsilon f(P) := \int_{\mathbb{R}^{d+1}} K_\varepsilon(P-Q) f(Q)\, dQ;$$

in particular, T_ε is associated to the kernel K_ε.

Proposition 2.4. *Let K be a kernel on $\mathbb{R}^{d+1}$, which satisfies standard estimates on $\mathbb{R}^{d+1}_>$ and on $\mathbb{R}^{d+1}_<$ of degree $-(d+1)$. Moreover, assume that for the surface integral of K over the unit sphere in $\mathbb{R}^{d+1}$ holds*

$$\int_{\partial B_1(0)} K(Q)\, d\omega = 0. \tag{10}$$

Then for every $1 < q < \infty$ the operators T_ε are uniformly bounded on $L^q(\mathbb{R}^{d+1})$ with respect to $\varepsilon > 0$. Moreover,

$$Tf(P) := \lim_{\varepsilon \to 0^+} T_\varepsilon f(P) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^{d+1}} K_\varepsilon(P-Q) f(Q)\, dQ \tag{11}$$

exists almost everywhere and $\lim_{\varepsilon \to 0^+} T_\varepsilon f = Tf$ in $L^q(\mathbb{R}^{d+1})$ norm. In particular, T is continuous on $L^q(\mathbb{R}^{d+1})$.

Proof. From (4a) follows that K is bounded by A_4 on the unit sphere. Thus all assumptions of the classical theorem of Calderón and Zygmund [3] are fulfilled and the assertion follows. □

Proposition 2.5. *Let K be a kernel on $\mathbb{R}^{d+1}$, which satisfies the same assumption as in Proposition 2.4. Then the operators T_ε, T are of weak type $(1,1)$ uniformly with respect to ε.*

Proof. We want to use Corollary I.7.1 in [21, p. 33]. Thus we have to verify condition (10) there, which in our context reads: there exists a constant $C > 0$ such that

$$\int_{|Q| \geqslant 2r} \bigl|K(Q-P) - K(Q)\bigr|\, dQ \leqslant C, \tag{12}$$

whenever $|P| < r$. For such $P = (x,t)$ and $Q = (y,s)$ we see that $Q, Q-P \in \mathbb{R}^{d+1}_<$ if $s < -r$ and that $Q, Q-P \in \mathbb{R}^{d+1}_>$ if $s > r$. Thus we can rewrite the left-hand side of (12) as

$$\int_{\substack{|Q| \geqslant 2r \\ s > r}} \bigl|K(Q-P) - K(Q)\bigr|\, dQ + \int_{\substack{|Q| \geqslant 2r \\ s < -r}} \bigl|K(Q-P) - K(Q)\bigr|\, dQ + \int_{\substack{|Q| \geqslant 2r \\ |s| \leqslant r}} \bigl|K(Q-P) - K(Q)\bigr|\, dQ =: I_1 + I_2 + I_3.$$

Using (4b) and $|P| < r$ one easily computes (cf. (8))

$$I_1 + I_2 \leqslant c(A_4, d, \delta) r^\delta \int\limits_{|Q| \geqslant 2r} \frac{dQ}{|Q|^{d+1+\delta}} \leqslant c(A_4, d, \delta), \tag{13}$$

where we have also used that $2|Q - P| > |Q|$. Using again this fact and (4a) we get similarly as in Lemma 2.3 (cf. (9))

$$\begin{aligned} I_3 &\leqslant c(A_4, d) \int\limits_{\substack{|Q|>2r \\ |s| \leqslant r}} \frac{dQ}{|Q|^{d+1}} \leqslant c(A_4, d) \sum_{j=1}^{\infty} \frac{1}{(2^j r)^{d+1}} \int\limits_{\substack{2^j r \leqslant |Q| < 2^{j+1} r \\ |s| \leqslant r}} dQ \\ &\leqslant c(A_4, d) \sum_{j=1}^{\infty} \frac{1}{2^j} \leqslant c(A_4, d). \end{aligned} \tag{14}$$

From (13) and (14) we immediately get (12) and thus Corollary I.7.1 in [21] implies that T_ε are of weak type $(1, 1)$ uniformly with respect to ε. That the same holds true for T now follows easily (cf. Remark 4.4 in [5]). □

Corollary 2.6. *Let K be a homogeneous kernel of degree $-(d+1)$ on $\mathbb{R}^{d+1}$, which satisfies the same assumptions as in Proposition* 2.4. *Let T be the operator defined by* (11). *Then, for all s_1, s_2 with $0 < s_1 < 1 < s_2$, there exists a constant $A_6 = A_6(s_1, s_2) > 0$ such that for all $f \in C_0^\infty(\mathbb{R}^d)$ and $P \in \mathbb{R}^{d+1}$ holds*

$$\left(M_1^\#\left(|Tf|^{s_1}\right)\right)^{1/s_1}(P) \leqslant A_6 M_{s_2} f(P). \tag{15}$$

Proof. Proposition 2.5 implies that T is of weak type $(1, 1)$. Thus we can proceed exactly as in the proof of Theorem 2.1 of [2]. However, in the last step we use our condition (D_{s_2}), which holds due to Lemma 2.3, instead of condition (D) in [2] to obtain the desired assertion. □

Theorem 2.7. *Let K be a homogeneous kernel of degree $-(d+1)$ on $\mathbb{R}^{d+1}$, which satisfies the same assumptions as in Proposition* 2.4. *Let T be the operator defined by* (11). *Let p be a bounded exponent with $p^- > 1$ and $0 < s_1 < 1 < s_2 < p^-$ such that $p, (p/s_1)', p/s_2 \in \mathcal{P}(\mathbb{R}^{d+1})$. Then T is a bounded operator on $L^{p(\cdot)}(\mathbb{R}^{d+1})$, i.e., there exists a constant $A_7 > 0$, such that*

$$\|Tf\|_{L^{p(\cdot)}(\mathbb{R}^{d+1})} \leqslant A_7 \|f\|_{L^{p(\cdot)}(\mathbb{R}^{d+1})}.$$

Proof. Since $p \in \mathcal{P}(\mathbb{R}^{d+1})$ and $0 < s_1 < 1$, there holds $p/s_1 \in \mathcal{P}(\mathbb{R}^{d+1})$ by Remark 2.3 in [5]. Thus it follows from Theorem 3.6 in [5] that for all $g \in L^{p(\cdot)}(\mathbb{R}^{d+1})$ holds

$$\|g\|_{L^{p(\cdot)}(\mathbb{R}^{d+1})} \leqslant C \left\| M_1^\# g \right\|_{L^{p(\cdot)}(\mathbb{R}^{d+1})}. \tag{16}$$

Let $f \in C_0^\infty(\mathbb{R}^{d+1})$. Then $Tf \in L^{p^-}(\mathbb{R}^{d+1}) \cap L^{p^+}(\mathbb{R}^{d+1})$ due to Proposition 2.4, which implies $Tf \in L^{p(\cdot)}(\mathbb{R}^{d+1})$ and $(Tf)^{s_1} \in L^{p(\cdot)/s_1}(\mathbb{R}^{d+1})$. This justifies the following calculations:

$$\begin{aligned}
\|Tf\|_{p(\cdot)} &= \big\| |Tf|^{s_1} \big\|_{p(\cdot)/s_1}^{1/s_1} \\
&\overset{(16)}{\leqslant} C \big\| M_1^\#\big(|Tf|^{s_1}\big)\big\|_{p(\cdot)/s_1}^{1/s_1} = C \big\| \big(M_1^\#\big(|Tf|^{s_1}\big)\big)^{1/s_1}\big\|_{p(\cdot)} \\
&\overset{(15)}{\leqslant} C \|M_{s_2} f\|_{p(\cdot)} = C \big\| M\big(|f|^{s_2}\big)\big\|_{p(\cdot)/s_2}^{1/s_2} \\
&\leqslant C \big\| |f|^{s_2}\big\|_{p(\cdot)/s_2}^{1/s_2} = C\|f\|_{p(\cdot)}, \qquad (17)
\end{aligned}$$

where we used in the last line $p/s_2 \in \mathcal{P}(\mathbb{R}^{d+1})$. Since $C_0^\infty(\mathbb{R}^{d+1})$ is dense in $L^{p(\cdot)}(\mathbb{R}^{d+1})$, this proves the theorem. □

In order to transfer the statements of Proposition 2.4 to the spaces $L^{p(\cdot)}(\mathbb{R}^{d+1})$ we need a modification of a classical result for the maximal truncated operator T_* (cf. Proposition I.7.2 in [21]), which is defined by

$$T_* f(P) := \sup_{\varepsilon > 0} \big|T_\varepsilon f(P)\big|.$$

Proposition 2.8. *Let K be a kernel on $\mathbb{R}^{d+1}$, which satisfies the same assumptions as in Proposition* 2.4. *Let T be the operator defined by* (11) *and let* $0 < s_1 \leqslant 1 < s_2$. *Then there exists a constant* $A_8 = A_8(s_1, s_2) > 0$, *such that*

$$T_* f(P) \leqslant A_8\big(M_{s_1}(Tf)(P) + M_{s_2} f(P)\big)$$

for all $f \in C_0^\infty(\mathbb{R}^{d+1})$ *and all* $P \in \mathbb{R}^{d+1}$.

Proof. Let us fix $P_0 \in \mathbb{R}^{d+1}$, $\varepsilon > 0$ and $f \in C_0^\infty(\mathbb{R}^{d+1})$. We decompose f as $f = f_1 + f_2$, where $f_1 := f\chi_{B_\varepsilon(P_0)}$ and $f_2 := f\chi_{\mathbb{R}^{d+1}\setminus B_\varepsilon(P_0)}$. By definition of K_ε we have $Tf_2(P_0) = T_\varepsilon f(P_0)$. We will prove that for all $P \in B_{\kappa\varepsilon}(P_0)$, $0 < \kappa < 1/2$, and $1 < s_2$ holds

$$\big|Tf_2(P_0) - Tf_2(P)\big| \leqslant C M_{s_2} f(P_0). \qquad (18)$$

Indeed, the left-hand side of (18) is bounded by

$$\begin{aligned}
&\int\limits_{|Q-P_0|>\varepsilon} \big|K(P_0 - Q) - K(P - Q)\big|\big|f(Q)\big|\, dQ \\
&\quad = \int\limits_{|Q|>\varepsilon} \big|K(Q) - K(P - P_0 + Q)\big|\big|f(P_0 - Q)\big|\, dQ. \qquad (19)
\end{aligned}$$

The domain of integration in the last integral is split again into three parts, namely $E_1 := \{Q = (y, s) \mid |Q| > \varepsilon,\ s > \kappa\varepsilon\}$, $E_2 := \{Q = (y, s) \mid |Q| > \varepsilon,\ s < -\kappa\varepsilon\}$ and $E_3 := \{Q = (y, s) \mid |Q| > \varepsilon,\ |s| \leqslant \kappa\varepsilon\}$. Note that $E_1 \subset \mathbb{R}^{d+1}_>$ and $E_2 \subset \mathbb{R}^{d+1}_<$ and thus we can use (4b) on these sets. Let us denote again the integrals on the right-hand side of (19) over E_i by I_i, $i = 1, 2, 3$. Since $|P - P_0| < \kappa\varepsilon < \frac{1}{2}|Q|$ we obtain similarly as in (8) (carefully tracking the dependencies on ε, cf. (13)) that

$$I_1 + I_2 \leqslant c(A_4, \delta, d, \kappa) Mf(P_0) \leqslant c(A_4, \delta, d, \kappa) M_{s_2} f(P_0).$$

For the term I_3 we proceed as in (9), carefully using the definition of E_3, to derive

$$I_3 \leqslant c(A_4, d, \kappa) M_{s_2} f(P_0).$$

The last two inequalities prove (18). Thus we have for all $P \in B_{\kappa\varepsilon}(P_0)$

$$\left|T_\varepsilon f(P_0)\right| \leqslant \left|Tf(P)\right| + \left|Tf_1(P)\right| + C M_{s_2} f(P_0). \tag{20}$$

Due to Proposition 2.4 we can now proceed exactly as in the proof of Proposition I.7.2 in [21] to show that there exists a point $P \in B_{\kappa\varepsilon}(P_0)$ such that

$$\left|Tf(P)\right| + \left|Tf_1(P)\right| \leqslant c\big(M_{s_1}(Tf)(P) + Mf(P)\big). \tag{21}$$

From (20), (21) and $Mf \leqslant M_{s_2} f$ we obtain the assertion of the proposition. □

Corollary 2.9. *Let K be a homogeneous kernel of degree $-(d+1)$ on $\mathbb{R}^{d+1}$, which satisfies the same assumptions as in Proposition* 2.4. *Let p be a bounded exponent with $p^- > 1$ and $0 < s_1 < 1 < s_2 < p^-$ such that $p, (p/s_1)', p/s_2 \in \mathcal{P}(\mathbb{R}^{d+1})$. Then T_* is bounded on $L^{p(\cdot)}(\mathbb{R}^{d+1})$, i.e., there exists a constant $A_9 > 0$, such that*

$$\|T_* f\|_{L^{p(\cdot)}(\mathbb{R}^{d+1})} \leqslant A_9 \|f\|_{L^{p(\cdot)}(\mathbb{R}^{d+1})}.$$

Proof. Let $f \in C_0^\infty(\mathbb{R}^{d+1})$ with $\|f\|_{p(\cdot)} \leqslant 1$. By Theorem 2.7 we have $\|Tf\|_{p(\cdot)} \leqslant A_7$. Since $p \in \mathcal{P}(\mathbb{R}^{d+1})$, this implies $\|M(Tf)\|_{p(\cdot)} \leqslant C$. In (17) we have shown that $\|M_{s_2} f\|_{p(\cdot)} \leqslant C\|f\|_{p(\cdot)} \leqslant C$. Now Proposition 2.8 with $s_1 = 1$ implies $\|T_* f\|_{p(\cdot)} \leqslant C$. Since $T_*(\lambda f) = |\lambda| T_*(f)$ and $C_0^\infty(\mathbb{R}^{d+1})$ is dense in $L^{p(\cdot)}(\mathbb{R}^{d+1})$, this proves the corollary. □

Theorem 2.10. *Let K be a homogeneous kernel of degree $-(d+1)$ on $\mathbb{R}^{d+1}$, which satisfies the same assumptions as in Proposition* 2.4. *Let p be a bounded exponent with $p^- > 1$ and let $0 < s_1 < 1 < s_2 < p^-$ be such that $p, (p/s_1)', p/s_2 \in \mathcal{P}(\mathbb{R}^{d+1})$. Then the operators T_ε are uniformly bounded on $L^{p(\cdot)}(\mathbb{R}^{d+1})$ with respect to $\varepsilon > 0$. Moreover,*

$$Tf(P) = \lim_{\varepsilon\to 0^+} T_\varepsilon f(P) = \lim_{\varepsilon\to 0^+} \int\limits_{\mathbb{R}^{d+1}} K_\varepsilon(P-Q) f(Q)\, dQ$$

exists almost everywhere and $\lim_{\varepsilon\to 0^+} T_\varepsilon f = Tf$ in $L^{p(\cdot)}(\mathbb{R}^{d+1})$ norm. In particular, T, T_ε are uniformly continuous in $L^{p(\cdot)}(\mathbb{R}^{d+1})$ with respect to ε.

Proof. Due to Corollary 2.9 the operator T_* is bounded on $L^{p(\cdot)}(\mathbb{R}^{d+1})$. Since $|T_\varepsilon f(P)| \leqslant T^* f(P)$ for all $f \in L^{p(\cdot)}(\mathbb{R}^{d+1})$ and all $P \in \mathbb{R}^{d+1}$ by definition of T_*, there follows that the operators T_ε are uniformly bounded on $L^{p(\cdot)}(\mathbb{R}^{d+1})$ with respect to $\varepsilon > 0$. Now fix $f \in L^{p(\cdot)}(\mathbb{R}^{d+1})$. Then for all $\delta > 0$ there exists $g \in C_0^\infty(\mathbb{R}^{d+1})$ such that $\|f-g\|_{p(\cdot)} < \delta$ (cf. Corollary 2.5 in [5]). By Proposition 2.4 there holds $\lim_{\varepsilon\to 0^+} T_\varepsilon g = Tg$ almost everywhere. Since $|T_\varepsilon g| \leqslant T^* g \in L^{p(\cdot)}(\mathbb{R}^{d+1})$ by Corollary 2.9, there follows by the dominated convergence theorem that $Tg \in L^{p(\cdot)}(\mathbb{R}^{d+1})$ and $\lim_{\varepsilon\to 0^+} \rho_p(T_\varepsilon g - Tg) = 0$, which is equivalent to $\lim_{\varepsilon\to 0^+} \|T_\varepsilon g - Tg\|_{p(\cdot)} = 0$. Thus

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \|T_\varepsilon f - Tf\|_{p(\cdot)} \\
&\quad \leqslant \lim_{\varepsilon\to 0^+} \big(\|T_\varepsilon g - Tg\|_{p(\cdot)} + \big\|T_\varepsilon(f-g)\big\|_{p(\cdot)} + \big\|T(f-g)\big\|_{p(\cdot)}\big) \\
&\quad \leqslant \big\|T_*(f-g)\big\|_{p(\cdot)} + \big\|T(f-g)\big\|_{p(\cdot)} \\
&\quad \leqslant C\|f-g\|_{p(\cdot)} \leqslant C\delta,
\end{aligned} \tag{22}$$

where we used Corollary 2.9 and Theorem 2.7. Since $\delta > 0$ was arbitrary, this proves the theorem. □

As a consequence of Theorem 2.10 we can prove the following proposition, which is the analogue of Lemma 3.2 in [1].

Proposition 2.11. *Let K be a homogeneous kernel of degree $-(d+1)$ on $\mathbb{R}^{d+1}_>$, which satisfies standard estimates of degree $-(d+1)$ and is nonnegative on $\mathbb{R}^{d+1}_>$. Let p be a bounded exponent with $p^- > 1$ on $\mathbb{R}^{d+1}_>$ which is extended to $\mathbb{R}^{d+1}$ by an even reflection, i.e., $p(x,t) := p(x,-t)$, $t < 0$. Let $0 < s_1 < 1 < s_2 < p^-$ be such that $p, (p/s_1)', p/s_2 \in \mathcal{P}(\mathbb{R}^{d+1})$. Let G be a measurable function defined on $\mathbb{R}^{d+1}_>$ which satisfies for all $P \in \mathbb{R}^{d+1}_>$*

$$\big|G(P)\big| < K(P). \tag{23}$$

Consider the function

$$u(x,t) := \int_{\mathbb{R}^d} \int_0^\infty G(x-y, t+s) v(y,s)\, dy\, ds, \tag{24}$$

where $v \in L^{p(\cdot)}(\mathbb{R}^{d+1}_>)$. Then $u(P)$ exists for a.e. $P = (x,t) \in \mathbb{R}^{d+1}_>$ and there exists a constant $A_{10} > 0$ such that

$$\|u\|_{L^{p(\cdot)}(\mathbb{R}^{d+1}_>)} \leqslant A_{10}\|v\|_{L^{p(\cdot)}(\mathbb{R}^{d+1}_>)}. \tag{25}$$

Proof. We extend K and G to $\mathbb{R}^{d+1}$ by an odd reflection, i.e., $K(x,t) := -K(x,-t)$ and $G(x,t) := -G(x,-t)$, $t < 0$. Let us extend $v \in L^{p(\cdot)}(\mathbb{R}^{d+1}_>)$ to $\mathbb{R}^{d+1}$ by $v(t,x) = 0$, $t < 0$. Moreover, we denote $\tilde{v}(y,s) := v(y,-s) \in L^{p(\cdot)}(\mathbb{R}^{d+1})$ and set for all $(x,t) \in \mathbb{R}^{d+1}_>$

$$u_1(x,t) := \iint_{\mathbb{R}^{d+1}} K(x-y, t-s)\big|\tilde{v}(y,s)\big|\, dy\, ds. \tag{26}$$

Since K and p satisfy the assumptions of Theorem 2.10 we get

$$\|u_1\|_{L^{p(\cdot)}(\mathbb{R}^{d+1}_>)} \leqslant \|u_1\|_{L^{p(\cdot)}(\mathbb{R}^{d+1})} \leqslant C\|\tilde{v}\|_{L^{p(\cdot)}(\mathbb{R}^{d+1})} = C\|v\|_{L^{p(\cdot)}(\mathbb{R}^{d+1}_>)}. \tag{27}$$

Thus for almost all $(x,t) \in \mathbb{R}^{d+1}_>$ the function $u_1(x,t)$ is finite. Moreover, we have for almost all $(x,t) \in \mathbb{R}^{d+1}_>$

$$\int_{\mathbb{R}^d}\int_0^\infty \left|G(x-y,t+s)v(y,s)\right|dy\,ds$$
$$\leqslant \int_{\mathbb{R}^d}\int_0^\infty K(x-y,t+s)\left|v(y,s)\right|dy\,ds$$
$$= \iint_{\mathbb{R}^{d+1}} K(x-y,t+s)\left|v(y,s)\right|dy\,ds$$
$$= \iint_{\mathbb{R}^{d+1}} K(x-y,t-s)\left|\tilde{v}(y,s)\right|dy\,ds = u_1(x,t) < \infty. \tag{28}$$

Since the integrand in (26) is nonnegative, the last estimate proves that the integral in (24) is well defined for almost all $(x,t)\in\mathbb{R}^{d+1}_>$. From the definition of u (cf. (24)) and (28) we see that for all $P\in\mathbb{R}^{d+1}_>$ holds $|u(P)|\leqslant u_1(P)$, which together with (27) implies

$$\|u\|_{L^{p(\cdot)}(\mathbb{R}^{d+1}_>)} \leqslant \|u_1\|_{L^{p(\cdot)}(\mathbb{R}^{d+1}_>)} \leqslant C\|V\|_{L^{p(\cdot)}(\mathbb{R}^{d+1}_>)},$$

which proves the proposition. □

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