



Geometric properties of boundary sections of solutions to the Monge–Ampère equation and applications

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Received 31 May 2012; accepted 25 October 2012

Available online 13 November 2012

Communicated by Cédric Villani

Abstract

In this paper, we establish several geometric properties of boundary sections of convex solutions to the Monge–Ampère equation: the engulfing and separating properties and volume estimates. As applications, we prove a covering lemma of Besicovitch type, a covering theorem and a strong type p – p estimate for the maximal function corresponding to boundary sections. Moreover, we show that the Monge–Ampère setting forms a space of homogeneous type.

Published by Elsevier Inc.

Keywords: Monge–Ampère equation; Engulfing property; Covering theorem; Maximal function; Boundary section; Localization theorem

1. Introduction

In recent years, there has been a growing interest in studying boundary regularity of solutions to the Monge–Ampère equation and its linearization, that is, the linearized Monge–Ampère equation. Solutions of many important problems in Analysis and Geometry require a deep understanding of boundary behaviors of the above Monge–Ampère type equations. Among those, one can mention the problems of global regularity of the affine maximal surface equation

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¹ T. Nguyen gratefully acknowledges the support provided by NSF grant DMS-0901449.

[26,27,18] and Abreu's equation in the context of existence of Kähler metric of constant scalar curvature [10–13,28]. In these papers, the properties of boundary cross sections of solutions to the Monge–Ampère equation play an important role as those of interior sections in the landmark paper [3] where Caffarelli discovered surprising interior $W^{2,p}$ estimates for solutions to the Monge–Ampère equation with right hand side being continuous and bounded away from 0 and ∞ .

The notion of sections (or cross sections) of solutions to the Monge–Ampère equation

$$\det D^2\phi = g \quad \text{in } \Omega \quad (1.1)$$

was first introduced and studied by Caffarelli [2–5]. Sections are defined as sublevel sets of convex solutions after subtracting their supporting hyperplanes. Understanding the geometry of sections is essential in obtaining sharp regularity properties for solutions of (1.1). As a matter of fact, the structure of Eq. (1.1) is ultimately related to that of sections of its solutions, and by studying the shape of interior sections Caffarelli derived fundamental interior regularity estimates for (1.1) in the above mentioned papers. When the right hand side of the Monge–Ampère equation is only bounded, sections of solutions in the sense of Aleksandrov can have degenerate geometry. However, in many applications in Analysis and Geometry involving equations of Monge–Ampère type, we would like these sections to have properties similar to Euclidean balls as in uniformly elliptic equations. This is the case of interior sections through the work of Caffarelli [2,4,5], Caffarelli and Gutiérrez [6,7] and Gutiérrez and Huang [15]. The case of boundary sections is less well understood. However, thanks to Savin's Localization Theorem [22,23] at the boundary for solutions of (1.1), we expect many properties of interior sections hold also for boundary ones. This is the subject of our present paper.

The purpose of this paper is to investigate several important geometric properties of boundary sections of convex solutions to the Monge–Ampère equation (1.1) with right hand side bounded away from 0 and ∞ and with smooth boundary data: engulfing and separating properties, and volume estimates. As applications, we prove a covering lemma of Besicovitch type and employ it to prove a covering theorem and a strong type p – p estimate for the maximal function with respect to boundary sections. Moreover, we introduce a quasi-distance induced by boundary sections and show that the structure of our Monge–Ampère equation gives rise to a space of homogeneous type. This allows us to place the Monge–Ampère setting in a more general context where many real analytic problems have been studied, see [8,9]. Our results are boundary version of those established by Caffarelli and Gutiérrez [6,7], Gutiérrez and Huang [15] and Aimar, Forzani and Toledano [1] for interior sections of solutions to Eq. (1.1) (see also the book by Gutiérrez [14]). The results in this paper are crucial for our studies in [19,20] about boundary regularity for solutions to the linearized Monge–Ampère equation

$$\text{trace}(\Phi D^2u) = f \quad \text{in } \Omega, \quad (1.2)$$

where $\Phi := (\det D^2\phi) (D^2\phi)^{-1}$ with ϕ being a convex solution of (1.1). In [19,20], we investigate Eq. (1.2) and establish global $W^{2,p}$ and $W^{1,p}$ estimates for its solutions which are boundary version of interior estimates obtained recently in [16,17].

The rest of the paper is organized as follows. We state our main results in Section 2. In Section 3, we recall the main tool to study geometric properties of boundary sections of solutions to the Monge–Ampère equation: the Localization Theorem at the boundary for solutions. Geometric properties of boundary sections are established in Section 4. In Section 5, we prove a

Besicovitch-type covering lemma, a covering theorem and a strong type p - p estimate for the maximal function corresponding to boundary sections. Finally, in Section 6, we show that the Monge–Ampère setting forms a space of homogeneous type.

2. Statement of the main results

The results in this paper hold under the following global information on the convex domain Ω and the convex function ϕ . We assume there exists $\rho > 0$ such that

$$\Omega \subset B_{1/\rho}, \text{ and for each } y \in \partial\Omega \text{ there is a ball } B_\rho(z) \subset \Omega \text{ that is tangent to } \partial\Omega \text{ at } y. \quad (2.3)$$

Let $\phi : \overline{\Omega} \rightarrow \mathbb{R}$, $\phi \in C^{0,1}(\overline{\Omega}) \cap C^2(\Omega)$ be a convex function satisfying

$$\det D^2\phi = g, \quad 0 < \lambda \leq g \leq \Lambda \text{ in } \Omega. \quad (2.4)$$

Assume further that on $\partial\Omega$, ϕ separates quadratically from its tangent planes, namely

$$\rho|x - x_0|^2 \leq \phi(x) - \phi(x_0) - \nabla\phi(x_0) \cdot (x - x_0) \leq \rho^{-1}|x - x_0|^2, \quad \forall x, x_0 \in \partial\Omega. \quad (2.5)$$

The section of ϕ centered at $x \in \overline{\Omega}$ with height h is defined by

$$S_\phi(x, h) := \{y \in \overline{\Omega} : \phi(y) < \phi(x) + \nabla\phi(x) \cdot (y - x) + h\}.$$

If $x \in \partial\Omega$, then we call $S_\phi(x, h)$ a boundary section. For $x \in \Omega$, we denote by $\bar{h}(x)$ the maximal height of all sections of ϕ centered at x and contained in Ω , that is,

$$\bar{h}(x) := \sup\{h \geq 0 \mid S_\phi(x, h) \subset \Omega\}.$$

In this case, $S_\phi(x, \bar{h}(x))$ is called the maximal interior section of ϕ with center $x \in \Omega$. In what follows, we will drop the dependence on ϕ of sections when no confusion arises.

We denote by $c, \bar{c}, C, C_1, C_2, \theta_0, \theta_*, \dots$, positive constants depending only on $\rho, \lambda, \Lambda, n$, and their values may change from line to line whenever there is no possibility of confusion. We refer to such constants as *universal constants*.

Our first result is the engulfing property of sections $\{S(x, t)\}$.

Theorem 2.1. *Assume that the convex domain Ω and the convex function ϕ satisfy (2.3)–(2.5). There exists $\theta_* > 0$ depending only on ρ, λ, Λ and n such that if $y \in S(x, t)$ with $x \in \overline{\Omega}$ and $t > 0$, then $S(x, t) \subset S(y, \theta_*t)$.*

In this case, we say that the sections $\{S(x, t)\}$ of ϕ satisfy the engulfing property with the constant θ_* .

The engulfing property of sections will be shown to be equivalent to the separating property of sections as stated in the following.

Proposition 2.2.

- (i) Assume that the convex domain Ω and the convex function ϕ satisfy (2.3)–(2.5). Let θ_* be the constant in Theorem 2.1. Then, the sections $\{S(x, t)\}$ of ϕ satisfy the separating property with the constant θ_*^2 , namely, if $y \notin S(x, t)$, then

$$S\left(y, \frac{t}{\theta_*^2}\right) \cap S\left(x, \frac{t}{\theta_*^2}\right) = \emptyset.$$

- (ii) Conversely, assume that the sections $\{S(x, t)\}$ of a convex function ϕ defined on a convex domain Ω satisfy the separating property with the constant θ . Then the sections $\{S(x, t)\}$ satisfy the engulfing property with the constant θ^2 .

A key in the proof of Theorem 2.1 is a dichotomy for sections of solutions to the Monge–Ampère equation: any section is either an interior section or included in a boundary section with comparable height. Thus, when dealing with sections, we can focus our attention to only interior sections and boundary sections. The precise statement is as follows.

Proposition 2.3. Assume that the convex domain Ω and the convex function ϕ satisfy (2.3)–(2.5). Let $S(x_0, t_0)$ be a section of ϕ with $x_0 \in \overline{\Omega}$ and $t_0 > 0$. Then one of the following is true:

- (i) $S(x_0, 2t_0)$ is an interior section, that is, $S(x_0, 2t_0) \subset \Omega$;
- (ii) $S(x_0, 2t_0)$ is included in a boundary section with comparable height, that is, there exists $z \in \partial\Omega$ such that

$$S(x_0, 2t_0) \subset S(z, \bar{c}t_0).$$

Here $\bar{c} > 1$ is a constant depending only on ρ, λ, Λ and n .

As an application of the dichotomy of sections, we obtain the following volume growth of sections.

Corollary 2.4. Assume that the convex domain Ω and the convex function ϕ satisfy (2.3)–(2.5). Then, there exist constants c_0, C_1, C_2 depending only on ρ, λ, Λ and n such that for any section $S_\phi(x, t)$ with $x \in \overline{\Omega}$ and $t \leq c_0$, we have

$$C_1 t^{n/2} \leq |S_\phi(x, t)| \leq C_2 t^{n/2}. \tag{2.6}$$

By exploiting the geometric properties of boundary sections, we obtain the following covering lemma of Besicovitch type.

Lemma 2.5. Assume that the convex domain Ω and the convex function ϕ satisfy (2.3)–(2.5). Let $A \subset \overline{\Omega}$ and suppose that for each $x \in A$ a section $S(x, t)$ is given such that t is bounded by a fixed number M . If we denote by \mathcal{F} the family of all these sections, then there exists a countable subfamily of \mathcal{F} , $\{S(x_k, t_k)\}_{k=1}^\infty$, with the following properties:

- (i) $A \subset \bigcup_{k=1}^{\infty} S(x_k, t_k)$;
- (ii) $x_k \notin \bigcup_{j < k} S(x_j, t_j), \forall k \geq 2$;
- (iii) The family $\{S(x_k, \frac{t_k}{\alpha})\}_{k=1}^{\infty}$ is disjoint, where $\alpha = 2\theta_*^2$, and θ_* is the engulfing constant in Theorem 2.1;
- (iv) There exists a constant $K > 0$ depending only on ρ, λ, Λ and n such that

$$\sum_{k=1}^{\infty} \chi_{S(x_k, (1-\varepsilon)t_k)}(x) \leq K \log \frac{1}{\varepsilon} \quad \text{for all } 0 < \varepsilon < 1.$$

Our next result is the following covering theorem.

Theorem 2.6. Assume that the convex domain Ω and the convex function ϕ satisfy (2.3)–(2.5). Let $\mathcal{O} \subset \overline{\Omega}$ be a Lebesgue measurable set and $\varepsilon > 0$ small. Suppose that for each $x \in \mathcal{O}$ a section $S(x, t_x)$ is given with

$$\frac{|S(x, t_x) \cap \mathcal{O}|}{|S(x, t_x)|} = \varepsilon.$$

Then if $\sup \{t_x : x \in \mathcal{O}\} < +\infty$, there exists a countable subfamily of sections $\{S(x_k, t_k)\}_{k=1}^{\infty}$ satisfying

- (i) $\mathcal{O} \subset \bigcup_{k=1}^{\infty} S(x_k, t_k)$.
- (ii) $|\mathcal{O}| \leq \sqrt{\varepsilon} |\bigcup_{k=1}^{\infty} S(x_k, t_k)|$.

As an application of the covering lemma, we have the following global strong type p - p estimate for the maximal function with respect to sections.

Theorem 2.7. Assume that the convex domain Ω and the convex function ϕ satisfy (2.3)–(2.5). For $f \in L^1(\Omega)$, define

$$\mathcal{M}(f)(x) := \sup_{t > 0} \frac{1}{|S_{\phi}(x, t)|} \int_{S_{\phi}(x, t)} |f(y)| dy, \quad \forall x \in \Omega.$$

Then we have:

- (i) There exists a constant C depending only on ρ, λ, Λ and n such that

$$|\{x \in \Omega : \mathcal{M}(f)(x) > \beta\}| \leq \frac{C}{\beta} \int_{\Omega} |f(y)| dy, \quad \forall \beta > 0.$$

- (ii) For any $1 < p < \infty$, there exists C depending only on $p, \rho, \lambda, \Lambda$ and n such that

$$\left(\int_{\Omega} |\mathcal{M}(f)(x)|^p dx \right)^{\frac{1}{p}} \leq C \left(\int_{\Omega} |f(y)|^p dy \right)^{\frac{1}{p}}.$$

Finally, we obtain that the function $d : \overline{\Omega} \times \overline{\Omega} \rightarrow [0, \infty)$, defined by

$$d(x, y) := \inf\{r > 0: x \in S_\phi(y, r) \text{ and } y \in S_\phi(x, r)\}, \quad \forall x, y \in \overline{\Omega},$$

is a quasi-distance on $\overline{\Omega}$. Moreover, $(\overline{\Omega}, d, |\cdot|)$ is a space of homogeneous type where $|\cdot|$ denotes the n -dimensional Lebesgue measure restricted to $\overline{\Omega}$. The precise statements of these are given in Section 6.

3. Sections of the Monge–Ampère equation and the Localization Theorem

In this section, we recall the main tool to study geometric properties of boundary sections of solutions to the Monge–Ampère equation: the Localization Theorem at the boundary for solutions to the Monge–Ampère equation (Theorem 3.1). Properties of solutions under suitable rescalings and global regularity for gradient will also be discussed. Throughout this section, we assume that the convex domain Ω and the convex function ϕ satisfy (2.3)–(2.5).

3.1. The Localization Theorem

We now focus on sections centered at a point on the boundary $\partial\Omega$ and describe their geometry. Assume this boundary point to be 0 and by (2.3), we can also assume that

$$B_\rho(\rho e_n) \subset \Omega \subset \{x_n \geq 0\} \cap B_{\frac{1}{\rho}}, \tag{3.7}$$

where $\rho > 0$ is the constant given by condition (2.3). After subtracting a linear function, we can assume further that

$$\phi(0) = 0, \quad \nabla\phi(0) = 0. \tag{3.8}$$

Let us denote

$$S(h) := S_\phi(0, h).$$

If the boundary data has quadratic growth near the hyperplane $\{x_n = 0\}$ then, as $h \rightarrow 0$, $S(h)$ is equivalent to a half-ellipsoid centered at 0. This is the content of the Localization Theorem proved by Savin in [22,23]. Precisely, this theorem reads as follows.

Theorem 3.1 (Localization Theorem). (See [22,23].) *Assume that Ω satisfies (3.7) and ϕ satisfies (2.4), (3.8), and*

$$\rho|x|^2 \leq \phi(x) \leq \rho^{-1}|x|^2 \quad \text{on } \partial\Omega \cap \{x_n \leq \rho\}.$$

Then there exists a constant $k > 0$ depending only on ρ, λ, Λ and n such that for each $h \leq k$ there is an ellipsoid E_h of volume $\omega_n h^{n/2}$ satisfying

$$kE_h \cap \overline{\Omega} \subset S(h) \subset k^{-1}E_h \cap \overline{\Omega}.$$

Moreover, the ellipsoid E_h is obtained from the ball of radius $h^{1/2}$ by a linear transformation A_h^{-1} (sliding along the $x_n = 0$ plane)

$$A_h E_h = h^{1/2} B_1, \quad \det A_h = 1,$$

$$A_h(x) = x - \tau_h x_n, \quad \tau_h = (\tau_1, \tau_2, \dots, \tau_{n-1}, 0),$$

with $|\tau_h| \leq k^{-1} |\log h|$.

The ellipsoid E_h , or equivalently the linear map A_h , provides useful information about the behavior of ϕ near the origin. From Theorem 3.1 we also control the shape of sections that are tangent to $\partial\Omega$ at the origin.

Proposition 3.2. *Let ϕ and Ω satisfy the hypotheses of the Localization Theorem 3.1 at the origin. Assume that for some $y \in \Omega$ the section $S_\phi(y, h) \subset \Omega$ is tangent to $\partial\Omega$ at 0 for some $h \leq c$ with c universal, that is, $\partial S_\phi(y, h) \cap \partial\Omega = \{0\}$. Then there exists a small positive constant $k_0 < k$ depending only on ρ, λ, Λ and n such that*

$$\nabla\phi(y) = a e_n \quad \text{for some } a \in [k_0 h^{1/2}, k_0^{-1} h^{1/2}],$$

$$k_0 E_h \subset S_\phi(y, h) - y \subset k_0^{-1} E_h, \quad k_0 h^{1/2} \leq \text{dist}(y, \partial\Omega) \leq k_0^{-1} h^{1/2},$$

with E_h and k the ellipsoid and constant defined in the Localization Theorem 3.1.

Proposition 3.2 is a consequence of Theorem 3.1 and was proved in [24].

The quadratic separation from tangent planes on the boundary for solutions to the Monge–Ampère equation is a crucial assumption in the Localization Theorem 3.1. This is the case for solutions to the Monge–Ampère with the right hand side bounded away from 0 and ∞ and smooth boundary data as proved in [23, Proposition 3.2]. In particular, the quadratic separation property holds if

$$\phi|_{\partial\Omega}, \quad \partial\Omega \in C^3, \quad \text{and} \quad \Omega \text{ is uniformly convex.}$$

3.2. Properties of the rescaled functions

Let ϕ and Ω satisfy the hypotheses of the Localization Theorem 3.1 at the origin. We know that for all $h \leq k$, $S(h)$ satisfies

$$k E_h \cap \overline{\Omega} \subset S(h) \subset k^{-1} E_h,$$

with A_h being a linear transformation and

$$\det A_h = 1, \quad E_h = A_h^{-1} B_{h^{1/2}}, \quad A_h x = x - \tau_h x_n,$$

$$\tau_h \cdot e_n = 0, \quad \|A_h^{-1}\|, \|A_h\| \leq k^{-1} |\log h|.$$

This gives

$$\overline{\Omega} \cap B_{ch^{1/2}/|\log h|}^+ \subset S(h) \subset B_{Ch^{1/2}|\log h|}^+ \subset B_{Ch^{1/4}}^+. \tag{3.9}$$

We denote the rescaled functions by

$$\phi_h(x) := \frac{\phi(h^{1/2}A_h^{-1}x)}{h}.$$

The function ϕ_h is continuous and is defined in $\overline{\Omega}_h$ with $\Omega_h := h^{-1/2}A_h\Omega$, and solves the Monge–Ampère equation

$$\det D^2\phi_h = g_h(x), \quad \lambda \leq g_h(x) := g(h^{1/2}A_h^{-1}x) \leq \Lambda.$$

The section at height 1 for ϕ_h centered at the origin satisfies $S_{\phi_h}(0, 1) = h^{-1/2}A_hS(h)$, and by the Localization Theorem we obtain

$$B_k \cap \overline{\Omega}_h \subset S_{\phi_h}(0, 1) \subset B_{k-1}^+.$$

Some properties of the rescaled function ϕ_h were established in [21, Lemma 4.2]. For later use, we record them here.

Lemma 3.3. *If $h \leq c$, then*

(a) *for any $x, x_0 \in \partial\Omega_h \cap B_{2/k}$ we have*

$$\frac{\rho}{4}|x - x_0|^2 \leq \phi_h(x) - \phi_h(x_0) - \nabla\phi_h(x_0)(x - x_0) \leq 4\rho^{-1}|x - x_0|^2,$$

(b) *if $r \leq c$ small, we have*

$$|\nabla\phi_h| \leq Cr|\log r|^2 \quad \text{in } \overline{\Omega}_h \cap B_r.$$

3.3. Global regularity

We note that if ϕ satisfies (2.4)–(2.5), then for any $\mathbf{v} \in \mathbb{R}^n$ the function $\tilde{\phi}(x) := \phi(x) + \mathbf{v} \cdot x$ also satisfies (2.4)–(2.5) with the same constants. As a result, under our hypotheses the gradient of ϕ is not bounded by any universal constant. Nevertheless, the oscillation of $\nabla\phi$ is globally bounded thanks to [21, Proposition 2.6]. We record this result and its direct consequence in the next lemma.

Lemma 3.4. *Assume that the convex domain Ω and the convex function ϕ satisfy (2.3)–(2.5). Then there exist constants $\alpha \in (0, 1)$ and $C > 0$ depending only on ρ, λ, Λ and n such that*

$$[\nabla\phi]_{C^\alpha(\overline{\Omega})} \leq C.$$

As a consequence, there is a universal constant $M > 0$ satisfying

$$S_\phi(x_0, M) \supset \overline{\Omega} \quad \text{for all } x_0 \in \overline{\Omega}.$$

Proof. The global C^α -estimate for the gradient is from [21, Proposition 2.6]. Now let $x_0 \in \overline{\Omega}$. Then for any $x \in \overline{\Omega}$, we have

$$\phi(x) - \phi(x_0) - \nabla\phi(x_0) \cdot (x - x_0) \leq [\nabla\phi]_{C^\alpha(\overline{\Omega})} |x - x_0|^{1+\alpha} \leq C(2\rho^{-1})^{1+\alpha} =: M.$$

Thus $\overline{\Omega} \subset S_\phi(x_0, M)$ and the lemma follows. \square

4. Geometric properties of sections

In this section, we establish several important properties of boundary sections of solutions to the Monge–Ampère equation. Unless otherwise stated, the convex domain Ω and the convex function ϕ are assumed to satisfy (2.3)–(2.5). Under these conditions, we will show that sections of ϕ satisfy a dichotomy, the engulfing and separating properties, and their volumes have expected growth.

4.1. Dichotomy of sections and volume growth

In this subsection, we shall prove the volume growth of sections and a dichotomy for sections of solutions to the Monge–Ampère equation: any section is either an interior section or included in a boundary section with comparable height. We begin with the proof of Proposition 2.3.

Proof of Proposition 2.3. It suffices to consider the case $x_0 \in \Omega$. Let $S(x_0, \bar{h}(x_0))$ be the maximal interior section with center x_0 , and let $\bar{h} := \bar{h}(x_0)$. If $t_0 \leq \bar{h}/2$ then (i) is satisfied. We now consider the case $\bar{h}/2 < t_0$ and show that (ii) holds. Without loss of generality, we assume that $\Omega \subset \mathbb{R}_+^n$, $\partial S_\phi(x_0, \bar{h})$ is tangent to $\partial\Omega$ at 0, and $\phi(0) = \nabla\phi(0) = 0$. It follows that $0 = \phi(x_0) - \nabla\phi(x_0) \cdot x_0 + \bar{h}$ yielding

$$S_\phi(x_0, t) = \{x \in \overline{\Omega} : \phi(x) < \nabla\phi(x_0) \cdot x + t - \bar{h}\}, \quad \forall t > 0. \tag{4.10}$$

Next, we have

Claim. *There exists a small constant $c > 0$ depending only on ρ, λ, Λ and n such that if $\bar{h}/2 < t_0 \leq c$, then*

$$S_\phi(x_0, 2t_0) \subset S_\phi(0, t^*) \quad \text{with } t^* := k_0^{-4}\bar{h} + 2(2t_0 - \bar{h}). \tag{4.11}$$

Indeed, if c is small, we have, by Proposition 3.2

$$\nabla\phi(x_0) = ae_n \quad \text{for some } a \in [k_0\bar{h}^{1/2}, k_0^{-1}\bar{h}^{1/2}], \tag{4.12}$$

where $k_0 > 0$ depends only on ρ, λ, Λ and n . It follows from (4.10) and (4.12) that

$$S_\phi(x_0, 2t_0) = \{x \in \overline{\Omega} : \phi(x) < ax_n + 2t_0 - \bar{h}\}.$$

Let us choose $c > 0$ small enough such that

$$k_0^{-4}\bar{h} + 2(2t_0 - \bar{h}) \leq 2k_0^{-4}c + 4c \leq k,$$

where k is the constant in the Localization Theorem 3.1. With this choice of c , we are going to show that (4.11) holds. Suppose otherwise that (4.11) is not true. Then, using the convexity of the sets $S_\phi(x_0, 2t_0)$ and $S_\phi(0, t^*)$ and the fact that their closures both contain 0, we can find a point $x \in \Omega$ such that $x \in \partial S_\phi(0, t^*) \cap S_\phi(x_0, 2t_0)$. At this point x , we have

$$k_0^{-4}\bar{h} + 2(2t_0 - \bar{h}) = t^* = \phi(x) < ax_n + 2t_0 - \bar{h} \leq k_0^{-1}\bar{h}^{1/2}x_n + 2t_0 - \bar{h}.$$

Hence,

$$\frac{k_0^{-4}\bar{h} + 2t_0 - \bar{h}}{k_0^{-1}\bar{h}^{1/2}} < x_n.$$

This together with the Localization Theorem 3.1 applied to $S_\phi(0, k_0^{-4}\bar{h} + 2(2t_0 - \bar{h}))$ gives

$$\frac{k_0^{-4}\bar{h} + 2t_0 - \bar{h}}{k_0^{-1}\bar{h}^{1/2}} < k^{-1}(k_0^{-4}\bar{h} + 2(2t_0 - \bar{h}))^{1/2} \leq k_0^{-1}(k_0^{-4}\bar{h} + 2(2t_0 - \bar{h}))^{1/2},$$

or

$$k_0^{-4}\bar{h} + 2t_0 - \bar{h} < k_0^{-2}\bar{h}^{1/2}(k_0^{-4}\bar{h} + 2(2t_0 - \bar{h}))^{1/2}.$$

Squaring, we get

$$k_0^{-8}\bar{h}^2 + 2k_0^{-4}\bar{h}(2t_0 - \bar{h}) + (2t_0 - \bar{h})^2 < k_0^{-4}\bar{h}(k_0^{-4}\bar{h} + 2(2t_0 - \bar{h}))$$

and as a consequence, $(2t_0 - \bar{h})^2 < 0$. This is a contradiction and hence the claim is proved.

Let $\bar{c} := \max\{2k_0^{-4} + 4, M/c\}$, where M is the universal constant given by Lemma 3.4. Then from the claim and since

$$t^* \leq (2k_0^{-4} + 4)t_0 \leq \bar{c}t_0,$$

we see that (ii) holds if $\bar{h}/2 < t_0 \leq c$.

In the case $c < t_0$, by using Lemma 3.4 we obtain

$$S_\phi(x_0, 2t_0) \subset \bar{\Omega} \subset S_\phi(0, M) \subset S_\phi\left(0, \frac{M}{c}t_0\right) \subset S_\phi(0, \bar{c}t_0)$$

and thus (ii) also holds true. \square

As an application of the dichotomy of sections, we obtain their volume growth.

Proof of Corollary 2.4. Let $c_0 := k/\bar{c}$, where k, \bar{c} are the constants in Theorem 3.1 and Proposition 2.3 respectively. Let $S(x_0, t_0)$ be a section with $t_0 \leq c_0$. If $x_0 \in \partial\Omega$ then by the Localization Theorem 3.1, we get the desired result.

Now, we suppose that $x_0 \in \Omega$ and let $S(x_0, \bar{h}(x_0))$ be the maximal interior section with center x_0 . For simplicity, we denote $\bar{h} := \bar{h}(x_0)$. If $t_0 \leq \bar{h}/2$, then the result follows from the volume growth of interior sections; see [14, Corollary 3.2.4]. Therefore, it remains to consider the case $\bar{h}/2 < t_0 \leq c_0$.

In this case, by Proposition 2.3 we have $S(x_0, 2t_0) \subset S(z, \bar{c}t_0)$ for some $z \in \partial\Omega$. Hence, by applying Theorem 3.1 we obtain the second inequality in (2.6). To prove the first inequality in (2.6), we first note that, if $t_0 < 2\bar{h}$ then

$$|S(x_0, t_0)| \geq \left| S\left(x_0, \frac{\bar{h}}{2}\right) \right| \geq C_0 \bar{h}^{n/2} \geq C_1 t_0^{n/2}.$$

Here, the second inequality follows from the volume growth of interior sections; see [14, Corollary 3.2.4]. Next, suppose that $t_0 \geq 2\bar{h}$. Without loss of generality, we assume that $\Omega \subset \mathbb{R}_+^n$, $\partial S(x_0, \bar{h})$ is tangent to $\partial\Omega$ at 0, and $\phi(0) = \nabla\phi(0) = 0$. Then by exactly the same arguments as in the proof of Proposition 2.3, we get for some positive number a

$$S(x_0, t_0) = \{x \in \bar{\Omega} : \phi(x) < ax_n + t_0 - \bar{h}\}.$$

Using this and the fact $x_n \geq 0$ in Ω , we can conclude that

$$S(x_0, t_0) \supset \{x \in \bar{\Omega} : \phi(x) < t_0 - \bar{h}\} \supset \left\{ x \in \bar{\Omega} : \phi(x) < \frac{t_0}{2} \right\} = S_\phi\left(0, \frac{t_0}{2}\right)$$

which together with the Localization Theorem 3.1 yields the first inequality in (2.6). \square

4.2. Engulfing and separating properties of sections

In this subsection, we will establish two important tools: the engulfing and separating properties of sections, Theorem 2.1 and Proposition 2.2, respectively. These properties are equivalent.

As a first step in the proof of Theorem 2.1, we prove the engulfing property when the center x lies on the boundary, that is, $S_\phi(x, t)$ is a boundary section. Without loss of generality, we assume that x is the origin and we write $S(t)$ for $S_\phi(0, t)$. Furthermore, we assume that ϕ and Ω satisfy the hypotheses of the Localization Theorem 3.1 at the origin.

Lemma 4.1. *There exists $\theta_0 > 0$ depending only on ρ, λ, Λ and n such that if $X \in S(t)$ with $t > 0$, then*

$$S(t) \subset S_\phi(X, \theta_0 t).$$

Proof. Let $t \leq c_0$ with $c_0 \leq k$ to be chosen later, where $k = k(\rho, \lambda, \Lambda, n)$ is the small constant in the Localization Theorem 3.1. Let us consider $h \in [t, k]$. Let A_h be the linear transformation associated with the section $S(h)$ as determined by the above theorem. Let

$$\phi_h(z) := \frac{\phi(h^{1/2}A_h^{-1}z)}{h} \quad \text{for } z \in \Omega_h := h^{-1/2}A_h\Omega.$$

For $X, Y \in S(t)$, we define

$$x := h^{-1/2} A_h X, \quad y := h^{-1/2} A_h Y.$$

Then

$$S_{\phi_h}(0, 1) = h^{-1/2} A_h S(h); \quad x, y \in S_{\phi_h}\left(0, \frac{t}{h}\right)$$

and furthermore,

$$\frac{1}{h} [\phi(Y) - \phi(X) - \nabla\phi(X) \cdot (Y - X)] = \phi_h(y) - \phi_h(x) - \nabla\phi_h(x) \cdot (y - x).$$

By Lemma 3.3(a), ϕ_h also satisfies the hypotheses of the Localization Theorem 3.1 in $S_{\phi_h}(0, 1)$. Hence, by (3.9), we have for C universal

$$|x|, |y| \leq C \left(\frac{t}{h}\right)^{1/2} \left| \log\left(\frac{t}{h}\right) \right|.$$

Now, we take $h > 0$ satisfying $t/h = 1/M_1$ with $M_1 > 1$ is chosen so that

$$C \left(\frac{t}{h}\right)^{1/2} \left| \log\left(\frac{t}{h}\right) \right| = C M_1^{-1/2} \log M_1 \leq c,$$

where c is determined by Lemma 3.3. Given this choice of M_1 , it suffices to take $c_0 := \frac{k}{M_1}$ so as to have $h \in [t, k]$. Then, by Lemma 3.3(b),

$$|\nabla\phi_h(x)| \leq c |\log c|^2.$$

Thus,

$$\frac{1}{h} [\phi(Y) - \phi(X) - \nabla\phi(X) \cdot (Y - X)] = \phi_h(y) - \phi_h(x) - \nabla\phi_h(x) \cdot (y - x) \leq \frac{t}{h} + 2c^2 |\log c|^2$$

implying

$$Y \in S_{\phi}(X, (1 + 2M_1 c^2 |\log c|^2)t).$$

Hence for any $X \in S(t)$ with $t \leq c_0$, we get

$$S(t) \subset S_{\phi}(X, (1 + 2M_1 c^2 |\log c|^2)t).$$

In the case $X \in S(t)$ with $t > c_0$, then by using Lemma 3.4 we obtain

$$S(t) \subset \overline{\Omega} \subset S_{\phi}(X, M) \subset S_{\phi}\left(X, \frac{M}{c_0}t\right).$$

Therefore, by taking $\theta_0 := \max\{1 + 2M_1 c^2 |\log c|^2, M/c_0\}$, we see that $S(t) \subset S_\phi(X, \theta_0 t)$ for any $t > 0$. \square

We are now ready to give the proof of Theorem 2.1.

Proof of Theorem 2.1. By Lemma 4.1, it remains to consider the case $x \in \Omega$. Let $S(x, \bar{h}(x))$ be the maximal interior section with center x .

If $t < \bar{h}(x)/2$ then $S(x, 2t)$ is an interior section and the result follows from the engulfing properties of interior sections of the Monge–Ampère equation with bounded right hand side (see the proof of Theorem 3.3.7 in [14]), namely,

$$S(x, t) \subset S(y, \theta t)$$

for some θ depending only on λ, A and n .

Now, consider the case $\bar{h}(x)/2 \leq t$. Then Proposition 2.3 implies that

$$S(x, 2t) \subset S(z, \bar{c}t)$$

for some $z \in \partial\Omega$. Since $y \in S(z, \bar{c}t)$, by the engulfing property of boundary sections from Lemma 4.1, we have

$$S(z, \bar{c}t) \subset S(y, \theta_0 \bar{c}t).$$

Therefore the result follows with $\theta_* := \max\{\theta, \theta_0 \bar{c}\}$ noting that $S(x, t) \subset S(z, \bar{c}t)$. \square

Finally, we prove the separating property of sections.

Proof of Proposition 2.2. (i) Suppose that $y \notin S(x, t)$. If $z \in S(y, \frac{t}{\theta_*^2}) \cap S(x, \frac{t}{\theta_*^2})$, then, by Theorem 2.1,

$$S\left(y, \frac{t}{\theta_*^2}\right) \cup S\left(x, \frac{t}{\theta_*^2}\right) \subset S\left(z, \frac{t}{\theta_*}\right).$$

This implies that $y, x \in S(z, \frac{t}{\theta_*})$ and also by the engulfing property that $S(z, \frac{t}{\theta_*}) \subset S(x, t)$. Therefore, $y \in S(x, t)$ which is a contradiction and so $S(y, \frac{t}{\theta_*^2}) \cap S(x, \frac{t}{\theta_*^2}) = \emptyset$.

(ii) Suppose $y \in S(x, t)$. We need to prove that

$$S(x, t) \subset S(y, \theta^2 t). \tag{4.13}$$

Suppose that it is not true. Then, we can find $z \in S(x, t)$ such that

$$z \notin S(y, \theta^2 t). \tag{4.14}$$

Since $y, z \in S(x, t)$, it follows from the separating property that

$$x \in S(y, \theta t) \cap S(z, \theta t). \tag{4.15}$$

Applying the separating property to (4.14), we get $S(z, \theta t) \cap S(y, \theta t) = \emptyset$. This is a contradiction to (4.15). Hence (4.13) holds as desired. \square

5. A covering theorem and maximal function estimates

In this section we establish a covering lemma of Besicovitch type. Using this lemma, we prove a covering theorem for boundary sections and derive a global strong type p - p estimate for the maximal function. Our results extend those in [6] where interior sections are considered. We begin with the proof of Lemma 2.5.

Proof of Lemma 2.5. We may assume that $M = \sup\{t: S(x, t) \in \mathcal{F}\}$. Let us first consider the family

$$\mathcal{F}_0 := \left\{ S(x, t) \in \mathcal{F}: \frac{M}{2} < t \leq M \right\} \quad \text{and} \quad A_0 := \{x: S(x, t) \in \mathcal{F}_0\}.$$

Pick $S(x_1, t_1) \in \mathcal{F}_0$ such that $t_1 > 3M/4$. If $A_0 \subset S(x_1, t_1)$, we stop. Otherwise, the set

$$\{t: S(x, t) \in \mathcal{F}_0 \text{ and } x \in A_0 \setminus S(x_1, t_1)\}$$

is nonempty and we let α_2 denote its supremum. Pick t_2 in this set such that $t_2 > 3\alpha_2/4$, and let $S(x_2, t_2)$ be the corresponding section. We then have $x_2 \notin S(x_1, t_1)$ and $t_1 > 3M/4 \geq 3\alpha_2/4 \geq 3t_2/4$. Again, if $A_0 \subset S(x_1, t_1) \cup S(x_2, t_2)$ we stop. Otherwise, we continue the process. As a result, we have constructed a family, possibly infinite, which we denote by

$$\mathcal{F}'_0 = \{S(x_i^0, t_i^0)\}_{i=1}^\infty \quad \text{with } x_j^0 \in A_0 \setminus \bigcup_{i<j} S(x_i^0, t_i^0).$$

We next consider the family

$$\mathcal{F}_1 := \left\{ S(x, t) \in \mathcal{F}: \frac{M}{4} < t \leq \frac{M}{2} \right\} \quad \text{and} \quad A_1 := \left\{ x: S(x, t) \in \mathcal{F}_1 \text{ and } x \notin \bigcup_{i=1}^\infty S(x_i^0, t_i^0) \right\}.$$

We repeat the above construction for the set A_1 and obtain a family of sections denoted by

$$\mathcal{F}'_1 = \{S(x_i^1, t_i^1)\}_{i=1}^\infty \quad \text{with } x_j^1 \in A_1 \setminus \bigcup_{i<j} S(x_i^1, t_i^1).$$

We continue this process and in the k th-stage we consider the family

$$\mathcal{F}_k := \left\{ S(x, t) \in \mathcal{F}: \frac{M}{2^{k+1}} < t \leq \frac{M}{2^k} \right\}$$

and

$$A_k := \{x: S(x, t) \in \mathcal{F}_k \text{ and } x \notin \text{sections previously selected}\}.$$

In the same way as before, we obtain a family of sections denoted by

$$\mathcal{F}'_k = \{S(x_i^k, t_i^k)\}_{i=1}^\infty \quad \text{with } x_j^k \in A_k \setminus \bigcup_{i < j} S(x_i^k, t_i^k).$$

We are going to show that the collection of all sections in all generations $\mathcal{F}'_k, k \geq 0$, is the family that satisfies the conclusion of the lemma.

Claim 1. *The overlapping in each generation \mathcal{F}'_k is at most κ , where κ depends only on ρ, λ, Λ and n .*

To show this, let us suppose that

$$z \in S(x_{j_1}^k, t_{j_1}^k) \cap \dots \cap S(x_{j_N}^k, t_{j_N}^k),$$

with $S(x_{j_i}^k, t_{j_i}^k) \in \mathcal{F}'_k$ and $j_1 < j_2 < \dots < j_N$. For simplicity we set $x_{j_i}^k = x_i, t_{j_i}^k = t_i$, and let t_0 be the maximum of all these t_i . Then, by the engulfing property of sections in Theorem 2.1, we have

$$\bigcup_{i=1}^N S(x_i, t_i) \subset S(z, \theta_* t_0). \tag{5.16}$$

For any $l > i$, as $x_l \notin S(x_i, t_i)$ we obtain from the separating property of sections in Proposition 2.2 that

$$S\left(x_l, \frac{t_l}{2\theta_*^2}\right) \cap S\left(x_i, \frac{t_i}{2\theta_*^2}\right) = \emptyset.$$

Since

$$\frac{M}{2^{k+1}} < t_i \leq \frac{M}{2^k},$$

we then conclude that $t_i > t_0/2$ and thus

$$S\left(x_l, \frac{t_l}{2\theta_*^2}\right) \cap S\left(x_i, \frac{t_i}{2\theta_*^2}\right) = \emptyset; \quad S\left(x_l, \frac{t_0}{2\theta_*^2}\right) \cap S\left(x_i, \frac{t_0}{2\theta_*^2}\right) = \emptyset. \tag{5.17}$$

Combining this fact with (5.16), we obtain

$$\sum_{i=1}^N \left| S\left(x_i, \frac{t_0}{2\theta_*^2}\right) \right| \leq |S(z, \theta_* t_0)|. \tag{5.18}$$

Let c_0 be the universal constant in Corollary 2.4. If $\theta_* t_0 \leq c_0$, then it follows from (5.18) and the volume estimates (2.6) that

$$\sum_{i=1}^N C_1 \left(\frac{t_0}{2\theta_*^2} \right)^{n/2} \leq C_2 (\theta_* t_0)^{n/2}$$

and thus

$$N \leq N_1 := \frac{C_2}{C_1} 2^{n/2} \theta_*^{3n/2}.$$

In the case $\theta_* t_0 > c_0$, then (5.18) implies $\sum_{i=1}^N |S(x_i, \frac{c_0}{2\theta_*^3})| \leq |\Omega| \leq |B_{1/\rho}|$. So by applying (2.6) again, we get

$$\sum_{i=1}^N C_1 \left(\frac{c_0}{2\theta_*^3} \right)^{n/2} \leq |B_{1/\rho}|$$

yielding

$$N \leq N_2 := \frac{|B_{1/\rho}|}{C_1 c_0^{n/2}} 2^{n/2} \theta_*^{3n/2}.$$

Therefore the overlapping in each generation \mathcal{F}'_k is at most κ , where $\kappa := \max\{N_1, N_2\}$.

Claim 2. *The family $\mathcal{F}'_k = \{S(x_i^k, t_i^k)\}_{i=1}^\infty$ is actually finite.*

Indeed, by Claim 1

$$\sum_i \chi_{S(x_i^k, t_i^k)}(x) \leq \kappa$$

and hence by integrating over Ω we obtain

$$\sum_i |S(x_i^k, t_i^k)| \leq \kappa |\Omega|.$$

Note that $M/2^{k+1} < t_i^k$. Therefore if we let $a := \min\{M/2^{k+1}, c_0\}$, then it follows from the above inequality and Corollary 2.4 that

$$\sum_i C_1 a^{n/2} \leq \sum_i |S(x_i^k, a)| \leq \kappa |\Omega|,$$

implying that the number of terms in the sum is finite and Claim 2 is proved.

From Claim 2 and our construction we get $A_k \subset \bigcup_{i=1}^\infty S(x_i^k, t_i^k)$ and thus (i) holds. Also since each generation \mathcal{F}'_k has a finite number of members, by relabeling the indices of all members of all generations \mathcal{F}'_k we obtain (ii).

In order to prove property (iii), let $x_i \neq x_j$. If $S(x_i, t_i)$ and $S(x_j, t_j)$ belong to the same generation, then $S(x_i, t_i/\alpha) \cap S(x_j, t_j/\alpha) = \emptyset$ by (5.17), where $\alpha := 2\theta_*^2$. On the other hand, suppose $S(x_i, t_i) \in \mathcal{F}'_k$ and $S(x_j, t_j) \in \mathcal{F}'_{k+p}$ for some $p \geq 1$. Then, by construction, $x_j \notin S(x_i, t_i)$ and so

$S(x_j, t_i/\theta_*^2) \cap S(x_i, t_i/\theta_*^2) = \emptyset$ by the separating property in Proposition 2.2. Since $t_i > t_j$, this gives $S(x_j, t_j/\theta_*^2) \cap S(x_i, t_i/\theta_*^2) = \emptyset$ and so (iii) is proved.

Claim 3. Assume $0 < r \leq c_0$. Then the number of sections $S(x, t)$ in the family $\{S(x_k, t_k)\}$ with $t \geq r$ is bounded by a constant N depending only on $r, \rho, \lambda, \Lambda$ and n .

To see this, let us denote this subfamily of sections by $\{S(x_k, t_k)\}_{k \in I}$. Then by using property (iii) and Corollary 2.4 we obtain

$$|B_{\frac{1}{\rho}}| \geq |\Omega| \geq \left| \bigcup_{k \in I} S\left(x_k, \frac{t_k}{\alpha}\right) \right| = \sum_{k \in I} \left| S\left(x_k, \frac{t_k}{\alpha}\right) \right| \geq \sum_{k \in I} \left| S\left(x_k, \frac{r}{\alpha}\right) \right| \geq \sum_{k \in I} C_1 \left(\frac{r}{\alpha}\right)^{n/2}.$$

Thus the number of elements in I is bounded by $N := C_1^{-1}(\alpha r^{-1})^{n/2} |B_{1/\rho}|$.

We next estimate the overlapping of sections belonging to different generations. Let $0 < \varepsilon < 1$ and

$$z \in \bigcap_i S(x_{r_i}^{e_i}, (1 - \varepsilon)t_{r_i}^{e_i}), \tag{5.19}$$

where $e_1 < e_2 < \dots$, $M2^{-(e_i+1)} < t_{r_i}^{e_i} \leq M2^{-e_i}$, and for simplicity in the notation we set $x_i = x_{r_i}^{e_i}$ and $t_i = t_{r_i}^{e_i}$. Our aim is to show that the number of sections in (5.19) is not more than $C \log \frac{1}{\varepsilon}$. We only need to consider $\varepsilon < 1 - \frac{1}{\alpha}$ since otherwise the sections are disjoint by (iii). Let $r_0 := \min\{\frac{k}{c\theta_*M_1}, \frac{c}{c\theta_*M_1}, c_0\}$, where c is the constant in Lemma 3.3 and M_1 is chosen as in the proof of Lemma 4.1. Then by Claim 3 and in view of our purpose, we can assume without loss of generality that $t_i \leq r_0$ for all i appearing in (5.19). Now for any $j > i$, we claim that

$$e_j - e_i \leq C \log \frac{1}{\varepsilon}, \tag{5.20}$$

where $C > 0$ depends only on ρ, λ, Λ and n . In particular, the number of members in (5.19) is at most $C \log \frac{1}{\varepsilon}$, which together with Claim 1 gives (iv) as desired.

To prove the claim, observe first that by the engulfing property of sections in Theorem 2.1 we have

$$S(x_j, t_j) \subset S(z, \theta_* t_j) \quad \text{and} \quad S(x_i, t_i) \subset S(z, \theta_* t_i). \tag{5.21}$$

Let $S(z, \bar{h}(z))$ be the maximal interior section with center z when z is an interior point of Ω . The case $z \in \partial\Omega$ will be dealt with briefly at the end of the proof. We then consider the following possibilities:

Case 1. $\theta_* t_i < \bar{h}(z)$. Then both sections $S(x_i, t_i)$ and $S(x_j, t_j)$ are interior sections and (5.20) follows from the proof in [14, Lemma 6.5.2]. We include the proof here for the sake of completeness. Let T be an affine map normalizing the section $S(x_i, t_i)$. Then since $t_i > t_j$, by [14, Theorem 3.3.8], there exists ε_1 depending on ρ, λ, Λ and n such that

$$T(S(x_j, t_j)) \subset B\left(Tx_j, K_1 \left(\frac{t_j}{t_i}\right)^{\varepsilon_1}\right).$$

By construction $x_j \notin S(x_i, t_i)$ and hence by [14, Corollary 3.3.6] we obtain

$$B(Tx_j, C\varepsilon^n) \cap T(S(x_i, (1 - \varepsilon)t_i)) = \emptyset.$$

We deduce from the above two relations that

$$C\varepsilon^n < |Tx_j - Tz| \leq K_1 \left(\frac{t_j}{t_i}\right)^{\varepsilon_1} \leq K_1 2^{\varepsilon_1} 2^{(e_i - e_j)\varepsilon_1}$$

implying (5.20).

Case 2. $\theta_*t_i \geq \bar{h}(z)$ and $\theta_*t_j \leq \bar{h}(z)/2$. We can assume that $\Omega \subset \mathbb{R}_+^n$, $\partial S(z, \bar{h}(z))$ is tangent to $\partial\Omega$ at 0, and $\phi(0) = \nabla\phi(0) = 0$. Then, by Proposition 2.3, we know that

$$S(z, \theta_*t_i) \subset S(0, \bar{c}\theta_*t_i).$$

Let M_1 be chosen as in the proof of Lemma 4.1 and denote

$$C_0 := \bar{c}\theta_*M_1; \quad h := C_0t_i.$$

Then $h \leq k$ where k is the constant in the Localization Theorem 3.1. We use the notation as in Section 3.2. We rescale ϕ by

$$\phi_h(x) := \frac{\phi(h^{1/2}A_h^{-1}x)}{h}$$

where A_h is the linear map in the Localization Theorem 3.1. The function ϕ_h is continuous and is defined in $\bar{\Omega}_h$ with $\Omega_h := h^{-1/2}A_h\Omega$, and solves the Monge–Ampère equation

$$\det D^2\phi_h = g_h(x), \quad \lambda \leq g_h(x) := g(h^{1/2}A_h^{-1}x) \leq \Lambda.$$

The section at height 1 for ϕ_h centered at the origin satisfies $S_{\phi_h}(0, 1) = h^{-1/2}A_hS(h)$, and by the Localization Theorem we obtain

$$B_k \cap \bar{\Omega}_h \subset S_{\phi_h}(0, 1) \subset B_{k-1}^+.$$

Let $T := h^{-1/2}A_h$. Due to our assumptions

$$\theta_*t_j \leq \bar{h}(z)/2 < C_0t_i = h,$$

the section $S_{\phi_h}(Tz, \theta_*t_j/h) = T(S(z, \theta_*t_j))$ is an interior section of ϕ_h in $S_{\phi_h}(0, 1)$. Therefore, from the proof of [14, Theorem 3.3.8], we find ε_1 depending on ρ, λ, Λ and n such that

$$S_{\phi_h}\left(Tz, \frac{\theta_*t_j}{C_0t_i}\right) \subset B\left(Tz, K\left(\frac{\theta_*t_j}{C_0t_i}\right)^{\varepsilon_1}\right).$$

Thus, by recalling (5.21), we obtain

$$|Tx_j - Tz| \leq K_1 \left(\frac{t_j}{t_i}\right)^{\varepsilon_1}. \tag{5.22}$$

We have

$$Tx_i, Tx_j, Tz \in T(S(z, \theta_*t_i)) \subset T(S(0, \bar{c}\theta_*t_i)) = S_{\phi_h}\left(0, \frac{\bar{c}\theta_*t_i}{C_0t_i}\right) = S_{\phi_h}\left(0, \frac{1}{M_1}\right).$$

By Lemma 3.3(a), ϕ_h also satisfies the hypotheses of the Localization Theorem 3.1 in $S_{\phi_h}(0, 1)$. Hence, by (3.9), Tx_i, Tx_j, Tz belong to $B(0, c)$. As a result, we obtain from Lemma 3.3(b) that

$$|\nabla\phi_h(Tx_i)|, |\nabla\phi_h(w)| \leq c|\log c|^2 \tag{5.23}$$

for any $w \in [Tx_j, Tz]$. Since $Tx_j \notin T(S(x_i, t_i)) = S_{\phi_h}(Tx_i, \frac{1}{C_0})$, we have

$$\frac{1}{C_0} \leq \phi_h(Tx_j) - \phi_h(Tx_i) - \nabla\phi_h(Tx_i) \cdot (Tx_j - Tx_i).$$

By rewriting the above right hand side in the form

$$\phi_h(Tz) - \phi_h(Tx_i) - \nabla\phi_h(Tx_i) \cdot (Tz - Tx_i) + \phi_h(Tx_j) - \phi_h(Tz) - \nabla\phi_h(Tx_i) \cdot (Tx_j - Tz)$$

and using the fact that $Tz \in T(S(x_i, (1 - \varepsilon)t_i)) = S_{\phi_h}(Tx_i, \frac{1-\varepsilon}{C_0})$, we obtain

$$\begin{aligned} \frac{1}{C_0} &\leq \frac{1-\varepsilon}{C_0} + \phi_h(Tx_j) - \phi_h(Tz) - \nabla\phi_h(Tx_i) \cdot (Tx_j - Tz) \\ &\leq \frac{1-\varepsilon}{C_0} + C|Tx_j - Tz| \end{aligned} \tag{5.24}$$

where (5.23) is used to obtain the last inequality. It follows from (5.22) and (5.24) that

$$\varepsilon \leq C|Tx_j - Tz| \leq C\left(\frac{t_j}{t_i}\right)^{\varepsilon_1} \leq C2^{(e_i - e_j)\varepsilon_1}$$

giving (5.20).

Case 3. $\theta_*t_i \geq \bar{h}(z)$ and $\theta_*t_j > \bar{h}(z)/2$. We can assume that $\Omega \subset \mathbb{R}_+^n$, $\partial S(z, \bar{h}(z))$ is tangent to $\partial\Omega$ at 0, and $\phi(0) = \nabla\phi(0) = 0$. Then Proposition 2.3 gives

$$S(z, \theta_*t_j) \subset S(0, \bar{c}\theta_*t_j) \quad \text{and} \quad S(z, \theta_*t_i) \subset S(0, \bar{c}\theta_*t_i).$$

Let C_0, h, ϕ_h and T be defined as in Case 2. By Lemma 3.3(a), ϕ_h also satisfies the hypotheses of the Localization Theorem 3.1 in $S_{\phi_h}(0, 1)$. By this theorem and (3.9), we have

$$T(S(z, \theta_* t_j)) \subset T(S(0, C_0 t_j)) = S_{\phi_h} \left(0, \frac{t_j}{t_i} \right) \subset B \left(0, K \left(\frac{t_j}{t_i} \right)^{\frac{1}{4}} \right),$$

as we can assume $t_j/t_i < k$ to prove (5.20). It follows that

$$|Tx_j - Tz| \leq 2K \left(\frac{t_j}{t_i} \right)^{\frac{1}{4}}.$$

We infer from this and the same estimate as (5.24) that

$$\varepsilon \leq C \left(\frac{t_j}{t_i} \right)^{\frac{1}{4}} \leq C 2^{\frac{1}{4}(e_i - e_j)}$$

giving (5.20).

Finally, we remark that when z in (5.19) is a boundary point of Ω , say $0 \in \partial\Omega$ where $\Omega \subset \mathbb{R}_+^n$, then we also obtain (5.20) exactly as in Case 3 of the interior points. Thus the proof of the lemma is complete. \square

With the help of Lemma 2.5, we are able to give the proof of the covering theorem.

Proof of Theorem 2.6. Let $0 < \mu < 1/2$ be arbitrary. By applying Lemma 2.5 to the family $\mathcal{F} := \{S(x, t_x)\}_{x \in \mathcal{O}}$, there exists a countable subfamily, denoted by $\{S(x_k, t_k)\}_{k=1}^\infty$, such that $\mathcal{O} \subset \bigcup_{k=1}^\infty S(x_k, t_k)$ and

$$\sum_{k=1}^\infty \chi_{S(x_k, (1-\mu)t_k)}(x) \leq K \log \frac{1}{\mu}. \tag{5.25}$$

Let us write S_k for $S(x_k, t_k)$ and S_k^μ for $S(x_k, (1 - \mu)t_k)$. Then we have

$$|\mathcal{O}| = \left| \mathcal{O} \cap \bigcup_{k=1}^\infty S_k \right| = \lim_{N \rightarrow \infty} \left| \mathcal{O} \cap \bigcup_{k=1}^N S_k \right| \leq \limsup_{N \rightarrow \infty} \sum_{k=1}^N |\mathcal{O} \cap S_k| = \varepsilon \limsup_{N \rightarrow \infty} \sum_{k=1}^N |S_k|.$$

Moreover, by the doubling property in Lemma 6.1(ii), we get

$$|S_k| \leq C \left| S \left(x_k, \frac{t_k}{2} \right) \right| \leq C |S(x_k, (1 - \mu)t_k)| = C |S_k^\mu|.$$

Therefore,

$$|\mathcal{O}| \leq C\varepsilon \limsup_{N \rightarrow \infty} \sum_{k=1}^N |S_k^\mu|. \tag{5.26}$$

Next let $n_N^\mu(x)$ be the overlapping function for the family $\{S_k^\mu\}_{k=1}^N$ as in the proof of [14, Theorem 6.3.3], that is,

$$n_N^\mu(x) := \begin{cases} \#\{k: x \in S_k^\mu\} & \text{if } x \in \bigcup_{k=1}^N S_k^\mu, \\ 1 & \text{if } x \notin \bigcup_{k=1}^N S_k^\mu. \end{cases}$$

Then

$$\chi_{\bigcup_{k=1}^N S_k^\mu}(x) = \frac{1}{n_N^\mu(x)} \sum_{k=1}^N \chi_{S_k^\mu}(x)$$

and $n_N^\mu(x) \leq K \log \frac{1}{\mu}$ by (5.25), and hence

$$\begin{aligned} \sum_{k=1}^N |S_k^\mu| &= \int_{\Omega} n_N^\mu(x) \frac{1}{n_N^\mu(x)} \sum_{k=1}^N \chi_{S_k^\mu}(x) dx \leq K \log \frac{1}{\mu} \int_{\Omega} \frac{1}{n_N^\mu(x)} \sum_{k=1}^N \chi_{S_k^\mu}(x) dx \\ &= K \log \frac{1}{\mu} \int_{\Omega} \chi_{\bigcup_{k=1}^N S_k^\mu}(x) dx = K \log \frac{1}{\mu} \left| \bigcup_{k=1}^N S_k^\mu \right|. \end{aligned}$$

We infer from this and (5.26) that

$$|\mathcal{O}| \leq CK\varepsilon \log \frac{1}{\mu} \left| \bigcup_{k=1}^{\infty} S_k^\mu \right| \quad \text{for all } 0 < \mu < 1/2.$$

By choosing $\mu > 0$ such that $\log \frac{1}{\mu} = 1/(CK\sqrt{\varepsilon})$, we obtain (ii) as desired. \square

We end this section by establishing some global estimates for the maximal function with respect to sections. The proof is based on the covering lemma (Lemma 2.5) and the standard method.

Proof of Theorem 2.7. Let $A_\beta := \{x \in \Omega: \mathcal{M}(f)(x) > \beta\}$ and M be the constant in Lemma 3.4. By Lemma 3.4, we have

$$\frac{1}{|S_\phi(x, t)|} \int_{S_\phi(x, t)} |f(y)| dy = \frac{1}{|\Omega|} \int_{\Omega} |f(y)| dy, \quad \forall t \geq M,$$

which implies that

$$\mathcal{M}(f)(x) = \sup_{t \leq M} \frac{1}{|S_\phi(x, t)|} \int_{S_\phi(x, t)} |f(y)| dy, \quad \forall x \in \Omega.$$

Therefore for each $x \in A_\beta$, we can find $t_x \leq M$ satisfying

$$\frac{1}{|S_\phi(x, t_x)|} \int_{S_\phi(x, t_x)} |f(y)| dy \geq \beta.$$

Consider the family $\{S_\phi(x, 2t_x)\}$. Then by Lemma 2.5, there exists a countable subfamily $\{S_\phi(x_k, 2t_k)\}_k$ such that $A_\beta \subset \bigcup_k S_\phi(x_k, 2t_k)$ and $\sum_k \chi_{S_\phi(x_k, (1-\epsilon)2t_k)}(x) \leq C \log \frac{1}{\epsilon}$ for every $0 < \epsilon < 1/2$. In particular,

$$|A_\beta| \leq \sum_k |S_\phi(x_k, 2t_k)| \leq C \sum_k |S_\phi(x_k, t_k)| \leq C \sum_k |S_\phi(x_k, (1-\epsilon)2t_k)|$$

noting that $|S_\phi(x_k, 2t_k)| \leq C |S_\phi(x_k, t_k)|$ by the doubling property in Lemma 6.1(ii). But as

$$\beta \leq \frac{1}{|S_\phi(x_k, t_k)|} \int_{S_\phi(x_k, t_k)} |f(y)| dy \leq \frac{C}{|S_\phi(x_k, (1-\epsilon)2t_k)|} \int_{S_\phi(x_k, (1-\epsilon)2t_k)} |f(y)| dy,$$

we conclude that

$$\begin{aligned} |A_\beta| &\leq \frac{C}{\beta} \sum_k \int_{S_\phi(x_k, (1-\epsilon)2t_k)} |f(y)| dy = \frac{C}{\beta} \sum_k \int_\Omega \chi_{S_\phi(x_k, (1-\epsilon)2t_k)}(y) |f(y)| dy \\ &= \frac{C}{\beta} \int_\Omega \sum_k \chi_{S_\phi(x_k, (1-\epsilon)2t_k)}(y) |f(y)| dy \leq \frac{C \log \frac{1}{\epsilon}}{\beta} \int_\Omega |f(y)| dy. \end{aligned}$$

Thus we have proved the weak type 1–1 estimate in (i). This together with the obvious inequality $\|\mathcal{M}(f)\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)}$ and the Marcinkiewicz interpolation lemma (see Theorem 5 in [25, p. 21]) yields the strong type p – p estimate in (ii). Alternatively, (ii) can be obtained by using the same arguments as in the proof of [29, Theorem 2.8.2]. \square

6. Quasi-distance and space of homogeneous type

In this section we will introduce a quasi-distance d induced by sections of solutions ϕ to the Monge–Ampère equation in Ω . Moreover, we show that $(\bar{\Omega}, d, \mu)$ is a space of homogeneous type, where $\mu := \det D^2\phi dx$ is the Monge–Ampère measure. We begin with the following simple lemma.

Lemma 6.1. *Assume that the convex domain Ω and the convex function ϕ satisfy (2.3)–(2.5). For all $x \in \bar{\Omega}$ and $t > 0$, we have*

- (i) *if $y \in S(x, t)$, then $S(y, t) \subset S(x, \theta_*^2 t)$;*
- (ii) *$|S(x, 2t)| \leq C |S(x, t)|$.*

Here θ_* is the engulfing constant and C depends only on ρ, λ, Λ and n .

Proof. If $y \in S(x, t)$, then $x \in S(x, t) \subset S(y, \theta_* t)$ by Theorem 2.1. By applying again the engulfing property, we obtain $S(y, \theta_* t) \subset S(x, \theta_*^2 t)$ which gives (i).

To prove the doubling property (ii), let c_0 be the universal constant in Corollary 2.4. If $2t \leq c_0$, then (ii) follows from the volume growth given by Corollary 2.4. Now assume $2t > c_0$. Then we have

$$|S(x, 2t)| \leq |\Omega| \leq |B_{\frac{1}{\rho}}| = CC_1 \left(\frac{c_0}{2}\right)^{\frac{n}{2}} \leq C \left|S\left(x, \frac{c_0}{2}\right)\right| \leq C|S(x, t)|,$$

where the third inequality is by Corollary 2.4. The proof is thus complete. \square

Let $\Omega \subset \mathbb{R}^n$ be a convex set and $\phi \in C(\overline{\Omega})$ be a convex function. We define a function $d : \overline{\Omega} \times \overline{\Omega} \rightarrow [0, \infty)$ by

$$d(x, y) := \inf\{r > 0: x \in S_{\phi}(y, r) \text{ and } y \in S_{\phi}(x, r)\}, \quad \forall x, y \in \overline{\Omega}. \tag{6.27}$$

Also the induced d -ball with center $x \in \overline{\Omega}$ and radius $r > 0$ is given by

$$B_d(x, r) := \{y \in \overline{\Omega}: d(x, y) < r\}.$$

The next result is the boundary version of that in [1, Section 3] where interior sections are considered.

Theorem 6.2. *Assume that the convex domain Ω and the convex function ϕ satisfy (2.3)–(2.5). Let $d : \overline{\Omega} \times \overline{\Omega} \rightarrow [0, \infty)$ be defined by (6.27). Then the function d satisfies*

- (i) $d(x, y) = d(y, x)$ for all $x, y \in \overline{\Omega}$;
- (ii) $d(x, y) = 0$ if and only if $x = y$;
- (iii) $d(x, y) \leq \theta_*^2[d(x, z) + d(z, y)]$ for all $x, y, z \in \overline{\Omega}$.

In addition, we have

$$S_{\phi}\left(x, \frac{r}{2\theta_*^2}\right) \subset B_d(x, r) \subset S_{\phi}(x, r) \tag{6.28}$$

for all $x \in \overline{\Omega}$ and $r > 0$. Here $\theta_* > 1$ is the engulfing constant given by Theorem 2.1.

Proof. The theorem follows from Lemmas 1 and 2 in [1] with $K := \theta_*^2$ provided that the following four conditions are satisfied:

- (a) $\bigcap_{r>0} S_{\phi}(x, r) = \{x\}$ for every $x \in \overline{\Omega}$;
- (b) $\bigcup_{r>0} S_{\phi}(x, r) = \overline{\Omega}$ for every $x \in \overline{\Omega}$;
- (c) for each $x \in \overline{\Omega}$, the map $r \mapsto S_{\phi}(x, r)$ is nondecreasing in r ;
- (d) for any $y \in S_{\phi}(x, r)$, we have $S_{\phi}(x, r) \subset S_{\phi}(y, \theta_*^2 r)$ and $S_{\phi}(y, r) \subset S_{\phi}(x, \theta_*^2 r)$.

Observe that (b) holds by Lemma 3.4 and (c) is obvious. On the other hand, property (d) is a consequence of Theorem 2.1 and Lemma 6.1(i).

To verify (a), it suffices to show that $\bigcap_{r>0} S_{\phi}(x, r) \subset \{x\}$. First, we consider the case x is a boundary point of Ω . Then, by (3.9), we have

$$\bigcap_{r>0} S_{\phi}(x, r) \subset \bigcap_{r>0} B(x, Cr^{1/4}) = \{x\}.$$

Now, consider the case x is an interior point of Ω . Let $\tilde{x} \in \bigcap_{r>0} S_\phi(x, r)$. Then $\phi(\tilde{x}) < \phi(x) + \nabla\phi(x) \cdot (\tilde{x} - x) + r$ for every $r > 0$. It follows that

$$\phi(\tilde{x}) = \phi(x) + \nabla\phi(x) \cdot (\tilde{x} - x),$$

that is, the supporting hyperplane $z = \phi(x) + \nabla\phi(x) \cdot (y - x)$ touches the graph of ϕ at both x and \tilde{x} . Since Ω and ϕ satisfy (2.3)–(2.5), ϕ is $C^{1,\alpha}$ on the boundary $\partial\Omega$ for all $\alpha \in (0, 1)$ as observed in [21, Lemma 4.1]. In fact, we have for all $x_0 \in \partial\Omega$ and for all x in $\overline{\Omega}$ close to x_0 ,

$$|\phi(x) - \phi(x_0) - \nabla\phi(x_0) \cdot (x - x_0)| \leq C|x - x_0|^2(\log|x - x_0|)^2.$$

Consequently, by Caffarelli's Localization Theorem [2], we know that ϕ is strictly convex in Ω . Therefore, we infer that $\tilde{x} = x$ and so property (a) is proved. \square

It follows from properties (i)–(iii) in Theorem 6.2 that d is a quasi-distance on $\overline{\Omega}$ and $(\overline{\Omega}, d)$ is a quasi-metric space. Moreover, as a consequence of Lemma 6.1(ii) and (6.28) we obtain the following doubling property for d -balls:

$$|B_d(x, 2r)| \leq |S(x, 2r)| \leq C \left| S\left(x, \frac{r}{2\theta_*^2}\right) \right| \leq C|B_d(x, r)| \quad \text{for all } x \in \overline{\Omega} \text{ and } r > 0,$$

where C depends only on ρ, λ, Λ and n . Thus, $(\overline{\Omega}, d, |\cdot|)$ is a doubling quasi-metric space and hence it is a space of homogeneous type; see [8, Remark on p. 67]. We refer readers to [8,9] for some results and analysis on this type of spaces.

Acknowledgment

The authors would like to thank the referee for carefully reading the paper and for useful comments which improve the presentation of the manuscript.

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